AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(8): 19189–19216.
DOI: 10.3934/math.2025858
Received: 06 May 2025
Revised: 16 July 2025
Accepted: 29 July 2025
Published: 22 August 2025

*Research article*

# Detecting jumps in stochastic volatility jump-diffusion models via the power variation approach

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**Abstract:** It is well-known that pretesting the presence of the jump component in an underlying price process is crucial for modeling this process. In this paper, we propose a consistent test for jump intensity of the conditional Poisson process in a stochastic volatility jump diffusion model. Theoretically, we derive the infill and long-span asymptotic properties of realized power variation under some suitable conditions, and verify the asymptotic size and power of the proposed test. Furthermore, the finite-sample performance of our proposed test is illustrated through simulation analysis, and an application to real price series provides empirical evidence of significant jump intensities.

**Keywords:** high-frequency data; jump intensity; long time span jump test; power variation
**Mathematics Subject Classification:** 62M20, 62P05

## 1. Introduction

In the financial markets, dramatic price changes of an asset might occur, which are commonly called jumps of the price. A continuous diffusion process is inadequate for modeling price processes with jumps. So quite a few researchers focus on constructing appropriate jump diffusion process models to capture the dynamics of the asset prices. Over the past few decades, great efforts have been made by researchers to address this issue, and some important results are reported in [1,2] and references therein. Meanwhile, jump diffusion models have also been widely used in risk measurement and management (see, e.g., [3, 4]), assets allocation (see, e.g., [5, 6]) and derivatives pricing (see, e.g., [7, 8]).

It is well known that, unlike continuous diffusion models, jump diffusion models possess clearly different theoretical properties and meanings in applications, so it is of great significance to statistically pretest the existence of jumps for an underlying price process. In the past few decades, there appeared a number of jump tests using high-frequency price series. In particular, the realized volatility measure has been a widely applied tool to carry out jump tests, including realized power variation tests [9, 10], realized (threshold) bipower variation tests [11–13], and nearest neighbor truncated realized volatility tests [14], among others. Besides, Jiang and Oomen [15] proposed a new jump test called the swap variance test, where the test statistic reflects the cumulative gain of a variance swap replication strategy and has significant differences in the presence of jumps or not. These tests are powerful in examining the existence of jumps in an observed path of the underlying price process (see, e.g., [16]), but they cannot identify the precise locations, sizes, and directions of detected jumps. To solve the problem, Andersen et al. [17] defined an intraday scaled return realization to identify a jump's arrival time and size. After that, Lee and Mykland [18] used the standardized return realizations to yield not only the locations and sizes but also the directions of jumps. As an extension of [18], Xue et al. [19] proposed a wavelet method and demonstrated that the property of wavelets is superior for detecting jumps. Since then, some wavelet-type filtering algorithms have been applied to locate and scale jumps (see, e.g., [20, 21]). Most recently, deep learning-based methods have emerged for jump detection. For instance, Chen and Zhang [22] developed a six-layer convolution neural network to identify jump occurrences over both entire time intervals and individual time points, and verified that it outperforms the test of [18]. Overall, these jump tests can not only test the presence of jumps, but also advance knowledge of jump arrival dynamics in observed intraday price series.

Tests for jumps in the observed intraday price series also bring some implications in modeling the underlying price processes with continuous diffusion models or jump diffusion models. However, they cannot be regarded as specification tests for nonzero jump intensity, which is [23]'s so-called inconsistency problem. Theoretically, if the probability of no intraday jumps occurring in a sample path is positive, then the power of the tests will not tend to one, no matter how high the sampling frequency is (see Section 3.5.2 in [24]). In practice, it is shown that the empirical power of the previous jump tests from high-frequency data is not adequate to test small jump intensity of a compound Poisson process (see also [25, 26]). In such cases, Corradi et al. [23] defined a jump intensity test (the CSS test hereafter) by utilizing realized third moments, or so-called tricity. The properties of tricity are completely different for high-frequency data from a diffusion process with jumps or not, while a long time span is required to obtain nontrivial asymptotic power against jump intensity of both the Poisson process and the Hawkes model.

In this paper, we shall apply the realized power variation to test the presence of jump intensity in a stochastic volatility jump diffusion model. Our paper makes three significant contributions compared to the existing literature. First, we establish the infill and long-span asymptotics of realized power variation for stochastic volatility jump diffusion models. Second, we propose a nonparametric jump intensity test named the long-span power variation test (for short, the LSPV test), while the jump intensity relies on the volatility of the asset price process. Finally, we conduct a comprehensive Monte Carlo analysis for the robustness of our test in comparison to the CSS test.

The rest of the paper is organized as follows: In Section 2, we introduce the model setup, and derive the two-dimensional asymptotic properties for realized power variation and the LSPV test statistic. Section 3 reports the simulation findings to examine the robustness of the LSPV test in comparison with the CSS test. Section 4 applies the testing methodology to stock price time series, stock index time series and exchange rate time series. We conclude in Section 5, and all the proofs are collected in an Appendix.

## 2. The setting and the jump intensity test

### 2.1. The setting

Let $X := (X_t)_{t\in\mathbb{R}^+}$ denote an efficient log-price process defined on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in\mathbb{R}^+}, \mathbb{P})$. We consider a stochastic volatility jump diffusion model:

$$dX_t = b(X_t)dt + \sigma_t dW_t + dJ_t, \tag{2.1}$$

$$d\sigma_t = \tilde{\mu}(\sigma_t)dt + \tilde{\sigma}(\sigma_t)dW_t^*, \tag{2.2}$$

where $\sigma := (\sigma_t)_{t\in\mathbb{R}^+}$ is the volatility process; $b(\cdot)$, $\tilde{\mu}(\cdot)$ and $\tilde{\sigma}(\cdot)$ are measurable functions defined on $\mathbb{R}$, ensuring the existence and uniqueness of the solution $X$ and the nonnegativity of $\sigma$; $dW_t^* = \rho dW_t + \sqrt{1-\rho^2}d\widetilde{W}_t$, $-1 \le \rho \le 1$, $W = (W_t)_{t\in\mathbb{R}^+}$ and $\widetilde{W} = (\widetilde{W}_t)_{t\in\mathbb{R}^+}$ are two independent standard Brownian motions.

Assume that the jump process $J := (J_t)_{t\in\mathbb{R}^+}$ is a pure jump process of the form:

$$J_t = \sum_{i=1}^{N_t} Z_i,$$

where $N := (N_t)_{t\in\mathbb{R}^+}$ is a conditional Poisson process [27] with stochastic intensity $\{\lambda(\sigma_t) : t \in \mathbb{R}^+\}$. The jump sizes $\{Z_i\}$ are independently and identically distributed (i.d.d.) with the same distribution $\mu_Z(\cdot)$ and independent of the processes $N$ and $\sigma$. Let $\mu(dt, dz)$ denote the counting measure of $J$:

$$\mu([0,t], U) := \sum_{0\le s\le t} 1_{\{\Delta J_s \in U\}} = \sum_{i=1}^{N_t} 1_{\{Z_i\in U\}}, \quad t > 0,\ U \in \mathcal{B}(\mathbb{R}).$$

Accordingly, $J$ can be specified as

$$J_t = \int_0^t \int_{\mathbb{R}} z\,\mu(ds, dz).$$

Assume that $\mathbb{E}\,\mu([0,t], U) = \mathbb{E}\int_0^t \int_U \lambda(\sigma_s)\mu_Z(dz)ds < \infty$ for each $t \in \mathbb{R}^+$ and $U \in \mathcal{B}(\mathbb{R})$. By the Doob-Meyer decomposition, $\mu(ds, dz)$ admits a compensator denoted by

$$v(ds, dz) := \lambda(\sigma_s)\mu_Z(dz)ds. \tag{2.3}$$

In the sequel, we consider testing whether the process $X$ is a diffusion or not, i.e., whether the jump intensity function $\lambda(\cdot)$ is zero or not. Thus, the hypotheses of interest in this paper are

$$H_0 : \lambda(\cdot) = 0 \ \ and \ \ H_1 : \lambda(\cdot) > 0.$$

### 2.2. Infill and long-span asymptotic results

Assume that $X$ is observed at equally spaced time points $i\Delta$ $(0 < \Delta < 1)$, $i = 0, 1, ..., n$, $n = T/\Delta$, with $T \to \infty$ and $\Delta \to 0$. We introduce the definitions of realized power variation and the truncation version [9]:

$$\begin{aligned} \hat{B}(p, T, \Delta) &:= \sum_{i=1}^{T/\Delta} |\Delta_i^n X|^p,\ p > 0, \\ \mathrm{Tr}\hat{B}(p, T, \Delta) &:= \sum_{i=1}^{T/\Delta} |\Delta_i^n X|^p 1_{\{|\Delta_i^n X| \le \gamma\Delta^\theta\}}, \quad \gamma > 0,\ 0 < \theta < \frac{1}{2}, \end{aligned} \tag{2.4}$$

where $\Delta_i^n X := X_{i\Delta} - X_{(i-1)\Delta}$. To deduce two-dimensional asymptotic results of the statistics in (2.4), the following assumptions are needed, which also ensure the existence and uniqueness of solutions to (2.1) and (2.2).

**Assumption 2.1.** *(a) $\tilde{\mu}(\cdot)$ and $\tilde{\sigma}(\cdot)$ satisfy the following conditions:*

$$|\tilde{\mu}(y) - \tilde{\mu}(y')| \le \kappa(|y - y'|),$$
$$|\tilde{\sigma}(y) - \tilde{\sigma}(y')| \le \varrho(|y - y'|), \tag{2.5}$$
$$|\tilde{\mu}(y)| + |\tilde{\sigma}(y)| \le C_1(1 + |y|), \tag{2.6}$$

*for any $y, y' \in \mathbb{R}$, where $C_1$ is a positive constant, $\kappa(\cdot)$ and $\varrho(\cdot)$ are nonnegative, strictly increasing, and concave with $\kappa(0) = 0$ and $\varrho(0) = 0$, satisfying $\int_{(0,\varepsilon)} \kappa^{-1}(u)du = \infty$ and $\int_{(0,\varepsilon)} \varrho^{-2}(u)du = \infty$ for any $\varepsilon > 0$, respectively. Furthermore, $\tilde{\mu}(\cdot)$ and $\tilde{\sigma}(\cdot)$ ensure the nonnegativity of the volatility process $\sigma$.*

*(b) $b(\cdot)$ satisfies the Lipschitz and linear growth conditions, and $\lambda(\cdot)$ satisfies the linear growth condition:*

$$|b(u) - b(u')| \le L\,|u - u'|\,, \tag{2.7}$$
$$|b(u)| + |\lambda(u)| \le C_2(1 + |u|), \tag{2.8}$$

*for any $u$, $u' \in \mathbb{R}$, where $L$ and $C_2$ are positive constants.*

*(c) There exists $p > 2$ such that $\sup_{t\in\mathbb{R}^+} \mathbb{E}|\sigma_t|^{2p} < \infty$, $\sup_{t\in\mathbb{R}^+} \mathbb{E}|X_t|^{2p} < \infty$, and $\mathbb{E}|Z_1|^{2p} < \infty$.*

*(d) $\sigma$ is an ergodic process with a unique invariant probability measure $\pi$, satisfying $0 < \pi_p < \infty$, for the $p$ in (c), where $\pi_p := \int_{\mathbb{R}} |x|^p \,\pi(dx)$.*

Regarding Assumption 2.1, we further explain as follows.

**Remark 2.1.** *Assumption 2.1(a) is sufficient for the existence and uniqueness of a strong solution $\sigma$ to (2.2) by Yamada-Watanabe theorem (Proposition 5.2.13 of [28]). In terms of sufficient conditions of global nonnegativity of $\sigma$, one can refer to [29–31] and references therein.*

**Remark 2.2.** *By Assumption 2.1(b) and the conditions that $\sup_{0\le s\le t} \mathbb{E}\,\sigma_s^2 < \infty$, $\mathbb{E}\,|Z_1| < \infty$, (2.1) has a unique solution $X$ (see Theorem 5.1.2 of [32] and Theorem 2.32 of [33]).*

**Remark 2.3.** *The first two inequalities of Assumption 2.1(c) can be checked by the moment conditions that $\sup_{t\in\mathbb{R}^+} \mathbb{E}|X_t|^r < \infty$ and $\sup_{t\in\mathbb{R}^+} \mathbb{E}|\sigma_t|^r < \infty$, for any $r \in \mathbb{Z}^+$, which are the standard condition in statistical inference for an ergodic diffusion (see [34]) or for an ergodic jump-diffusion [35, 36]. Their sufficient conditions are provided by [37] for diffusion models and [38] for jump-diffusion models.*

**Remark 2.4.** *With the ergodic property in Assumption 2.1(d), the ergodic theorem of $\sigma$ holds: $\int_0^T |\sigma_t|^p dt/T \xrightarrow{a.s.} \pi_p$, $T \to \infty$. It has been proven for a path-dependent diffusion (see e.g., Proposition 3.3 of [39], Theorems V.53.1 and V.53.5 of [40]). In addition, $\pi_\lambda := \int_{\mathbb{R}} \lambda(x)\pi(dx) < \infty$ holds, which yields $\int_0^T \lambda(\sigma_t)dt/T \xrightarrow{a.s.} \pi_\lambda$.*

**Remark 2.5.** *A typical example of a volatility process satisfying Assumption 2.1 is the square-root diffusion process (see [41]).*

**Theorem 2.1.** *Suppose that Assumption 2.1 holds with $p > 2$.*

(1) *Under $H_0$:*

$$\frac{1}{T}\hat{B}(p,T,\Delta)\Delta^{1-\frac{p}{2}} \xrightarrow{\mathbb{P}} \pi_p\, d_p, \tag{2.9}$$

*as $T \to \infty$ and then $\Delta \to 0$, or $T \to \infty$ and $\Delta \to 0$ simultaneously, where $d_p := \mathbb{E}|D|^p$, $D \sim N(0,1)$.*

(2) *Under $H_1$:*

$$\frac{1}{T}\hat{B}(p,T,\Delta) \xrightarrow{\mathbb{P}} \pi_\lambda\, \mathbb{E}|Z_1|^p, \tag{2.10}$$

*as $T \to \infty$ and then $\Delta \to 0$, or $T \to \infty$ and $\Delta \to 0$ simultaneously.*

(3) *Under $H_0$ or $H_1$, for the parameter $\frac{1}{2} - \frac{1}{p} < \theta < \frac{1}{2}$ in the the threshold level:*

$$\frac{1}{T}\mathrm{Tr}\hat{B}(p,T,\Delta)\Delta^{1-\frac{p}{2}} \xrightarrow{\mathbb{P}} \pi_p\, d_p, \tag{2.11}$$

*as $T \to \infty$ and then $\Delta \to 0$, or $T \to \infty$ and $\Delta \to 0$ simultaneously.*

Theorem 2.1 shows that $\hat{B}(p,T,\Delta)$ carries information about the diffusive volatility, the probability of arrival, or the features of the jump size distribution. Furthermore, under the two-dimensional asymptotic scheme, we have

$$\hat{\mathrm{T}}(p,k,T,\Delta) \xrightarrow{\mathbb{P}} \begin{cases} k^{\frac{p}{2}-1}, & \lambda(\cdot) = 0, \\ 1, & \lambda(\cdot) > 0, \end{cases}$$

where

$$\hat{\mathrm{T}}(p,k,T,\Delta) := \frac{\hat{B}(p,T,k\Delta)}{\hat{B}(p,T,\Delta)}, \quad k > 1.$$

To obtain the critical region of the jump intensity test based on $\hat{\mathrm{T}}(p,k,T,\Delta)$, we need additional assumptions.

**Assumption 2.2.** *Assume that the following conditions hold:*

(a) *$T\sqrt{\Delta} \to 0$.*

(b) *There exists $p \geq 2$ such that $\sup_{t\in\mathbb{R}^+}\mathbb{E}|\sigma_t|^{4p} < \infty$, $\sup_{t\in\mathbb{R}^+}\mathbb{E}|X_t|^{4p} < \infty$, $\mathbb{E}|Z_1|^{4p} < \infty$, and $0 < \pi_{2p} < \infty$.*

(c) *The function $\varrho(\cdot)$ in (2.5) is of the form $\varrho(x) = \alpha x^\beta$, $\alpha > 0$, $\beta \in [\frac{1}{2}, 1]$.*

**Theorem 2.2.** *Suppose that Assumptions 2.1 and 2.2 hold with $p \geq 2$. Under $H_0$, we have*

$$\frac{\sqrt{T}}{\sqrt{\Delta}}(\hat{\mathrm{T}}(p,k,T,\Delta) - k^{\frac{p}{2}-1}) \xrightarrow{d} N(0,e^2), \tag{2.12}$$

*as $\Delta \to 0$, $T \to \infty$ simultaneously, where $e^2 = G(p,k)\pi_{2p}/\pi_p^2$, $G(p,k) = \Big(k^{p-2}(1+k)\,d_{2p} + k^{p-2}(k-1)\,d_p^2 - 2k^{\frac{p}{2}-1}\ell_{k,p}\Big)/d_p^2$, $\ell_{k,p} = \mathbb{E}\Big(|D_1|^p|D_1 + \sqrt{k-1}D_2|^p\Big)$, and $D_1$ and $D_2$ are independent standard normal variables.*

**Remark 2.6.** *Assumption 2.2(a) and (c) ensure that the Euler approximation error in the proof of Assumption 2.2 is negligible (see the convergence of (A.32) in the Appendix for details).*

*2.3. The test statistic for nonzero jump intensity*

From (2.11) and (2.12), we obtain a feasible test statistic:

$$\check{\mathrm{T}}(p,k,T,\Delta) := \sqrt{\frac{T}{\Delta}}\frac{(\hat{\mathrm{T}}(p,k,T,\Delta) - k^{\frac{p}{2}-1})}{\hat{e}},$$

where

$$\hat{e} := \sqrt{G(p,k)\frac{\hat{\pi}_{2p}}{\hat{\pi}_p^2}}, \quad \hat{\pi}_p := \frac{\Delta^{1-\frac{p}{2}}}{Td_p}\mathrm{Tr}\hat{B}(p,T,\Delta),$$

and the following result holds for the jump intensity test.

**Corollary 2.1.** *Suppose that Assumptions 2.1 and 2.2 hold with $p > 2$, $\frac{1}{2} - \frac{1}{p} < \theta < \frac{1}{2}$. Let $T \to \infty$ and $\Delta \to 0$ simultaneously. Under $H_0$, we have*

$$\check{\mathrm{T}}(p,k,T,\Delta) \xrightarrow{d} N(0,1), \tag{2.13}$$

*and* $\lim_{\substack{T\to\infty \\ \Delta\to 0}} \mathbb{P}\left(\check{\mathrm{T}}(p,k,T,\Delta) < z_\alpha | H_0\right) < \alpha$. *Under $H_1$, we have*

$$\lim_{\substack{T\to\infty \\ \Delta\to 0}} \mathbb{P}(\check{\mathrm{T}}(p,k,T,\Delta) < Z_\alpha | H_1) = 1, \tag{2.14}$$

*where $\check{\mathrm{T}}(p,k,T,\Delta) \asymp_{\mathbb{P}} -\sqrt{T/\Delta}$. Here, $z_\alpha$ denotes the $\alpha$-quantile of $N(0,1)$, and the notation $x_n \asymp_{\mathbb{P}} y_n$ means that $x_n$ is of the same order as $y_n$ in probability, i.e., $x_n = O_p(y_n)$ and $y_n = O_p(x_n)$.*

## 3. Finite-sample performance

In this section, we study the finite-sample performance of our test and compare it with the CSS test. We implement experiments for different choices of sampling period $T$ together with a fixed sampling interval $\Delta$, or vice versa. The number of Monte Carlo replications for every experiment is 1000. Data in our experiments are generated from the following stochastic volatility model:

$$\begin{cases} dX_t &= bdt + \sigma_t dW_t + dJ_t, \\ d\sigma_t &= k_v(\theta_v - \sigma_t)dt + \zeta\sqrt{\sigma_t}dW_t^*, \end{cases} \tag{3.1}$$

where $b \in \mathbb{R}$, $k_v, \theta_v, \zeta > 0$, $J_t = \sum_{i=1}^{N_t} Z_i$; $\{Z_i\}$ are i.i.d. with jump density function $f_Z(z)$; $N$ is a Poisson process with intensity $\lambda$; $[W, W^*]_t = \rho t$. Sample paths are simulated using the Euler discretization scheme.

We now select the threshold levels for the two tests, respectively. The threshold level of the LSPV test is of the form $\tau(\Delta)_1 = \gamma\Delta^\theta$, $\gamma > 0$, $1/2 - 1/p < \theta < 1/2$. In practical applications, the parameter $\theta$ is typically set near 1/2 to avoid an excessively loose threshold. Thus, we set $\theta = 0.47$. As regards the parameter $\gamma$, we follow the rationale illustrated in [42] and set $\gamma = 5\hat{\sigma}^{(n,1)}$, where $\hat{\sigma}^{(n,1)} = \sqrt{\sum_{i=1}^n |\Delta_i^n X|^2 1_{\{|\Delta_i^n X|^2 \le \Delta^{0.99}\}}/T}$ is the threshold estimator of the long-run average spot volatility [43]. To ensure the rate conditions for the truncation level of the CSS test, we use the form $\tau(\Delta)_2 = \hat{\sigma}^{(n,2)} T^{-0.251}$ as specified in [23], where $\hat{\sigma}^{(n,2)} = \sqrt{\sum_{i=1}^n |\Delta_i^n X - (X_T - X_\Delta)/n|^2/T}$.

### 3.1. Finite-sample distribution of $\check{T}(p, k, T, \Delta)$

To investigate the finite-sample distribution of the LSPV test statistic $\check{T}(p, k, T, \Delta)$ under $H_0$ and $H_1$, we set the model parameters $(b, k_v, \theta_v, \zeta, \rho, \lambda) = (0.1, 5, 0.4, 0.5, -0.5, 0.5)$ and the jump density function $f_Z(z) = N(0, 1.5^2)$. For the parameters in $\check{T}(p, k, T, \Delta)$, we set $p = 4$, $k = 2$, and $T \in \{10, 50, 100\}$ together with $\Delta = 1/78$ (5 min) and $\Delta = 1/156$ (2.5 min).

First, Figure 1 presents the kernel density estimators of $\check{T}(p, k, T, \Delta)$. Under $H_0$, the finite-sample distribution of $\check{T}(p, k, T, \Delta)$ is approximately standard normal, which is consistent with (2.13) in Corollary 2.1. Furthermore, for a fixed sampling interval $\Delta$, $\check{T}(p, k, T, \Delta)$ with a larger time span $T$ makes a better distinction between the continuous diffusion model and the jump diffusion model. In addition, compared with the case of $\Delta = 1/78$, the distribution of $\check{T}(p, k, T, \Delta)$ for $\Delta = 1/156$ exhibits a more significant difference for models with jumps or not. Thus, when the time span and the sampling frequency increase, the test statistic $\check{T}(p, k, T, \Delta)$ would better discriminate a jump diffusion model from a continuous diffusion model, which demonstrates (2.14) in Corollary 2.1.

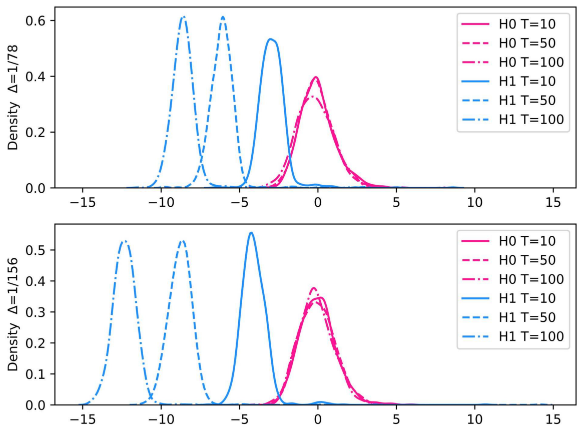



**Figure 1.** Finite-sample distributions of the studentized test statistic $\check{T}(p, k, T, \Delta)$. Notes: Presented are Monte Carlo distributions of the LSPV test statistic, using data generating process (3.1) with $\lambda = 0$ (pink lines), $\lambda = 0.5$ and $f_Z(z) = N(0, 1.5^2)$ (blue lines). In either continuous or discontinuous cases, the time spans are $T = 10$ (solid lines), $T = 50$ (dashed lines) and $T = 100$ (dash-dotted lines) together with $\Delta = 1/78$ and $\Delta = 1/156$.

### 3.2. Robustness analysis

In the sequel, we study the robustness of the LSPV and CSS tests over different choices of sampling schemes and model specifications. Suppose that $X$ follows model (3.1), and the parameters $b, k_v, \theta_v, \zeta, \rho$, $p$, and $k$ are the same as those in Section 3.1. To select appropriate values for jump intensity $\lambda$ and jump size variance in the compound Poisson process, we introduce the definition of the proportion of return variance due to jumps [44]:

$$r := \frac{V_d}{V_c + V_d}, \tag{3.2}$$

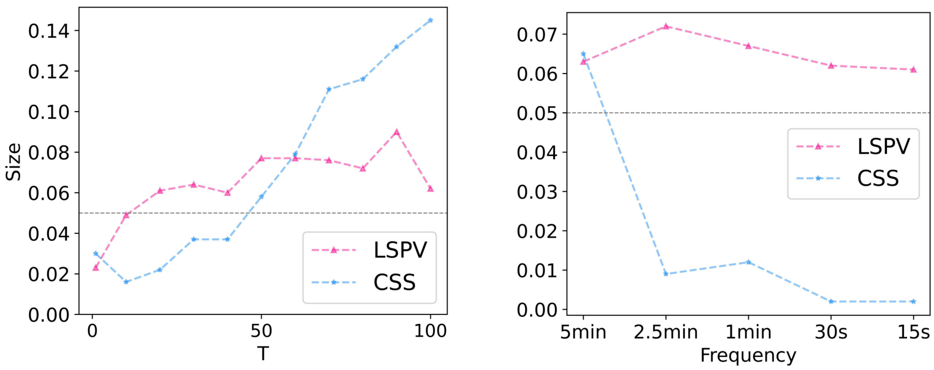
where $V_c := \lim_{T\to\infty} \int_0^T \sigma_t^2 dt/T$ and $V_d := \lim_{T\to\infty} Var(J_T)/T = \lambda\sigma_Z^2$, represent the long-run variance of diffusion and jump components, respectively.

### 3.2.1. Sampling schemes

We shall compare the finite-sample size and power performance of the two tests across different sampling schemes. Suppose that the jump intensity parameter $\lambda \in \{0.1, 0.4, 0.7\}$ and the jump density function $f_Z(z) = N(0, \sigma_Z^2)$. Since the square-root diffusion process $\sigma$ in (3.1) with $\theta_v, k_v, \zeta > 0$ is ergodic with an invariant distribution $\pi$ being a Gamma law $\Gamma(2k_v\theta_v/\zeta^2, \zeta^2/(2k_v))$ (see [41]), we have $V_c = \int_{\mathbb{R}^+} x^2\pi(dx) = \theta_v^2 + \theta_v\zeta^2/(2k_v) = 0.17$. Then, we set $\sigma_Z^2 = 0.2$ such that the contribution of the jump component to the total variance $r$ ranges from 10% to 50%, depending on different choices of $\lambda$. We set $\Delta$=1/78 with an increasing time span $T$ ranging from 1 to 100, and then fix $T = 50$ days with a decreasing $\Delta$ ranging from 1/78 (5 min) to 1/4680 (15s). Note that the CSS test statistic includes an additional parameter, $T_1$, which we set to $10T$.

Figure 2 demonstrates that the LSPV test gives stable sizes near the nominal 5% level. For instance, the sizes stay close to 5% for $T$ =80, 90, and 100 days when $\Delta = 1/78$, as well as for various sampling frequencies with $T$ fixed at 50 days. In contrast, the CSS test yields relatively unstable sizes across different $T$ and $\Delta$ values. These variations may be due to changes in truncation thresholds as $T$ or $\Delta$ varies. Specifically, as $T$ increases, the threshold becomes tighter, and more increments would be removed, leading to a smaller acceptance region. Conversely, when the sampling frequency increases, more increments would be sufficiently small to be retained, resulting in a larger acceptance region.

Furthermore, some findings of Figure 3 are illustrated as follows: First, subfigures from left to right show that the LSPV and CSS tests become more powerful as $\lambda$ increases, due to more frequent occurrences of jumps. Second, the upper row of subfigures in Figure 3 shows that the powers of the LSPV and CSS tests approach one as T increases. In particular, when the jump volatility accounts for more than 30% of the total variance of returns (i.e., $r > 30\%$), the powers of the two tests reach 95% for $T = 50$ days. Finally, subfigures in the lower row of Figure 3 show that when $r > 30\%$, the powers exceed 95% in the case of 2.5-minute frequency, and get larger when the sampling frequency increases. These results imply that the jump part could be better disentangled from the continuous part of a sample path when time span and sampling frequency increase.




**(a)** Size results for time span $T \in \{$ 1, 10, 20,..., 100$\}$ and sampling interval $\Delta$=1/78 (5 min).

**(b)** Size results for $\Delta = \{$ 1/78, 1/156, 1/390, 1/780, 1/1560$\}$ (5 min, 2.5 min, 1 min, 30s, 15s) and $T = 50$ days.

**Figure 2.** Empirical sizes. Note: Figure 2 presents estimated sizes of the LSPV and CSS tests using 5% critical values for different sampling schemes.

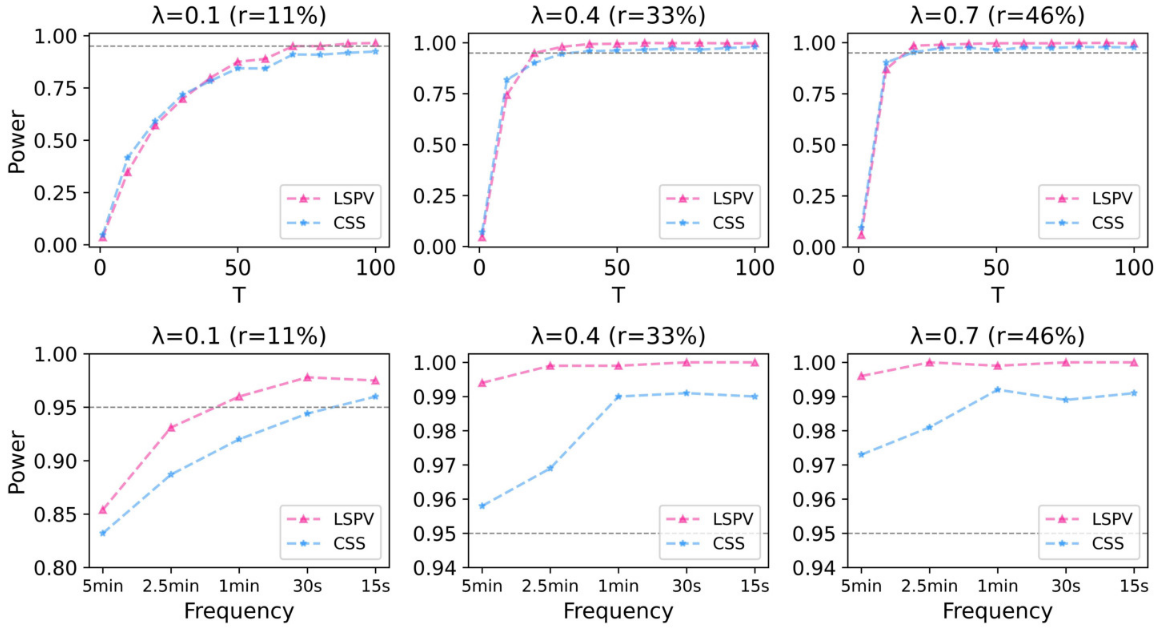



**Figure 3.** Empirical powers. Note: Rejection frequencies of the LSPV and CSS tests are presented using 5% critical values. The first row of subfigures provides power results for time span $T \in \{1, 10, 20, ..., 100\}$ and sampling interval $\Delta = 1/78$ (5 min), while the second row of subfigures reveals power results for $\Delta = \{1/78, 1/156, 1/390, 1/780, 1/1560\}$ (5 min, 2.5 min, 1 min, 30s, 15s) and $T = 50$ days. We set the jump intensity $\lambda \in \{0.1, 0.4, 0.7\}$ and jump density function $f_Z(z) = N(0, 0.2)$ such that the proportions of return variance due to jumps $r$ (see (3.2)) are lower than 50%.

In summary, our test exhibits stable sizes close to the level of significance over various sampling schemes, while the powers generally increase as $T$ grows. These findings suggest that our jump test yields adequate sizes over different permutations of $T$ and $\Delta$, whereas an expanding $T$ is required for a consistent test against jump intensity of the Poisson process.

3.2.2. Jump intensity processes

In this section, we verify the power robustness of the LSPV and CSS tests against different jump intensity models, including a positive constant intensity $\lambda_t = \lambda$, an affine jump intensity process $\lambda_t = \lambda_1 \sigma_t$, an exponential affine jump intensity process $\lambda_t = e^{-\lambda_2 \sigma_t}$, and a Hawkes model $d\lambda_t = a(\lambda_3 - \lambda_t)dt + bdN_t$, where $(\lambda, \lambda_1, \lambda_2, \lambda_3, a, b)$=(0.2, 0.5, 4, 0.2, 1, 0.4). Here, the volatility-dependent models are used to depict linear, nonlinear, positive, and negative relations between $\lambda_t$ and $\sigma_t$ (see [45]), while the Hawkes model characterizes the self-excitation in jumps (see [23]). Let the jump density function be $f_Z(z) = N(0, 0.2)$. If we use the long-run mean $\theta_v = 0.4$ of the volatility process $\sigma$ to approximate the value of spot volatility $\sigma_t$, then all the proportions of return variance due to jumps in (3.2) for the above jump intensity models are roughly 20%. We set $T = 50$ days with $\Delta \in \{1/156, 1/78\}$ (2.5 min, 5 min). For comparison purposes, we also consider the case of $T = 1$ day with $\Delta \in \{1/23400, 1/78\}$ (1s, 5 min).

Table 1 displays the testing results for different jump intensity processes with a significance level of $\alpha = 5\%$. When $T = 50$, the powers of the LSPV and CSS tests are robust to different jump intensity models. In particular, for $T = 50$ with regard to $\Delta \in \{1/156, 1/78\}$, the powers of the LSPV tests are bigger than the CSS test. However, when $T = 1$, the two tests have insufficient powers against different jump intensity models, no matter how high the frequency is. These results imply again that expanding the time span $T$ is essential for fully identifying the presence of a jump component in a Poisson-type jump diffusion model.

**Table 1.** Empirical powers for different jump intensity processes.

| | Constant | | Affine | | Exponential affine | | Hawkes | |
|---|---|---|---|---|---|---|---|---|
| ($\Delta$,$T$) | LSPV | CSS | LSPV | CSS | LSPV | CSS | LSPV | CSS |
| (1/23400,1) | 0.238 | 0.444 | 0.175 | 0.484 | 0.418 | 0.526 | 0.227 | 0.520 |
| (1/78,1) | 0.029 | 0.051 | 0.043 | 0.073 | 0.066 | 0.080 | 0.034 | 0.085 |
| (1/156,50) | 0.994 | 0.967 | 0.994 | 0.967 | 0.996 | 0.979 | 0.993 | 0.968 |
| (1/78,50) | 0.974 | 0.922 | 0.965 | 0.910 | 0.994 | 0.947 | 0.978 | 0.947 |

Notes: Table 1 gives estimated powers of the 5% level for the LSPV and CSS tests against a constant intensity, an affine intensity process, an exponential affine intensity process, and a Hawkes model. For each jump model, the proportion of return variance due to jumps $r$ in (3.2) is nearly 20%.

### 3.2.3. Jump size distributions

In the sequel, we examine the power robustness of the LSPV and CSS tests for different jump size distributions. We set $\lambda = 0.1$ together with $f_Z(z) = N(0, 0.2)$, $N(0.5, 0.2)$, and $5e^{-5z}1_{\{z\geq 0\}}$ such that the proportions of return variance due to jumps are around 11%. We set $\Delta = 1/78$ and $T \in \{50, 60, 70, 80\}$ for comparison purposes.

Table 2 shows that the LSPV test exhibits more stable powers over various jump density functions compared to the CSS test. For example, when the jump density function $f_Z(z) = N(0.5, 0.2)$ is replaced by $f_Z(z) = N(0, 0.2)$, the powers of the CSS test fall down more rapidly than that of the LSPV test. This is because the CSS test statistic is of order $1/\Delta$ or $\sqrt{T}/\Delta$ with respect to jump distribution being symmetric or not, corresponding to the order $-\sqrt{T/\Delta}$ of the LSPV test statistic. In this regards, our test is more robust to different jump size distributions compared to the CSS test.

**Table 2.** Empirical powers for different jump size distributions.

| | N(0,0.2) | | N(0.5,0.2) | | E(5) | |
|---|---|---|---|---|---|---|
| ($\Delta$,$T$) | LSPV | CSS | LSPV | CSS | LSPV | CSS |
| (1/78,50) | 0.862 | 0.840 | 0.945 | 0.962 | 0.853 | 0.915 |
| (1/78,60) | 0.898 | 0.853 | 0.978 | 0.980 | 0.917 | 0.960 |
| (1/78,70) | 0.925 | 0.892 | 0.985 | 0.996 | 0.952 | 0.980 |
| (1/78,80) | 0.947 | 0.898 | 0.988 | 0.995 | 0.965 | 0.988 |

Notes: The estimated powers of the 5% level for the LSPV and CSS tests are presented with respect to different jump density functions. In each case, the proportion of return variance due to jumps $r$ in (3.2) is around 11%.

3.2.4. Microstructure noise

In the sequel, we assume that a log-price process is given by

$$Y_{i\Delta} = X_{i\Delta} + \varepsilon_{i\Delta},\ i = 1, ..., n,$$

where the process $X_t$ follows model (3.1) and $\varepsilon$ are i.d.d. from the $N(0, \varpi^2)$. We set $\lambda = 0.1$ and $f_Z(z) = N(0, 0.2)$ such that $r$ in (3.2) approximates 11%. Furthermore, we set $\varpi \in \{0.0001, 0.0005, 0.001\}$ under microstructure noise, referring to estimation results for real stock price series (see e.g., [46,47]). Let $\Delta = 1/78$ and $T \in$\{50, 60, 70, 80\} for comparison purposes. Columns from left to right in Table 3 show that the LSPV and CSS tests exhibit stable sizes and powers over different levels of microstructure noise at a 5-minute sampling frequency. This is expected since sampling sparsely at a lower frequency alleviates the adverse impact of microstructure noise.

**Table 3.** Empirical sizes and powers for different standard deviations of microstructure noise.

| | | $\varpi = 0.0001$ | | $\varpi = 0.0005$ | | $\varpi = 0.001$ | |
|---|---|---|---|---|---|---|---|
| | ($\Delta$,$T$) | LSPV | CSS | LSPV | CSS | LSPV | CSS |
| size | (1/78,50) | 0.073 | 0.059 | 0.076 | 0.066 | 0.062 | 0.062 |
| | (1/78,60) | 0.064 | 0.074 | 0.070 | 0.076 | 0.058 | 0.079 |
| | (1/78,70) | 0.069 | 0.095 | 0.062 | 0.093 | 0.081 | 0.093 |
| | (1/78,80) | 0.062 | 0.101 | 0.075 | 0.111 | 0.068 | 0.107 |
| power | (1/78,50) | 0.867 | 0.839 | 0.856 | 0.807 | 0.862 | 0.829 |
| | (1/78,60) | 0.894 | 0.860 | 0.904 | 0.850 | 0.913 | 0.827 |
| | (1/78,70) | 0.932 | 0.903 | 0.934 | 0.881 | 0.932 | 0.892 |
| | (1/78,80) | 0.944 | 0904 | 0.946 | 0.887 | 0.960 | 0.910 |

Notes: Table 3 provides estimated sizes and powers of the 5% level for the LSPV and CSS tests over different standard deviations of microstructure noise. Under $H_1$, in each case, the proportion of return variance due to jumps $r$ in (3.2) is approximately 11%.

## 4. Empirical analysis

In the following empirical study, we apply the LSPV and CSS tests to the price series from Dukascopy Bank, to ascertain whether the underlying price models are diffusions or jump diffusions. Specifically, we analyze (i) two stock prices, including Apple Inc. (AAPL) and Amazon.com, Inc. (AMZ); (ii) two stock indexes, including DAX 40 Index (DAX40) and Nikkei 225 Index (N225); (iii) two exchange rates, including the exchange rate of New Zealand Dollars to one Swiss franc (NZDCHF) and U.S. Dollars to one Yen (USDJPY). We apply the threshold levels specified in the simulation study. In the sampling scheme, we set $T \approx 60$, $T_1 = 10T$, and $\Delta = 1/156$ to ensure the robustness of the two tests verified in the simulations.

The steps involved in sampling and processing data for the specified jump tests are as follows. First, we collect trade data at a 1-second frequency from August 01, 2023, to October 31, 2023. Data from May 01, 2021, to July 31, 2023, is also sampled to improve the robustness of the CSS test against the leverage effect. In each trading day, the trading period runs from 16:30 to 23:00, with the first post-16:30 trade discarded to avoid overnight effects (see [47]). Second, we use the previous-tick sampling scheme to obtain equally-spaced trade data (see [48]). Third, to measure the level of the microstructure noise, we use the 1-second log-price data between May 01, 2021, and October 31, 2023, to estimate the standard deviation $\varpi$ of the microstructure noise (see [49]). Finally, the 2.5-minute trade data are sampled from the tick-by-tick log-price series to examine the null of zero jump intensity for the underlying price processes.

The estimations and testing results are presented in Table 4. Since the estimations of $\varpi$ are sufficiently small, we conclude from our simulation analysis in Section 3.2.4 that there exists few microstructure effects on the two jump tests. For all the price series, the LSPV and CSS tests reject the null of diffusions at the 5% significance level. Thus, the testing results confirm the presence of jump intensity, meaning the jump arrival rate is significant for each underlying price model. However, the CSS test yields significantly lower p-values compared with the LSPV test. For instance, the LSPV test yields p-values above 0.1% for the two exchange rate time series, while the p-values for the CSS test are both close to zero. These results are likely due to jump asymmetry, as illustrated in Section 3.2.3.

**Table 4.** Jump test statistics for stock prices, stock indexes, and exchange rates.

| | AAPL | AMZ | DAX40 | N225 | NZDCHF | USDJPY |
|---|---|---|---|---|---|---|
| $\hat{\varpi}$ | 8.30e-5 | 5.65e-4 | 5.40e-5 | 6.00e-5 | 3.20e-5 | 2.80e-5 |
| LSPV test | 0.000*** | 0.000*** | 0.000*** | 0.003*** | 0.001*** | 0.004*** |
| CSS test | 0.000*** | 0.000*** | 0.000*** | 0.000*** | 0.000*** | 0.000*** |

Notes: Entries are jump intensity test statistics for six randomly selected price series from Dukascopy Bank sampling at a 2.5-minute frequency from August 01, 2023, to October 31, 2023. The symbols *** and ** indicate rejections at 1% and 5% significance levels, respectively; "ae-b" denotes $a \times 10^{-b}$, where $a, b > 0$.

## 5. Conclusions

This paper develops the infill and long-span asymptotic theory of realized power variation for stochastic volatility jump diffusion models, thereby proposing a novel jump intensity test. Monte Carlo experiments confirm that the proposed test exhibits adequate and stable sizes and powers over different sampling schemes and model specifications. The empirical findings confirm the prevalence of jump intensities in the underlying price processes.

Our testing method targets general stochastic volatility jump-diffusion models. For future research, it might be of interest to develop a kernel function-based jump test for specific model specifications (see e.g., [50]). Furthermore, given empirical evidence of rough volatility in recent literature (see for instance, [51]), it might be worthwhile to generalize our testing method to fractional processes with an additive jump component and develop jump-robust inference on the Hurst parameter [52].

## Author contributions

Zhongyu Chen: Conceptualization, Formal analysis, Methodology, Investigation, Data curation, Software, Writing; Juliang Yin: Conceptualization (supporting), Formal analysis (supporting), Funding acquisition, Supervision, Writing (supporting). All authors have read and approved the final version of the manuscript for publication.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Acknowledgments

This work was supported by the National Natural Science Foundation of China under Grant No. 61973096.

## Conflict of interest

The authors declare that there are no competing interests.

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## Appendix

In the following, $K$, $K_p$ and $K_q$ denote some constants that are independent of $\Delta$, $T$ and $i$, $i = 1, ..., \frac{T}{\Delta}$, while $K_p$ and $K_q$ depend on $p > 0$ and $q > 0$, respectively. These constants may change from line to line. $\mathbb{E}_i^n$ denotes the conditional expectation given $\mathscr{F}_{i\Delta}$. Note that in [9], proofs for realized power variation rely on sample path properties over a fixed time horizon $[0, T]$. Most of these proof techniques become unfeasible as $T \to \infty$, particularly for the standard localization procedure via a stopping time. We therefore substitute the local boundedness conditions used therein with Assumption 1(a),(b) and Assumption 2(b),(c). Below are some moment estimates for jump diffusion processes.

**Lemma A.1.** *Under $H_0$, suppose that the linear growth conditions* (2.6) *and* (2.8) *hold, and there exists $p \geq 2$ such that* $\sup_{t\in\mathbb{R}^+} \mathbb{E}\,|\sigma_t|^p < \infty$ *and* $\sup_{t\in\mathbb{R}^+} \mathbb{E}\,|X_t|^p < \infty$. *Then we have*

$$\sup_{(i-1)\Delta\leq s\leq i\Delta} \mathbb{E}\left|\sigma_s - \sigma_{(i-1)\Delta}\right|^p \leq K_p\Delta^{\frac{p}{2}}, \tag{A.1}$$

$$\mathbb{E}\left|\Delta_i^n X - \sigma_{(i-1)\Delta}\Delta_i^n W\right|^p \leq K_p\Delta^p, \tag{A.2}$$

$$\mathbb{E}\left|\Delta_i^n X\right|^p \leq K_p\Delta^{\frac{p}{2}}, \tag{A.3}$$

$$\mathbb{E}\left|\sigma_{(i-1)\Delta}\Delta_i^n W\right|^p \leq K_p\Delta^{\frac{p}{2}}. \tag{A.4}$$

*Proof.* By the $C_r$ inequality, Hölder's inequality, the Burkholder-Davis-Gundy inequality and (2.6), for

$(i-1)\Delta \le s \le i\Delta$, we have

$$\begin{aligned}\mathbb{E}\left|\sigma_s - \sigma_{(i-1)\Delta}\right|^p &= \mathbb{E}\Big|\int_{(i-1)\Delta}^{s} \tilde{\mu}(\sigma_u)du + \int_{(i-1)\Delta}^{s} \tilde{\sigma}(\sigma_u)dW_u^*\Big|^p \\ &\le K_p\,\mathbb{E}\Big|\int_{(i-1)\Delta}^{s} \tilde{\mu}(\sigma_u)du\Big|^p + K_p\,\mathbb{E}\Big|\int_{(i-1)\Delta}^{s} \tilde{\sigma}(\sigma_u)dW_u^*\Big|^p \\ &\le K_p\,\Delta^{p-1}\int_{(i-1)\Delta}^{i\Delta} \mathbb{E}\left|\tilde{\mu}(\sigma_u)\right|^p du + K_p\,\mathbb{E}\Big(\int_{(i-1)\Delta}^{i\Delta} \tilde{\sigma}^2(\sigma_u)du\Big)^{\frac{p}{2}} \\ &\le K_p \sup_{t\in\mathbb{R}^+}\mathbb{E}\,|\sigma_t|^p(\Delta^p + \Delta^{\frac{p}{2}}) \le K_p\Delta^{\frac{p}{2}},\end{aligned}$$

and then the required assertion (A.1) follows. By (2.8) and (A.1), we have

$$\begin{aligned}\mathbb{E}\left|\Delta_i^n X - \sigma_{(i-1)\Delta}\Delta_i^n W\right|^p &= \mathbb{E}\Big|\int_{(i-1)\Delta}^{i\Delta} b(X_s)ds + \int_{(i-1)\Delta}^{i\Delta}\Big(\sigma_s - \sigma_{(i-1)\Delta}\Big)dW_s\Big|^p \\ &\le K_p \sup_{t\in\mathbb{R}^+}\mathbb{E}\,|X_t|^p\Delta^p + K_p \sup_{(i-1)\Delta\le s\le i\Delta}\mathbb{E}\left|\sigma_s - \sigma_{(i-1)\Delta}\right|^p\Delta^{\frac{p}{2}} \\ &\le K_p\Delta^p,\end{aligned}$$

which gives (A.2). Similarly, it is easy to derive (A.3) and (A.4). □

**Lemma A.2.** *Under $H_1$, assume that* (2.8) *holds, and there exists $p \ge 1$ such that* $\sup_{t\in\mathbb{R}^+}\mathbb{E}\,|\sigma_t|^p < \infty$, $\mathbb{E}\,|Z_1|^p < \infty$. *Let $0 < q \le p$, we have*

$$\mathbb{E}\left|J_{i\Delta} - J_{(i-1)\Delta}\right|^q \le K_q\Delta. \tag{A.5}$$

*Proof.* If $0 < q \le 1$, by (2.8) and the $C_r$ inequality, we have

$$\begin{aligned}\mathbb{E}\left|J_{i\Delta} - J_{(i-1)\Delta}\right|^q &= \mathbb{E}\left|\sum_{i=1}^{\Delta_i^n N} Z_i\right|^q \le \mathbb{E}\sum_{i=1}^{\Delta_i^n N}|Z_i|^q \\ &= \mathbb{E}\int_{(i-1)\Delta}^{i\Delta}\int_{\mathbb{R}}|z|^q\lambda(\sigma_s)\mu_Z(dz)ds \\ &\le C_2\left(1 + \sup_{t\in\mathbb{R}^+}\mathbb{E}\sigma_t\right)\mathbb{E}\,|Z_1|^q\Delta \le K_q\Delta.\end{aligned}$$

If $1 < q \le p$, then there exists $M_q \in \{0, 1, 2, ...\}$ such that $2^{M_q} < q \le 2^{M_q+1}$. With the compensator $\nu$ of $\mu$ defined in (2.3), by applying the $C_r$ inequality and the Burkholder-Davis-Gundy inequality repeatedly, we have

$$\begin{aligned}\mathbb{E}\left|J_{i\Delta} - J_{(i-1)\Delta}\right|^q &= \mathbb{E}\left|\int_{(i-1)\Delta}^{i\Delta}\int_{\mathbb{R}} z\,\nu(ds,dz) + \int_{(i-1)\Delta}^{i\Delta}\int_{\mathbb{R}} z(\mu-\nu)(ds,dz)\right|^q \\ &\le K_q\Big[\sum_{j=0}^{M_q}\mathbb{E}\Big(\int_{(i-1)\Delta}^{i\Delta}\int_{\mathbb{R}}|z|^{2^j}\nu(ds,dz)\Big)^{\frac{q}{2^j}} + \mathbb{E}\Big(\int_{(i-1)\Delta}^{i\Delta}\int_{\mathbb{R}}|z|^{2^{M_q+1}}\mu(ds,dz)\Big)^{\frac{q}{2^{M_q+1}}}\Big].\end{aligned}$$

For $j = 0, 1, \ldots, M_q$, we have

$$\begin{aligned}\mathbb{E}\Big(\int_{(i-1)\Delta}^{i\Delta}\int_{\mathbb{R}}|z|^{2^j}\nu(ds,dz)\Big)^{\frac{q}{2^j}} &= \mathbb{E}\Big(\int_{(i-1)\Delta}^{i\Delta}\int_{\mathbb{R}}|z|^{2^j}\lambda(\sigma_s)ds\mu_Z(dz)\Big)^{\frac{q}{2^j}}\\ &= \Big(\mathbb{E}\,|Z_1|^{2^j}\Big)^{\frac{q}{2^j}}\mathbb{E}\Big(\int_{(i-1)\Delta}^{i\Delta}\lambda(\sigma_s)ds\Big)^{\frac{q}{2^j}}\\ &\le K_q\mathbb{E}\,|Z_1|^q\sup_{t\in\mathbb{R}^+}\mathbb{E}\sigma_t^p\,\Delta^{\frac{q}{2^j}}\\ &\le K_q\Delta^{\frac{q}{2^j}} \le K_q\Delta.\end{aligned}$$

Furthermore, we can easily observe that

$$\mathbb{E}\Big(\int_{(i-1)\Delta}^{i\Delta}\int_{\mathbb{R}}|z|^{2^{M_q+1}}\mu(ds,dz)\Big)^{\frac{q}{2^{M_q+1}}} = \mathbb{E}\left(\sum_{i=1}^{\Delta_i^n N}|Z_i|^{2^{M_q+1}}\right)^{\frac{q}{2^{M_q+1}}} \le \mathbb{E}\sum_{i=1}^{\Delta_i^n N}|Z_i|^q \le K_q\Delta.$$

Thus, the proof is completed. □

*Proof of Theorem 2.1.* **Step 1.** To prove (2.9), it is enough to show that

$$\frac{1}{T}\sum_{i=1}^{T/\Delta}\left|\Delta_i^n X\right|^p\Delta^{1-\frac{p}{2}} - \frac{1}{T}\sum_{i=1}^{T/\Delta}\left|\sigma_{(i-1)\Delta}\Delta_i^n W\right|^p\Delta^{1-\frac{p}{2}} \xrightarrow{\mathbb{P}} 0, \tag{A.6}$$

$$\frac{1}{T}\sum_{i=1}^{T/\Delta}\left|\sigma_{(i-1)\Delta}\Delta_i^n W\right|^p\Delta^{1-\frac{p}{2}} - \frac{1}{T}\sum_{i=1}^{T/\Delta}\mathbb{E}_{i-1}^n\left|\sigma_{(i-1)\Delta}\Delta_i^n W\right|^p\Delta^{1-\frac{p}{2}} \xrightarrow{\mathbb{P}} 0, \tag{A.7}$$

$$\frac{1}{T}\sum_{i=1}^{T/\Delta}\mathbb{E}_{i-1}^n\left|\sigma_{(i-1)\Delta}\Delta_i^n W\right|^p\Delta^{1-\frac{p}{2}} - \frac{1}{T}\int_0^T\sigma_s^p ds d_p \xrightarrow{\mathbb{P}} 0, \tag{A.8}$$

$$\frac{1}{T}\int_0^T\sigma_s^p ds d_p \xrightarrow{\mathbb{P}} \pi_p d_p, \tag{A.9}$$

as $T \to \infty$ and then $\Delta \to 0$, or $T \to \infty$ and $\Delta \to 0$ simultaneously. To prove (A.6), we need the following inequality:

$$||a|^r - |b|^r| \le |a-b|\left(|a|^{r-1} + |b|^{r-1}\right),\ r \ge 1,\ a, b \in \mathbb{R}, \tag{A.10}$$

which can be found in [53]. By (A.10), the Cauchy-Schwarz inequality, (A.2)–(A.4), we obtain

$$\begin{aligned}\mathbb{E}\Big||\Delta_i^n X|^p - |\sigma_{(i-1)\Delta}\Delta_i^n W|^p\Big| &\le K_p\sqrt{\mathbb{E}\left|\Delta_i^n X - \sigma_{(i-1)\Delta}\Delta_i^n W\right|^2}\times\sqrt{\mathbb{E}\left|\Delta_i^n X\right|^{2p-2} + \mathbb{E}\left|\sigma_{(i-1)\Delta}\Delta_i^n W\right|^{2p-2}}\\ &\le K_p\Delta^{\frac{1+p}{2}}.\end{aligned}$$

(A.6) then follows.

For (A.7), denote $\xi_i^n := \frac{1}{T}|\sigma_{(i-1)\Delta}\Delta_i^n W|^p \Delta^{1-\frac{p}{2}}$. Since $\xi_i^n - \mathbb{E}_{i-1}^n \xi_i^n, i = 1, ..., \frac{T}{\Delta}$ is a martingale difference sequence (i.e., $\mathbb{E}_{i-1}^n(\xi_i^n - \mathbb{E}_{i-1}^n \xi_i^n) = 0$), we have

$$\mathbb{E}\left(\sum_{i=1}^{T/\Delta}(\xi_i^n - \mathbb{E}_{i-1}^n \xi_i^n)\right)^2 = \sum_{i=1}^{T/\Delta}\mathbb{E}\left(\xi_i^n - \mathbb{E}_{i-1}^n \xi_i^n\right)^2.$$

By Chebyshev's inequality and the $C_r$ inequality, it is enough to show that $\sum_{i=1}^{T/\Delta}\mathbb{E}(\xi_i^n)^2 \to 0$. This holds by the fact that

$$\sum_{i=1}^{T/\Delta}\mathbb{E}\left(\xi_i^n\right)^2 \le K_p \frac{\Delta}{T}.$$

For (A.8), since $\mathbb{E}_{i-1}^n|\sigma_{(i-1)\Delta}\Delta_i^n W|^p = \left|\sigma_{(i-1)\Delta}\right|^p \mathbb{E}_{i-1}^n \left|\Delta_i^n W\right|^p = \sigma_{(i-1)\Delta}^p \Delta^{\frac{p}{2}} d_p$, it is enough to show that

$$\frac{1}{T}\sum_{i=1}^{T/\Delta}\int_{(i-1)\Delta}^{i\Delta}\mathbb{E}\left|\sigma_s^p - \sigma_{(i-1)\Delta}^p\right| ds \to 0. \tag{A.11}$$

By (A.10), the Cauchy-Schwarz inequality and (A.1), we have

$$\begin{aligned}\mathbb{E}\left|\sigma_s^p - \sigma_{(i-1)\Delta}^p\right| &\le K_p \sqrt{\mathbb{E}\left|\sigma_s - \sigma_{(i-1)\Delta}\right|^2}\sqrt{\mathbb{E}\left|\sigma_s\right|^{2p-2} + \mathbb{E}\left|\sigma_{(i-1)\Delta}\right|^{2p-2}} \\ &\le K_p \Delta^{\frac{1}{2}}.\end{aligned}$$

(A.11) then follows and we obtain (A.8). Finally, we obtain (A.9) immediately by the ergodic theorem of $\sigma$ under the condition $0 < \pi_p < \infty$.

**Step 2.** Denote $X_t^c := X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s$, then $X_t = X_t^c + J_t$. For the result (2.10), by the Minkowski inequality, it is enough to show that

$$\frac{1}{T}\sum_{i=1}^{T/\Delta}|\Delta_i^n J|^p \xrightarrow{\mathbb{P}} \pi_\lambda \mathbb{E}\left|Z_1\right|^p, \tag{A.12}$$

$$\frac{1}{T}\sum_{i=1}^{T/\Delta}|\Delta_i^n X^c|^p \xrightarrow{\mathbb{P}} 0, \tag{A.13}$$

as $T \to \infty$ and then $\Delta \to 0$, or $T \to \infty$ and $\Delta \to 0$ simultaneously. We obtain (A.13) immediately by (A.3) and the condition $p > 2$.

To prove (A.12) , it is sufficient to prove that

$$\frac{1}{T}\sum_{i=1}^{T/\Delta}|\Delta_i^n J|^p - \frac{1}{T}\sum_{s\le T}|\Delta J_s|^p \xrightarrow{\mathbb{P}} 0, \tag{A.14}$$

$$\frac{1}{T}\sum_{s\le T}|\Delta J_s|^p - \frac{1}{T}\sum_{i=1}^{T/\Delta}\mathbb{E}_{i-1}^n\int_{(i-1)\Delta}^{i\Delta}\int_{\mathbb{R}}|z|^p \mu(ds, dz) \xrightarrow{\mathbb{P}} 0, \tag{A.15}$$

$$\frac{1}{T}\sum_{i=1}^{T/\Delta}\mathbb{E}_{i-1}^{n}\int_{(i-1)\Delta}^{i\Delta}\int_{\mathbb{R}}|z|^{p}\mu(ds,dz)-\frac{1}{T}\int_{0}^{T}\lambda(\sigma_s)ds\,\mathbb{E}|Z_1|^{p}\xrightarrow{\mathbb{P}}0, \tag{A.16}$$

$$\frac{1}{T}\int_{0}^{T}\lambda(\sigma_s)ds\,\mathbb{E}|Z_1|^{p}-\pi_{\lambda}\mathbb{E}|Z_1|^{p}\xrightarrow{\mathbb{P}}0, \tag{A.17}$$

where $\Delta J_s := J_s - J_{s^-}$ refers to the jump size at time $s$.

To prove (A.14), we have

$$\begin{aligned}\frac{1}{T}\mathbb{E}\Big|\sum_{i=1}^{T/\Delta}|\Delta_i^n J|^p-\sum_{s\le T}|\Delta J_s|^p\Big| &\le\frac{1}{T}\sum_{i=1}^{T/\Delta}\mathbb{E}\Big||\Delta_i^n J|^p-\int_{(i-1)\Delta}^{i\Delta}\int_{\mathbb{R}}|z|^p\mu(ds,dz)\Big|\\ &=\frac{1}{T}\sum_{i=1}^{T/\Delta}\mathbb{E}\Big||\Delta_i^n J|^p-\int_{(i-1)\Delta}^{i\Delta}\int_{\mathbb{R}}|z|^p\mu(ds,dz)\Big|1_{\{\Delta_i^n N\ge 2\}}\\ &\le\frac{\sqrt{2}}{T}\sum_{i=1}^{T/\Delta}\sqrt{\mathbb{E}|\Delta_i^n J|^{2p}+\mathbb{E}\Big(\int_{(i-1)\Delta}^{i\Delta}\int_{\mathbb{R}}|z|^p\mu(ds,dz)\Big)^2}\times\sqrt{\mathbb{P}(\Delta_i^n N\ge 2)}.\end{aligned} \tag{A.18}$$

From (A.5), we know $\mathbb{E}|\Delta_i^n J|^{2p} \le K_p\Delta$. By the $C_r$ inequality and the Burkholder-Davis-Gundy inequality, we obtain

$$\mathbb{E}\Big(\int_{(i-1)\Delta}^{i\Delta}\int_{\mathbb{R}}|z|^p\mu(ds,dz)\Big)^2\le K_p\Delta. \tag{A.19}$$

Moreover, according to the definition of the conditional Poisson process [27], we have

$$\begin{aligned}\mathbb{P}(\Delta_i^n N\ge 2)&=\mathbb{E}\Big[exp\Big(-\int_{(i-1)\Delta}^{i\Delta}\lambda(\sigma_s)ds\Big)\sum_{n=2}^{\infty}\frac{\Big(\int_{(i-1)\Delta}^{i\Delta}\lambda(\sigma_s)ds\Big)^n}{n!}\Big]\\ &\le\mathbb{E}\Big(\int_{(i-1)\Delta}^{i\Delta}\lambda(\sigma_s)ds\Big)^2\le K\Delta^2.\end{aligned}$$

Thus, from (A.18), we have

$$\frac{1}{T}\mathbb{E}\Big|\sum_{i=1}^{T/\Delta}|\Delta_i^n J|^p-\sum_{s\le t}|\Delta J_s|^p\Big|\le K_p\Delta^{\frac{1}{2}}.$$

This implies (A.14) holds.

Since the left side of (A.15) is the sum of a martingale difference sequence, similar to the proof of (A.7), it is sufficient to show that $\frac{1}{T^2}\sum_{i=1}^{T/\Delta}\mathbb{E}\,(\int_{(i-1)\Delta}^{i\Delta}\int_{\mathbb{R}}|z|^p\ \mu(ds,dz))^2\to 0$. This is easily seen from (A.19) that

$$\frac{1}{T^2}\sum_{i=1}^{T/\Delta}\mathbb{E}\Big(\int_{(i-1)\Delta}^{i\Delta}\int_{\mathbb{R}}|z|^p\mu(ds,dz)\Big)^2\le K_p\frac{1}{T}.$$

(A.15) then follows.

For (A.16), under (b) and (c) in Assumption 2.1, we have

$$\mathbb{E}\int_0^t\int_{\mathbb{R}}|z|^p\mu(ds,dz)=\mathbb{E}\int_0^t\lambda(\sigma_s)ds\mathbb{E}|Z_1|^p<\infty.$$

From the Doob-Meyer decomposition, $\int_0^t\int_{\mathbb{R}}|z|^p\mu(ds,dz)-\int_0^t\lambda(\sigma_s)ds\mathbb{E}|Z_1|^p$ is a martingale. Thus,

$$\mathbb{E}_{i-1}^n\int_{(i-1)\Delta}^{i\Delta}\int_{\mathbb{R}}|z|^p\mu(ds,dz)=\mathbb{E}_{i-1}^n\int_{(i-1)\Delta}^{i\Delta}\lambda(\sigma_s)ds\mathbb{E}|Z_1|^p,$$

and the left side of (A.16) is a martingale difference sequence. Similar to the proof of (A.15), we obtain (A.16). Finally, (A.17) holds by the ergodic theorem of $\sigma$ given that $0<\pi_\lambda<\infty$.

**Step 3.** By (2.9), we have

$$\frac{1}{T}\sum_{i=1}^{T/\Delta}\left|\Delta_i^nX^c\right|^p\Delta^{1-\frac{p}{2}}\xrightarrow{\mathbb{P}}\pi_pd_p,$$

as $T\to\infty$ and then $\Delta\to 0$, or $T\to\infty$ and $\Delta\to 0$ simultaneously. Thus, to prove (2.11), it is enough to show that

$$\frac{1}{T}\sum_{i=1}^{T/\Delta}\left|\left|\Delta_i^nX\right|^pI_{\{|\Delta_i^nX|\le\gamma\Delta^\theta\}}-\left|\Delta_i^nX^c\right|^p\right|\Delta^{1-\frac{p}{2}}\xrightarrow{\mathbb{P}}0.$$

From the inequality $\left||a+b|^r1_{\{|a+b|\le c\}}-|a|^r\right|\le K_r\left(\frac{|a|^{r+1}}{c}+c^r(\frac{|b|}{c}\wedge 1)^2+|a|^{r-1}(|b|\wedge c)\right)$, $r\ge 2$, $0<c<1$ and $a,b\in\mathbb{R}$ (see [9]), it is equivalent to prove that

$$\frac{1}{T}\Delta^{1-\frac{p}{2}}\sum_{i=1}^{T/\Delta}\frac{\left|\Delta_i^nX^c\right|^{p+1}}{\gamma\Delta^\theta}\xrightarrow{\mathbb{P}}0, \tag{A.20}$$

$$\frac{1}{T}\Delta^{1-\frac{p}{2}}\left(\gamma\Delta^\theta\right)^p\sum_{i=1}^{T/\Delta}\left(\frac{\left|\Delta_i^nJ\right|}{\gamma\Delta^\theta}\wedge 1\right)^2\xrightarrow{\mathbb{P}}0, \tag{A.21}$$

$$\frac{1}{T}\Delta^{1-\frac{p}{2}}\sum_{i=1}^{T/\Delta}\left|\Delta_i^nX^c\right|^{p-1}\left(\left|\Delta_i^nJ\right|\wedge\gamma\Delta^\theta\right)\xrightarrow{\mathbb{P}}0. \tag{A.22}$$

We immediately obtain (A.20) under the condition $0<\theta<1/2$. To prove (A.21), for any arbitrarily small $\epsilon\in(0,1)$, we have

$$\mathbb{E}\left(\frac{\left|\Delta_i^nJ\right|}{\gamma\Delta^\theta}\wedge 1\right)^2\le\mathbb{E}\left(\frac{\left|\Delta_i^nJ\right|}{\gamma\Delta^\theta}\wedge 1\right)^\epsilon\le\mathbb{E}\left|\frac{\Delta_i^nJ}{\gamma\Delta^\theta}\right|^\epsilon\le K\Delta^{1-\epsilon\theta},$$

where in the last inequality we have used (A.5), and the positive constant $K=C_2\left(1+\sup_{t\in\mathbb{R}^+}\mathbb{E}\sigma_t\right)\mathbb{E}\left|Z_1\right|\vee 1$ is independent of $\epsilon$. By letting $\epsilon\to 0$, we have

$$\mathbb{E}\left(\frac{\left|\Delta_i^nJ\right|}{\gamma\Delta^\theta}\wedge 1\right)^2\le K\Delta. \tag{A.23}$$

Thus, (A.21) holds under the condition $\theta>\frac{1}{2}-\frac{1}{p}$.

As for (A.22), by applying the Cauchy-Schwarz inequality to the $L^1$ form of the right hand side of (A.22), together with (A.23), we have

$$\frac{1}{T}\Delta^{1-\frac{p}{2}}\sum_{i=1}^{T/\Delta}\mathbb{E}\left|\Delta_i^n X^c\right|^{p-1}\left(\left|\Delta_i^n J\right|\wedge\gamma\Delta^\theta\right)\le\frac{1}{T}\Delta^{1-\frac{p}{2}}\sum_{i=1}^{T/\Delta}\left[\mathbb{E}\left(\Delta_i^n X^c\right)^{2(p-1)}\right]^{\frac{1}{2}}\left[\mathbb{E}\left(\frac{\left|\Delta_i^n J\right|}{\gamma\Delta^\theta}\wedge 1\right)^2\right]^{\frac{1}{2}}\gamma\Delta^\theta$$
$$\le K\Delta^\theta.$$

(A.22) then follows. □

*Proof of Theorem 2.2.* Since the function $h(x,y) := y/x$ is a continuously differentiable function on $(0,\infty)\times(0,\infty)$, by the mean value theorem, (2.9), (A.9) and Slutsky's theorem, it is sufficient to show that

$$\boldsymbol{S}_{T,\Delta} := \sqrt{\frac{T}{\Delta}}\left(\begin{pmatrix}\frac{1}{T}\sum_{i=1}^{T/\Delta}|\Delta_i^n X|^p\Delta^{1-\frac{p}{2}}\\ \frac{1}{T}\sum_{i=1}^{T/(k\Delta)}|X_{ki\Delta}-X_{k(i-1)\Delta}|^p\Delta^{1-\frac{p}{2}}\end{pmatrix}-\begin{pmatrix}\frac{1}{T}\int_0^T\sigma_s^p ds d_p\\ k^{\frac{p}{2}-1}\frac{1}{T}\int_0^T\sigma_s^p ds d_p\end{pmatrix}\right)\xrightarrow{d}\boldsymbol{\Sigma Z}, \tag{A.24}$$

as $\Delta\to 0$ and $T\to\infty$ simultaneously, where

$$\boldsymbol{\Sigma}^\top\boldsymbol{\Sigma}=\begin{pmatrix}d_{2p}-d_p^2 & \ell_{k,p}-k^{\frac{p}{2}}d_p^2\\ \ell_{k,p}-k^{\frac{p}{2}}d_p^2 & k^{p-1}(d_{2p}-d_p^2)\end{pmatrix}\pi_{2p}=:\begin{pmatrix}g_{11}^2 & g_{12}^2\\ g_{21}^2 & g_{22}^2\end{pmatrix},$$

$\boldsymbol{Z}$ is a two-dimensional standard normal random vector. To unify the lower and upper limits of summation signs of $\boldsymbol{S}_{T,\Delta}$, let

$$U_i^n(1)=\frac{1}{T}\sum_{j=k(i-1)+1}^{ki}\left|\Delta_j^n X\right|^p\Delta^{1-\frac{p}{2}}-\frac{1}{T}\int_{k(i-1)\Delta}^{ki\Delta}\sigma_s^p ds d_p,$$
$$U_i^n(2)=\frac{1}{T}\left|X_{ki\Delta}-X_{k(i-1)\Delta}\right|^p\Delta^{1-\frac{p}{2}}-k^{\frac{p}{2}-1}\frac{1}{T}\int_{k(i-1)\Delta}^{ki\Delta}\sigma_s^p ds d_p,$$

then $\boldsymbol{S}_{T,\Delta}=\sqrt{\frac{T}{\Delta}}\sum_{i=1}^{T/(k\Delta)}\begin{pmatrix}U_i^n(1)\\ U_i^n(2)\end{pmatrix}$. We decompose $U_i^n(l)=\tilde{U}_i^n(l)+R_i^n(l)+\tilde{R}_i^n(l)$, $l=1,2$, where

$$\tilde{R}_i^n(1)=\frac{1}{T}\sum_{j=k(i-1)+1}^{ki}\left(\left|\Delta_j^n X\right|^p-\left|\sigma_{k(i-1)\Delta}\Delta_j^n W\right|^p\right)\Delta^{1-\frac{p}{2}},$$
$$\tilde{R}_i^n(2)=\frac{1}{T}\left(\left|X_{ki\Delta}-X_{k(i-1)\Delta}\right|^p-\left|\sigma_{k(i-1)\Delta}(W_{ki\Delta}-W_{k(i-1)\Delta})\right|^p\right)\Delta^{1-\frac{p}{2}},$$
$$\tilde{U}_i^n(1)=\frac{1}{T}\sum_{j=k(i-1)+1}^{ki}\left(\left|\sigma_{k(i-1)\Delta}\Delta_j^n W\right|^p-\mathbb{E}_{k(i-1)}^n\left|\sigma_{k(i-1)\Delta}\Delta_j^n W\right|^p\right)\Delta^{1-\frac{p}{2}},$$
$$\tilde{U}_i^n(2)=\frac{1}{T}\Big(\left|\sigma_{k(i-1)\Delta}(W_{ki\Delta}-W_{k(i-1)\Delta})\right|^p$$
$$-\mathbb{E}_{k(i-1)}^n\left|\sigma_{k(i-1)\Delta}(W_{ki\Delta}-W_{k(i-1)\Delta})\right|^p\Big)\Delta^{1-\frac{p}{2}},$$
$$R_i^n(1)=\frac{1}{T}\sum_{j=k(i-1)+1}^{ki}\mathbb{E}_{k(i-1)}^n\left|\sigma_{k(i-1)\Delta}\Delta_j^n W\right|^p\Delta^{1-\frac{p}{2}}-\frac{1}{T}\int_{k(i-1)\Delta}^{ki\Delta}\sigma_s^p ds d_p,$$

$$R_i^n(2) = \frac{1}{T}\mathbb{E}^n_{k(i-1)\Delta}\left|\sigma_{k(i-1)\Delta}(W_{ki\Delta} - W_{k(i-1)\Delta})\right|^p \Delta^{1-\frac{p}{2}} - k^{\frac{p}{2}-1}\frac{1}{T}\int_{k(i-1)\Delta}^{ki\Delta} \sigma_s^p ds d_p.$$

Thus, by Slutsky's theorem, it is enough to show that

$$\sqrt{\frac{T}{\Delta}} \sum_{i=1}^{T/(k\Delta)} \begin{pmatrix} \tilde{U}_i^n(1) \\ \tilde{U}_i^n(2) \end{pmatrix} \xrightarrow{d} \mathbf{\Sigma Z}, \tag{A.25}$$

$$\sqrt{\frac{T}{\Delta}} \sum_{i=1}^{T/(k\Delta)} R_i^n(l) \xrightarrow{\mathbb{P}} 0,\ l = 1, 2, \tag{A.26}$$

$$\sqrt{\frac{T}{\Delta}} \sum_{i=1}^{T/(k\Delta)} \tilde{R}_i^n(l) \xrightarrow{\mathbb{P}} 0,\ l = 1, 2. \tag{A.27}$$

For (A.25), by applying the Multi-dimensional Martingale Central Limit Theorem (see [54]), it is enough to prove that

$$\frac{T}{\Delta} \sum_{i=1}^{T/(k\Delta)} \mathbb{E}^n_{k(i-1)} \tilde{U}_i^n(l)\tilde{U}_i^n(r) \xrightarrow{\mathbb{P}} g_{rl}^2, r, l = 1, 2, \tag{A.28}$$

$$\left(\frac{T}{\Delta}\right)^2 \sum_{i=1}^{T/(k\Delta)} \mathbb{E}^n_{k(i-1)}\left(\tilde{U}_i^n(l)\right)^4 \xrightarrow{\mathbb{P}} 0, l = 1, 2. \tag{A.29}$$

We first prove the case $r = l = 1$ of (A.28). From the proof of (A.8) and (A.9) under the condition $0 < \pi_{2p} < \infty$, we have

$$\begin{aligned}
\frac{T}{\Delta} \sum_{i=1}^{T/(k\Delta)} \mathbb{E}^n_{k(i-1)} \tilde{U}_i^n(1)^2 &= \frac{1}{T} \sum_{i=1}^{T/(k\Delta)} \Big\{\mathbb{E}^n_{k(i-1)}\Big(\sum_{j=k(i-1)+1}^{ki} \Big|\frac{1}{\sqrt{\Delta}}\sigma_{k(i-1)\Delta}\Delta_j^n W\Big|^p\Big)^2 \\
&\quad - \Big[\mathbb{E}^n_{k(i-1)}\Big(\sum_{j=k(i-1)+1}^{ki} \Big|\frac{1}{\sqrt{\Delta}}\sigma_{k(i-1)\Delta}\Delta_j^n W\Big|^p\Big)\Big]^2\Big\}\Delta \\
&= (d_{2p} - d_p^2)\left(\frac{1}{T}\sum_{i=1}^{T/(k\Delta)} \sigma^{2p}_{k(i-1)\Delta} k\Delta\right) \xrightarrow{\mathbb{P}} g_{11}^2.
\end{aligned}$$

Similarly, for the case $r = 1$ and $l = 2$ of (A.28), we have

$$\begin{aligned}
\frac{T}{\Delta} \sum_{i=1}^{T/(k\Delta)} \mathbb{E}^n_{k(i-1)} \tilde{U}_i^n(1)^2 &= \left(\frac{1}{T}\sum_{i=1}^{T/(k\Delta)} \sigma^{2p}_{k(i-1)\Delta}\Delta\right) \times \mathbb{E}\left\{\left(\sum_{j=1}^{k} \left|z_j\right|^p - d_p\right)\left(\left|\sum_{j=1}^{k} z_j\right|^p - k^{\frac{p}{2}} d_p\right)\right\} \\
&= \left(\ell_{k,p} - k^{\frac{p}{2}} d_p^2\right)\left(\frac{1}{T}\sum_{i=1}^{T/(k\Delta)} \sigma^{2p}_{k(i-1)\Delta} k\Delta\right) \xrightarrow{\mathbb{P}} g_{12}^2,
\end{aligned}$$

where $z_1, ...z_k$ are independent standard normal random variables. Next, we prove the case $l = 1$ of (A.29). Under Assumption 2.2(b), we have

$$\mathbb{E}\left|\left(\frac{T}{\Delta}\right)^2 \sum_{i=1}^{T/(k\Delta)} \mathbb{E}^n_{k(i-1)}\left(\tilde{U}_i^n(1)\right)^4\right| \le K_p \sup_{t\in\mathbb{R}^+} \mathbb{E}|\sigma_t|^{4p}\frac{\Delta}{T} \to 0.$$

For other cases of (A.28) and (A.29), one can similarly prove them.

As for (A.26), without loss of generation, we show the case $k = 1$ holds:

$$I := \sqrt{\frac{T}{\Delta}}\Big(\frac{1}{T}\int_0^T \sigma_s^p ds - \frac{1}{T}\sum_{i=1}^{T/\Delta}\sigma_{(i-1)\Delta}^p \Delta\Big) \xrightarrow{\mathbb{P}} 0. \tag{A.30}$$

For the sake of convenience, we denote $f(x) = |x|^p$ , $p \geq 2$. Then we decompose

$$\begin{aligned} I =& \sqrt{\frac{T}{\Delta}}\frac{1}{T}\sum_{i=1}^{T/\Delta}\int_{(i-1)\Delta}^{i\Delta} f'(\sigma_{(i-1)\Delta})(\sigma_s - \sigma_{(i-1)\Delta})\, ds \\ &+ \sqrt{\frac{T}{\Delta}}\frac{1}{T}\sum_{i=1}^{T/\Delta}\int_{(i-1)\Delta}^{i\Delta} \Big(f'(\xi) - f'(\sigma_{(i-1)\Delta})\Big)\Big(\sigma_s - \sigma_{(i-1)\Delta}\Big)\, ds \\ =:& I_1 + I_2, \end{aligned}$$

and

$$\begin{aligned} I_1 =& \sqrt{\frac{T}{\Delta}}\frac{1}{T}\sum_{i=1}^{T/\Delta} f'(\sigma_{(i-1)\Delta})\int_{(i-1)\Delta}^{i\Delta}\int_{(i-1)\Delta}^{s}\tilde{\mu}(\sigma_u)\, du\, ds \\ &+ \sqrt{\frac{T}{\Delta}}\frac{1}{T}\sum_{i=1}^{T/\Delta} f'(\sigma_{(i-1)\Delta})\int_{(i-1)\Delta}^{i\Delta}\int_{(i-1)\Delta}^{s}\tilde{\sigma}(\sigma_u)\, dW_u^*\, ds \\ =:& I_1(1) + I_1(2), \end{aligned}$$

where $\xi$ lies on the segment between $\sigma_{(i-1)\Delta}$ and $\sigma_{i\Delta}$, and $f'$ denotes the first derivative of $f$. Under Assumption 2.2(a), we have $\mathbb{E}|I_1(1)| \leq K_p\sqrt{T\Delta} \to 0$, which implies $I_1(1) \xrightarrow{\mathbb{P}} 0$. For $I_1(2)$, from the definition of conditional expectation and the Fubini's theorem, it is easy to obtain

$$\mathbb{E}_{i-1}^n\Big(f'(\sigma_{(i-1)\Delta})\int_{(i-1)\Delta}^{i\Delta}\int_{(i-1)\Delta}^{s}\tilde{\sigma}(\sigma_u)dW_u^* ds\Big) = 0.$$

Thus, $I_1(2)$ is the sum of a martingale difference sequence. Since

$$\begin{aligned} &\sum_{i=1}^{T/\Delta}\mathbb{E}\Big(\sqrt{\frac{T}{\Delta}}\frac{1}{T}f'(\sigma_{(i-1)\Delta})\int_{(i-1)\Delta}^{i\Delta}\int_{(i-1)\Delta}^{s}\tilde{\sigma}(\sigma_u)dW_u^* ds\Big)^2 \\ \leq& K_p\Delta\sqrt{\sup_{t\in\mathbb{R}^+}\mathbb{E}|\sigma_t|^{4(p-1)}}\sqrt{\sup_{t\in\mathbb{R}^+}\mathbb{E}|\sigma_t|^4} \leq K_p\Delta, \end{aligned}$$

we have $I_1(2) \xrightarrow{\mathbb{P}} 0$. We now prove $I_2 \xrightarrow{\mathbb{P}} 0$. Considering that $f \in \mathbb{C}^2(\mathbb{R})$ and $|f''(x)| \leq K|x|^{p-2}$, for

$p \geq 2$, by applying the mean value theorem again, we have

$$\begin{aligned}
\mathbb{E}|I_2| &\leq \sqrt{\frac{T}{\Delta}}\frac{1}{T}\sum_{i=1}^{T/\Delta}\int_{(i-1)\Delta}^{i\Delta}\mathbb{E}\left|f''(\tilde{\xi})(\sigma_s-\sigma_{(i-1)\Delta})\right|^2 ds\\
&\leq \sqrt{\frac{T}{\Delta}}\frac{1}{T}\sum_{i=1}^{T/\Delta}\int_{(i-1)\Delta}^{i\Delta}\sqrt{\mathbb{E}\left|f''(\tilde{\xi})\right|^4}\sqrt{\mathbb{E}\left|\sigma_s-\sigma_{(i-1)\Delta}\right|^4}ds\\
&\leq \sqrt{\frac{T}{\Delta}}\frac{1}{T}\sum_{i=1}^{T/\Delta}\int_{(i-1)\Delta}^{i\Delta}\sqrt{\sup_{t\in\mathbb{R}^+}\mathbb{E}\left|\sigma_t\right|^{4(p-2)}}\sqrt{\mathbb{E}|\sigma_s-\sigma_{(i-1)\Delta}|^4}ds\\
&\leq K\sqrt{T\Delta},
\end{aligned}$$

where $\tilde{\xi}$ lies on the segment between $\sigma_{(i-1)\Delta}$ and $\xi$, and $f''$ is the second derivative of $f$. Under Assumption 2.2(a), $I_2 \xrightarrow{\mathbb{P}} 0$. Thus, (A.30) holds, and we obtain (A.26).

Regarding (A.27), without loss of generation, let $k = 1$. Thus, we only need to prove

$$\sqrt{\frac{T}{\Delta}}\frac{1}{T}\sum_{i=1}^{T/\Delta}\left(\left|\Delta_i^n X\right|^p - \left|\sigma_{(i-1)\Delta}\Delta_i^n W\right|^p\right)\Delta^{1-\frac{p}{2}} \xrightarrow{\mathbb{P}} 0. \tag{A.31}$$

Note that [9] only considered the case of $\Delta \to 0$, so their standard localization procedure and the dominated convergence theorem is unfeasible when $T \to \infty$. In the following, we utilize the assumptions imposed on $b(\cdot)$, $\tilde{\mu}(\cdot)$ and $\tilde{\sigma}(\cdot)$ to complete the proof of (A.31). We first denote the left side of (A.31) by $II$ and decompose it:

$$\begin{aligned}
II &= \sqrt{\frac{T}{\Delta}}\sum_{i=1}^{T/\Delta}\frac{1}{T}f'(\sigma_{(i-1)\Delta}\Delta_i^n W)(\Delta_i^n X-\sigma_{(i-1)\Delta}\Delta_i^n W)\Delta^{1-\frac{p}{2}}\\
&\quad+\sqrt{\frac{T}{\Delta}}\sum_{i=1}^{T/\Delta}(f'(\eta)-f'(\sigma_{(i-1)\Delta}\Delta_i^n W))(\Delta_i^n X-\sigma_{(i-1)\Delta}\Delta_i^n W)\Delta^{1-\frac{p}{2}}\\
&=: II_1 + II_2,
\end{aligned}$$

where $\eta$ lies on the segment between $\Delta_i^n X$ and $\sigma_{(i-1)\Delta}\Delta_i^n W$. Let

$$\begin{aligned}
II_1 &= \sqrt{\frac{T}{\Delta}}\sum_{i=1}^{T/\Delta}\frac{1}{T}f'(\sigma_{(i-1)\Delta}\Delta_i^n W)\, b(X_{(i-1)\Delta})\Delta^{2-\frac{p}{2}}\\
&\quad+\sqrt{\frac{T}{\Delta}}\sum_{i=1}^{T/\Delta}\frac{1}{T}f'(\sigma_{(i-1)\Delta}\Delta_i^n W)\int_{(i-1)\Delta}^{i\Delta}\left(b(X_s)-b(X_{(i-1)\Delta})\right)ds\Delta^{1-\frac{p}{2}}\\
&\quad+\sqrt{\frac{T}{\Delta}}\sum_{i=1}^{T/\Delta}\frac{1}{T}f'(\sigma_{(i-1)\Delta}\Delta_i^n W)\int_{(i-1)\Delta}^{i\Delta}\tilde{\sigma}(\sigma_{(i-1)\Delta})(W_s^*-W_{(i-1)\Delta}^*)dW_s\Delta^{1-\frac{p}{2}}\\
&\quad+\sqrt{\frac{T}{\Delta}}\sum_{i=1}^{T/\Delta}\frac{1}{T}f'(\sigma_{(i-1)\Delta}\Delta_i^n W)\int_{(i-1)\Delta}^{i\Delta}\Big[\int_{(i-1)\Delta}^{s}\tilde{\mu}(\sigma_u)du+\int_{(i-1)\Delta}^{s}\Big(\tilde{\sigma}(\sigma_u)\\
&\quad-\tilde{\sigma}(\sigma_{(i-1)\Delta})\Big)dW_u^*\Big]dW_s\Delta^{1-\frac{p}{2}}\\
&=: II_1(1)+II_1(2)+II_1(3)+II_1(4).
\end{aligned}$$

Thus, it is sufficient to prove

$$II_1(m) \xrightarrow{\mathbb{P}} 0, m = 1, 2, 3, 4, \tag{A.32}$$

$$II \xrightarrow{\mathbb{P}} 0. \tag{A.33}$$

For the case $m = 1$ of (A.32), since $W_t, t \geq s$ is independent of $\mathscr{F}_s$, and $f'(\cdot)$ is an odd function for $p \geq 2$, we have

$$\mathbb{E}_{i-1}^n \left(f'\left(\sigma_{(i-1)\Delta}\Delta_i^n W\right) b\left(X_{(i-1)\Delta}\right)\right) = 0.$$

Then, $II_1(1)$ is the sum of a martingale difference sequence, and $II_1(1) \xrightarrow{\mathbb{P}} 0$ is implied by

$$\frac{T}{\Delta}\sum_{i=1}^{T/\Delta}\mathbb{E}\left(\frac{1}{T}f'\left(\sigma_{(i-1)\Delta}\Delta_i^n W\right) b\left(X_{(i-1)\Delta}\right)\Delta^{2-\frac{p}{2}}\right)^2 \leq K_p\sqrt{\sup_{t\in\mathbb{R}^+}\mathbb{E}|\sigma_t|^{4(p-1)}}\sqrt{\sup_{t\in\mathbb{R}^+}\mathbb{E}|X_t|^4}\,\Delta \leq K_p\Delta.$$

For the case $m = 2$ of (A.32), using (2.7) and (A.3), we have

$$\mathbb{E}|II_1(2)| \leq K_p\sqrt{\frac{\Delta}{T}}\sum_{i=1}^{T/\Delta}\sqrt{sup_{t\in\mathbb{R}^+}\mathbb{E}|\sigma_t|^{2(p-1)}\int_{(i-1)\Delta}^{i\Delta}\mathbb{E}|X_s - X_{(i-1)\Delta}|^2 ds} \leq K_p\sqrt{T\Delta}.$$

Then, Assumption 2.2(a) implies $II_1(2) \xrightarrow{\mathbb{P}} 0$. For the case $m = 3$ of (A.32), we now deduce that, for each $i$,

$$\mathbb{E}_{i-1}^n(A_i) := \mathbb{E}_{i-1}^n\left(f'\left(\sigma_{(i-1)\Delta}\Delta_i^n W\right)\int_{(i-1)\Delta}^{i\Delta}\tilde{\sigma}(\sigma_{(i-1)\Delta})\left(W_s^* - W_{(i-1)\Delta}^*\right)dW_s\right) = 0. \tag{A.34}$$

For simplicity, we firstly define

$$\begin{aligned}
\mathbb{E}_{i-1}^n(A_i(M)) =&\mathbb{E}_{i-1}^n\Big\{f'\left(\sigma_{(i-1)\Delta}\Delta_i^n W\right)\sum_{j=1}^{M}\Big[\tilde{\sigma}\left(\sigma_{(i-1)\Delta}\right)\Big(W^*_{(i-1)\Delta+\frac{(j-1)\Delta}{M}} - W^*_{(i-1)\Delta}\Big)\Big(W_{(i-1)\Delta+\frac{j\Delta}{M}} - W_{(i-1)\Delta+\frac{(j-1)\Delta}{M}}\Big)\Big]\Big\}\\
=&\mathbb{E}\Big\{f'\left(x\Delta_i^n W\right)\sum_{j=1}^{M}\Big[\tilde{\sigma}(x)\Big(\rho\Big(W_{(i-1)\Delta+\frac{(j-1)\Delta}{M}} - W_{(i-1)\Delta}\Big) + \sqrt{1-\rho^2}\\
&\times\Big(\widetilde{W}_{(i-1)\Delta+\frac{(j-1)\Delta}{M}} - \widetilde{W}_{(i-1)\Delta}\Big)\Big)\Big(W_{(i-1)\Delta+\frac{j\Delta}{M}} - W_{(i-1)\Delta+\frac{(j-1)\Delta}{M}}\Big)\Big]\Big\}\Big|_{x=\sigma_{(i-1)\Delta}},
\end{aligned} \tag{A.35}$$

where $M$ denotes the number of segments within the time interval $((i-1)\Delta, i\Delta)$. The second equality of (A.35) holds by the properties of independent increments of $W$ and $W^*$. Next, we denote

$$B_j = W_{(i-1)\Delta+\frac{j\Delta}{M}} - W_{(i-1)\Delta+\frac{(j-1)\Delta}{M}}, \quad \widetilde{B}_j = \widetilde{W}_{(i-1)\Delta+\frac{j\Delta}{M}} - \widetilde{W}_{(i-1)\Delta+\frac{(j-1)\Delta}{M}}.$$

Then, we have

$$\begin{aligned}
\mathbb{E}_{i-1}^n(A_i(M)) =&\rho\left\{\tilde{\sigma}(x)\mathbb{E}\left[f'\left(x\left(B_1+\cdots+B_M\right)\right)\sum_{j=1}^{M}\left(B_1+\cdots+B_{j-1}\right)B_j\right]\right\}\Big|_{x=\sigma_{(i-1)\Delta}}\\
&+\sqrt{1-\rho^2}\left\{\tilde{\sigma}(x)\mathbb{E}\left[f'\left(x\left(B_1+\cdots+B_M\right)\right)\sum_{j=1}^{M}\left(\widetilde{B}_1+\cdots+\widetilde{B}_{j-1}\right)B_j\right]\right\}\Big|_{x=\sigma_{(i-1)\Delta}}.
\end{aligned} \tag{A.36}$$

Since $\{B_j\}_{j=1}^M$ and $\{\widetilde{B}_j\}_{j=1}^M$ are independent random vectors with zero means, the second term of the right side of (A.36) equals zero. In addition, considering that $\{B_j\}_{j=1}^M$ follows a multivariate standard normal distribution, and $f'$ is an odd function, we have

$$
\begin{aligned}
&-\mathbb{E}f'(x(B_1+\cdots+B_M))\sum_{j=1}^{M}\rho\left(B_1+\cdots+B_{j-1}\right)B_j\\
&=\mathbb{E}f'\left(x(B_1+\cdots+B_M)\right)\sum_{j=1}^{M}\rho\left(B_1+\cdots+B_{j-1}\right)B_j.
\end{aligned}
$$

Thus, the first term of the right side of (A.36) also equals zero, and we obtain $\mathbb{E}_{i-1}^n(A_i(M)) = 0$. Since $A_i(M) \xrightarrow{\mathbb{P}} A_i$, $M \to \infty$, and $\{A_i(M)\}_{M=1}^\infty$ is uniformly integrable due to the fact that $\mathbb{E}(A_i(M))^2 \le K\Delta^{p+1} < \infty$, the $L^1$- Convergence Criterion implies (A.34). Then, similar to the proof of $II_1(1) \xrightarrow{\mathbb{P}} 0$, it is easy to obtain $II_1(3) \xrightarrow{\mathbb{P}} 0$. For the case $m = 4$ of the result (A.32), using (2.5) and Assumption 2.2(c), we have

$$
\begin{aligned}
\mathbb{E}|II_1(4)| \le{}& K_p\sqrt{\sup_{t\in\mathbb{R}^+}\mathbb{E}|\sigma_t|^{2p}}\frac{1}{\sqrt{T}}\sum_{i=1}^{T/\Delta}\bigg(\int_{(i-1)\Delta}^{i\Delta}\Big[\mathbb{E}\Big(\int_{(i-1)\Delta}^{s}\tilde{\mu}(\sigma_u)du\Big)^2\\
&+\mathbb{E}\Big(\int_{(i-1)\Delta}^{s}(\tilde{\sigma}(\sigma_u)-\tilde{\sigma}(\sigma_{(i-1)\Delta}))dW_u^*\Big)^2\Big]ds\bigg)^{\frac{1}{2}}\\
\le{}& K_p\sqrt{T\Delta^\beta}\le K_p\sqrt{T\Delta^{\frac{1}{2}}},
\end{aligned}
$$

with $\beta \in [\frac{1}{2}, 1]$, $0 < \Delta < 1$. Thus, the condition $T\sqrt{\Delta} \to 0$ implies that $II_1(4) \xrightarrow{\mathbb{P}} 0$. Finally, for the result (A.33), by applying the mean value theorem again, the Cauchy-Schwarz inequality, $|f''(x)| \le K|x|^{p-2}$ for $p \ge 2$ and (A.2), we have

$$
\begin{aligned}
\mathbb{E}|II| \le{}& \sqrt{\frac{\Delta}{T}}\sum_{i=1}^{T/\Delta}\mathbb{E}\left[\left|f''(\tilde{\eta})\right|\left|\Delta_i^nX-\sigma_{(i-1)\Delta}\Delta_i^nW\right|^2\right]\Delta^{-\frac{p}{2}}\\
\le{}& K_p\sqrt{\frac{\Delta}{T}}\sum_{i=1}^{T/\Delta}\sqrt{\mathbb{E}\left|\Delta_i^nX-\sigma_{(i-1)\Delta}\Delta_i^nW\right|^4}\frac{1}{\Delta}\\
\le{}& K_p\sqrt{T\Delta},
\end{aligned}
$$

where $\tilde{\eta}$ lies on the segment between $\eta$ and $\sigma_{(i-1)\Delta}\Delta_i^nW$. The reult (A.33) then follows, and we obtain (A.31). □

*Proof of Corollary 2.1.* By (2.11) and (2.12), we obtain (2.13) and thus $\lim_{\Delta\to0,T\to\infty}\mathbb{P}\Big(\check{\mathrm{T}}(p,k,T,\Delta) < z_\alpha|H_0\Big) < \alpha$. For (2.14) under $H_1$, by (2.10), we have

$$
\frac{\hat{\mathrm{T}}(p,k,T,\Delta)-k^{\frac{p}{2}-1}}{\hat{e}} \xrightarrow{\mathbb{P}} \frac{1-k^{\frac{p}{2}-1}}{e}. \tag{A.37}
$$

Note that $1 - k^{\frac{p}{2}-1} < 0$, for $p > 2$, $k > 1$. Thus, from (A.37), one sees that

$$\begin{aligned}\mathbb{P}\Big(\check{\mathrm{T}}(p,k,T,\Delta) \leq z_\alpha\Big) &\geq \mathbb{P}\Big(\Big\{\check{\mathrm{T}}(p,k,T,\Delta) \leq z_\alpha\Big\} \cap \Big\{\Big|\frac{\hat{\mathrm{T}}(p,k,T,\Delta) - k^{\frac{p}{2}-1}}{\hat{e}} - \frac{1-k^{\frac{p}{2}-1}}{e}\Big| \leq \varepsilon\Big\}\Big) \\ &\geq \mathbb{P}\Big(\Big\{\sqrt{\frac{\Delta}{T}}(\varepsilon + \frac{1-k^{\frac{p}{2}}}{e}) \leq z_\alpha\Big\} \cap \Big\{\Big|\frac{\hat{\mathrm{T}}(p,k,T,\Delta) - k^{\frac{p}{2}-1}}{\hat{e}} - \frac{1-k^{\frac{p}{2}-1}}{e}\Big| \leq \varepsilon\Big\}\Big) \\ &= \mathbb{P}\Big(\Big|\frac{\hat{\mathrm{T}}(p,k,T,\Delta) - k^{\frac{p}{2}-1}}{\hat{e}} - \frac{1-k^{\frac{p}{2}-1}}{e}\Big| \leq \varepsilon\Big) \to 1,\end{aligned}$$

where the last equality holds for small enough $\varepsilon > 0$ and large enough $\frac{T}{\Delta}$. Similarly, by using (A.37), it is easy to get $\check{\mathrm{T}}(p,k,T,\Delta) \asymp_{\mathbb{P}} -\sqrt{\frac{T}{\Delta}}$. □