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Research article

Attracting sets in sup-norms for mild solutions of impulsive-perturbed parabolic semilinear problems

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Abstract: In this paper, we investigate the qualitative behavior of an evolutionary problem that consists of a semilinear parabolic equation whose trajectories undergo instantaneous impulsive perturbations at the moments when some integral functional reaches a certain threshold value. The key object is the uniform attractor of the corresponding impulsive infinite-dimensional dynamical system. The novelty of this study is the analysis of mild solutions in the phase space of continuous functions. Under general assumptions on the impulsive parameters, we prove that this problem generates an impulsive dynamical system, and its trajectories have a compact uniform attractor with respect to the supremum norm (sup-norm).

Keywords: impulsive dynamical systems; semilinear parabolic equations; mild solutions; uniform attractors; sup-norm; compact limit dynamics; impulsive perturbations; phase space; global attractors
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1. Introduction

The evolution of many mechanical systems exhibits a combination of continuous and discontinuous behaviors [1]. A general exposition of such hybrid systems was presented in [2], where it was pointed out, that one of the most effective mathematical tools to investigate such processes is the theory of differential equations with jumps or impulsive differential equations.

The qualitative theory of impulsive dynamical systems has subsequently developed in several directions. In [3], questions of stability and asymptotic stability for impulsive semidynamical systems were investigated. Applied problems, such as heat conductivity with self-regulated pulse maintenance, were analyzed in [4], providing one of the first practical motivations for the study of impulsive effects. A systematic framework for impulsive differential equations was presented in the monograph [5], which laid the foundations of the modern theory, while further theoretical results and diverse applications were developed in [1]. Since then, impulsive systems have remained an active area of research, with applications ranging from control theory and mechanics to biological and economic models. In most of these studies, trajectories are assumed to undergo impulsive perturbations whenever they reach a fixed subset of the phase space. Within this framework, various aspects of the theory of ω-limit sets in finite-dimensional phase spaces have been investigated [6–9]. Moreover, recent results have extended the theory to impulsive systems with delay [10, 11], further broadening the scope of applications and mathematical techniques.

In infinite-dimensional dissipative dynamical systems, global attractors play a key role in describing the qualitative behavior of trajectories [12–14]. The transfer of the basic constructions of the global attractors theory to infinite-dimensional impulsive systems was performed in [15–18]. It should be noted that all the aforementioned results concern the limit behavior of weak solutions in the L^2-norms. The novelty of the present paper is obtaining results on the compact limit dynamics of impulsive trajectories in supremum norms (sup-norms). Since the main task is to prove the existence of a compact attracting set in the phase space, obtaining the property of attraction in the sup-norm gives much more information about the behavior of trajectories than the fact of attraction in the L^2-norm. For a linear heat equation in a one-dimensional spatial domain, this fact was established in [19] for the ω-limit sets of individual trajectories based on the explicit formula of the impulsive semigroup. In the present work, we consider weakly nonlinear parabolic equations and prove the existence of a compact set that uniformly attracts all trajectories with respect to the bounded initial data.

The paper is organized as follows: the second section presents the setting of the problem, some auxiliary statements, and a comparative analysis with previous results; in the third section, under general assumptions on impulsive parameters, we prove that the mild solutions of an impulsive perturbed semilinear parabolic problem generate an impulsive dynamical system in the phase space of continuous functions; and in the fourth section, under an additional assumption, we show that this impulsive dynamical system possesses a nonempty compact uniform attractor in the sup-norm.

2. Setting of the problem

We consider the following semilinear problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = \varepsilon \cdot f(u), \quad t > 0, \quad x \in \Omega, \\ u|_{\partial\Omega} = 0, \\ u|_{t=0} = u_0, \end{cases} \tag{2.1}$$

where $u = u(t, x)$ is an unknown function,

$$\triangle u(t, x) = \sum_{i=1}^{d} \frac{\partial^2 u(t, x)}{\partial x_i^2}, \ \ \Omega \subset \mathbb{R}^d,$$

$d \geqslant 1$ is a bounded domain, $\varepsilon \geqslant 0$ is a small parameter, $f : \mathbb{R} \mapsto \mathbb{R}$ is locally Lipschitz function such that $f(0) = 0$, and

$$\exists\, c > 0, \quad \forall s \in \mathbb{R}, \quad |f(s)| \leqslant c.$$

We consider (2.1) in the phase space as follows:

$$X = \mathbb{C}_0(\Omega) = \{v \in \mathbb{C}(\bar{\Omega}) \mid v = 0 \ \text{ on } \partial\Omega\}.$$

The space X is equipped with the sup-norm as follows:

$$\|v\|_X = \sup_{x \in \Omega} |v(x)|.$$

We will understand the solution of (2.1) in the mild sense [20] (i.e., a function $u \in \mathbb{C}([0,T]; X)$), $u(0) = u_0$ is called a (mild) solution of (2.1) on $(0,T)$ if $\forall t \in [0,T]$ and

$$u(t) = S(t)u_0 + \varepsilon \int_0^t S(t-s) f(u(s))ds,$$

where $S(t)$ is a C_0-semigroup of contractions generated by $-\Delta$ in X.

Now, we introduce an impulsive problem.

For a given $R > 0$, $\psi \in L^2(\Omega)$, we consider the impulsive set $M \subset X$ as follows:

$$M = \left\{ v \in X \mid v(x) \geqslant 0, \int_\Omega v(x)\psi(x)dx = R \right\}. \tag{2.2}$$

At the moment of meeting the phase point $u(t)$ with the set M, it undergoes an impulsive perturbation and instantly finds itself in a new position as follows:

$$Iu = u + \varphi, \tag{2.3}$$

where $\varphi \in X$ is a given function, $\varphi(x) \geqslant 0$, and $\int_\Omega \varphi(x)\psi(x)dx = \varphi_0 > 0$. After that, the phase point continues to move along the trajectory of (2.1) until the next meeting with M, and so on.

Assuming $u(\cdot)$ to be right-continuous at the moments of jumps, we denote the point of the impulsive trajectory at the moment of time $t \geqslant 0$ that started from u_0 at $t = 0$ by $G(t, u_0)$.

First, we prove that G generated by (2.1)–(2.3) for sufficiently small $\varepsilon \geqslant 0$ is an impulsive dynamical system (see Section 3).

In the sequel, the phrase "for sufficiently small $\varepsilon \geqslant 0$" will mean that a certain property holds for every $\varepsilon \in (0, \varepsilon_1)$, where $\varepsilon_1 > 0$ only depends on the constants of the problems (2.1)–(2.3).

Second, under some additional assumptions on impulsive parameters, we prove that the impulsive dynamical system G possesses a compact *uniform attractor* in X for sufficiently small $\varepsilon \geqslant 0$, i.e., there exists a compact set $\Theta \subset X$ such that $\forall r > 0$,

$$\sup_{\|u_0\|_X \leqslant r} \operatorname{dist}_X (G(t, u_0), \Theta) \to 0 \quad \text{as} \quad t \to \infty, \tag{2.4}$$

and Θ is the minimal among all closed sets that satisfies (2.4).

Now, we briefly discuss the established results for problem (2.1)–(2.3).

Due to the inequality

$$\varepsilon \cdot f(s) \cdot s \leq \delta s^2 + c_\delta,$$

where $\delta = \delta(\varepsilon) > 0$ can be chosen arbitrarily small with an appropriate choice of ε. From [21] and Theorem 4.1, we can conclude that the $\forall u_0 \in X$ problem (2.1) has a unique global mild solution $u \in \mathbb{C}([0, +\infty); X)$, and the corresponding semigroup $V : [0, +\infty) \times X \mapsto X$ and

$$V(t, u_0) = S(t)u_0 + \varepsilon \int_0^t S(t-s) f(u(s)) ds \tag{2.5}$$

possesses a global attractor in X, i.e., there exists a compact set $A \subset X$ such that

$$\forall r > 0, \quad \sup_{\|u_0\|_X \leqslant r} \operatorname{dist}_X(V(t, u_0), A) \to 0 \quad \text{as} \quad t \to \infty, \forall t \geqslant 0, \quad V(t, A) = A.$$

This result remains true under much more general conditions on f, which includes multi-valued case [21]. Of course, a global attractor is a uniform attractor because invariance implies minimality. However, in general, we cannot expect invariance in impulsive problems [18].

There are several works [16, 17, 22] where attracting sets are investigated for problem (2.1) with different types of impulsive perturbations. However, in these papers, the asymptotic behavior of impulsive trajectories was studied in L^2-norms. In particular, in [22], the existence of uniform attractor Θ in the phase space $L^2(\Omega)$ was proved with additional property $G(t, \Theta \setminus M) = \Theta \setminus M$, $t \geq 0$ for a problem similar to (2.1)–(2.3) for sufficiently small $\varepsilon > 0$. However, the key estimates on L^2-norms used in [22] do not apply to sup-norms. In the present work, we use the comparison principle [23] to derive a priori estimates in the phase space X.

3. Design of impulsive dynamical system

An abstract impulsive dynamical system G [18] consists of the continuous semigroup $V : [0, +\infty) \times X \mapsto X$ and the impulsive parameters $M \subset X$ (impulsive set) and $I : M \mapsto X$ (impulsive map), which satisfy the following properties:

$$M \text{ is closed}, \quad M \cap IM = \emptyset, \tag{3.1}$$

$$\forall u_0 \in M, \quad \exists \tau = \tau(u_0), \quad \forall t \in (0, \tau), \quad V(t, u_0) \notin M. \tag{3.2}$$

For $u \in X$, we denote the following:

$$M^+(u) = \bigcup_{t>0} V(t, u) \cap M.$$

Then, either $M^+(u_0) = \emptyset$, and in this case

$$G(t, u_0) = V(t, u_0), \quad t \in [0, +\infty),$$

or

$$\exists s_0 = s(u_0) > 0, \quad \text{such that} \quad \forall t \in (0, s_0), \quad V(t, u_0) \notin M, \quad V(s_0, u_0) \in M, \tag{3.3}$$

and in this case

$$G(t,u_0)=\begin{cases}V(t,u_0), & t\in(0,s_0),\\ IV(s_0,u_0), & t=s_0.\end{cases}$$

We continue this procedure by considering $u_1^+ = IV(s_0, u_0)$ as a new initial point. As a result, we have a finite or infinite number of impulsive points as follows:

$$\{u_{n+1}^+ = IV(s_n, u_n^+)\}_{n\geqslant 0}\,,\quad u_0^+ = u_0.$$

For $T_{n+1} := \sum_{k=0}^{n} s_k$, $T_0 := 0$, we obtain the following:

$$G(t,u_0)=\begin{cases} V(t-T_n, u_n^+), & t\in(T_n,T_{n+1}),\\ u_{n+1}^+, & t=T_{n+1}.\end{cases} \tag{3.4}$$

To avoid the “beating” (or Zeno) effect [24], when an infinite number of impulsive moments accumulate over a finite interval, we should verify that $\forall u_0 \in X$ $G(\cdot, u_0)$ either has no more than finite impulses or that

$$T_n\to\infty \quad \text{as} \quad n\to\infty. \tag{3.5}$$

That if conditions (3.1), (3.2) and (3.5) hold, then the mapping $G : [0, +\infty)\times X \mapsto X$, given by (3.4), is a semigroup with right-continuous trajectories, and such G is called an *impulsive dynamical system* [9, 18].

Lemma 1. *Assume that ψ is a solution of the following spectral problem:*

$$\begin{cases} -\Delta\psi=\lambda\psi,\\ \psi\mid_{\partial\Omega}=0,\end{cases}\qquad \lambda>0. \tag{3.6}$$

Then, problems (2.1)–(2.3) *generate impulsive dynamical systems.*

Proof. The continuous semigroup $V : [0, +\infty)\times X \mapsto X$ is given by (2.5). Denote the scalar product in the space $L^2(\Omega)$ by $(\cdot,\cdot)$. The set M, given by (2.2), is closed, and

$$\forall u\in M \quad (Iu,\psi)=(u+\varphi,\psi)=R+\varphi_0>R;$$

thus, $M \cap IM = \varnothing$.

Let $u_0 \in M$ and $u(t) = V(t, u_0)$. We consider the following function:

$$g(t) := (u(t),\psi) = (V(t,u_0),\psi) = (u(0),\psi)e^{-\lambda t} + \varepsilon\int_0^t e^{-\lambda(t-s)}F(s)ds, \tag{3.7}$$

where

$$F(s) = (f(u(s)),\psi)\,,\quad |F(s)|\leqslant c\|\psi\|_{L^1}.$$

Thus, $g \in \mathbb{C}^1$, $g(0) = R$ and

$$g'(0) = -\lambda(u(0),\psi)+\varepsilon F(0) = -\lambda R+\varepsilon F(0) < -\frac{\lambda R}{2}$$

for sufficiently small $\varepsilon > 0$.

Therefore, $\exists \tau = \tau(u(0), \varepsilon) > 0$ such that

$$\forall t \in (0, \tau) \quad g(t) < R \quad \Rightarrow \quad Eq\ (3.2).$$

Let $u_0 \in IM$. Then, $u_0(x) \geqslant 0$ and

$$u(t,x) \geqslant 0, \quad \forall t \geqslant 0$$

due to the maximum principle [23]. Moreover,

$$(u_0, \psi) = R + \varphi_0 > R.$$

From (3.7), $\exists s > 0 : u(t) \notin M$, $t \in (0, s)$, $u(s) \in M$. Therefore,

$$R = (R + \varphi_0)e^{-\lambda s} + \varepsilon \int_0^s e^{-\lambda(s-p)} F(p) dp. \tag{3.8}$$

Thus,

$$(R + \varphi_0)e^{-\lambda s} < R + \varepsilon \cdot \int_0^s e^{-\lambda(s-p)} |F(p)| dp,$$

and

$$s > \bar{s} = \frac{1}{\lambda} \ln\left(1 + \frac{\varphi_0 - \delta}{R}\right) \tag{3.9}$$

for fixed $\delta \in (0, \varphi_0)$ for sufficiently small ε. Estimate (3.9) means that the distance between adjacent moments of impulses is not less than a certain constant value. Thus, impulses do not accumulate on a finite interval, and every impulsive trajectory exists on $[0, +\infty)$. Therefore, (3.9) implies (3.5). The lemma is proven. □

Remark 1. *Additionally, using* (3.8) *we obtain the following:*

$$s < \hat{s} = \frac{1}{\lambda} \ln\left(1 + \frac{\varphi_0 + \delta}{R}\right).$$

4. Attracting sets for impulsive dynamical systems (2.1)–(2.3)

We start with the general result about ω-limit sets for semigroups.

Lemma 2 ([14])**.** *Assume that the semigroup $G : \mathbb{R}_+ \times X \mapsto X$ satisfies the following properties:*

$$\forall r > 0 \quad \textit{the set} \quad \left\{ \bigcup_{t \geqslant 0} G(t, u_0) \,\middle|\, \|u_0\|_X \leqslant r \right\} \quad \textit{is bounded, and} \tag{4.1}$$

$$\forall r > 0\ \forall t_m \nearrow \infty \quad \textit{the set} \quad \left\{ G(t_m, u_0^{(m)}) \,\middle|\, \|u_0^{(m)}\|_X \leqslant r \right\} \quad \textit{is precompact in } X. \tag{4.2}$$

Then, $\forall r > 0$, the ω-limit set

$$\Theta_r = \bigcap_{T>0} \overline{\bigcup_{t \geqslant T} G(t, B_r)}, \ \textit{where } B_r = \{\|u\|_X \leq r\|\},$$

is nonempty and compact, and

$$\sup_{\|u_0\|_X \leqslant r} \operatorname{dist}_X (G(t,u_0), \Theta_r) \to 0, \quad t \to \infty.$$

Moreover, if

$$\exists r_0 > 0 \ \forall r > 0 \ \exists T = T(r) \text{ such that } \left\{ \bigcup_{t \geqslant T} G(t,u_0) \,\middle|\, \|u_0\|_X \leqslant r \right\} \subset B_{r_0},$$

then Θ_{r_0} is the uniform attractor.

The next result provides sufficient conditions on the continuous semigroup $V : \mathbb{R}_+ \times X \mapsto X$ and the impulsive parameters $M \subset X$, $I : M \mapsto X$, which guarantee that the corresponding impulsive dynamical system G, given by (3.4)*, possesses a nonempty, compact and attracting set.*

Lemma 3. *Assume that V, M and I satisfy* (3.1)*,* (3.2)*, and* (3.5)*, the corresponding impulsive dynamical system G, given by* (3.4)*, satisfies* (4.1)*, and there exists a compactly embedded space $Y \Subset X$ such that*

$\forall t > 0 \ \forall r > 0 \ \exists c(t,r) > 0$, *such that*

$$\forall u_0 : \quad \|u_0\|_X \leqslant r \quad \Rightarrow \quad \|V(t,u_0)\|_Y \leqslant c(t,r) \tag{4.3}$$

and $\exists \bar{u} \in X \ \forall r > 0 \ \exists c(r) > 0$ *such that*

$$\forall u \in M \cap Y : \ \|u\|_Y \leqslant r \quad \Rightarrow \quad \|Iu - \bar{u}\|_Y \leqslant c(r); \tag{4.4}$$

$$\forall r > 0 \ \exists s_r > 0 \ \forall u \in I(M \cap Y), \ \|u\|_X \leq r \quad \Rightarrow \quad s(u) \geqslant s_r, \tag{4.5}$$

where $s(u)$ is given by (3.3)*. Then, condition* (4.2) *holds.*

Proof. Let

$$\xi_m = G\left(t_{m,} u_0^{(m)}\right), \ \ t_m \nearrow \infty, \ \ \left\|u_0^{(m)}\right\|_X \leqslant r.$$

According to (3.4), we have impulsive points

$$\left\{u_{n+1}^{(m)+} = IV\left(s_n^{(m)}, u_n^{(m)+}\right)\right\}, \ \ u_0^{(m)+} = u_0^m$$

and moments of impulses

$$T_{n+1}^{(m)} = \sum_{k=0}^{n} s_k^{(m)}, \qquad T_0 = 0.$$

If $t_m < s_0^{(m)}$, then

$$\xi_m = V\left(t_m, u_0^{(m)}\right) = V\left(1, V\left(t_m - 1, u_0^{(m)}\right)\right).$$

Thus, $\|\xi_m\|_Y \leqslant c(1, D(r))$, where $D(r) = \sup_{t \geqslant 0, \ \|u_0\|_X \leq r} \|G(t,u_0)\|_X$ is a finite number due to (4.1). Therefore, $\{\xi_m\}$ is precompact in X.

If $t_m = s_0^{(m)}$, then

$$\xi_m = IV\left(t_m, u_0^{(m)}\right),$$

and due to (4.4),

$$\|\xi_m - \bar{u}\|_Y \leqslant C(C(1, D(r))),$$

which means that $\{\xi_m\}$ is precompact in X.

Finally, if $t_m > s_0^{(m)}$, then

$$\xi_m = V\left(t_m - T_n, u_n^+\right) \quad \text{or} \quad \xi_m = u_{n+1}^+$$

for some $n = n(m) \geqslant 1$. Due to (4.1), $\|u_n^+\|_X \leqslant D(r)$. Thus,

$$u_{n+1}^+ = IV\left(s_n, u_n^+\right)$$

due to the equality. By (4.5), we get that $s_n \geqslant s_{D(r)}$. Thus,

$$u_{n+1}^+ = IV\left(s_{D(r)}, V\left(s_n - s_{D(r)}, u_n^+\right)\right).$$

From (4.3) and (4.4), we deduce that

$$\left\|u_{n+1}^+ - \bar{u}\right\|_Y \leqslant c\left(c\left(s_{D(r)}, D(r)\right)\right).$$

Thus, in this case, $\{u_n^+\}$ is precompact in X.

Since $\xi_m = V\left(t_m - T_{n(m)}, u_{n(m)}^+\right)$, and due to (3.9) and Remark 1, we have that $\xi_m = V\left(\tau_m, u_{n(m)}^+\right)$, where

$$u_{n(m)}^+ \to \eta \quad \text{in} \quad X, \quad \tau_m \to \tau \geqslant 0$$

up to subsequence.

Therefore, a continuity of V implies the precompactness of $\{\xi_m\}$. The lemma is proven. □

Before the formulation of the main result, we need some properties of the semigroup V, which is defined in (2.5).

Let $\lambda_1 > 0$ be the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. For $\lambda \in [0, \lambda_1)$, the C_0-semigroup of contractions generated by $-\Delta - \lambda \cdot I$ in X is denoted as $S_\lambda(t)$, $S_0(t) = S(t)$.

Then, for $L = \exp\left\{\frac{\lambda_1 |\Omega|^{\frac{2}{d}}}{4\pi}\right\} > 1$, we have the following [25]:

$$\forall t \geqslant 0, \qquad \|S_\lambda(t)\|_{L(X,X)} \leqslant L e^{-(\lambda_1 - \lambda)t}. \tag{4.6}$$

Moreover, $\forall\, T > 0$, $\exists C > 0$, $\alpha \in (0, 1)$, and $\delta \in \left(\frac{1}{2}, 1\right)$ such that

$$\forall t \in (0, T], \quad \forall u_0 \in X, \quad \|S_\lambda(t) u_0\|_{C^{1+\alpha}} \leqslant \frac{C}{t^\delta} \|u_0\|_X .$$

In particular, for every mild solution of (2.1), $\forall t \in (0, T]$ and

$$\|u(t)\|_{C^{1+\alpha}} \leqslant \frac{C}{t^\delta} \|u_0\|_X + \frac{C t^{1-\delta}}{1 - \delta} \varepsilon c. \tag{4.7}$$

According to [21, 26], every mild solution $u(t)$ of (2.1) is a weak solution of (2.1), i.e., $u \in L^2\left(0, T; H_0^1(\Omega)\right)$, $\forall\, v \in H_0^1(\Omega)$ $\forall \eta \in C_0^\infty(0, T)$, and

$$\int_0^T (u(t), v)\eta_t dt + \int_0^T (u(t), v)_{H_0^1} \eta dt = \varepsilon \int_0^T (f(u(t)), v)\eta dt.$$

Moreover,

$$\forall \tau > 0, \quad u \in \mathbb{C}\left([\tau, T] : H_0^1(\Omega)\right) \cap L^2\left(\tau, T; H^2 \cap H_0^1\right), \quad u_t \in L^2\left(\tau, T; L^2(\Omega)\right).$$

Therefore, $\forall K > 0$ and

$$\frac{1}{2}\frac{d}{dt}\left\|(u \mp K)_\pm(t)\right\|_{L^2}^2 + \left\|(u \mp K)_\pm(t)\right\|_{H_0^1}^2 = \varepsilon \cdot \int_\Omega (u(t,x) \mp K)_\pm \cdot f(u(t,x))dx \tag{4.8}$$

for almost all $t \in (\tau, T)$, where

$$v_+ = \begin{cases} v, & v \geqslant 0, \\ 0, & v \leqslant 0, \end{cases}$$

and $v_- = v - v_+$ are truncation functions [13].

Lemma 4. *For sufficiently small $\varepsilon \geqslant 0$ and $\forall r > R$, we have the following:*

$$\|u_0\|_X \leqslant r \quad \Rightarrow \quad \forall t \geqslant 0, \quad \|V(t, u_0)\|_X \leqslant r. \tag{4.9}$$

Proof. Let us rewrite (2.1) with a new operator $-\Delta - \lambda I$ and with a new nonlinear term $-\lambda u + \varepsilon f(u)$, $\lambda \in (0, \lambda_1)$.

Then, for sufficiently small $\varepsilon > 0$,

$$s \cdot (-\lambda s + \varepsilon f(s)) = -\lambda s^2 + \varepsilon s f(s) \leqslant -\lambda s^2 + \varepsilon \cdot c \cdot |s| < 0, \quad |s| > \frac{R}{2}. \tag{4.10}$$

Thus, choosing $K = r$, $r > R$ in (4.8) and from (4.8) and Poincaré inequality, we obtain

$$\frac{d}{dt}\|(u-K)_+\|_{L^2}^2 + \nu_\lambda \cdot \|(u-K)_+\|_{L^2}^2 \leq 0$$

for some $\nu_\lambda > 0$ and for almost all $t > \tau$. For $\forall t \geq \tau$,

$$\int_\Omega (u(t,x) - K)_+^2 dx \leqslant e^{-\nu_\lambda \cdot (t-\tau)} \cdot \int_\Omega (u(\tau, x) - K)_+^2 dx.$$

After passing to the limit as $\tau \to 0$, we obtain the following:

$$\forall t \geqslant 0 \qquad \int_\Omega (u(t,x) - K)_+^2 dx \leqslant e^{-\nu_\lambda \cdot t} \cdot \int_\Omega (u_0(x) - K)_+^2 dx. \tag{4.11}$$

The same inequality holds for $(u + K)_-$.

Inequality (4.11) implies (4.9). The lemma is proven. □

In all further considerations, we assume that $\lambda = \lambda_1$ in (3.6) is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ and $\psi = \psi_1$ is the corresponding eigenfunction.

Additionally, without a loss of generality, we assume that $\psi_1(x) \geqslant 0$, $\forall x \in \Omega$.

Now, we are ready to prove the main result.

Theorem 4.1. *Assume that* (3.6) *holds and one of the following assumptions holds:*

$$\lambda_1 \cdot \frac{|\Omega|^{\frac{2}{d}}}{4\pi} < \ln\left(1 + \frac{\varphi_0}{R}\right) \tag{4.12}$$

or

$$\varphi(x) \leqslant c_1\psi_1(x), \quad \forall x \in \Omega, \quad f(s) \leqslant c_2 \cdot s, \quad \forall s \geqslant 0. \tag{4.13}$$

Then, for sufficiently small $\varepsilon > 0$, the impulsive dynamical systems (2.1)–(2.3) *possess a nonempty, compact uniform attractor $\Theta \subset X$, which is invariant in the following sense:*

$$\forall t \geqslant 0, \quad G(t, \Theta \setminus M) = \Theta \setminus M. \tag{4.14}$$

Proof. Let us prove the dissipativity conditions in both cases for (4.12) and (4.13).

Assume that (4.12) takes place, $\|u_0\|_X \leqslant r$ and $r > R$. According to (4.9), $\forall t \geqslant 0$ $\|V(t, u_0)\|_X \leq r$.

Passing to the shift differential operator and the corresponding nonlinear term (see (4.10)), we can achieve the following property:

$$f(u)u \leqslant 0, \quad |u| \geqslant \frac{R}{2}.$$

This property guarantees [21] that if $\|u_0\|_X \leq r$ and $M^+(u_0) = \emptyset$, then

$$\exists T = T(r), \quad \forall t \geqslant T, \quad G(t, u_0) = V(t, u_0) \subset B_1(0).$$

Let $s_0 > 0$ at the first moment of intersection with M. Then, due to (2.5),

$$\|V(s_0, u_0)\|_X \leq Le^{-\lambda s_0}\|u_0\|_X + \varepsilon \cdot L \cdot \frac{c}{\lambda},$$

where $L = e^{\frac{\lambda|\Omega|^{\frac{2}{d}}}{4\pi}}$. Then,

$$\left\|u_1^+\right\|_X \leqslant Le^{-\lambda s_0} r + \|\varphi\|_X + \frac{\varepsilon Lc}{\lambda}.$$

After the second impulse moment, using (3.9), we obtain the following:

$$\begin{aligned}\left\|u_2^+\right\|_X &\leqslant \left\|V(s_1, u_1^+)\right\|_X + \|\varphi\|_X \\ &\leqslant Le^{-\lambda\bar{s}}\left(Le^{-\lambda s_0} \cdot r + \|\varphi\|_X + \frac{\varepsilon Lc}{\lambda}\right) + \|\varphi\|_X + \frac{\varepsilon Lc}{\lambda} \\ &= L^2 e^{-\lambda(\bar{s}+s_0)} \cdot r + \left(\|\varphi\|_X + \frac{\varepsilon Lc}{\lambda}\right)\left(1 + Le^{-\lambda\bar{s}}\right).\end{aligned}$$

After the n-th step, we obtain the following:

$$\left\|u_n^+\right\|_X \leqslant L^n \cdot e^{-\lambda n\bar{s}} \cdot e^{\lambda\bar{s}} \cdot e^{-\lambda s_0} \cdot r + \left(\|\varphi\|_X + \frac{\varepsilon Lc}{\lambda}\right)\left(1 + L^{n-1}e^{-\lambda(n-1)\bar{s}}\right). \tag{4.15}$$

Due to inequality (4.12),

$$L \cdot e^{-\lambda\bar{s}} = e^{\frac{\lambda|\Omega|^{\frac{2}{d}}}{4\pi}} \cdot e^{-\ln\left(1+\frac{\varphi_0}{R}\right)} < 1. \tag{4.16}$$

Combining (2.5), (4.15), and (4.16), we conclude that (4.1) holds. Moreover, for $r_0 = 1 + \frac{\|\varphi\|_X + \frac{\varepsilon Lc}{\lambda}}{1 - Me^{-\lambda\bar{s}}}$,

$$\forall r > 0, \quad \forall u_0, \|u_0\|_X \leqslant r, \quad \exists T = T(r), \quad \forall t \geqslant T, \quad \|G(t, u_0)\|_X \leqslant r_0. \tag{4.17}$$

Assume that (4.13) holds. Then, after the appropriate choice of ε, we can achieve the following:

$$\varepsilon f(s) \leqslant \lambda s, \quad s \geqslant 0, \quad \text{where} \quad \lambda < \lambda_1.$$

Thus, let $\|u_0\|_X \leq r$, $r > R$, and $s_0 > 0$ be the first moment of intersection with M. Then, for $u(t) = V(t, u_0)$, we have the following:

$$u(s_0, x) \geqslant 0, \quad u(s_0, x) = R\psi + \alpha(x), \quad \text{where} \quad \|\alpha\|_X \leqslant r.$$

Therefore, due to (4.13),

$$0 \leq u_1^+(x) \leq (R + c_1)\,\psi + \alpha(x) =: \bar{u}_1^+(x).$$

Due to the maximum principle for $S_\lambda(t)$ (see (4.6)), we have the following:

$$V(t, u_1^+) \leqslant S_\lambda(t)\bar{u}_1^+ = (R + c_1)\, e^{-\overline{\lambda}_1 t}\psi + S_\lambda(t)\alpha, \text{ where } \overline{\lambda}_1 = \lambda_1 - \lambda.$$

Thus,

$$0 \leq u(s_1, x) \leqslant (R + c_1)e^{-\bar{\lambda}_1 s_1}\psi + S_\lambda(s_1)\,\alpha(x),$$
$$0 \leqslant u_2^+(x) \leqslant \left((R + c_1)e^{-\bar{\lambda}_1 s_1} + c_1\right)\psi(x) + S_\lambda(s_1)\,\alpha(x).$$

On the n-th step, using $s_i \geqslant \bar{s}$ we obtain the following:

$$0 \leqslant u_n^+(x) \leqslant \left((R + c_1)e^{-\bar{\lambda}_1(n-1)\bar{s}} + c_1\left(1 + e^{-\bar{\lambda}_1\bar{s}} + \ldots + e^{-\bar{\lambda}_1(n-2)\bar{s}}\right)\right)\psi(x)$$
$$+ S_\lambda(s_1 + \ldots + s_{n-1})\,\alpha(x). \tag{4.18}$$

Using the inequality (4.6), from (4.18), we obtain the property (4.17) with the following:

$$r_0 = 1 + \frac{c_1}{1 - e^{\bar{\lambda}_1\bar{s}}}.$$

The inequality (4.7) implies (4.3) with $Y = C^1(\Omega)$. Then, Lemmas 2 and 3 guarantee the existence of the uniform attractor Θ. □

We need the following result to prove equality (4.14).

Lemma 5 ([16, 22]). *Assume that G satisfies* (3.1), (3.2)*, and* (3.5)*, and possesses a compact uniform attractor Θ.*

Assume that the following properties hold:

$$I : M \mapsto X \quad \textit{is continuous}, \quad \forall u \in IM, \quad s(u) < \infty, \tag{4.19}$$

where $s(u)$ is given by (3.3).

For $u_n \to u_0 \in \Theta \setminus M$,

$$\begin{aligned} &\textit{if } s(u_n) < \infty, \quad \textit{then} \quad s(u_0) < \infty \textit{ and } s(u_n) \to s(u_0), \\ &\textit{if } s(u_n) = \infty, \quad \textit{then for} \quad t_n \nearrow \infty \textit{ and } \xi = \lim V(t_n, u_n) \textit{ we have } s(\xi) = \infty; \end{aligned} \tag{4.20}$$

for $u_n \to u_0 \in \Theta \cap M$,

$$\textit{either} \quad s(u_n) = \infty \quad \textit{or} \quad s(u_n) \to 0; \tag{4.21}$$

for any bounded $\{u_n\} \subset X$, $\forall t_n \nearrow \infty$ up to subsequence

$$V(t_n, u_n) \to \xi \notin M. \tag{4.22}$$

Then, (4.14) *holds.*

Thus, let us prove (4.14). Property (4.19) was already proven in Lemma 1.

Due to the maximum principle, one has the following:

$$\int_{\Omega}(u(t_n,x)-M_1)_t^2\,dx \leqslant e^{-\delta t_n}\int_{\Omega}(u(0,x)-M_1)_r^2\,dx, \tag{4.23}$$

where M_1 is taken from the inequality $f(u)u \le 0$, $|u| > M_1$. By choosing ε, we can make M_1 as small as we need. It means that every limit in (4.22) satisfies $|\xi(x)| \leqslant M_1 < R$ for $M_1 < R$. Thus, $\xi \notin M$ and (4.22) holds.

Let us prove (4.20) and (4.21).

Let $u_n(t) = V\left(t, u_0^n\right)$, $u_0^n \to u_0 \notin M$, and $s^n = s\left(u_0^n\right) < \infty$. Then, $u_n(s^n, x) \geqslant 0$ and $g_n(s_n) = (u_n(s_n), \psi) = R$. Thus,

$$\left(u_0^n, \psi\right) e^{-\lambda s^n} + \varepsilon\int_0^{s^n} e^{-\lambda(s^n - s)}F_n(s)ds = R. \tag{4.24}$$

If $\left\|u_0^n\right\|_X \leqslant r$, $r > R$, then

$$re^{-\lambda s^n} \geqslant \frac{R}{2} \quad \Rightarrow s^n \leqslant \frac{1}{\lambda}\ln\left(\frac{2r}{R}\right)$$

for sufficiently small ε and up to subsequence $s^n \to s \geqslant 0$.

Passing to the limit in (4.24), we obtain the following:

$$(u(s), \psi) = R, \quad u(s,x) \geqslant 0.$$

If $s = 0$, then $u_0 \in M$; thus, $s > 0$ and $s(u_0) < \infty$.

The equality $s = s(u_0)$ follows from the fact that a trajectory that starts from M will not turn back to M.

Indeed, if $u_0 \in M$, then $g(0) = R$ and

$$g(t) < Re^{-\lambda t} + \frac{R}{4}$$

for sufficiently small $\varepsilon > 0$. Therefore, we have that $g(t) < \frac{R}{2}$ for $t \geqslant \frac{1}{\lambda}\ln(4)$. From the other side, $g'(t) < 0$ for $t \in \left[0, \frac{1}{\lambda}\ln 4\right]$, thus

$$\forall t > 0, \quad g(t) < R.$$

The second part of (4.20) is the consequence of (4.23).

Finally, let $u_0^n \to u_0 \in M$ and $s\left(u_0^n\right) < \infty$. Then, $s_n = s\left(u_0^n\right) \to s \geqslant 0$.

Assume that $s > 0$. Then, after passing to the limit in (4.24), we obtain $u(s) \in M$, which is a contradiction. The Theorem is proven.

Example 1. *Assume that $\Omega = (0,\pi)$, $f(s) = \sin s$, $\psi(x) \equiv 1$, $R = 1$, and $\varphi(x) = \sin x$. Then, we have the following impulsive problem in the phase space $X = \mathbb{C}_0(0,\pi)$:*

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2}u = \varepsilon\cdot\sin u, \quad t > 0, \quad x \in (0,\pi), \\ u|_{x=0} = u|_{x=\pi} = 0, \\ u|_{t=0} = u_0, \end{cases} \tag{4.25}$$

$$M = \left\{ v \in X \mid v(x) \geqslant 0, \int_0^\pi v(x)dx = 1 \right\}. \tag{4.26}$$

At the moment of meeting the phase point $u(t) \in X$ with the set M, it undergoes an impulsive perturbation and instantly finds itself in a new position as follows:

$$(Iu)(x) = u(t,x) + \sin x. \tag{4.27}$$

The inclusion $u(t) \in X$ means that $u(t)$ is a continuous function of x (i.e., $u(t)(x) = u(t,x)$). Moreover, in this particular situation, the formula for the semigroup $S(t)$ is available [13]; thus,

$$u(t) = S(t)u_0 + \varepsilon \int_0^t S(t-s) \sin u(s) ds,$$

where

$$(S(t)v)(x) = \frac{2}{\pi} \sum_{k=1}^{\infty} \left(\int_0^\pi v(x) \sin kx dx \right) e^{-k^2 t} \sin kx.$$

Then, assumption (4.12) of Theorem 4.1 holds (it turns into an obvious inequality $\frac{\pi}{4} < \ln 3$). Thus, for sufficiently small $\varepsilon > 0$, the impulsive dynamical systems (4.25)–(4.27) *possess a nonempty, compact uniform attractor $\Theta \subset X$, which satisfies the equality (4.14).*

5. Conclusions

In this paper, we investigated the qualitative behavior of mild solutions for a semilinear parabolic equation subject to impulsive perturbations triggered by an integral functional threshold. The key novelty of our work lies in the study of impulsive dynamical systems within the phase space of continuous functions equipped with the sup-norm, thus extending previous results that primarily focused on L^2-norms.

Under general assumptions on the impulsive parameters, we established that the problem generates an infinite-dimensional impulsive dynamical system. Furthermore, we proved the existence of a compact uniform attractor in the sup-norm, which captures the long-term dynamics of the system. Our analysis relied on a combination of the semigroup theory, comparison principles, and careful estimates to ensure the avoidance of the Zeno effect (infinite impulses in finite time).

The results hold under two distinct sets of conditions: (a) a spectral constraint linking the domain size, the first eigenvalue of the Laplacian, and the impulsive magnitude (Theorem 4.1, condition (4.12)); and (b) bounds on the impulsive function φ and the nonlinearity f (Theorem 4.1, condition (4.13)).

Additionally, an important invariance property (4.14) was demonstrated, thereby showing that the attractor remains unchanged under the flow outside the impulsive set M. This work opens avenues for further research, such as extending the framework to more general nonlinearities or multi-dimensional impulsive effects.

Author contributions

Oleksiy V. Kapustyan: Supervision, conceptualization, investigation, methodology, writing original draft preparation, writing-review and editing; Svetlana M. Temesheva: investigation, writing original draft preparation, and editing; Agila B. Tleulessova: formal analysis, resourses and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. A. M. Samoilenko, N. A. Perestyuk, *Impulsive differential equations*, Singapore: World Scientific, 1995. https://doi.org/10.1142/2892
2. R. Goebel, R. G. Sanfelice, A. R. Teel, *Hybrid dynamical systems: modeling, stability, and robustness*, Princtone University Press, 2012. https://doi.org/10.23943/princeton/9780691153896.001.0001
3. S. K. Kaul, Stability and asymptotic stability in impulsive semidynamical systems, *J. Appl. Math. Stochastic Anal.*, **7** (1994), 509–523. http://doi.org/10.1155/s1048953394000390
4. A. D. Myshkis, Heat conductivity with self-regulated pulse maintenance, *Autom. Remote Control*, **56** (1995), 179–186.
5. V. Lakshmikantham, D. Bainov, P. S. Simeonov, *Theory of impulsive differential equations*, World Scientific, Singapore, 1989. http://doi.org/10.1142/0906
6. M. U. Akhmet, Perturbations and Hopf bifurcation of the planar discontinuous dynamical system, *Nonlinear Anal.*, **60** (2005), 163–178. http://doi.org/10.1016/S0362-546X(04)00347-5
7. K. Ciesielski, On stability in impulsive dynamical systems, *Bull. Polish Academy Sci. Math.*, **52** (2004), 81–91.

8. E. M. Bonotto, M. Federson, Limit sets and the Poincaré-Bendixson theorem in impulsive semidynamical systems, *J. Differ. Equations*, **244** (2008), 2334–2349. http://doi.org/10.1016/j.jde.2008.02.007

9. E. M. Bonotto, Flows of characteristic 0^+ in impulsive semidynamical systems, *J. Math. Anal. Appl.*, **332** (2007), 81–96. http://doi.org/10.1016/j.jmaa.2006.09.076

10. K. Shah, I. Ahmad, J. J. Nieto, G. U. Rahman, T. Abdeljawad, Qualitative investigation of nonlinear fractional coupled pantograph impulsive differential equations, *Qual. Theory Dyn. Syst.*, **21** (2022), 131. http://doi.org/10.1007/s12346-022-00665-z

11. T. Abdeljawad, K. Shah, M. S. Abdo, F. Jarad, An analytical study of fractional delay impulsive implicit systems with Mittag-Leffler law, *Appl. Comput. Math.*, **22** (2023), 31–44. http://doi.org/10.30546/1683-6154.22.1.2023.31

12. J. C. Robinson, *Infinite-dimensional dynamical systems*, An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors, Cambridge University Press, 2001.

13. R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, 2 Eds., Springer, 1997. http://doi.org/10.1007/978-1-4612-0645-3

14. V. V. Chepyzhov, M. I. Vishik, *Attractors for equations of mathematical physics*, American Mathematical Society, 2002.

15. E. M. Bonotto, M. C. Bortolan, A. N. Carvalho, R. Czaja, Global attractors for impulsive dynamical systems – a precompact approach, *J. Differ. Equations*, **259** (2015), 2602–2625. http://doi.org/10.1016/j.jde.2015.03.033

16. S. Dashkovskiy, O. Kapustyan, Y. Perestyuk, Stability of uniform attractors of impulsive multivalued semiflows, *Nonlinear Anal.*, **40** (2021), 101025. http://doi.org/10.1016/j.nahs.2021.101025

17. S. Dashkovskiy, O. A. Kapustian, O. V. Kapustyan, N. V. Gorban, Attractors for multivalued impulsive systems: existence and applications to reaction-diffusion system, *Math. Probl. Eng.*, **2021** (2021), 7385450. http://doi.org/10.1155/2021/7385450

18. O. V. Kapustyan, M. O. Perestyuk, Global attractors in impulsive infinite-dimensional systems, *Ukr. Math. J.*, **68** (2016), 583–598. http://doi.org/10.1007/s11253-016-1243-0

19. O. V. Kapustyan, O. A. Kapustian, I. Korol, B. Rubino, Uniform attractor of impulse-perturbed reaction-diffusion system, *Math. Mech. Complex Syst.*, **11** (2023), 45–55. http://doi.org/10.2140/memocs.2023.11.45

20. A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, Vol. 44, Springer, 1983. http://doi.org/10.1007/978-1-4612-5561-1

21. P. Feketa, O. V. Kapustyan, O. A. Kapustian, I. Korol, Global attractors of mild solutions semiflow for semilinear parabolic equation without uniqueness, *Appl. Math. Lett.*, **135**(2022), 108435. http://doi.org/10.1016/j.aml.2022.108435

22. O. Kapustyan, O. Kapustian, I. Korol, B. Rubino, Attracting sets of impulse-perturbed heat equation in the space of continuous functions, *Miskolc Math. Notes*, **25** (2024), 317–327. https://doi.org/10.18514/MMN.2024.4213

23. J. Valero, A weak comparison principle for reaction-diffusion systems, *J. Funct. Spaces*, **2012** (2012), 679465. http://doi.org/10.1155/2012/679465

24. S. Dashkovskiy, P. Feketa, Asymptotic properties of Zeno solutions, *Nonlinear Anal.*, **30** (2018), 256–265. http://doi.org/10.1016/j.nahs.2018.06.005

25. A. Haraux, *Nonlinear evolution equations-global behavior of solutions*, Vol. 841, Springer, 1981. http://doi.org/10.1007/bfb0089606

26. J. M. Ball, Strongly continuous semigroups, weak solutions, and the variation of constants formula, *Proc. Amer. Math. Soc.*, **63** (1977), 370–373. http://doi.org/10.1090/S0002-9939-1977-0442748-6

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