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Research article

Asymptotic and oscillation properties of solutions of differential equations in the Canonical case

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Abstract: The existence of oscillatory solutions to fourth-order differential equations with multiple delays is investigated in this paper. The comparison method and the Riccati method are used to create new oscillation conditions. The results we have got, when compared to some studies in the literature, not only enhance the conditions for the oscillation of the examined equations, but they also expand some of the results that have already been published. Some examples are provided to illustrate the results.

Keywords: oscillatory solutions; fourth-order differential equations; several delays
Mathematics Subject Classification: 34C10, 34K11

1. Introduction

The oscillation conditions for the following sorts of fourth-order differential equation solutions are examined in this article:

$$\left(u(s)\left(z'''(s)\right)^{\kappa_1}\right)' + \sum_{i=1}^{j} \varsigma_i(s)\, z^{\kappa_2}\left(\xi_i(s)\right) = 0, \tag{1.1}$$

and

$$\left(u(s)\left(z'''(s)\right)^{\kappa_1}\right)' + \sigma(s)\left(z'''(s)\right)^{\kappa_1} + \sum_{i=1}^{j} \varsigma_i(s)\, z^{\kappa_2}\left(\xi_i(s)\right) = 0, \tag{1.2}$$

where κ_1 and κ_2 are quotients of odd positive integers, $u,\ \varsigma_i \in C([s_0,\infty),[0,\infty))$, $u(s) > 0$, $\varsigma_i(s) > 0$, $\xi_i(s) \in C([s_0,\infty),\mathbb{R})$, $\xi_i(s) \le s$, $\lim_{s\to\infty} \xi_i(s) = \infty, i = 1,2,3,...$ Moreover, we study (1.1) under the canonical case

$$\int_{s_0}^{\infty} \frac{1}{u^{1/\kappa_1}(\zeta)} d\zeta = \infty, \tag{1.3}$$

and (1.2) under the conditions $\sigma \in C([s_0,\infty),[0,\infty))$, $u'(s)+\sigma(s) \geq 0$ and under canonical case

$$\int_{s_0}^{\infty}\left[\frac{1}{u(\zeta)}\exp\left(-\int_{s_0}^{\zeta}\frac{\sigma(x)}{u(x)}dx\right)\right]^{1/\kappa_1}d\zeta=\infty, \tag{1.4}$$

Definition 1. *The solution to (1.1) or (1.2) is said to be oscillatory if it changes its sign an infinite number of times on* $[s_0,\infty)$ *for every* s_0. *Otherwise, this solution is considered nonoscillatory.*

Definition 2. *The Eq (1.1) or (1.2) is oscillatory if it has oscillatory solutions.*

Differential equations (DEs), a fundamental branch of mathematics, specify the connections between variables and the rates of change. They are therefore an essential tool for understanding complex physical, biological, and engineering models. In mathematical analysis and the applied sciences, equations with higher orders and complex nonlinear properties are fascinating. Fourth-order differential equations are particularly important in many scientific and engineering applications because of their ability to describe complex oscillatory behavior. Researchers are still interested in this family of equations because they provide a thorough theoretical basis for understanding and controlling complex dynamic occurrences (see [1–3]).

In addition to their theoretical significance, delay differential equations (DDEs) have a wide range of applications, including the analysis of networks with lossless transmission lines, which are used in high-speed computers to connect switching circuits [4]. As a result, the study of DDEs is crucial in many engineering and physical fields. DDEs are crucial for the mathematical modeling of systems that depend on the values of present, past, or future variables. They are characterized by the existence of the highest-order derivative of the unknown function both now and at a later or future time (see [5,6]).

The applications of differential equations extend to include most areas of life, such as mathematical biology, mathematical economics, and climatology. Modern control theory also relies heavily on differential equations, where methods have been developed to design control systems and analyze their stability [7]. It has been noted that a broad research interest has emerged related to the study of the oscillatory and asymptotic properties of second-order differential equations (see [8–10]).

Because of the efforts of early pioneers like Alexander Lyapunov and Henri Poincare, the field of analyzing oscillatory processes in differential equations has a long history and is well established. Because of its oscillating nature, a certain family of differential equations has caught the attention of numerous academics (see [11]).

Many researchers have focused on understanding the oscillatory nature of differential equations in different orders: Almarri et al. [12] and Philos [13] presented important results on the oscillation of delay equations with damping term in the even-order using advanced techniques for analyzing nonlinear effects; Althubiti [14] provided exact conditions for the oscillation of equations with infinite coefficients at the third order; Graef et al. [15] presented sophisticated findings on the oscillation of delay equations at the fourth order using new methodologies; the authors in [16–18] extended the understanding of the oscillation of a class of nonlinear delay equations.

Grace et al. [19] studied the same Eq (1.1) but with another approach in an attempt to improve the previous results. Equation (1.1) where $\kappa_1=\kappa_2$, was examined by Zhang et al. [20], and some oscillation criteria were obtained.
Baculikova et al. [21] studied the oscillatory properties of equation

$$\left[u(s)\left(z^{(n-1)}(s)\right)^{\kappa_1}\right]'+\varsigma(s)f(z(\tau(s)))=0.$$

Their analyses concentrated on using exacting analytical and comparative methods to construct oscillation criteria. By elucidating the effects of delay and nonlinear structure on the qualitative characteristics of solutions, these investigations made a substantial contribution to the theory. They also proved that the oscillation criterion of this equation is coupled to the oscillation of the following equation:

$$y'(s)+\varsigma(s)f\left(\frac{\delta\tau^{n-1}(s)}{(n-1)!u^{\frac{1}{\kappa_1}}(\tau(s))}\right)f\left(y^{\frac{1}{\kappa_1}}(\tau(s))\right)=0.$$

In [22], Moaaz et al. examined a more general nonlinear form

$$\left(u(s)\left(z'''(s)\right)^{\kappa_1}\right)'+\varsigma(s)z^{\kappa_1}(\xi(s))=0. \tag{1.5}$$

Their work provided broader oscillation criteria under more flexible structural assumptions. They obtained new criteria with one condition that ensures the oscillation of the studied equation in canonical form. Their results extend, complete, and simplify some results in previous studies.

Elabbasy et al. [23] proved that Eq (1.2) where $\kappa_1 = \kappa_2 = 1$ is oscillatory, if there exist functions $\eta, \vartheta \in C^1([\nu_0,\infty),\mathbb{R})$ such that

$$\int_{s_0}^{\infty}\left(\eta(\zeta)\varsigma(\zeta)\frac{\varepsilon}{2}\xi^2(\zeta)-\frac{1}{4\eta(\zeta)u(\zeta)}\left[\frac{\eta'(\zeta)}{\eta(\zeta)}-\frac{\sigma(\zeta)}{u(\zeta)}\right]^2\right)d\zeta=\infty,$$

for some $\varepsilon \in (0,1)$, and

$$\int_{s_0}^{\infty}\left[\vartheta(\zeta)\int_{\zeta}^{\infty}\left[\frac{1}{u(\upsilon)}\int_{\upsilon}^{\infty}\varsigma(\nu)\left(\frac{\xi^2(\nu)}{\nu^2}\right)d\nu\right]d\upsilon-\frac{(\vartheta'(\zeta))^2}{4\vartheta(\zeta)}\right]d\zeta=\infty,$$

under the condition (1.4). They were initially interested in introducing some new monotonic properties for the solutions of Eq (1.2) and then used these properties to obtain new oscillatory conditions that ensure that all solutions of Eq (1.2) are oscillatory. Their results complement and extend some related results in the literature.

From the above, we note that there are many studies that have been interested in studying the fourth-order DEs in different forms, whether in the canonical case or in the non-canonical case. We also know that there are few studies that have been interested in studying the DE (1.1) and (1.2), and most of them were interested in the non-canonical case. As a result, the aim of this paper was to study the DE (1.1) and (1.2) in the canonical case and to find new criteria that expand some of the previous studies. We discussed some examples to illustrate the effectiveness of our main criteria.

Comparing these findings from earlier research with our own, we see that this work stands out for the following reasons:

1. Variations in the exponents of the first and second terms of the differential equation, κ_1 and κ_2, under study impact our findings and expand their range of use.
2. Our findings are applicable to multiple delay arguments, without assumption on about boundedness of delays.
3. We can apply our conclusions to a range of differential equations, such as ordinary ($\xi_i(s) = s$), linear ($\kappa_1 = \kappa_2 = 1$), half-linear ($\kappa_1 = \kappa_2$), and Emden-Fowler equations, due to their universality.

For convenience, we denote

$$G(s):=\frac{\pi^{\kappa_2}}{6^{\kappa_2}}\frac{\sum_{i=1}^{j}\varsigma_i(s)\xi_i^{3\kappa_2}(s)}{u^{\kappa_2/\kappa_1}(\xi_i(s))},$$

and

$$\widetilde{u}(s) := \pi^{\kappa_2/\kappa_1} \int_s^\infty \left(\frac{1}{u(x)} \int_x^\infty \sum_{i=1}^{j} \varsigma_i(\zeta) \left(\frac{\xi_i(\zeta)}{\zeta} \right)^{\kappa_2} \mathrm{d}\zeta \right)^{1/\kappa_1} \mathrm{d}x,$$

where $\pi \in (0,1)$.

2. Main results

Using the comparison technique and Riccati approach with a variety of substitutions, this section presents criteria that guarantee the oscillatory behavior of solutions to equations . These specifications are derived from a thorough analysis that eventually concentrates on positive solutions while accounting for the equation's particular structure. In order to simplify the proof without sacrificing generality, we assume that the functional inequalities hold for every sufficiently big s.

Lemma 1. *[24] Let $h \in C^n([s_0,\infty),(0,\infty))$. Suppose that $h^{(n)}(s)$ is of a fixed sign, such that, for all $s \geq s_1$,*

$$h^{(n-1)}(s)\, h^{(n)}(s) \leq 0.$$

If we have $\lim_{s\to\infty} h(s) \neq 0$, then

$$h(s) \geq \frac{\pi}{(n-1)!} s^{n-1} \left| h^{(n-1)}(s) \right|,$$

for every $\pi \in (0,1)$ and $s \geq s_\pi$.

Lemma 2. *[25] If the function z satisfies $z^{(i)}(s) > 0$, $i = 0,1,...,n$, and $z^{(n+1)}(s) < 0$, then*

$$\frac{z(s)}{s^n/n!} \geq \frac{z'(s)}{s^{n-1}/(n-1)!}.$$

Lemma 3. *[26] Suppose that x and $V > 0$ are constants. Then*

$$xy - Vy^{(\kappa_1+1)/\kappa_1} \leq \frac{\kappa_1^{\kappa_1}}{(\kappa_1+1)^{\kappa_1+1}} x^{\kappa_1+1} V^{-\kappa_1}, \tag{2.1}$$

where κ_1 is a quotient of odd positive integers

2.1. Oscillation results for Eq (1.1)

Lemma 4. *Suppose that (1.3) holds. Let $z(s)$ be an eventually positive solution of (1.1); then $z' > 0$ and $z''' > 0$.*

Proof. Let $z(s)$ be a positive solution of (1.1); then $z(s) > 0$ and $z(\xi_i(s)) > 0$ for $s \geq s_1$. From (1.1), we obtain

$$\left(u(s)(z'''(s))^{\kappa_1}\right)' = -\sum_{i=1}^{j} \varsigma_i(s)\, z^{\kappa_2}(\xi_i(s)) < 0.$$

As a result, $u(s)(z'''(s))^{\kappa_1}$ is decreasing by one sign. Thus, we see that

$$z'''(s) > 0.$$

From the positive of $z(s)$, using (1.1), we see that

$$\left(u(s)\left(z'''(s)\right)^{\kappa_1}\right)' = u'(s) + \kappa_1 u(s)\left(z'''(s)\right)^{\kappa_1-1} z^{(4)}(s) \leq 0.$$

This implies that $z^{(4)}(s) \leq 0$; hence $z'(s) > 0,\ s \geq s_1$. Therefore, the proof is complete. □

Theorem 1. *Suppose that (1.3) holds. If the differential equation*

$$x'(s) + G(s)\, x^{\kappa_2/\kappa_1}(\xi_i(s)) = 0 \tag{2.2}$$

is oscillatory for some $\pi \in (0,1)$, *then (1.1) is oscillatory.*

Proof. Let $z(s)$ be a positive solution of (1.1). From Lemma 4, we find that

$$x(s) := u(s)\left(z'''(s)\right)^{\kappa_1} > 0,$$

which with (1.1) gives

$$x'(s) + \sum_{i=1}^{j} \varsigma_i(s)\, z^{\kappa_2}(\xi_i(s)) = 0. \tag{2.3}$$

We have $\lim_{s\to\infty} z(s) \neq 0$ since z is positive and increasing. Consequently, Lemma 1 gives us

$$z^{\kappa_2}(\xi_i(s)) \geq \frac{\pi^{\kappa_2}}{6^{\kappa_2}} \xi_i^{3\kappa_2}(s)\left(z'''(\xi_i(s))\right)^{\kappa_2}, \tag{2.4}$$

for all $\pi \in (0,1)$. By (2.3) and (2.4), we see that

$$x'(s) + \frac{\pi^{\kappa_2}}{6^{\kappa_2}} \sum_{i=1}^{j} \varsigma_i(s)\, \xi_i^{3\kappa_2}(s)\left(z'''(\xi_i(s))\right)^{\kappa_2} \leq 0.$$

Since x is a positive solution of the differential inequality, we observe that

$$x'(s) + G(s)\, x^{\kappa_2/\kappa_1}(\xi_i(s)) \leq 0.$$

The related Eq (2.2) likewise has a positive solution in light of [27, Theorem 1], which is contradictory. There is proof of the theorem. □

Corollary 1. *If* $\kappa_1 = \kappa_2$, *(1.3) holds. If*

$$\liminf_{s\to\infty} \int_{\xi_i(s)}^{s} G(\zeta)\, \mathrm{d}\zeta > \frac{1}{\mathrm{e}}, \tag{2.5}$$

for some $\pi \in (0,1)$, *then (1.1) is oscillatory.*

Proof. The oscillation of (1.1) is implied by [28, Theorem 2.1.1], as is widely known. □

Lemma 5. *Let* $z(s)$ *be a positive solution of (1.1). If*

$$\int_{s_0}^{\infty} \left(M^{\kappa_2-\kappa_1} \eta(s) \sum_{i=1}^{j} \varsigma_i(s) \frac{\xi_i^{3\kappa_1}(s)}{s^{3\kappa_1}} - \frac{2^{\kappa_1}}{(\kappa_1+1)^{\kappa_1+1}} \frac{u(s)\left(\eta'(s)\right)^{\kappa_1+1}}{\varepsilon^{\kappa_1} s^{2\kappa_1} \eta^{\kappa_1}(s)} \right) \mathrm{d}\zeta = \infty, \tag{2.6}$$

for some $\varepsilon \in (0,1)$, *and* $\eta \in C^1([\nu_0,\infty),\mathbb{R})$, *then* $z'' < 0$.

Proof. On the other hand, assuming that $z''(s) > 0$, we use Lemmas 1 and 2 to obtain

$$\frac{z(\xi_i(s))}{z(s)} \geq \frac{\xi_i^3(s)}{s^3}, \tag{2.7}$$

and

$$z'(s) \geq \frac{\varepsilon}{2} s^2 z'''(s). \tag{2.8}$$

A function ψ is defined by

$$\psi(s) := \eta(s) \frac{u(s)(z'''(s))^{\kappa_1}}{z^{\kappa_1}(s)} > 0.$$

By differentiating and using (2.7) and (2.8), we obtain

$$\psi'(s) \leq \frac{\eta'(s)}{\eta(s)}\psi(s) - \eta(s)\sum_{i=1}^{j} \varsigma_i(s)\frac{\xi_i^{3\kappa_1}(s)}{s^{3\kappa_1}} z^{\kappa_2-\kappa_1}(\xi_i(s)) - \frac{\kappa_1\varepsilon}{2}\frac{s^2}{\eta^{1/\kappa_1}(s)u^{1/\kappa_1}(s)}\psi^{1+1/\kappa_1}(s). \tag{2.9}$$

Since $z'(s) > 0$, there exist an $s_2 \geq s_1$ and a constant $M > 0$ such that $z(s) > M$, for all $s \geq s_2$. Using the inequality (2.1) with $x = \eta'/\eta, V = \kappa_1\varepsilon s^2/\left(2u^{1/\kappa_1}(s)\eta^{1/\kappa_1}(s)\right)$ and $y = \psi$, we obtain

$$\psi'(s) \leq -M^{\kappa_2-\kappa_1}\eta(s)\sum_{i=1}^{j}\varsigma_i(s)\frac{\xi_i^{3\kappa_1}(s)}{s^{3\kappa_1}} + \frac{2^{\kappa_1}}{(\kappa_1+1)^{\kappa_1+1}}\frac{u(s)(\eta'(s))^{\kappa_1+1}}{\varepsilon^{\kappa_1}s^{2\kappa_1}\eta^{\kappa_1}(s)}.$$

This implies that

$$\int_{s_1}^{s}\left(M^{\kappa_2-\kappa_1}\eta(s)\sum_{i=1}^{j}\varsigma_i(s)\frac{\xi_i^{3\kappa_1}(s)}{s^{3\kappa_1}} - \frac{2^{\kappa_1}}{(\kappa_1+1)^{\kappa_1+1}}\frac{u(s)(\eta'(s))^{\kappa_1+1}}{\varepsilon^{\kappa_1}s^{2\kappa_1}\eta^{\kappa_1}(s)}\right)\mathrm{d}\zeta \leq \psi(s_1),$$

which contradicts (2.6). There is proof of the lemma. □

Theorem 2. *Let $\kappa_2 \geq \kappa_1$. If*

$$y''(s) + M^{\kappa_2-\kappa_1}\widetilde{u}(s)y(s) = 0 \tag{2.10}$$

is oscillatory, then (1.1) is oscillatory.

Proof. Let $z(s)$ be a positive solution of (1.1). From Lemmas 1 and 4, we have that

$$z'(s) > 0,\ z''(s) < 0 \text{ and } z'''(s) > 0, \tag{2.11}$$

for $s \geq s_2$, and we are integrating the Eq (1.1) from s to l, we have

$$u(l)(z'''(l))^{\kappa_1} = u(s)(z'''(s))^{\kappa_1} - \int_s^l \sum_{i=1}^{j}\varsigma_i(\zeta)z^{\kappa_2}(\xi_i(\zeta))\,\mathrm{d}\zeta. \tag{2.12}$$

Using Lemma 1 with (2.11), we obtain

$$\frac{z(\xi_i(s))}{z(s)} \geq \pi\frac{\xi_i(s)}{s},$$

for all $\pi \in (0,1)$, which with (2.12) gives

$$u(l)\left(z'''(l)\right)^{\kappa_1} - u(s)\left(z'''(s)\right)^{\kappa_1} + \pi^{\kappa_2}\int_s^l \sum_{i=1}^{j} \varsigma_i(\zeta)\left(\frac{\xi_i(\zeta)}{\zeta}\right)^{\kappa_2} z^{\kappa_2}(\zeta)\,\mathrm{d}\zeta \le 0.$$

It follows, by $z' > 0$, that

$$u(l)\left(z'''(l)\right)^{\kappa_1} - u(s)\left(z'''(s)\right)^{\kappa_1} + \pi^{\kappa_2} z^{\kappa_2}(s)\int_s^l \sum_{i=1}^{j} \varsigma_i(\zeta)\left(\frac{\xi_i(\zeta)}{\zeta}\right)^{\kappa_2} \mathrm{d}\zeta \le 0. \tag{2.13}$$

Taking $l \to \infty$, we have

$$-u(s)\left(z'''(s)\right)^{\kappa_1} + \pi^{\kappa_2} z^{\kappa_2}(s)\int_s^\infty \sum_{i=1}^{j} \varsigma_i(\zeta)\left(\frac{\xi_i(\zeta)}{\zeta}\right)^{\kappa_2} \mathrm{d}\zeta \le 0,$$

that is

$$z'''(s) \ge \frac{\pi^{\kappa_2/\kappa_1}}{u^{1/\kappa_1}(s)} z^{\kappa_2/\kappa_1}(s)\left(\int_s^\infty \sum_{i=1}^{j} \varsigma_i(\zeta)\left(\frac{\xi_i(\zeta)}{\zeta}\right)^{\kappa_2} \mathrm{d}\zeta\right)^{1/\kappa_1}.$$

Integrating the above inequality from s to ∞, we obtain

$$-z''(s) \ge \pi^{\kappa_2/\kappa_1} z^{\kappa_2/\kappa_1}(s)\int_s^\infty \left(\frac{1}{u(x)}\int_x^\infty \sum_{i=1}^{j} \varsigma_i(\zeta)\left(\frac{\xi_i(\zeta)}{\zeta}\right)^{\kappa_2} \mathrm{d}\zeta\right)^{1/\kappa_1} \mathrm{d}x,$$

hence

$$z''(s) \le -\widetilde{u}(s)\, z^{\kappa_2/\kappa_1}(s). \tag{2.14}$$

Now, if we define ϖ by

$$\varpi(s) = \frac{z'(s)}{z(s)},$$

then $\varpi(s) > 0$ for $s \ge s_1$, and

$$\varpi'(s) = \frac{z''(s)}{z(s)} - \left(\frac{z'(s)}{z(s)}\right)^2.$$

By using (2.14) and definition of $\varpi(s)$, we obtain

$$\varpi'(s) \le -\widetilde{u}(s)\frac{z^{\kappa_2/\kappa_1}(s)}{z(s)} - \varpi^2(s). \tag{2.15}$$

Given that $z'(s) > 0$, there is a constant $M > 0$ such that, for any $s \ge s_2$, where s_2 is sufficiently large, $z(s) \ge M$. Then, (2.15) becomes

$$\varpi'(s) + \varpi^2(s) + M^{\kappa_2-\kappa_1}\widetilde{u}(s) \le 0, \tag{2.16}$$

by [29], we find that (2.10) is nonoscillatory if (2.16) holds, which is a contradiction. The proof is finished. □

Theorem 3. *Let $\kappa_2 \geq \kappa_1$ and $\xi_i'(s) > 1$, (1.3) and (2.6) hold, for some $\varepsilon \in (0,1)$. If*

$$\left(\frac{1}{\xi_i'(s)} y'(s)\right)' + M^{\kappa_2/\kappa_1 - 1} u(s) y(s) = 0 \tag{2.17}$$

is oscillatory, then (1.1) is oscillatory.

Proof. We get (2.12) by using the same steps as in the proof of Theorem 2. $\xi_i'(s) \geq 0$ and $z'(s) \geq 0$ hence imply that

$$u(l)\left(z'''(l)\right)^{\kappa_1} - u(s)\left(z'''(s)\right)^{\kappa_1} + z^{\kappa_2}\left(\xi_i(s)\right) \int_s^l \sum_{i=1}^{j} \varsigma_i(\zeta)\, \mathrm{d}\zeta \leq 0. \tag{2.18}$$

Thus, (2.11) becomes

$$z''(s) \leq -u(s)\, z^{\kappa_2/\kappa_1}\left(\xi_i(s)\right). \tag{2.19}$$

Now, if we define χ by

$$\chi(s) = \frac{z'(s)}{z(\xi_i(s))},$$

then $\chi(s) > 0$ for $s \geq s_1$, and

$$\begin{aligned} \chi'(s) &= \frac{z''(s)}{z(\xi_i(s))} - \frac{z'(s)}{z^2(\xi_i(s))} z'(\xi_i(s))\, \xi_i'(s) \\ &\leq \frac{z''(s)}{z(\xi_i(s))} - \xi_i'(s)\left(\frac{z'(s)}{z(\xi_i(s))}\right)^2. \end{aligned}$$

By using (2.19) and the definition of $\chi(s)$, we see that

$$\chi'(s) + M^{\kappa_2/\kappa_1 - 1} u(s) + \xi_i'(s)\chi^2(s) \leq 0. \tag{2.20}$$

Using [29], we see that (2.17) is nonoscillatory if (2.20) holds, which is a contradiction. The proof is finished. □

Numerous findings exist about the oscillation of (2.10) and (2.17), including Philos type, Hille and Nehari types, and others. We have the following corollary based on [30, 31], respectively.

Corollary 2. *If*

$$\lim_{s\to\infty} \frac{1}{H(s,s_0)} \int_{s_0}^{s} \left(H(s,\zeta)\widetilde{u}(\zeta) - \frac{1}{4}h^2(s,\zeta)\right) \mathrm{d}\zeta = \infty,$$

or

$$\liminf_{s\to\infty} s \int_s^\infty \widetilde{u}(\zeta)\, \mathrm{d}\zeta > \frac{1}{4}, \tag{2.21}$$

then (1.1) is oscillatory.

Corollary 3. *Assume that $\kappa_2 = \kappa_1$, (1.3), and (2.6) hold, for $\kappa \in (0, 1/4]$ such that*

$$s^2 \widetilde{u}(\zeta) \geq \kappa,$$

and

$$\limsup_{s\to\infty} \left(s^{\kappa-1} \int_{s_0}^{s} \zeta^{2-\kappa} \widetilde{u}(\zeta)\, \mathrm{d}\zeta + s^{1-\widetilde{\kappa}} \int_s^\infty \zeta^{\widetilde{\kappa}} \widetilde{u}(\zeta)\, \mathrm{d}\zeta\right) > 1,$$

where $\widetilde{\kappa} = \frac{1}{2}\left(1 - \sqrt{1-4\kappa}\right)$, then (1.1) is oscillatory.

2.2. Oscillation results for Eq (1.2)

The notations utilized in this paper are shown below:

$$\epsilon_{s_0}(s) := \exp\left(\int_{s_0}^{s} \frac{\sigma(x)}{u(x)} \mathrm{d}x\right),$$

and

$$\widehat{u}(s) := \varepsilon_1^{\kappa_2/\kappa_1} \int_s^\infty \left(\frac{1}{u(x)\,\epsilon_{s_0}(s)} \int_x^\infty \epsilon_{s_0}(s) \sum_{i=1}^{j} \varsigma_i(\zeta) \left(\frac{\xi_i(\zeta)}{\zeta}\right)^{\kappa_2} \mathrm{d}\zeta \right)^{1/\kappa_1} \mathrm{d}x,$$

where $\varepsilon_1 \in (0,1)$. We find some oscillation conditions for (1.2) by converting it into the form (1.1). It is not difficult to see that

$$\begin{aligned} \frac{1}{\epsilon_{s_0}(s)} \frac{\mathrm{d}}{\mathrm{d}s} \left(\varepsilon(s)\,u(s)\left(z'''(s)\right)^{\kappa_1}\right) &= \frac{1}{\epsilon_{s_0}(s)} \left[\epsilon_{s_0}(s)\left(u(s)\left(z'''(s)\right)^{\kappa_1}\right)' + \epsilon'_{s_0}(s)\,u(s)\left(z'''(s)\right)^{\kappa_1}\right] \\ &= \left(u(s)\left(z'''(s)\right)^{\kappa_1}\right)' + \frac{\epsilon'_{s_0}(s)}{\epsilon_{s_0}(s)} u(s)\left(z'''(s)\right)^{\kappa_1}, \\ &= \left(u(s)\left(z'''(s)\right)^{\kappa_1}\right)' + \sigma(s)\left(z'''(s)\right)^{\kappa_1}, \end{aligned}$$

which with (1.2) gives

$$\left(\epsilon_{s_0}(s)\,u(s)\left(z'''(s)\right)^{\kappa_1}\right)' + \epsilon_{s_0}(s) \sum_{i=1}^{j} \varsigma_i(s)\, z^{\kappa_2}(\xi_i(s)) = 0.$$

Corollary 4. *Assume that $\kappa_1 = \kappa_2$, (1.4) holds. If*

$$\liminf_{s\to\infty} \int_{\xi_i(s)}^{s} \widehat{G}(\zeta)\,\mathrm{d}\zeta > \frac{1}{\mathrm{e}},$$

for some $\pi \in (0,1)$, where

$$\widehat{G}(s) := \frac{\pi^{\kappa_2}}{6^{\kappa_2}} \frac{\epsilon_{s_0}(s) \sum_{i=1}^{j} \varsigma_i(s)\, \xi_i^{3\kappa_2}(s)}{\epsilon_{s_0}^{\kappa_2/\kappa_1}(\xi_i(s))\, u^{\kappa_2/\kappa_1}(\xi_i(s))},$$

then (1.2) is oscillatory.

Corollary 5. *Assume that $\kappa_2 = \kappa_1$, (1.4), and*

$$\int_{s_0}^{\infty} \left(M^{\kappa_2-\kappa_1} \eta(s)\,\epsilon_{s_0}(s) \sum_{i=1}^{j} \varsigma_i(s) \frac{\xi_i^{3\kappa_1}(s)}{s^{3\kappa_1}} - \frac{2^{\kappa_1}}{(\kappa_1+1)^{\kappa_1+1}} \frac{u(s)\,\epsilon_{s_0}(s)\left(\eta'(s)\right)^{\kappa_1+1}}{\varepsilon^{\kappa_1} s^{2\kappa_1} \eta^{\kappa_1}(s)} \right) \mathrm{d}\zeta = \infty, \tag{2.22}$$

hold, for some $\varepsilon \in (0,1)$. If

$$\lim_{s\to\infty} \frac{1}{H(s,s_0)} \int_{s_0}^{s} \left(H(s,\zeta)\,\widehat{u}(\zeta) - \frac{1}{4} h^2(s,\zeta) \right) \mathrm{d}\zeta = \infty,$$

or

$$\liminf_{s\to\infty} \int_s^\infty \widehat{u}(\zeta)\,\mathrm{d}\zeta > \frac{1}{4},$$

then (1.2) is oscillatory.

Corollary 6. *Assume that $\kappa_2 = \kappa_1$, (1.4), and (2.22) hold, for $\kappa \in (0, 1/4]$ such that*

$$s^2\widehat{u}(\zeta) \geq \kappa,$$

and

$$\limsup_{s\to\infty}\left(s^{\kappa-1}\int_{s_0}^{s}\zeta^{2-\kappa}\widehat{u}(\zeta)\,\mathrm{d}\zeta + s^{1-\widetilde{\kappa}}\int_{s}^{\infty}\zeta^{\widetilde{\kappa}}\widehat{u}(\zeta)\,\mathrm{d}\zeta\right) > 1,$$

where $\widetilde{\kappa}$ defined as Corollary 3, then (1.2) is oscillatory.

Example 1. *Let the equation be:*

$$\left(s^3\left(z'''(s)\right)^3\right)' + \frac{1}{s^3}z^3(\nu s) + \frac{\varsigma_{0-s^4}}{s^7}z^3(\nu s) = 0, s \geq 1, \varsigma_0 > 0, \tag{2.23}$$

where $\kappa_1 = \kappa_2 = 3$, $u(s) = s^3$, $\nu \in (0, 1]$, $\xi_i(s) = \nu s$, $\varsigma_1(s) = 1/s^3$ and $\varsigma_2(s) = \varsigma_0/s^7$. Thus, it is easy to see that

$$G(s) = \frac{\pi^3\nu^6}{6^3}\frac{\varsigma_0}{s},$$

and

$$\widetilde{u}(s) = \pi\left(\frac{\varsigma_0}{6}\right)^{1/3}\nu\frac{1}{2s^2}.$$

From Corollary 1, we see (2.23) is oscillatory if

$$\varsigma_0 > \frac{6^3}{e\left(\ln\frac{1}{\nu}\right)\nu^6}. \tag{2.24}$$

From Corollary 2, we get that Conditions (2.6) and (2.21) become

$$\varsigma_0 > \left(\frac{3^4}{2}\right)\frac{1}{\nu^9},$$

and

$$\varsigma_0 > 6\left(\frac{1}{4\nu}\right)^3,$$

respectively. Thus, Eq (2.23) is oscillatory if

$$\varsigma_0 > \max\left\{\left(\frac{3^4}{2}\right)\frac{1}{\nu^9}, 6\left(\frac{1}{4\nu}\right)^3\right\} = \left(\frac{3^4}{2}\right)\frac{1}{\nu^9}. \tag{2.25}$$

Example 2. *Consider the equation:*

$$\left(s\left(z'''(s)\right)\right)' + z'''(s) + \frac{6}{s}z\left(\frac{1}{2}s\right) - \frac{6s-\varsigma_0}{s^2}z\left(\frac{1}{2}s\right) = 0, s \geq 1, \varsigma_0 > 0, \tag{2.26}$$

where $\kappa_1 = \kappa_2 = 1$, $u(s) = s$, $\xi_i(s) = \frac{1}{2}s$, $\varsigma_1(s) = 6/s$, and $\varsigma_2(s) = (6s - \varsigma_0)/s^2$. Thus, it is easy to see that

$$\int_{s_0}^{\infty}\left[\frac{1}{u(\zeta)}\exp\left(-\int_{s_0}^{\zeta}\frac{\sigma(x)}{u(x)}dx\right)\right]^{1/\kappa_1}d\zeta$$

$$
\begin{aligned}
&= \int_{s_0}^{\infty}\left[\frac{1}{\zeta}\exp\left(-\int_{s_0}^{\zeta}\frac{dx}{x}\right)\right]d\zeta \\
&= \infty.
\end{aligned}
$$

From Corollary 4, we see that

$$
\begin{aligned}
\widehat{G}(s) &= \frac{\pi^{\kappa_2}}{6^{\kappa_2}}\frac{\epsilon_{s_0}(s)\sum_{i=1}^{j}\varsigma_i(s)\,\xi_i^{3\kappa_2}(s)}{\epsilon_{s_0}^{\kappa_2/\kappa_1}(\xi_i(s))\,u^{\kappa_2/\kappa_1}(\xi_i(s))} \\
&= \frac{4\pi\varsigma_0}{6s^3 8}.
\end{aligned}
$$

Thus, it is easy to see that

$$
\begin{aligned}
&\liminf_{s\to\infty}\int_{\xi_i(s)}^{s}\widehat{G}(\zeta)\,\mathrm{d}\zeta \\
&= \liminf_{s\to\infty}\int_{\xi_i(s)}^{s}\frac{4\pi\varsigma_0}{6\zeta^3 8}\mathrm{d}\zeta > \frac{1}{\mathrm{e}}.
\end{aligned}
$$

Thus, Eq (2.26) is oscillatory.

Remark 1. *By applying Eq (2.23) on [22] where $\nu = 1/2$, we obtain*

$$
\varsigma_0 > 20736.
$$

Consequently, our outcome enhances results [22].

3. Conclusions

In this article, the oscillation of solutions of Eqs (1.1) and (1.2) has been demonstrated. The comparison and Riccari method are used to create new criteria that are more effective than the existing criteria in literature. Additionally, some examples are given to show how our findings build upon those of [22]. For future work, our approach can be applied in studying the following neutral delay DEs:

$$
\left(u(s)\left(y'''(s)\right)^{\kappa_1}\right)' + \sigma(s)\left(y'''(s)\right)^{\kappa_1} + \varsigma(s)\,z^{\kappa_2}(\xi(s)) = 0,
$$

where $y(s) = x(s) + p(s)\,x(\varpi(s))$.

Use of Generative-AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no competing financial interest.

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