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Research article

On classical and sequential conformable fractional boundary value problems: new results via alternative fixed point method

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Abstract: This paper investigates the existence and uniqueness of solutions for two types of conformable fractional boundary value problems: a classical CFBVP of order $\gamma \in (\frac{3}{2}, 2]$, and a sequential conformable fractional boundary value problem (SCFBVP) of order $\delta \in (\frac{1}{2}, 1]$. By establishing new integral bounds for the Green's functions associated with both problems, we extend the results obtained by Z. Laadjal et al. (*Numerical Methods for Partial Differential Equations*, **40** (2024), e22760) by applying Rus's fixed point theorems. Furthermore, we establish an existence and uniqueness theorem for the SCFBVP based on the Banach fixed point theorem, which complements their findings. Our results improve their work by relaxing key assumptions and broadening applicability. Finally, we present a detailed numerical comparison between the results herein and the existing results, highlighting the advantages of our approach, followed by concluding remarks.

Keywords: conformable fractional calculus; fractional derivatives; boundary value problems; nonlinear equations; Rus's fixed point theorem; existence and uniqueness of solutions; hypergeometric function; numerical approximation; partial differential equations
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1. Introduction

Recently, Z. Laadjal, T. Abdeljawad, and F. Jarad [20] considered the following conformable fractional boundary value problem (CFBVP),

$$T_{\gamma}^{a}u(t) = -\mathrm{g}(t, u(t)); \quad a < t < b, \quad \gamma \in (1, 2], \tag{1.1}$$

where: T_{γ}^{a} represents the conformable fractional derivative of order γ, and $\mathrm{g} : [a, b]\times\mathbb{R} \to \mathbb{R}$ is assumed to be a continuous function. In their results, the CFBVP (1.1) is subjected to the following boundary conditions:

$$u(a) = M_1, \ \ u(b) = M_2; \ \ M_1, M_2 \in \mathbb{R}. \tag{1.2}$$

The authors of [20] established the existence and uniqueness results for the CFBVP (1.1) (1.2) using Banach's contraction mapping principle [10] and analyzed properties of the associated Green's function. They then used these properties to obtain a sharp lower bound for the eigenvalue problem. Furthermore, they also posed the following sequential conformable fractional boundary value problem (SCFBVP) as an open problem:

$$T^a_\delta T^a_\delta u(t) = -\mathfrak{h}(t, u(t)); \quad a < t < b, \quad \delta \in (1/2, 1], \tag{1.3}$$

subjected to the boundary condition (1.2).

Observe that when $\gamma = 2$, and $\delta = 1$, and $\mathfrak{g} = \mathfrak{h} := \mathfrak{f}$, then the CFBVP (1.1) coincides with the SCFBVP (1.3), which leads to the following classical two-point boundary value problems (BVPs):

$$u''(t) = -\mathfrak{f}(t, u(t)); \quad a < t < b. \tag{1.4}$$

Consequently, the results of [20] reduced to the result of Peterson and Kelley [16, Theorem 7.7] in which the existence and uniqueness results for BVPs (1.4), and (1.2) was also obtained via the use of Banach's contraction mapping principle [10].

Moreover, the earlier works by S. Almuthaybiri and C. Tisdell [4, 5] introduced Rus's fixed point theorem [23] in the study of boundary value problems. The approach developed therein offers a more suitable framework than classical tools like Banach's theorem [10] by relaxing key assumptions on the Lipschitz constants while ensuring existence and uniqueness of solutions. Since then, Rus's theorem [23] has gained increasing attention in the study of BVPs, and see for example, [6, 18, 21, 25], including the recent results involving Caputo and Riemann–Liouville derivatives [3, 7].

Motivated by these advantages, in this work, we focus on the conformable fractional derivative due to its growing importance in the study of fractional differential equations. Its structure allows for simpler formulation of initial and boundary value problems while preserving many essential properties of classical derivatives. This makes it particularly suitable for analytical techniques involving fixed-point theorems. Therefore, we apply Rus's fixed point theorem to the conformable fractional case—to the best of our knowledge, this is the first time Rus's theorem has been applied in this context. Our goal is to demonstrate the analytical advantages of this approach and to contribute to the development of solution methods for conformable fractional boundary value problems.

In particular, we consider both CFBVP (1.1) with $\gamma \in (\frac{3}{2}, 2]$ and SCFBVP (1.2). The central objective is to advance the results established in [20], particularly extending the existence and uniqueness results of [20] to a wider class of problems. We achieve this improvement in two key ways:

■ **(1):** Deriving new bounds for certain integrals involving the Green's functions corresponding to both CFBVP (1.1), (1.2), and SCFBVP (1.3), (1.2).

■ **(2):** These new bounds are then used to establish new existence and uniqueness results for both CFBVP (1.1), (1.2), and SCFBVP (1.3), (1.2). This is achieved by using Rus's fixed point theorem [23].

Let us now introduce the necessary definitions, useful results, and metrics to build a framework for interpreting both the above and the forthcoming content of this article.

Definition 1.1. [1] *Let $\gamma \in (n, n+1]$ and $\delta = \gamma - n$. Then, the (left) conformable fractional derivative starting from a of a function $\omega : [a, \infty) \to \mathbb{R}$ of order γ, $\omega^{(n)}(t)$ exists, is defined by*

$$(T_\gamma^a \omega)(t) = \lim_{\epsilon \to 0} \frac{\omega^{(n)}(t + \epsilon(t-a)^{1-\gamma}) - \omega^{(n)}(t)}{\epsilon}. \tag{1.5}$$

For the special case where $\gamma \in (0, 1]$, the conformable derivative is given by:

$$(T_\gamma^a \omega)(t) = \lim_{\epsilon \to 0} \frac{\omega(t + \epsilon(t-a)^{1-\gamma}) - \omega(t)}{\epsilon}. \tag{1.6}$$

Moreover, if $\omega^{(2)}(t)$ exists on (a, ∞), and $1/2 < \gamma \leq 1$. Then,

$$(T_\gamma^a T_\gamma^a \omega)(t) = T_{2\gamma}^a \omega(t) + (1-\gamma)(t-a)^{-\gamma} T_\gamma^a \omega(t).$$

Note that if we let $\gamma \to 1$, then we have $T_\gamma^a T_\gamma^a \omega(t) = T_2^a \omega(t) = \omega''(t)$.

The following is the definition for the conformable fractional integrals.

Definition 1.2. [1] *The (left) conformable fractional integral starting from a of a function $\omega : [a, \infty) \to \mathbb{R}$ of order $0 < \gamma \leq 1$ is defined by*

$$(\mathcal{I}_a^\gamma \omega)(t) = \int_a^t (s-a)^{\gamma-1} \omega(s)\, ds. \tag{1.7}$$

If γ and δ are as in Definition 1.1, then the general conformable fractional integral is given by

$$(\mathcal{I}_a^\gamma \omega)(t) = {}^R I_a^{n+1}[(t-a)^{\delta-1}\omega(t)] = \frac{1}{n!} \int_a^t (t-s)^n (s-a)^{\delta-1} \omega(s)\, ds. \tag{1.8}$$

where ${}^R I_a^{n+1}$ is the Riemann–Liouville fractional integral of order $n+1$.

Definition 1.3. [19] *We denote by $\mathcal{R}(z)$ the real part of z. The Riemann–Liouville fractional integral ${}^R I_a^\gamma \omega$ of order γ is defined by*

$${}^R I_a^\gamma \omega(t) = \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} \omega(s) ds, \ \forall\ t > a \text{ and } \mathcal{R}(\gamma) > 0. \tag{1.9}$$

For further details, the reader is referred to [1, 17, 19], which provide the foundational definitions and basic results related to conformable fractional calculus.

The following is the definition for the Euler gamma function.

Definition 1.4. [19] *We denote by $\mathbb{C}$ the set of complex numbers. The Euler gamma function $\Gamma(z)$ is defined by*

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx, \ \ \forall\, z \in \mathbb{C} \,|\, \mathcal{R}(z) > 0. \tag{1.10}$$

The following is the definition for the beta function.

Definition 1.5. [19] *The beta function is defined by the Euler integral of the first kind:*

$$B(z, w) = \int_0^1 x^{z-1} (1-x)^{w-1} dx, \tag{1.11}$$

$$(\mathcal{R}(z) > 0, \mathcal{R}(w) > 0).$$

The relation between the *beta* function and *gamma* function is given by

$$B(z,w)=\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \tag{1.12}$$

$$(z,w\notin\mathbb{Z}_0^-:=-\mathbb{N}=\{\dots,-n,\dots,-1,0\}).$$

The following is the definition of the Gauss hypergeometric function. As we shall see, applying Rus's fixed point theorem to the CFBVP (1.1) leads to the emergence of the Gauss hypergeometric function as a natural consequence, illustrating a fundamental relationship between the analysis of a fractional boundary value problem and the structure of hypergeometric functions, see [3, 7].

Definition 1.6. [19] *The Gauss hypergeometric function $_2F_1$ is defined in the unit disk as the sum of the hypergeometric series as follows:*

$$ {}_2F_1\left(\begin{matrix} q_1,q_2 \\ q_3\end{matrix};z\right)=\sum_{k=0}^{\infty}\frac{(q_1)_k(q_2)_k}{(q_3)_k}\frac{z^k}{k!}, \tag{1.13}$$

$$(q_1,q_2\in\mathbb{C};q_3\in\mathbb{C}\setminus\mathbb{Z}_0^-;\quad z\in\overline{B}:=\{z\in\mathbb{C}\mid |z|\leq 1\};\ \mathcal{R}(q_3-q_2-q_1)>0),$$

where $(z)_k$ is the *Pochhammer symbol*, defined for $z\in\mathbb{C}$ and $k\in\mathbb{N}^*:=\mathbb{N}\setminus\{0\}$ by

$$(z)_0=1 \text{ and } (z)_k=z(z+1)...(z+k-1),\ \ \forall\, k\in\mathbb{N}^*. \tag{1.14}$$

Moreover, when $q_1=-m$, $m=0,1,2,...$, and $q_3\neq 0,-1,-2,...$, the $_2F_1$ is a polynomial: that is,

$$ {}_2F_1\left(\begin{matrix} -m,q_2 \\ q_3\end{matrix};z\right)=\sum_{k=0}^{m}\frac{(-m)_k(q_2)_k}{(q_3)_k k!}z^k. \tag{1.15}$$

Rus's fixed point theorem [23] is now introduced, as it plays a central role in our analysis and in extending the results of [20].

Theorem 1.1. *Rus [23] Let χ be a nonempty set, and let d and $\varkappa$ be two metrics on χ such that (χ,d) forms a complete metric space. If the following conditions hold,*

(A_1) The mapping $\mathcal{T}:\chi\to\chi$ is continuous with respect to (d,χ),
(A_2) $d(\mathcal{T}u,\mathcal{T}v)\leq c\varkappa(u,v)$, for some $c>0$ and all $u,v\in\chi$,
(A_3) $\varkappa(\mathcal{T}u,\mathcal{T}v)\leq c_1\varkappa(u,v)$, for some $0<c_1<1$ and all $u,v\in\chi$,

then there is a unique $z\in\chi$ such that $\mathcal{T}z=z$.

Given that Rus's fixed point theorem involves two possibly non-equivalent metrics, with completeness assumed only for the first, we select these two metrics as follows:

We consider the space $\chi:=C([a,b])$. For functions $x,u\in\chi$ define

$$d(u,v):=\max_{t\in[a,b]}|u(t)-v(t)|, \tag{1.16}$$

and

$$\varkappa(u,v):=\left(\int_a^b|u(t)-v(t)|^2\,dt\right)^{1/2}, \tag{1.17}$$

then, (χ, d) is complete, but $(\chi, \varkappa)$ is not. Moreover, the following is a well-known important relationship between these two metrics on χ,

$$\varkappa(u, v) \leq (b-a)^{1/2} d(u, v), \quad \text{for all } u, v \in \chi. \tag{1.18}$$

Remark 1.1. *The results in [3, 4, 6, 7, 26] provide comprehensive details on Rus's theorem and the completeness assumptions for the dual metrics d and ϰ, and we refer interested readers to those works for further background.*

We now outline the structure of the remainder of the paper.

In Section 2, we develop integral estimates for Green's functions, divided into two parts: Subsection 2.1 addresses the CFBVP (1.1), (1.2) for $\gamma \in (\frac{3}{2}, 2]$, while Subsection 2.2 focuses on the SCFBVP (1.3), (1.2).

The estimates derived in Subsection 2.1 are then applied in Section 3 to establish our first main result: the existence and uniqueness of solutions to the CFBVP (1.1), (1.2) when $\gamma \in (\frac{3}{2}, 2]$ via Rus's fixed point theorem. In Section 4, a similar approach is employed, using the estimates from Subsection 2.2 to prove the existence and uniqueness results for the SCFBVP (1.3), (1.2). Additionally, we establish the existence and uniqueness results for the SCFBVP (1.3), (1.2) via the Banach fixed point theorem. In Section 5, we provide a detailed comparison between our main results obtained in Sections 3 and 4 and the results of [20]. We show that our findings significantly relax the assumptions required to establish the existence and uniqueness for both CFBVP (1.1), (1.2) and SCFBVP (1.3), (1.2). In Section 6, we conclude with some remarks, including the formulation of an open problem that naturally arises from the present study.

2. Estimates for some fractional integrals of Green's functions

In this section, we establish our first new bounds on the integrals of the Green's functions corresponding to both CFBVP (1.1), (1.2), and SCFBVP (1.3), (1.2). These bounds will play a key role in developing our main results in both Sections 3 and 4.

2.1. *Estimates for Green's function of* (1.1) *and* (1.2)

This subsection is devoted to deriving properties of the Green's function associated with the CFBVP (1.1) and (1.2).

Lemma 2.1. *[20, Lemma 7] Suppose that* $\mathfrak{g}$ *is a continuous function. A function* $u \in C[a, b]$ *is a solution of CFBVP* (1.1), (1.2) *if and only if u satisfies the integral equation*

$$u(t) = \xi(t) + \int_a^b \mathcal{G}_1(t, s)\mathfrak{g}(s, u(s))\, ds; \quad t \in [a, b], \tag{2.1}$$

where $\mathcal{G}_1$ is given explicitly by

$$\mathcal{G}_1(t, s) = \begin{cases} \frac{(t-a)(b-s)(s-a)^{\gamma-2}}{(b-a)} - (t-s)(s-a)^{\gamma-2}, & \text{for } a \leq s \leq t \leq b, \\ \frac{(t-a)(b-s)(s-a)^{\gamma-2}}{(b-a)}, & \text{for } a \leq t \leq s \leq b, \end{cases} \tag{2.2}$$

and ξ is defined by

$$\xi(t) := M_1 + \frac{(M_2 - M_1)(t-a)}{(b-a)}. \tag{2.3}$$

To avoid repetition of some expressions, we define

$$K_1(t,s,\gamma) := \frac{(t-a)^2(b-s)^2(s-a)^{2\gamma-4}}{(b-a)^2}, \quad K_2(t,s,\gamma) := (t-a)^2(s-a)^{2\gamma-4},$$

$$K_3(t,s,\gamma) := \frac{2(t-a)(t-s)(b-s)(s-a)^{2\gamma-4}}{(b-a)}.$$

We now establish the following useful result on the fractional integral of some functions, which will be used to derive the properties of the Green's function for the CFBVP (1.1), (1.2).

Lemma 2.2. *For all $t \in (a,b)$, where $a,b \in \mathbb{R}$ and $a < b$ and $\gamma \in (\frac{3}{2}, 2]$, we have*

$$\textbf{(i)}.\ J_1^\gamma(t) := \int_a^b K_1(t,s,\gamma)\,ds = \frac{2\Gamma(2\gamma-3)(b-a)^{2\gamma-3}}{\Gamma(2\gamma)}(t-a)^2, \tag{2.4}$$

$$\textbf{(ii)}.\ J_2^\gamma(t) := \int_a^t K_2(t,s,\gamma)\,ds = \frac{2\Gamma(2\gamma-3)}{\Gamma(2\gamma)}(t-a)^{2\gamma-1}, \tag{2.5}$$

$$\textbf{(iii)}.\ J_3^\gamma(t) := \int_a^t K_3(t,s,\gamma)\,ds = \left[\frac{2\Gamma(2\gamma-3)}{\Gamma(2\gamma-1)}\right](t-a)^{2\gamma-1}{}_2F_1\left(\begin{matrix}-1, 2\gamma-3\\ 2\gamma-1\end{matrix}; h(t)\right), \tag{2.6}$$

where $h(t) := \frac{t-a}{b-a}$.

$$\textbf{(iv)}.\ \int_a^b J_3^\gamma(t)dt = \left[\frac{(b-a)^{2\gamma}\Gamma(2\gamma-3)(6\gamma-1))}{\gamma\Gamma(2\gamma)(2\gamma+1)}\right]. \tag{2.7}$$

Proof: **(i).** Let $\gamma \in (\frac{3}{2}, 2]$ and by changing the integration variable from s to x with $s = a + x(b-a)$, we have

$$\begin{aligned} J_1^\gamma(t) &= \frac{(t-a)^2}{(b-a)^2}\int_0^1 (b-a-x(b-a))^2(b-a)^{2\gamma-4}x^{2\gamma-4}(b-a)dx \\ &= (b-a)^{2\gamma-3}(t-a)^2\int_0^1 x^{2\gamma-4}(1-x)^2dx. \end{aligned}$$

Above, we have the *beta function* with $z = 2\gamma-3$ and $w = 3$; that is,

$$J_1^\gamma(t) = (b-a)^{2\gamma-3}(t-a)^2 B(2\gamma-3, 3).$$

Thus, we obtained the result of **(i)**.

(ii). For the proof, we refer the reader to the formula in [24, (2.44), p. 40], as our case (2.5) is a special

case of that formula.
(iii). For the proof, we also refer the reader to the formula in [24, (2.46), p. 41], as our case (2.6) is a special case of that formula. We also refer to [27] for more details regarding the Gauss hypergeometric functions ${}_2F_1$ and ${}_3F_2$.
(iv). To prove (2.7), we denote the left-hand side of (2.7) by Λ_1, that is,

$$\Lambda_1 := \left[\frac{2\Gamma(2\gamma-3)}{\Gamma(2\gamma-1)}\right]\int_a^b (t-a)^{2\gamma-1}{}_2F_1\left(\begin{matrix}-1, 2\gamma-3\\ 2\gamma-1\end{matrix}\ ; h(t)\right) dt. \tag{2.8}$$

Taking into account (1.15), we express ${}_2F_1$ as a series, and [27, Lemma 2.2] can be used to justify changing the order of integration and summation, we have

$$\begin{aligned}
\Lambda_1 &:= \left[\frac{2\Gamma(2\gamma-3)}{\Gamma(2\gamma-1)}\right]\sum_{k=0}^{1}\frac{(-1)_k(2\gamma-3)_k}{(2\gamma-1)_k k!}\int_a^b (t-a)^{2\gamma-1}\left(\frac{t-a}{b-a}\right)^k dt\\
&= \left[\frac{2\Gamma(2\gamma-3)}{\Gamma(2\gamma-1)}\right]\sum_{k=0}^{1}\frac{(-1)_k(2\gamma-3)_k}{(2\gamma-1)_k k!}\left(\frac{1}{b-a}\right)^k\int_a^b (t-a)^{k+2\gamma-1}\, dt\\
&= (b-a)^{2\gamma}\left[\frac{2\Gamma(2\gamma-3)}{\Gamma(2\gamma-1)}\right]\sum_{k=0}^{1}\frac{(-1)_k(2\gamma-3)_k}{(2\gamma-1)_k k!}\frac{1}{(k+2\gamma)}\\
&= b-a)^{2\gamma}\left[\frac{2\Gamma(2\gamma-3)}{\Gamma(2\gamma-1)}\right]\left(\frac{1}{2\gamma}-\frac{2\gamma-3}{(2\gamma-1)(2\gamma+1)}\right)\\
&= \left[\frac{(b-a)^{2\gamma}\Gamma(2\gamma-3)(6\gamma-1))}{\gamma\Gamma(2\gamma)(2\gamma+1)}\right].
\end{aligned}$$

This completes the proof. □

We now state and prove the new estimates for the integral of the Green's function for the CFBVP (1.1), (1.2).

Proposition 2.1. *The function* $\mathcal{G}_1(t,s)$ *in* (2.2) *satisfies the following properties:*

$$(\mathbf{B_1}).\ \max_{t\in[a,b]}\left(\int_a^b |\mathcal{G}_1(t,s)|\, ds\right) = \frac{(b-a)^\gamma}{\gamma^{\frac{\gamma}{\gamma-1}+1}},\quad \text{for all } t\in[a,b],\quad 1<\gamma\le 2, \tag{2.9}$$

$$(\mathbf{B_2}).\ \max_{t\in[a,b]}\left(\int_a^b |\mathcal{G}_1(t,s)|^2\, ds\right) \le \left(\frac{4(b-a)^{2\gamma-1}}{\Gamma(2\gamma)}\right),\quad \text{for all } t\in[a,b],\quad \gamma\in(\frac{3}{2},2], \tag{2.10}$$

$$(\mathbf{B_3}).\ \int_a^b\left(\int_a^b |\mathcal{G}_1(t,s)|^2\, ds\right)dt = \mathcal{H}_1(a,b,\gamma),\quad \text{for all } t\in[a,b],\quad \gamma\in(\frac{3}{2},2], \tag{2.11}$$

where

$$\boxed{\mathcal{H}_1(a,b,\gamma) := (b-a)^{2\gamma}\left[\frac{2\Gamma(2\gamma-3)}{3\Gamma(2\gamma)}+\frac{\Gamma(2\gamma-3)}{\gamma\Gamma(2\gamma)}-\frac{\Gamma(2\gamma-3)(6\gamma-1))}{\gamma\Gamma(2\gamma)(2\gamma+1)}\right]}. \tag{2.12}$$

Proof: **(B_1).** For the proof, we refer to [20, Proposition 8].
(B_2). Let $\gamma\in(\frac{3}{2},2]$. For all $t\in[a,b]$, we have

$$\int_a^b |\mathcal{G}_1(t,s)|^2\, ds = \int_a^t |\mathcal{G}_1(t,s)|^2\, ds + \int_t^b |\mathcal{G}_1(t,s)|^2\, ds$$

$$
\begin{aligned}
&= \int_a^t (K_1(t,s,\gamma) + K_2(t,s,\gamma) - K_3(t,s,\gamma))\,ds + \int_t^b K_1(t,s,\gamma)ds \\
&= \int_a^b K_1(t,s,\gamma)ds + \int_a^t K_2(t,s,\gamma)ds - \int_a^t K_3(t,s,\gamma)ds \\
&= J_1^\gamma(t) + J_2^\gamma(t) - J_3^\gamma(t) \qquad (2.13)\\
&\le J_1^\gamma(b) + J_2^\gamma(b) \\
&= \frac{4(b-a)^{2\gamma-1}}{\Gamma(2\gamma)},
\end{aligned}
$$

where above we applied Lemma 2.2 to obtain (2.13). Thus, we have

$$
\max_{t\in[a,b]} \left(\int_a^b |\mathcal{G}_1(t,s)|^2 \, ds \right) \le \left(\frac{4(b-a)^{2\gamma-1}}{\Gamma(2\gamma)} \right),
$$

which yields the desired result of **(**B_2**)**.
(B_3**).** From (2.13), we have

$$
\begin{aligned}
\int_a^b \left(\int_a^b |\mathcal{G}_1(t,s)|^2 \, ds \right) dt &= \int_a^b \left(J_1^\gamma(t) + J_2^\gamma(t) \right) dt - \int_a^b J_3^\gamma(t)dt \\
&= \frac{2(b-a)^{2\gamma}\Gamma(2\gamma-3)}{3\Gamma(2\gamma)} + \frac{(b-a)^{2\gamma}\Gamma(2\gamma-3)}{\gamma\Gamma(2\gamma)} \\
&\quad - \left[\frac{(b-a)^{2\gamma}\Gamma(2\gamma-3)(6\gamma-1)}{\gamma\Gamma(2\gamma)(2\gamma+1)} \right], \qquad (2.14)
\end{aligned}
$$

above (2.7) was used to obtain (2.14). Thus, we have

$$
\boxed{\int_a^b \left(\int_a^b |\mathcal{G}_1(t,s)|^2 \, ds \right) dt = (b-a)^{2\gamma} \left[\frac{2\Gamma(2\gamma-3)}{3\Gamma(2\gamma)} + \frac{\Gamma(2\gamma-3)}{\gamma\Gamma(2\gamma)} - \frac{\Gamma(2\gamma-3)(6\gamma-1)}{\gamma\Gamma(2\gamma)(2\gamma+1)} \right]},
$$

which is the desired result of **(**B_3**)**. The proof is complete. □

2.2. *Estimates for Green's function of* (1.3) *and* (1.2)

We now turn to the derivation of key properties of the Green's function for the SCFBVP (1.3) and (1.2).

Lemma 2.3. *[2, Lemma 5] Suppose that* $\mathfrak{h}$ *is a continuous function. A function* $u \in C[a,b]$ *is a solution of SCFBVP* (1.3), (1.2) *if and only if* u *satisfies the integral equation*

$$
u(t) = \zeta(t) + \int_a^b \mathcal{G}_2(t,s)\mathfrak{h}(s,u(s))\,ds; \quad t \in [a,b], \qquad (2.15)
$$

where $\mathcal{G}_2$ is given explicitly by

$$
\mathcal{G}_2(t,s) = \begin{cases} \frac{(s-a)^{2\delta-1}}{\delta} - \frac{(t-a)^\delta (s-a)^{2\delta-1}}{\delta(b-a)^\delta}, & \text{for } a \le s \le t \le b, \\ \\ \frac{(t-a)^\delta (s-a)^{\delta-1}}{\delta} - \frac{(t-a)^\delta (s-a)^{2\delta-1}}{\delta(b-a)^\delta}, & \text{for } a \le t \le s \le b, \end{cases} \qquad (2.16)
$$

and ζ is defined by

$$\zeta(t) := M_1 + \frac{(M_2 - M_1)(t-a)^{\delta}}{(b-a)^{\delta}}. \tag{2.17}$$

To avoid repetition of some expressions, we define

$$R_1(s) := \frac{(s-a)^{4\delta-2}}{\delta^2}, \quad R_2(t,s) := \frac{2(s-a)^{4\delta-2}(t-a)^{\delta}}{\delta^2(b-a)^{\delta}}, \quad R_3(t,s) := \frac{(s-a)^{4\delta-2}(t-a)^{2\delta}}{\delta^2(b-a)^{2\delta}},$$

$$R_4(t,s) := \frac{(s-a)^{2\delta-2}(t-a)^{2\delta}}{\delta^2}, \text{ and } R_5(t,s) := \frac{2(s-a)^{3\delta-2}(t-a)^{2\delta}}{\delta^2(b-a)^{\delta}}.$$

We now establish the new estimates for the integral of the Green's function for the SCFBVP (1.3) and (1.2).

Proposition 2.2. *Let* $1/2 < \delta \le 1$; *then the function* $\mathcal{G}_2(t,s)$ *in* (2.16) *satisfies the following properties:*

$$(\mathbf{C_1}).\ \max_{t\in[a,b]}\left(\int_a^b |\mathcal{G}_2(t,s)|\,ds\right) = \frac{(b-a)^{2\delta}}{8\delta^2}, \text{ for all } t\in[a,b], \tag{2.18}$$

$$(\mathbf{C_2}).\ \max_{t\in[a,b]}\left(\int_a^b |\mathcal{G}_2(t,s)|^2\,ds\right) \le \left(\frac{2(b-a)^{4\delta-1}}{\delta(2\delta-1)(4\delta-1)}\right), \text{ for all } t\in[a,b], \tag{2.19}$$

$$(\mathbf{C_3}).\ \int_a^b\left(\int_a^b |\mathcal{G}_2(t,s)|^2\,ds\right)dt = \mathcal{H}_2(a,b,\delta), \text{ for all } t\in[a,b], \tag{2.20}$$

where

$$\boxed{\mathcal{H}_2(a,b,\delta) := \frac{2(b-a)^{4\delta}}{5\delta^2(3\delta-1)(4\delta-1)} + \frac{2(b-a)^{4\delta}}{(2\delta+1)(2\delta-1)(3\delta-1)(4\delta-1)} - \frac{(b-a)^{4\delta}}{2\delta^2(2\delta-1)(4\delta-1)}}. \tag{2.21}$$

Proof" (C_1). The function $\mathcal{G}_2(t,s)$ in (2.16) satisfies $\mathcal{G}_2 \ge 0$ on $[a,b]\times[a,b]$, see [2, Lemma 6]. Thus, we have by direct evaluation

$$\begin{aligned}\int_a^b |\mathcal{G}_2(t,s)|\,ds &= \int_a^t \left[\frac{(s-a)^{2\delta-1}}{\delta} - \frac{(t-a)^{\delta}(s-a)^{2\delta-1}}{\delta(b-a)^{\delta}}\right]ds \\ &\quad + \int_t^b \left[\frac{(t-a)^{\delta}(s-a)^{\delta-1}}{\delta} - \frac{(t-a)^{\delta}(s-a)^{2\delta-1}}{\delta(b-a)^{\delta}}\right]ds \\ &= \frac{1}{2\delta^2}\left[(t-a)^{\delta}(b-a)^{\delta} - (t-a)^{2\delta}\right].\end{aligned}$$

Now, we define

$$\int_a^b |\mathcal{G}_2(t,s)|\,ds = \frac{1}{2\delta^2}\left[(t-a)^{\delta}(b-a)^{\delta} - (t-a)^{2\delta}\right] := f(t),$$

then it is easy to see that the maximum of f on $[a,b]$ is attained when

$$t = a + \frac{b-a}{2^{1/\delta}}.$$

Thus, we have

$$\max_{t\in[a,b]} f(t) = \frac{1}{2\delta^2}\max_{t\in[a,b]}\left[(t-a)^{\delta}(b-a)^{\delta} - (t-a)^{2\delta}\right]$$
$$= \frac{(b-a)^{2\delta}}{8\delta^2}.$$

This completes the proof of (C_1).
(C_2). Let $1/2 < \delta \le 1$, and for all $t \in [a,b]$, we have

$$\begin{aligned}
&\int_a^b |\mathcal{G}_2(t,s)|^2\,ds = \int_a^t |\mathcal{G}_2(t,s)|^2\,ds + \int_t^b |\mathcal{G}_2(t,s)|^2\,ds\\
&= \int_a^t (R_1(s) - R_2(t,s) + R_3(t,s))\,ds + \int_t^b (R_4(t,s) - R_5(t,s) + R_3(t,s))\,ds\\
&= \int_a^t (R_1(s) - R_2(t,s))\,ds + \int_t^b (R_4(t,s) - R_5(t,s))\,ds + \int_a^b R_3(t,s)ds\\
&= \frac{(t-a)^{4\delta-1}}{\delta^2(4\delta-1)} - \frac{2(t-a)^{5\delta-1}}{\delta^2(b-a)^{\delta}(4\delta-1)} + \frac{(b-a)^{2\delta-1}(t-a)^{2\delta}}{\delta^2(2\delta-1)} - \frac{(t-a)^{4\delta-1}}{\delta^2(2\delta-1)}\\
&- \frac{2(b-a)^{2\delta-1}(t-a)^{2\delta}}{\delta^2(3\delta-1)} + \frac{2(t-a)^{5\delta-1}}{\delta^2(b-a)^{\delta}(3\delta-1)} + \frac{(b-a)^{2\delta-1}(t-a)^{2\delta}}{\delta^2(4\delta-1)}\\
&= \frac{2(t-a)^{5\delta-1}}{\delta(b-a)^{\delta}(3\delta-1)(4\delta-1)} + \frac{2(t-a)^{2\delta}(b-a)^{2\delta-1}}{(2\delta-1)(3\delta-1)(4\delta-1)} - \frac{2(t-a)^{4\delta-1}}{\delta(2\delta-1)(4\delta-1)} \qquad (2.22)\\
&\le \frac{2(b-a)^{5\delta-1}}{\delta(b-a)^{\delta}(3\delta-1)(4\delta-1)} + \frac{2(b-a)^{2\delta}(b-a)^{2\delta-1}}{(2\delta-1)(3\delta-1)(4\delta-1)}\\
&= \frac{2(b-a)^{4\delta-1}}{\delta(2\delta-1)(4\delta-1)}.
\end{aligned}$$

This completes the proof of (C_2).
(C_3). From (2.22), we have by direct evaluation

$$\int_a^b \left(\int_a^b |\mathcal{G}_2(t,s)|^2\,ds\right)dt = \mathcal{H}_2(a,b,\delta). \qquad (2.23)$$

The proof is complete. □

Remark 2.1. We remark that if $\gamma = 2$ and $\delta = 1$, then Lemma 2.1 coincides with Lemma 2.3; consequently, Proposition 2.1 precisely coincides with Proposition 2.2.

3. Existence and uniqueness results on (1.1) and (1.2)

This section is devoted to establishing our initial novel result on the existence and uniqueness of solutions to CFBVP (1.1) and (1.2). We prove this through an application of Theorem 1.1.

Theorem 3.1. *Suppose that* $\gamma \in (\frac{3}{2}, 2]$. *Let* $\mathfrak{g} : [a,b]\times\mathbb{R} \to \mathbb{R}$ *be continuous, and let* $K > 0$ *such that*

$$|g(t,u) - g(t,v)| \leq K|u - v|, \quad \text{for all } (t,u),(t,v) \in [a,b] \times \mathbb{R}. \tag{3.1}$$

If

$$K\sqrt{\mathcal{H}_1(a,b,\gamma)} < 1, \tag{3.2}$$

where $\mathcal{H}_1(a,b,\gamma)$ is defined by (2.12), then the CFBVP (1.1) and (1.2) admits a unique solution.
Proof: Based on Lemma 2.1, we define the operator $\mathcal{P}_1 : \chi \to \chi$ by

$$(\mathcal{P}_1 u)(t) := \int_a^b \mathcal{G}_1(t,s) g(s,u(s))\, ds + \xi(t), \quad t \in [a,b].$$

Now, Theorem 1.1 will be used to prove that there is a unique $u \in \chi$ such that $\mathcal{P}_1 u = u$, which is equivalent to proving that the CFBVP (1.1) and (1.2) have a unique solution.

Let us first consider the complete metric space $(\chi, d) = (C([a,b]), d)$ and the metric $(C([a,b]), \varkappa)$.

■ **Stage one:** In this stage, it will be shown that the condition (A_2) of Theorem 1.1 holds. For $u, v \in \chi$ and $t \in [a,b]$, consider

$$\begin{aligned}
|(\mathcal{P}_1 u)(t) - (\mathcal{P}_1 v)(t)| &\leq \int_a^b |\mathcal{G}_1(t,s)|\, |g(s,u(s)) - g(s,v(s))|\, ds \\
&\leq \int_a^b |\mathcal{G}_1(t,s)|\, K\, |u(s) - v(s)|\, ds \\
&\leq \left(\int_a^b |\mathcal{G}_1(t,s)|^2\, ds\right)^{1/2} K \left(\int_a^b |u(s) - v(s)|^2\, ds\right)^{1/2} \qquad (3.3) \\
&\leq K \max_{t\in[a,b]} \left(\int_a^b |\mathcal{G}_1(t,s)|^2\, ds\right)^{1/2} \varkappa(u,v) \\
&\leq K \left(\frac{4(b-a)^{2\gamma-1}}{\Gamma(2\gamma)}\right)^{1/2} \varkappa(u,v). \qquad (3.4)
\end{aligned}$$

Note that by Hölder's inequality [14, 22] and (3.1), we obtained (3.3) and by (2.10), we obtained (3.4). Thus by defining

$$c := K\left(\frac{4(b-a)^{2\gamma-1}}{\Gamma(2\gamma)})\right)^{1/2}, \tag{3.5}$$

we have

$$d(\mathcal{P}_1 u, \mathcal{P}_1 v) \leq c\varkappa(u,v), \quad \text{for some } c > 0 \text{ and all } u, v \in \chi, \tag{3.6}$$

which shows the condition (A_2) of Theorem 1.1 holds.

■ **Stage two:** We show that the condition (A_1) of Theorem 1.1 holds: that is, the operator $\mathcal{P}_1$ is continuous on (χ, d). Now, for all $u, v \in \chi$ we may apply (1.18) to (3.6) to obtain

$$d(\mathcal{P}_1 u, \mathcal{P}_1 v) \leq c\varkappa(u,v) \leq c(b-a)^{1/2} d(u,v).$$

Thus,

$$\forall\, \varepsilon > 0,\ \exists\, \Theta = \frac{\varepsilon}{c\,(b-a)^{1/2}} \text{ such that } d(u,v) < \Theta \implies d(\mathcal{P}_1 u, \mathcal{P}_1 v) < \varepsilon. \tag{3.7}$$

This proves that the condition (A_1) of Theorem 1.1 is satisfied.

■ **Stage three:** is to show that the condition (A_3) of Theorem 1.1 holds, so from (3.3), we consider for each $u, v \in \chi$

$$\left(\int_a^b |(\mathcal{P}_1 u)(t) - (\mathcal{P}_1 u)(t)|^2 \, dt\right)^{1/2} \leq K\left(\int_a^b \left(\int_a^b |\mathcal{G}_1(t,s)|^2 \, ds\right) dt\right)^{1/2} \varkappa(u,v),$$
$$= K\sqrt{\mathcal{H}_1(a,b,\gamma)}\varkappa(u,v), \tag{3.8}$$

that is

$$\varkappa(\mathcal{P}_1 u, \mathcal{P}_1 v) \leq K\sqrt{\mathcal{H}_1(a,b,\gamma)}\varkappa(u,v).$$

Above, we have used (2.11) to obtain (3.8). Thus, from (3.2), we have that $\mathcal{P}_1$ is contractive on $(\chi, \varkappa)$.

By Theorem 1.1, we conclude that the operator $\mathcal{P}_1$ has a unique fixed point in χ, that is equivalently shown by the CFBVP (1.1), (1.2) admitting a unique solution for $\gamma \in (\frac{3}{2}, 2]$. □

The following theorem was obtained by [20, Theorem 9], which proves the existence and uniqueness of solutions to the CFBVP (1.1), (1.2), using Banach fixed point theorem.

Theorem 3.2. *[20, Theorem 9] Suppose that* $1 < \gamma \leq 2$. *Let* $\mathfrak{g} : [a,b] \times \mathbb{R} \to \mathbb{R}$ *be continuous, and let* $K > 0$ *such that*

$$|\mathfrak{g}(t,u) - \mathfrak{g}(t,v)| \leq K|u - v|, \quad \text{for all } (t,u),(t,v) \in [a,b] \times \mathbb{R}. \tag{3.9}$$

If

$$K\frac{(b-a)^\gamma}{\gamma^{\frac{\gamma}{\gamma-1}+1}} < 1, \tag{3.10}$$

then the CFBVP (1.1), (1.2) admits a unique solution.

4. Existence and uniqueness results on (1.3) and (1.2)

In this section, we present our second novel result concerning the existence and uniqueness of solutions to the SCFBVP (1.3), (1.2), using Theorem 1.1.

Theorem 4.1. *Suppose that* $1/2 < \delta \leq 1$. *Let* $\mathfrak{h} : [a,b] \times \mathbb{R} \to \mathbb{R}$ *be continuous, and let* $K > 0$ *such that*

$$|\mathfrak{h}(t,u) - \mathfrak{h}(t,v)| \leq K|u - v|, \quad \text{for all } (t,u),(t,v) \in [a,b] \times \mathbb{R}. \tag{4.1}$$

If

$$K\sqrt{\mathcal{H}_2(a,b,\delta)} < 1, \tag{4.2}$$

where $\mathcal{H}_2(a,b,\delta)$ is defined by (2.21), then the SCFBVP (1.3), (1.2) admits a unique solution.

Proof: Since the argument follows a similar structure to that of Theorem 3.1, we briefly outline the core steps here, which is to apply the necessary modifications (e.g., replacing $\mathcal{G}_1$ with $\mathcal{G}_2$).

Based on Lemma 2.3, we define the operator $\mathcal{P}_2 : \chi \to \chi$ by

$$(\mathcal{P}_2 u)(t) := \int_a^b \mathcal{G}_2(t,s)\mathfrak{h}(s,u(s))\, ds + \zeta(t), \quad t \in [a,b].$$

■ Consider the complete metric space $(\chi, d) = (C([a,b]), d)$ and the metric $(C([a,b]), \varkappa)$. For $u, v \in \chi$ and $t \in [a,b]$, consider

$$
\begin{aligned}
|(\mathcal{P}_2 u)(t) - (\mathcal{P}_2 v)(t)| &\leq \int_a^b |\mathcal{G}_2(t,s)|\, |\mathfrak{h}(s,u(s)) - \mathfrak{h}(s,v(s))|\, ds \\
&\leq \int_a^b |\mathcal{G}_2(t,s)|\, K\, |u(s) - v(s)|\, ds \\
&\leq \left(\int_a^b |\mathcal{G}_2(t,s)|^2\, ds \right)^{1/2} K \left(\int_a^b |u(s) - v(s)|^2\, ds \right)^{1/2} \qquad (4.3)\\
&\leq K \max_{t \in [a,b]} \left(\int_a^b |\mathcal{G}_2(t,s)|^2\, ds \right)^{1/2} \varkappa(u,v) \\
&\leq K \left(\frac{2(b-a)^{4\delta-1}}{\delta(2\delta-1)(4\delta-1)} \right)^{1/2} \varkappa(u,v). \qquad (4.4)
\end{aligned}
$$

Note that by Hölder's inequality [14, 22] and (4.1), we obtained (4.3), and by (2.19), we obtained (4.4). Thus by defining

$$
c := K \left(\frac{2(b-a)^{4\delta-1}}{\delta(2\delta-1)(4\delta-1)} \right)^{1/2}, \qquad (4.5)
$$

we have

$$
d(\mathcal{P}_2 u, \mathcal{P}_2 v) \leq c\varkappa(u,v), \quad \text{for some } c > 0 \text{ and all } u, v \in \chi, \qquad (4.6)
$$

which shows the condition (A_2) of Theorem 1.1 holds.

■ For all $u, v \in \chi$ we may apply (1.18) to (4.6) to obtain

$$
d(\mathcal{P}_2 u, \mathcal{P}_2 v) \leq c\varkappa(u,v) \leq c(b-a)^{1/2} d(u,v).
$$

Thus,

$$
\forall\, \varepsilon > 0,\ \exists\, \Theta_1 = \frac{\varepsilon}{c\,(b-a)^{1/2}} \text{ such that } d(u,v) < \Theta_1 \implies d(\mathcal{P}_2 u, \mathcal{P}_2 v) < \varepsilon. \qquad (4.7)
$$

This proves that the condition (A_1) of Theorem 1.1 is satisfied.

■ From (4.3), we consider for each $u, v \in \chi$

$$
\begin{aligned}
\left(\int_a^b |(\mathcal{P}_2 u)(t) - (\mathcal{P}_2 u)(t)|^2\, dt \right)^{1/2} &\leq K \left(\int_a^b \left(\int_a^b |\mathcal{G}_2(t,s)|^2\ ds \right) dt \right)^{1/2} \varkappa(u,v), \\
&= K \sqrt{\mathcal{H}_2(a,b,\delta)}\, \varkappa(u,v), \qquad (4.8)
\end{aligned}
$$

that is

$$
\varkappa(\mathcal{P}_2 u, \mathcal{P}_2 v) \leq K \sqrt{\mathcal{H}_2(a,b,\delta)}\, \varkappa(u,v).
$$

Above, we have used (2.20) to obtain (4.8). Thus, from (4.2), we have shown that $\mathcal{P}_2$ is contractive on $(\chi, \varkappa)$.

By Theorem 1.1, we conclude that the operator $\mathcal{P}_2$ has a unique fixed point in χ, which is equivalently shown by the SCFBVP (1.3), (1.2) admitting a unique solution for $1/2 < \delta \leq 1$.

We now state and prove the following result, which establishes the existence and uniqueness of solutions to the SCFBVP (1.3),(1.2), using the Banach fixed point theorem. As mentioned earlier, the authors of [20] had posed the SCFBVP (1.3), (1.2) as an open problem. This complements the results of [20].

Theorem 4.2. *Suppose that* $1/2 < \delta \leq 1$. *Let* $\mathfrak{h} : [a,b] \times \mathbb{R} \to \mathbb{R}$ *be continuous, and let* $K > 0$ *such that*

$$|\mathfrak{h}(t,u) - \mathfrak{h}(t,v)| \leq K|u - v|, \quad \text{for all } (t,u),(t,v) \in [a,b] \times \mathbb{R}. \tag{4.9}$$

If

$$K\frac{(b-a)^{2\delta}}{8\delta^2} < 1, \tag{4.10}$$

then the SCFBVP (1.3), (1.2) admits a unique solution.

Proof: Consider the complete metric space $(\chi, d) = (C([a,b]), d)$, and the operator $\mathcal{P}_2 : \chi \to \chi$ defined as in the proof of Theorem 4.1. For $u, v \in \chi$ and $t \in [a,b]$, consider

$$\begin{aligned} |(\mathcal{P}_2 u)(t) - (\mathcal{P}_2 v)(t)| &\leq \int_a^b |\mathcal{G}_2(t,s)|\, |\mathfrak{h}(s,u(s)) - \mathfrak{h}(s,v(s))|\, ds \\ &\leq d(u,v)K \int_a^b |\mathcal{G}_2(t,s)|\, ds \\ &\leq K\frac{(b-a)^{2\delta}}{8\delta^2} d(u,v), \end{aligned} \tag{4.11}$$

where above we have used (2.18) to obtain (4.11). Now by taking the maximum of both sides of the inequality (4.11), we thus have for all $u, v \in \chi$

$$d(\mathcal{P}_2 u)(t), (\mathcal{P}_2 v)) \leq K\frac{(b-a)^{2\delta}}{8\delta^2} d(u,v),$$

and in the light of (4.10), we have by Banach's fixed point theorem [10] that $\mathcal{P}_2$ has a unique fixed point in χ, which is equivalently shown the SCFBVP (1.3), (1.2) admins a unique solution for $1/2 < \delta \leq 1$. □

4.1. Results on classical BVPs (1.4) *and* (1.2)

Remark 4.1. In view of Remark 2.1, we observe that when $\gamma = 2$ and $\delta = 1$, then Theorem 3.1 coincides with Theorem 4.1. This leads to the following result on classical BVPs (1.4) and (1.2), which aligns with the results in [3, 7].

Corollary 4.1. *Let* $\mathfrak{f} : [a,b] \times \mathbb{R} \to \mathbb{R}$ *be continuous, and let* $K > 0$ *such that*

$$|\mathfrak{f}(t,u) - \mathfrak{f}(t,v)| \leq K|u - v|, \quad \text{for all } (t,u),(t,v) \in [a,b] \times \mathbb{R}. \tag{4.12}$$

If

$$K\frac{(b-a)^2}{\sqrt{90}} < 1, \tag{4.13}$$

then BVPs (1.4), (1.2) admits a unique solution.

Moreover, when $\gamma = 2$ and $\delta = 1$, then Theorem 3.2 coincides with Theorem 4.2, and that becomes the result of Peterson and Kelley [16, Theorem 7.7].

5. Comparisons and numerical variations

In this section, we provide a detailed comparison between our main results obtained in Sections 3 and 4, and the results of [20]. We show that our results herein significantly relax the assumptions required for the existence and uniqueness of both CFBVP (1.1), (1.2) and SCFBVP (1.3), (1.2).

5.1. On (1.1) *and* (1.2)

Remark 5.1. Consider $1 < \gamma \le 2$ and $a < t < b$, based on the results of [20, Theorem 2], the uniqueness of solution to the CFBVP (1.1), (1.2) exists if the Lipschitz condition (3.9) holds, with the constant K satisfying (3.10), that is

$$K\mathcal{V}_1(a,b,\gamma) < 1, \tag{5.1}$$

where

$$\mathcal{V}_1(a,b,\gamma) := \frac{(b-a)^\gamma}{\gamma^{\frac{\gamma}{\gamma-1}+1}}. \tag{5.2}$$

Whereas, our Theorem 3.1 proves that the uniqueness of solution to the CFBVP (1.1), (1.2) exists if the constant K satisfies (3.2), that is

$$K\sqrt{\mathcal{H}_1(a,b,\gamma)} < 1, \tag{5.3}$$

where $\mathcal{H}_1(a,b,\gamma)$ is defined by (2.12) and $\gamma \in (\frac{3}{2}, 2]$.

A direct evaluation for any special case shows that condition (5.3) is strictly less restrictive than (5.1) for all $\gamma \in (\frac{3}{2}, 2]$. Therefore, for $\gamma \in (\frac{3}{2}, 2]$, our results extend to a wider class of problems than those addressed in [20, Theorem 7]. To substantiate this claim, we now present a comprehensive numerical comparison between $\mathcal{V}_1(a,b,\gamma)$ and $\sqrt{\mathcal{H}_1(a,b,\gamma)}$, focusing on the specific case, which is $a = 0$ and $b = 2$. In this setting, both $\mathcal{V}_1(0,2,\gamma)$ and $\sqrt{\mathcal{H}_1(0,2,\gamma)}$ are computed as functions of the parameter γ using Python 3. In particular, the following libraries were used: mpmath for arbitrary-precision arithmetic, NumPy for numerical operations, and Matplotlib for visualization. Decimal precision was set to 6 digits using `mp.dps = 6` in mpmath to ensure sufficient accuracy in our evaluation.

The results are plotted over $\gamma \in (\frac{3}{2}, 2]$, see Figure 1, and selected data points are reported in Table 1.

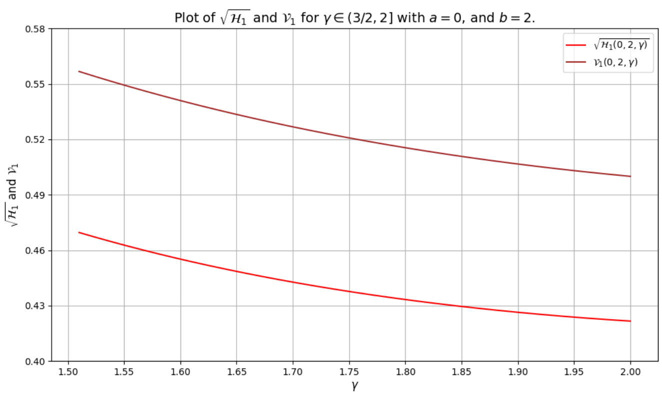

Figure 1. Comparison between square root of bounds from Rus's method ($\sqrt{\mathcal{H}_1(0,2,\gamma)}$) and Banach's method ($\mathcal{V}_1(0,2,\gamma)$) for varying $\gamma \in (3/2, 2]$.

Table 1. Evaluation of some special values of $\gamma \in (3/2, 2]$.

γ	1.51	1.6	1.7	1.8	1.9
$\mathcal{V}_1(0,2,\gamma)$	0.556757	0.541013	0.526793	0.515487	0.506668
$\sqrt{\mathcal{H}_1(0,2,\gamma)}$	0.469588	0.455188	0.442724	0.433302	0.426402

It is clear from both Figure 1 and Table 1, that

$$\sqrt{\mathcal{H}_1(0,2,\gamma)} < \mathcal{V}_1(0,2,\gamma), \text{ for all } \gamma \in (3/2, 2].$$

Consequently, for $\gamma \in (3/2, 2]$, our Theorem 3.1 applies to a wider range of problems than that of Theorem 3.2 obtained in [20, Theorem 7]. Therefore, Theorem 3.1 represents a significant improvement over the previous result that relies on $\mathcal{V}_1(a, b, \gamma)$.

5.2. *On* (1.3) *and* (1.2)

In this section, we show by numerical comparison that Theorem 4.1 (proven using Rus's fixed point theorem) is more applicable to a wider class of problems than Theorem 4.2 (proven using Banach fixed point theorem). Our numerical comparison follows an exact structure similar to that of the section above. Therefore, we provide only a brief overview here.

■ By Theorem 4.2, the uniqueness of the solution to the SCFBVP (1.3), (1.2) exists if the Lipschitz condition (4.9) holds, with the constant K satisfying (4.10), that is

$$K\mathcal{V}_2(a,b,\delta) < 1, \quad \text{for all } a < t < b, \ \delta \in (1/2, 1], \tag{5.4}$$

where,

$$\mathcal{V}_2(a,b,\delta) := \frac{(b-a)^{2\delta}}{8\delta^2}. \tag{5.5}$$

■ By Theorem 4.1 the uniqueness of the solution to the CFBVP (1.3), (1.2) exists if the constant K satisfies (4.2), that is

$$K\sqrt{\mathcal{H}_2(a,b,\delta)} < 1, \quad \text{for all } a < t < b, \ \delta \in (1/2, 1], \tag{5.6}$$

where $\mathcal{H}_2(a, b, \delta)$ is defined by (2.21).

Now, we choose a specific case that is when $a = 0$ and $b = 1$, so in this case, we again compute both $\mathcal{V}_2(0, 1, \delta)$ and $\sqrt{\mathcal{H}_2(0, 1, \delta)}$ as functions of the parameter δ using Python 3, where we used the exact libraries used above. We then generate a plot for $1/2 < \delta \leq 1$; see Figure 2 and some selected value of δ given in Table 2 as follows:

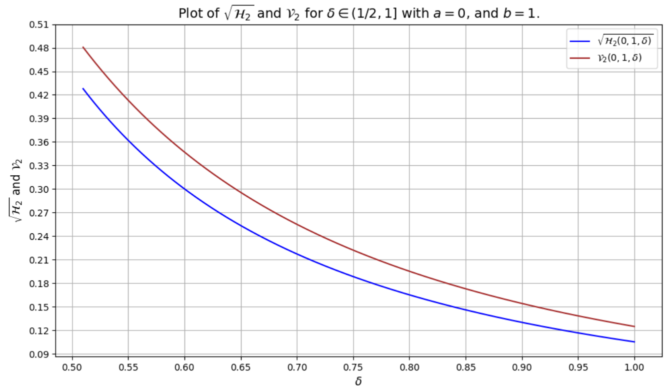

Figure 2. Comparison between square root of bounds from Rus's method ($\sqrt{\mathcal{H}_2(0, 1, \delta)}$) and Banach's method ($\mathcal{V}_2(0, 1, \delta)$) for varying $\delta \in (1/2, 1]$.

Table 2. Evaluation of some special values of $\delta \in (1/2, 1]$.

δ	0.51	0.6	0.7	0.8	0.9
$\mathcal{V}_2(0, 1, \delta)$	0.480584	0.347222	0.255102	0.1953125	0.154320
$\sqrt{\mathcal{H}_2(0, 1, \delta)}$	0.427796	0.300312	0.217350	0.165276	0.130224

Figure 2 and Table 2 show that

$$\sqrt{\mathcal{H}_2(0, 1, \delta)} < \mathcal{V}_2(0, 1, \delta), \text{ for all } \delta \in (1/2, 1].$$

This shows that Theorem 4.1 extends the applicability of the theory beyond that of Theorem 4.2.

6. Conclusions and open problems

In this work, we derived new integral bounds involving the Green's functions corresponding to both CFBVP (1.1), (1.2) and SCFBVP (1.3), (1.2). Then, we established new existence and uniqueness results for both types of problems by applying Rus's theorem (Theorem 1.1). This approach provided a sharper bound than those obtained via the classical contraction mapping, allowing us to improve on the results from [20]. Our findings relax previous assumptions and extend the applicability of existing results, as further demonstrated through numerical comparisons.

However, our results for CFBVP (1.1) and (1.2) are restricted to the range $\frac{3}{2} < \gamma \leq 2$ to ensure well-defined bounds. For example, the term $\Gamma(2\gamma - 3)$ appears in the derived bounds, and when $\gamma = \frac{3}{2}$, this leads to $\Gamma(0)$, which is undefined because the Gamma function has simple poles at all non-positive integers, including zero. Developing alternative approaches or introducing additional restrictions to handle this critical case and others may lead to interesting new results. This remains an open problem.

Future research directions include extending the analysis to cover the critical case $\gamma = \frac{3}{2}$ and exploring parameter ranges beyond those currently considered. Further investigation into alternative mathematical techniques may allow relaxation of the current restrictions and generalization to more complex fractional boundary value problems, for example [8, 13, 15]. Additionally, applying our theoretical ideas to practical numerical methods and real-world models, for example [9, 11, 12], may represent a promising avenue for continued study.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that he has no conflicts of interest.

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