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Research article

Best proximity point results of iterated function systems for α-ψ-contractions

Mustafa Aslantas[1,*], Ali Hussein Bachay[1,2], Hakan Sahin[3] and A. Duran Türkoğlu[4]

[1] Department of Mathematics, Faculty of Science, Çankırı Karatekin University, 18100 Çankırı, Turkey
[2] Department of Business Administration, College of Administration and Economics, University of Sumer, Iraq
[3] Department of Mathematics, Faculty of Engineering and Natural Sciences, Bursa Technical University, 16310 Bursa, Turkey
[4] Department of Mathematics, Faculty of Science, Gazi University, 06010 Ankara, Turkey

* **Correspondence:** Email: maslantas@karatekin.edu.tr.

Abstract: In this paper, we present two new concepts, called α-ψ-iterated function system and α-ψ-proximal iterated function system, by using α-ψ-contractions and α-ψ-proximal contractions. Hence, we extended and generalized some definitions existing in the literature. We present some results that determine the necessary conditions to obtain a fractal with an attractor in the mentioned systems. Finally, some interesting examples are presented to apply our results.

Keywords: iterated function systems; fractals; best proximity point
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1. Introduction

Fixed point theory is largely concerned with locating points that remain unchanged under a given mapping; i.e., a fixed point is a point $\varkappa$ such that $\eta(\varkappa) = \varkappa$. This notion is important in many fields, including dynamical systems, game theory, and nonlinear analysis, because the stability and behavior of systems under recurrent mappings are critical to understanding their long-term behavior. One of the most celebrated results in fixed point theory is the Banach Contraction Principle, also known as the Banach Fixed Point Theorem. This theorem, formulated by S.Banach [1] in 1922, provides a powerful and elegant method for finding fixed points of certain types of functions. According to the theorem, let $(\eth, \sigma)$ be a complete metric space, and $\eta : \eth \to \eth$ be a mapping satisfying $\sigma(\eta\varkappa, \eta\zeta) \leq k\sigma(\varkappa, \zeta)$ for all $\varkappa, \zeta \in \eth$ where $k \in (0, 1)$, which is called k-contraction mapping. Then, η has a unique fixed point $\varkappa^*$ in $\eth$ and for any initial point $\varkappa_0 \in \eth$, the sequence defined by $\varkappa_{n+1} = \eta(\varkappa_n)$ converges to $\varkappa^*$. The Banach Contraction Principle is central to many areas of analysis and applied mathematics. It has applications in solving differential equations, optimization problems, and proving the existence of solutions in various mathematical models. Therefore, there are many extension results in the literature [2–5]. One of these extensions was obtained by Samet, Vetro, and Vetro [6] by introducing α-ψ-contractive type mappings. On the other hand, the best proximity point theory was put forward by Basha and Veeramani [7] by considering non-self mappings in metric spaces. The concept of best proximity points emerged as a generalization of fixed point results. There is no fixed-point of the mapping $\eta : \mathfrak{R} \to \mathfrak{I}$ in the situation of $\mathfrak{R} \cap \mathfrak{I} = \emptyset$. Thus, it makes sense to check to see if there is a point $\check{s}$ in $\mathfrak{R}$, such that $\sigma(\varkappa, \eta\varkappa) = \sigma(\mathfrak{R}, \mathfrak{I})$ are known as the best proximity point of η. Many authors have explored this subject because the best proximity point theory incorporates the fixed-point theory in a particular situation, $\mathfrak{R} = \mathfrak{I} = \eth$ [8–15].

Fractals, on the other hand, are complex structures that show self-similarity at many scales. These complicated geometric structures are frequently created via iterative procedures in which simple rules are repeated to construct progressively elaborate patterns. Fractals are not confined to classical Euclidean geometry, as their structure is usually non-smooth and exhibits a form of self-repetition at every level of magnification. The connection between fixed point theory and fractals becomes evident when we consider iterative functions that generate fractals. In many fractal structures, such as the Mandelbrot set, the points that produce the fractals are determined by the behavior of sequences under iteration. Fixed points play a central role in determining the properties of these fractals, particularly in understanding which points are stable (remain fixed) under repeated iterations and which points lead to chaotic or divergent behavior. As a result, Hutchinson [16] investigated objects with the self-similar quality and developed an iterated function system (IFS), which is one of the most prevalent methods for creating these fractals. Barnsley [17] popularized the system by deriving a fundamental theorem based on the famous principle. Hence, there are many results in the literature [18–20]. Moreover, the relationship between fractal and best proximity theory was established by Altun et al. [21] and Aslantas et al. [22].

In this paper, we present two new concepts, called α-ψ-iterated function system and α-ψ-proximal iterated function system, using α-ψ-contractions and α-ψ-proximal contractions. Then, we present some results that determine the necessary conditions to obtain a fractal with an attractor in the mentioned systems. Finally, some interesting examples are presented to apply our results.

2. Prelimainaries

In this section, we give some basic definitions and theorems necessary for our work.

Let $(\eth, \sigma)$ be a metric space. Consider the following subsets throughout the paper

$$\mathfrak{R}_0 = \{\varkappa \in \mathfrak{R} : \sigma(\varkappa, \zeta) = \sigma(\mathfrak{R}, \mathfrak{I}) \text{ for some } \zeta \in \mathfrak{I}\}$$

and

$$\mathfrak{I}_0 = \{\zeta \in \mathfrak{I} : \sigma(\varkappa, \zeta) = \sigma(\mathfrak{R}, \mathfrak{I}) \text{ for some } \varkappa \in \mathfrak{R}\},$$

where $\sigma(\mathfrak{R}, \mathfrak{I}) = \inf\{\sigma(\varkappa, \zeta) : \varkappa \in \mathfrak{R} \text{ and } \zeta \in \mathfrak{I}\}$.

Definition 1. *Let $(\eth, \sigma)$ be a metric space and $\emptyset \neq \mathfrak{R}, \mathfrak{I} \subseteq \eth$. Then, we say that $(\mathfrak{R}, \mathfrak{I})$ have the P_σ -Property if*

$$\left.\begin{array}{l}\sigma(u_1, v_1) = \sigma(\mathfrak{R}, \mathfrak{I})\\ \sigma(u_2, v_2) = \sigma(\mathfrak{R}, \mathfrak{I})\end{array}\right\} \Longrightarrow \sigma(u_1, u_2) = \sigma(v_1, v_2)$$

for all $u_1, u_2 \in \mathfrak{R}$ and $v_1, v_2 \in \mathfrak{I}$.

Definition 2. *Let $(\eth, \sigma)$ be a metric space and $\emptyset \neq \mathfrak{R}, \mathfrak{I} \subseteq \eth$. Assume that $\eta : \mathfrak{R} \to \mathfrak{I}$ is a mapping and $\alpha : \mathfrak{R} \times \mathfrak{R} \to [0, \infty)$ is a function. If η satisfies*

$$\left.\begin{array}{l}\alpha(\varkappa_1, \varkappa_2) \geq 1\\ \sigma(u_1, \eta\varkappa_1) = \sigma(\mathfrak{R}, \mathfrak{I})\\ \sigma(u_2, \eta\varkappa_2) = \sigma(\mathfrak{R}, \mathfrak{I})\end{array}\right\} \Longrightarrow \alpha(u_1, u_2) \geq 1$$

for all $\varkappa_1, \varkappa_2, u_1, u_2 \in \mathfrak{R}$, then it is called α_σ-proximal admissible.

Let $\psi : [0, \infty) \to [0, \infty)$ be a nondecreasing function satisfying $\sum_{k=1}^{\infty} \psi^k(t) < \infty$ for each $t > 0$ and $\psi(0) = 0$. It can be seen that $\psi(t) < t$ for all $t > 0$. The set of the functions ψ will be denoted by Ψ.

Definition 3. *Let $(\eth, \sigma)$ be a metric space, $\emptyset \neq \mathfrak{R}, \mathfrak{I} \subseteq \eth$, $\eta : \mathfrak{R} \to \mathfrak{I}$ be a mapping and $\alpha : \mathfrak{R} \times \mathfrak{R} \to [0, \infty)$ be a function. Then, η is said to be an α-ψ-contraction mapping if there exists $\psi \in \Psi$ such that*

$$\alpha(\varkappa, \zeta)\sigma(\eta\varkappa, \eta\zeta) \leq \psi(\sigma(\varkappa, \zeta))$$

for all $\varkappa, \zeta \in \mathfrak{R}$.

If we take $\alpha(\varkappa, \zeta) = 1$ for all $\varkappa, \zeta \in \mathfrak{R}$ and $\psi(t) = kt$ for all $t \in [0, \infty)$ where $k \in [0, 1)$, it is a k-contraction mapping, which is a famous contraction.

Theorem 1. *Let $\emptyset \neq \mathfrak{R}, \mathfrak{I}$ be subsets of $\eth$, where $\mathfrak{R}$ is closed, $(\eth, \sigma)$ is a complete metric space, and $\mathfrak{R}_0 \neq \emptyset$. Let $\alpha : \mathfrak{R} \times \mathfrak{R} \to [0, \infty)$ be a function, $\eta : \mathfrak{R} \to \mathfrak{I}$ be a mapping and $\psi \in \Psi$. Suppose that the following statements hold.*

i) $\eta(\mathfrak{R}_0) \subseteq \mathfrak{I}_0$ and $(\mathfrak{R}, \mathfrak{I})$ has the P_σ-Property.
ii) η is α_σ-proximal admissible.
iii) There exists $\varkappa_0$ and $\varkappa_1$ in $\mathfrak{R}_0$, such that

$$\sigma(\varkappa_1, S\varkappa_0) = \sigma(\mathfrak{R}, \mathfrak{I}) \text{ and } \alpha(\varkappa_0, \varkappa_1) \geq 1.$$

iv) η is a continuous α-ψ-contraction.
Then, η has a best proimity point in $\mathfrak{R}$.

Definition 4. *Let $(\eth, \sigma)$ be a metric space, $\emptyset \neq \mathfrak{R}, \mathfrak{I} \subseteq \eth$. Then, the pair $(\mathfrak{R}, \mathfrak{I})$ is said to have the weak P_σ -Property if*

$$\left.\begin{array}{l}\sigma(u_1, v_1) = \sigma(\mathfrak{R}, \mathfrak{I})\\ \sigma(u_2, v_2) = \sigma(\mathfrak{R}, \mathfrak{I})\end{array}\right\} \Longrightarrow \sigma(u_1, u_2) \leq \sigma(v_1, v_2)$$

for all $u_1, u_2 \in \mathfrak{R}$ and $v_1, v_2 \in \mathfrak{I}$.

Similar to the proof of Theorem 1, we can prove the following result by taking the weak P_σ-Property instead of the P_σ-Property.

Theorem 2. *Let $\emptyset \neq \mathfrak{R}, \mathfrak{I}$ be subsets of $\eth$ where $\mathfrak{R}$ is closed, $(\eth, \sigma)$ be a complete metric space, and $\mathfrak{R}_0 \neq \emptyset$. Let $\alpha : \mathfrak{R} \times \mathfrak{R} \to [0, \infty)$ be a function $\eta : \mathfrak{R} \to \mathfrak{I}$ be a mapping and $\psi \in \Psi$. Suppose that the following statements are hold.*

i) $\eta(\mathfrak{R}_0) \subseteq \mathfrak{I}_0$ and $(\mathfrak{R}, \mathfrak{I})$ has the weak P_σ-Property.
ii) η is α_σ-proximal admissible.
iii) There exists $\varkappa_0$ and $\varkappa_1$ in $\mathfrak{R}_0$ such that

$$\sigma(\varkappa_1, S\varkappa_0) = \sigma(\mathfrak{R}, \mathfrak{I}) \textit{ and } \alpha(\varkappa_0, \varkappa_1) \geq 1.$$

iv) η is a continuous α-ψ-contraction.
Then, η has a best proimity point in $\mathfrak{R}$.

The subset $\mathfrak{I}$ is called approximately compact w.r.t. $\mathfrak{R}$ if there exists a subsequence $\{\zeta_{n_k}\}$ of $\{\zeta_n\}$ such that $\zeta_{n_k} \to \zeta \in \mathfrak{I}$ for any sequence $\{\zeta_n\}$ in $\mathfrak{I}$, satisfying $\sigma(\varkappa, \zeta_n) \to \sigma(\varkappa, \mathfrak{I})$ for some $\varkappa \in \mathfrak{R}$.

Definition 5. *Let $(\eth, \sigma)$ be a metric space, $\emptyset \neq \mathfrak{R}, \mathfrak{I} \subseteq \eth$, $\eta : \mathfrak{R} \to \mathfrak{I}$ be a mapping and $\alpha : \mathfrak{R} \times \mathfrak{R} \to [0, \infty)$ be a function. Then, η is said to be an α-ψ-proximal contraction mapping if there is $\psi \in \Psi$, satisfying*

$$\left.\begin{aligned} \sigma(u_1, \eta\varkappa_1) = \sigma(\mathfrak{R}, \mathfrak{I}) \\ \sigma(u_2, \eta\varkappa_2) = \sigma(\mathfrak{R}, \mathfrak{I}) \end{aligned}\right\} \Longrightarrow \alpha(u_1, u_2)\sigma(u_1, u_2) \leq \psi(\sigma(\varkappa_1, \varkappa_2))$$

for all $\varkappa_1, \varkappa_2, u_1, u_2 \in \mathfrak{R}$.

Theorem 3. *Let $\emptyset \neq \mathfrak{R}, \mathfrak{I}$ be subsets of $\eth$ where $\mathfrak{R}$ is closed, $(\eth, \sigma)$ be a complete metric space, and $\mathfrak{R}_0 \neq \emptyset$. Let $\alpha : \mathfrak{R} \times \mathfrak{R} \to [0, \infty)$ be a function, $\eta : \mathfrak{R} \to \mathfrak{I}$ be a mapping and $\psi \in \Psi$. Suppose that the following statements are hold.*

i) η is α_σ-proximal admissible and $\eta(\mathfrak{R}_0) \subseteq \mathfrak{I}_0$.
ii) $\mathfrak{I}$ is approimately compact with respect to $\mathfrak{R}$.
iii) There exists $\varkappa_0$ and $\varkappa_1$ in $\mathfrak{R}_0$ such that

$$\sigma(\varkappa_1, \eta\varkappa_0) = \sigma(\mathfrak{R}, \mathfrak{I}) \textit{ and } \alpha(\varkappa_0, \varkappa_1) \geq 1.$$

iv) η is a continuous α-ψ-proximal contraction.
Then, η has a best proimity point in $\mathfrak{R}$.

Proof. From (iii), there exists $\varkappa_0$ and $\varkappa_1$ in $\mathfrak{R}_0$ such that

$$\sigma(\varkappa_1, \eta\varkappa_0) = \sigma(\mathfrak{R}, \mathfrak{I}) \text{ and } \alpha(\varkappa_0, \varkappa_1) \geq 1.$$

Since $\eta(\mathfrak{R}_0) \subseteq \mathfrak{I}_0$, there exists $\varkappa_2 \in \mathfrak{R}_0$ such that $\sigma(\varkappa_2, \eta\varkappa_1) = \sigma(\mathfrak{R}, \mathfrak{I})$. Since η is α_σ-proximal admissible and α-ψ-proximal contraction, we have

$$\alpha(\varkappa_1, \varkappa_2) \geq 1$$

and

$$\alpha(\varkappa_1, \varkappa_2)\sigma(\varkappa_1, \varkappa_2) \leq \psi(\sigma(\varkappa_0, \varkappa_1)).$$

We can construct a sequence

$$\alpha(\varkappa_n, \varkappa_{n+1}) \geq 1$$

and

$$\alpha(\varkappa_n, \varkappa_{n+1})\sigma(\varkappa_n, \varkappa_{n+1}) \leq \psi(\sigma(\varkappa_{n-1}, \varkappa_n))$$

for all $n \geq 1$. Hence, as in the proof of Theorem 1, we can show that there exists a point $\varkappa$, such that $\sigma(\varkappa, \eta\varkappa) = \sigma(\mathfrak{R}, \mathfrak{I})$. □

Remark 1. *Let $\mathfrak{R}, \mathfrak{I}$ be nonempty subsets of complete metric space. If $\mathfrak{R}$ is closed and $\mathfrak{I}$ is approximately compact with respect to $\mathfrak{R}$, then $(\mathfrak{R}_0, \sigma)$ is a complete metric space.*

Hence, we give the generalization of Theorem 3 as follow:

Theorem 4. *Let $(\eth, \sigma)$ be a metric space, $(\mathfrak{R}, \sigma)$ be a complete metric space and $\mathfrak{R}_0 \neq \emptyset$. Let $\alpha : \mathfrak{R}\times \mathfrak{R} \to [0, \infty)$ be a function $\eta : \mathfrak{R} \to \mathfrak{I}$ be a mapping and $\psi \in \Psi$. Suppose that the following statements are hold.*

i) η is α_σ-proximal admissible and $\eta(\mathfrak{R}_0) \subseteq \mathfrak{I}_0$.

iii) There exists $\varkappa_0$ and $\varkappa_1$ in $\mathfrak{R}_0$, such that

$$\sigma(\varkappa_1, \eta\varkappa_0) = \sigma(\mathfrak{R}, \mathfrak{I}) \text{ and } \alpha(\varkappa_0, \varkappa_1) \geq 1.$$

iv) η is a continuous α-ψ-proximal contraction.
Then, η has a best proimity point in $\mathfrak{R}$.

The set the family of all nonempty compact subset of a metric space $(\eth, \sigma)$ will be denoted by $K(\eth)$. For all $\mathfrak{R}, \mathfrak{I} \in K(\eth)$, we define

$$D(\mathfrak{R}, \mathfrak{I}) = \sup\{\sigma(\varkappa, \mathfrak{I}) : \varkappa \in \mathfrak{I}\}$$

where

$$\sigma(\varkappa, \mathfrak{I}) = \inf\{\sigma(\varkappa, \zeta) : \zeta \in \mathfrak{I}\}.$$

$\hbar : K(\eth) \times K(\eth) \to [0, \infty)$ defined by

$$\hbar(\mathfrak{R}, \mathfrak{I}) = \max\{D(\mathfrak{R}, \mathfrak{I}), D(\mathfrak{I}, \mathfrak{R})\}$$

for all $\mathfrak{R}, \mathfrak{I} \in K(\eth)$ is a metric. If $(\eth, \sigma)$ is complete, then $(K(\eth), \hbar)$ is complete.

Lemma 1. *([17]) Let $(\eth, \sigma)$ be a metric space and $\{U_i\}_{i=1}^N$, $\{V_i\}_{i=1}^N$ be collections of subsets of $K(\eth)$. Hence, we get*

$$\hbar\left(\bigcup_{i=1}^N U_i, \bigcup_{i=1}^N V_i\right) \leq \max_{1\leq i\leq N} \hbar(U_i, V_i).$$

Theorem 5. *Let $(\eth, \sigma)$ be a complete metric space and $\{\eth; \eta_i, i = 1, 2, \cdots, N\}$ be an iterated function system. Then, $S : K(\eth) \to K(\eth)$ defined by*

$$S(C) = \bigcup_{i=1}^{\mathfrak{I}} \eta_i(C)$$

for all $C \in K(\eth)$ is also q-contraction mapping on complete metric space $K(\eth)$ where $q = \max\{q_i : i = 1, 2, \cdots, N\}$. Further, for arbitrary set $\mathfrak{I} \in K(\eth)$, it satisfies

$$\lim_{n\to\infty} S^n(\mathfrak{I}) = \mathfrak{R},$$

where $\mathfrak{R}$ is the attractor of the system.

Lemma 2. *Let $(\eth, \sigma)$ be a metric space and $\emptyset \neq \mathfrak{R}, \mathfrak{I} \subseteq \eth$ with $\mathfrak{R}_0 \neq \emptyset$. Hence, we get*

$$H\left(K(\mathfrak{R}), K(\mathfrak{I})\right) = \sigma(\mathfrak{R}, \mathfrak{I})$$

where $H\left(K(\mathfrak{R}), K(\mathfrak{I})\right) = \inf\left\{\hbar(U, V) : U \in K(\mathfrak{R}) \text{ and } V \in K(\mathfrak{I})\right\}$

Lemma 3. *Let $(\eth, \sigma)$ be a metric space and $\emptyset \neq \mathfrak{R}, \mathfrak{I} \subseteq \eth$ with $\mathfrak{R}_0 \neq \emptyset$. Hence, we get $\left(K(\mathfrak{R})\right)_0 \neq \emptyset$.*

Lemma 4. *Let $(\eth, \sigma)$ be a metric space and $\emptyset \neq \mathfrak{R}, \mathfrak{I} \subseteq \eth$ with $\mathfrak{R}_0 \neq \emptyset$. If the pair $(\mathfrak{R}, \mathfrak{I})$ has the P_σ-Property, then we have $\left(K(\mathfrak{R})\right)_0 = K(\mathfrak{R}_0)$.*

Lemma 5. *Let $(\eth, \sigma)$ be a metric space and $\emptyset \neq \mathfrak{R}, \mathfrak{I} \subseteq \eth$ with $\mathfrak{R}_0 \neq \emptyset$. Assume that the pair $(\mathfrak{R}, \mathfrak{I})$ has the P_σ-Property and $\eta_i : \mathfrak{R} \to \mathfrak{I}$ are continuous mappings satisfying $\eta_i(\mathfrak{R}_0) \subseteq \mathfrak{I}_0$ for all $i = 1, 2, \cdots, N$. Then, for the mapping $S : K(\mathfrak{R}) \to K(\mathfrak{I})$ defined as*

$$SU = \bigcup_{i=1}^{N} \eta_i(U) \tag{2.1}$$

we have $S\left(K(\mathfrak{R}_0)\right) \subseteq K(\mathfrak{I}_0)$.

Lemma 6. *Let $(\eth, \sigma)$ be a metric space and $\emptyset \neq \mathfrak{R}, \mathfrak{I} \subseteq \eth$ with $\mathfrak{R}_0 \neq \emptyset$. Hence, we get $\left(K(\mathfrak{R})\right)_0 \subseteq K(\mathfrak{R}_0)$.*

We can prove the following lemma as in [Lemma 5 in [21]].

Lemma 7. *Let $(\eth, \sigma)$ be a metric space, $\emptyset \neq \mathfrak{R}, \mathfrak{I} \subseteq \eth$ with $\mathfrak{R}_0 \neq \emptyset$ and $\eta_i : \mathfrak{R} \to \mathfrak{I}$ be continuous α-ψ_i-contraction mappings satisfying $\eta_i(\mathfrak{R}_0) \subseteq \mathfrak{I}_0$ for all $i = 1, 2, \cdots, N$. Then, $S : K(\mathfrak{R}) \to K(\mathfrak{I})$ defined by*

$$S(P) = \bigcup_{i=1}^{N} \eta_i(P)$$

satisfies $S\left((K(\mathfrak{R}))_0\right) \subseteq (K(\mathfrak{I}))_0$.

Lemma 8. *Let $(\eth, \sigma)$ be a complete metric space. Assume that $\emptyset \neq \mathfrak{R}, \mathfrak{I} \subseteq \eth$ with $\mathfrak{R}_0 \neq \emptyset$ where $\mathfrak{I}$ is approximately compact w.r.t. $\mathfrak{R}$ and $\mathfrak{R}$ is closed. Hence, $((K(\mathfrak{R}))_0, \hbar)$ is complete metric space.*

3. α-ψ-iterated function systems

We give the following definitions.

Definition 6. *Let $(\eth, \sigma)$ be a metric space, $\emptyset \neq \mathfrak{R}, \mathfrak{I} \subseteq \eth$ and $\eta_i : \mathfrak{R} \to \mathfrak{I}$ be a mapping for $i = 1, 2$. We say that (η_1, η_2) is $\alpha_{\eta_1\eta_2}$-proximal admissible if*

$$\left.\begin{array}{c}\alpha(\varkappa, \zeta) \geq 1\\ \sigma(u, \eta_1\varkappa) = \sigma(\mathfrak{R}, \mathfrak{I})\\ \sigma(v, \eta_2\varkappa) = \sigma(\mathfrak{R}, \mathfrak{I})\end{array}\right\} \Longrightarrow \alpha(u, v) \geq 1$$

for all $\varkappa, \zeta \in \mathfrak{R}$.

Definition 7. *Let $(\eth, \sigma)$ be a metric space and $\emptyset \neq \mathfrak{R}, \mathfrak{I} \subseteq \eth$. Then, the pair $(K(\mathfrak{R}), K(\mathfrak{I}))$ is said to have the weak $P_\hbar$-Property if*

$$\left.\begin{array}{c}\hbar(U_1, V_1) = H(K(\mathfrak{R}), K(\mathfrak{I}))\\ \hbar(U_2, V_2) = H(K(\mathfrak{R}), K(\mathfrak{I}))\end{array}\right\} \Longrightarrow \hbar(U_1, U_2) \leq \hbar(V_1, V_2)$$

for all $U_1, U_2 \in K(\mathfrak{R})$ and $V_1, V_2 \in K(\mathfrak{I})$.

Lemma 9. *Let $(\eth, \sigma)$ be a metric space and $\emptyset \neq \mathfrak{R}, \mathfrak{I} \subseteq \eth$ with $\mathfrak{R}_0 \neq \emptyset$. If the pair $(\mathfrak{R}, \mathfrak{I})$ has the weak P_σ-Property, then the pair $(K(\mathfrak{R}), K(\mathfrak{I}))$ has the weak $P_\hbar$-Property.*

Proof. Let $U_1, U_2 \in K(\mathfrak{R})$ and $V_1, V_2 \in K(\mathfrak{I})$, satisfying

$$\begin{aligned}\hbar(U_1, V_1) &= H(K(\mathfrak{R}), K(\mathfrak{I}))\\ \hbar(U_2, V_2) &= H(K(\mathfrak{R}), K(\mathfrak{I})).\end{aligned}$$

From Lemma 2, we get

$$\begin{aligned}\hbar(U_1, V_1) &= \sigma(\mathfrak{R}, \mathfrak{I})\\ \hbar(U_2, V_2) &= \sigma(\mathfrak{R}, \mathfrak{I}).\end{aligned}$$

Hence, from the definition of $\hbar$, we have

$$D(U_1, V_1) = \sigma(\mathfrak{R}, \mathfrak{I}), \tag{3.1}$$

$$D(V_1, U_1) = \sigma(\mathfrak{R}, \mathfrak{I}), \tag{3.2}$$

and

$$D(U_2, V_2) = \sigma(\mathfrak{R}, \mathfrak{I}), \tag{3.3}$$

$$D(V_2, U_2) = \sigma(\mathfrak{R}, \mathfrak{I}). \tag{3.4}$$

Then, from (3.1), for all $u_1 \in U_1$, there is $v_{u_1} \in V_1$, satisfying

$$\sigma(u_1, v_{u_1}) = \sigma(\mathfrak{R}, \mathfrak{I}).$$

Similarly, from (3.4), for all $v_2 \in V_2$, there is $u_{v_2} \in U_2$, satisfying

$$\sigma(v_2, u_{v_2}) = \sigma(\mathfrak{R}, \mathfrak{I}).$$

Since the pair $(\mathfrak{R}, \mathfrak{I})$ has the weak P-Property, we have

$$\sigma(u_1, u_{v_2}) \leq \sigma(v_{u_1}, v_2).$$

Hence, we get

$$\begin{aligned} \sigma(u_1, U_2) &\leq \sigma(u_1, u_{v_2}) \\ &\leq \sigma(v_{u_1}, v_2), \end{aligned}$$

for all $v_2 \in V_2$. Then, we have

$$\begin{aligned} \sigma(u_1, U_2) &\leq \inf \{\sigma(v_{u_1}, v_2) : v_2 \in V_2\} \\ &= \sigma(v_{u_1}, V_2) \\ &\leq D(V_1, V_2) \end{aligned}$$

for all $u_1 \in U_1$ and so, from last inequality we have

$$\begin{aligned} D(U_1, U_2) &= \sup \{\sigma(u_1, U_2) : u_1 \in U_1\} \\ &\leq D(V_1, V_2). \end{aligned}$$

Similarly, we can obtain

$$D(U_2, U_1) \leq D(V_2, V_1)$$

and so we have

$$\begin{aligned} \hbar(U_1, U_2) &= \max\{D(U_1, U_2), D(U_2, U_1)\} \\ &\leq \max\{D(V_1, V_2), D(V_2, V_1)\} \\ &= \hbar(V_1, V_2). \end{aligned}$$

This shows that the pair $(K(\mathfrak{R}), K(\mathfrak{I}))$ has the weak $P_\hbar$-Property. □

Lemma 10. *Let $(\eth, \sigma)$ be a metric space, and $\eta_i : \mathfrak{R} \to \mathfrak{I}$ be continuous for all $i = 1, 2, \cdots, N$. Then, $S : K(\mathfrak{R}) \to K(\mathfrak{I})$ defined by*

$$S(C) = \bigcup_{i=1}^{N} \eta_i(C)$$

is continuous.

Proof. Assume that $U_n \to U$ as $n \to \infty$. Then, we get

$$\begin{aligned} \lim_{n\to\infty} \hbar(U_n, U) &= 0 \Longrightarrow \lim_{n\to\infty} \max\{D(U_n, U), D(U, U_n)\} = 0 \\ &\Longrightarrow \lim_{n\to\infty} D(U_n, U) = 0 \text{ and } \lim_{n\to\infty} D(U, U_n) = 0. \end{aligned}$$

Since U and U_n for all $n \in \mathbb{N}$ are compact, for all $\varkappa \in U$, there exists $\varkappa_n \in U_n$, such that

$$\lim_{n\to\infty} \sigma(\varkappa, \varkappa_n) = 0.$$

Since η_i is continuous for all $1 \leq i \leq N$, we get

$$\lim_{n\to\infty} \sigma(\eta_i\varkappa, \eta_i\varkappa_n) = 0.$$

For all $\varkappa \in U$, we get

$$0 \leq \lim_{n\to\infty} D(\eta_i\varkappa, \eta_i(U_n)) \leq \lim_{n\to\infty} \sigma(\eta_i\varkappa, \eta_i\varkappa_n) = 0,$$

and so we have

$$\lim_{n\to\infty} D(\eta_i\varkappa, \eta_i(U_n)) = 0$$

for all $\varkappa \in U$. Hence, we have

$$\lim_{n\to\infty} D(\eta_i(U), \eta_i(U_n)) = \lim_{n\to\infty} \sup_{\varkappa\in U} D(\eta_i\varkappa, \eta_i(U_n)) = 0.$$

Similar way, we can show that

$$\lim_{n\to\infty} D(\eta_i(U_n), \eta_i(U)) = 0,$$

and so we have

$$\lim_{n\to\infty} \hbar(\eta_i(U_n), \eta_i(U)) = 0$$

for all $i = 1, 2, ..., N$. Hence, from Lemma 1, we get

$$\begin{aligned} \lim_{n\to\infty} \hbar(S U_n, S U) &= \lim_{n\to\infty} \hbar\left(\bigcup_{i=1}^{N} \eta_i(U_n), , \bigcup_{i=1}^{N} \eta_i(U)\right) \\ &\leq \lim_{n\to\infty} \max_{1\leq i\leq N} \hbar(\eta_i U_n, \eta_i U) = 0. \end{aligned}$$

Therefore, S is a continuous mapping. □

Lemma 11. *Let $(\eth, \sigma)$ be a metric space, $\emptyset \neq \mathfrak{R}, \mathfrak{I} \subseteq \eth$ with $\mathfrak{R}_0 \neq \emptyset$ and $\eta_i : \mathfrak{R} \to \mathfrak{I}$ be continuous mappings for all $i = 1, 2, \cdots, N$. If $\left(\eta_i, \eta_j\right)$ is an $\alpha_{\eta_i\eta_j}$ -proximal admisibble for all $i, j = 1, 2, \cdots, N$, then $S : K(\mathfrak{R}) \to K(\mathfrak{I})$ defined by*

$$S(C) = \bigcup_{i=1}^{N} \eta_i(C)$$

is $\alpha_\hbar$-proximal admisibble where $\alpha_\hbar : K(\mathfrak{R}) \times K(\mathfrak{I}) \to [0, \infty)$ is a function defined by

$$\alpha_\hbar(U, V) = \inf\{\alpha(\varkappa, \zeta) : \varkappa \in U \text{ and } \zeta \in V\}.$$

Proof. Assume that

$$\begin{gathered} \alpha_\hbar(U_1, U_2) \geq 1 \\ \hbar(V_1, S U_1) = H(K(\mathfrak{R}), K(\mathfrak{I})) \\ \hbar(V_2, S U_2) = H(K(\mathfrak{R}), K(\mathfrak{I})) \end{gathered}$$

for all $U_1, U_2, V_1, V_2 \in K(\mathfrak{R})$. We want to show that $\alpha_\hbar(V_1, V_2) \geq 1$. Then, we get

$$\begin{aligned} \alpha_\hbar(U_1, U_2) &\geq 1 \Longrightarrow \inf\{\alpha(\varkappa, \zeta) : \varkappa \in U_1 \text{ and } \zeta \in U_2\} \geq 1 \\ &\Longrightarrow \text{for all } \varkappa \in U_1 \text{ and } \zeta \in U_2,\ \alpha(\varkappa, \zeta) \geq 1. \end{aligned} \tag{3.5}$$

On the other hand, from Lemma 2,

$$\begin{aligned}\hbar(V_1, SU_1) &= H(K(\mathfrak{R}), K(\mathfrak{I})) \Longrightarrow \hbar(V_1, SU_1) = \sigma(\mathfrak{R}, \mathfrak{I}) \\ &\Longrightarrow \max\{D(V_1, SU_1), D(SU_1, V_1)\} = \sigma(\mathfrak{R}, \mathfrak{I}) \\ &\Longrightarrow D(V_1, SU_1) = \sigma(\mathfrak{R}, \mathfrak{I}) \text{ and } D(SU_1, V_1) = \sigma(\mathfrak{R}, \mathfrak{I}).\end{aligned}$$

Since $\hbar(V_2, SU_2) = H(K(\mathfrak{R}), K(\mathfrak{I}))$, we get

$$D(V_2, SU_2) = \sigma(\mathfrak{R}, \mathfrak{I}) \text{ and } D(SU_2, V_2) = \sigma(\mathfrak{R}, \mathfrak{I}).$$

Hence, using the compactness of V_1 and SU_1, for each $u \in V_1$ there exist $\varkappa_u \in U_1$ and $1 \leq i_0 \leq N$, such that

$$\sigma(u, \eta_{i_0}\varkappa_u) = \sigma(\mathfrak{R}, \mathfrak{I}). \tag{3.6}$$

Also, since $D(V_2, SU_2) = \sigma(\mathfrak{R}, \mathfrak{I})$, for each $u' \in V_2$, there exists $\zeta_{u'} \in U_2$ and $1 \leq j_0 \leq N$, such that

$$\sigma(u', \eta_{j_0}\zeta_{u'}) = \sigma(\mathfrak{R}, \mathfrak{I}). \tag{3.7}$$

From (3.5),since $\varkappa_u \in U_1$ and $\zeta_{u'} \in U_2$, we get

$$\alpha(\varkappa_u, \zeta_{u'}) \geq 1. \tag{3.8}$$

Since $\left(\eta_{i_0}, \eta_{j_0}\right)$ is $\alpha_{\eta_{i_0}\eta_{j_0}}$-proximal admissible, from (3.6)–(3.8), we get

$$\alpha(u, u') \geq 1,$$

for all $u \in V_1$ and $u' \in V_2$. Then, we have

$$\alpha_{\hbar}(V_1, V_2) = \inf\{\alpha(u, u') : u \in V_1 \text{ and } u' \in V_2\} \geq 1.$$

□

Now, we introduce the following definitions:

Definition 8. *Let $(\eth, \sigma)$ be a metric space, $\emptyset \neq \mathfrak{R}, \mathfrak{I} \subseteq \eth$, $\eta_i : \mathfrak{R} \to \mathfrak{I}$ be mapping for all $i = 1, 2, \cdots N$ and $\alpha : \mathfrak{R}\times\mathfrak{R} \to [0, \infty)$ be a function. If $\left(\eta_i, \eta_j\right)$ is an $\alpha_{\eta_i\eta_j}$ -proximal admisible for all $i, j = 1, 2, \cdots, N$, then η_i is said to be an α-ψ_i-contraction mapping if there exists $\psi_i \in \Psi$, such that*

$$\alpha(\varkappa, \zeta)\sigma(\eta_i\varkappa, \eta_i\zeta) \leq \psi_i(\sigma(\varkappa, \zeta))$$

for all $\varkappa, \zeta \in \mathfrak{R}$ and $i = 1, 2, \cdots, N$.

Definition 9. *Let $(\eth, \sigma)$ be a metric space, $\emptyset \neq \mathfrak{R}, \mathfrak{I} \subseteq \eth$ and $\eta_i : \mathfrak{R} \to \mathfrak{I}$ be mappings for all $i = 1, 2, \cdots, N$. If the metric space $(\eth, \sigma)$ is complete, $\mathfrak{R}$ is closed and $\eta_i : \mathfrak{R} \to \mathfrak{I}$ are continuous α-ψ_i-contraction mappings for all $i = 1, 2, \cdots, N$, then the system $\{\mathfrak{R}, \mathfrak{I}; \eta_i, i = 1, 2, \cdots, N\}$ is called α-ψ- iterated function system where $\psi(\varkappa) = \max_{1\leq i\leq N}\{\psi_i(\varkappa)\}$.*

Theorem 6. *Let* $\{\mathfrak{R}, \mathfrak{I}; \eta_i, i = 1, 2, \cdots, N\}$ *be an* α-ψ- *iterated function system,* $\mathfrak{R}_0 \neq \emptyset$ *and* $\eta_i(\mathfrak{R}_0) \subseteq \mathfrak{I}_0$ *for all* $1 \leq i \leq N$*. Assume that for all* $1 \leq i \leq N$*, there exist* $\varkappa_0, \varkappa_1 \in \mathfrak{R}_0$*, such that* $\sigma(\varkappa_1, \eta_i\varkappa_0) = \sigma(\mathfrak{R}, \mathfrak{I})$ *and* $\alpha(\varkappa_0, \varkappa_1) \geq 1$*. Then the mapping* $S : K(\mathfrak{R}) \to K(\mathfrak{I})$ *defined by*

$$S(C) = \bigcup_{i=1}^{N} \eta_i(C),$$

has a unique best proximity point C *in* $K(\mathfrak{R})$ *which is called best attractor of* α-ψ-*iterated function system.*

Proof. Since $\mathfrak{R}$ is a closed subset of $\supset$, then $K(\mathfrak{R})$ is a closed subset of the complete metric space $(K(\supset), \hbar)$. From Lemmas 3 and 5, we get $(K(\mathfrak{R}))_0 \neq \emptyset$ and $S((K(\mathfrak{R}))_0) \subseteq (K(\mathfrak{I}))_0$. Additionally, from Lemmas 9 and 11, the pair $(K(\mathfrak{R}), K(\mathfrak{I}))$ has the weak $P_\hbar$-Property and S is $\alpha_\hbar$-proximal admissible. Also, if we choose $P_0 = \{\varkappa_0\}$, $P_1 = \{\varkappa_1\} \in K(\mathfrak{R}_0)$, then since for all $1 \leq i \leq N$, $\sigma(\varkappa_1, \eta_i\varkappa_0) = \sigma(\mathfrak{R}, \mathfrak{I})$ and $\alpha(\varkappa_0, \varkappa_1) \geq 1$, we get

$$\hbar(P_1, SP_0) = \sigma(\mathfrak{R}, \mathfrak{I}) = H(K(\mathfrak{R}), K(\mathfrak{I}))$$

and

$$\alpha(P_0, P_1) = \alpha(\varkappa_0, \varkappa_1) \geq 1.$$

Now, we want to show that S is α-ψ-contraction, that is,

$$\alpha_\hbar(U_1, U_2)\hbar(SU_1, SU_2) \leq \psi(\hbar(U_1, U_2))$$

for all $U_1, U_2 \in K(\mathfrak{R})$ holds for some $\psi \in \Psi$. Let $U_1, U_2 \in K(\mathfrak{R})$ be arbitrary sets. Since U_1 is compact and η_i is continuous, there exists $u_1 \in U_1$, such that

$$D(\eta_i(U_1), \eta(U_2)) = \min_{u_2 \in U_2} \sigma(\eta_i u_1, \eta_i u_2).$$

Then,

$$\begin{aligned}
\alpha_\hbar(U_1, U_2) D(\eta_i(U_1), \eta_i(U_2)) &= \alpha_\hbar(U_1, U_2) \min_{u_2 \in U_2} \sigma(\eta_i u_1, \eta_i u_2) \\
&\leq \alpha_\hbar(u_1, u_2)\sigma(\eta_i u_1, \eta_i u_2) \\
&\leq \psi_i(\sigma(u_1, u_2))
\end{aligned}$$

for all $u_2 \in U_2$. Let $b \in U_2$, such that $\min_{u_2 \in U_2} \sigma(u_1, u_2) = \sigma(u_1, b)$.

$$\begin{aligned}
\alpha_\hbar(U_1, U_2) D(\eta_i(U_1), \eta_i(U_2)) &\leq \psi_i(\sigma(u_1, b)) \\
&= \psi_i\left(\min_{u_2 \in U_2} \sigma(u_1, u_2)\right) \\
&\leq \psi_i\left(\max_{u_1 \in U_1} \min_{u_2 \in U_2} \sigma(u_1, u_2)\right) \\
&= \psi_i(D(U_1, U_2)) \\
&\leq \psi(D(U_1, U_2)).
\end{aligned}$$

Also, we can find

$$\alpha_\hbar(U_1, U_2) D(\eta_i(U_2), \eta_i(U_1)) \leq \psi(D(U_2, U_1)).$$

Hence, we get

$$\begin{aligned}\alpha_\hbar(U_1,U_2)\hbar(\eta_i(U_1),\eta_i(U_2)) &= \max\left\{\begin{array}{c}\alpha_\hbar(U_1,U_2)D(\eta_i(U_1),\eta_i(U_2)),\\ \alpha_\hbar(U_1,U_2)D(\eta_i(U_2),\eta_i(U_1))\end{array}\right\}\\ &\le \max\{\psi(D(U_1,U_2)),\psi(D(U_2,U_1))\}\\ &\le \psi(\max\{D(U_1,U_2),D(U_2,U_1)\})\\ &= \psi(\hbar(U_1,U_2))\end{aligned}$$

for all $1 \le i \le N$. Then,

$$\begin{aligned}\alpha_\hbar(U_1,U_2)\hbar(SU_1,SU_2) &= \alpha_\hbar(U_1,U_2)\hbar\left(\bigcup_{i=1}^{n}\eta_i(U_1),\bigcup_{i=1}^{n}\eta_i(U_2)\right)\\ &\le \alpha_\hbar(U_1,U_2)\max_{1\le i\le N}\{\hbar(\eta_i(U_1),\eta_i(U_2))\}\\ &= \max_{1\le i\le \mathfrak{I}}\{\alpha_\hbar(U_1,U_2)\hbar(\eta_i(U_1),\eta_i(U_2))\}\\ &= \psi(\hbar(U_1,U_2))\end{aligned}$$

for all $U_1, U_2 \in K(\mathfrak{R})$. Hence, S is an α-ψ-contraction mapping. Additionally, from Lemma 10, S is continuous. Hence, since all hypothesis of Theorem 1 hold, then S has a best proximity point in $K(\mathfrak{R})$; i.e., the α-ψ-iterated function system has a best attractor. □

Example 1. *Let*

$$\eth = \left\{re^{i\gamma} : 1 \le r \text{ and } -\pi \le \gamma < \pi\right\}$$

be endowed with the metric $\sigma : \eth \times \eth \to \mathbb{R}$ *defined by*

$$\sigma(r_1e^{i\gamma_1}, r_2e^{i\gamma_2}) = |r_1 - r_2| + \min\{|\gamma_1-\gamma_2|, 2\pi - |\gamma_1-\gamma_2|\}.$$

Then, $(\mathbb{R}^2, \sigma)$ *is a complete metric space. Indeed, let*

$$\eth = \left\{re^{i\gamma} : 1 \le r \text{ and } -\pi \le \gamma < \pi\right\}$$

be endowed with the metric $\sigma : \eth \times \eth \to \mathbb{R}$ *defined by*

$$\sigma(r_1e^{i\gamma_1}, r_2e^{i\gamma_2}) = |r_1 - r_2| + \min\{|\gamma_1-\gamma_2|, 2\pi - |\gamma_1-\gamma_2|\}.$$

Assume that $\{r_ne^{i\gamma_n}\}$ *is a Cauchy sequence in* $\eth$*. Then, we get*

$$\lim_{n,m\to\infty}\{|r_n - r_m| + \min\{|\gamma_n-\gamma_m|, 2\pi-|\gamma_n-\gamma_m|\}\} = 0$$

which implies that

$$\lim_{n,m\to\infty}|r_n - r_m| = 0 \text{ and } \lim_{n,m\to\infty}\min\{|\gamma_n-\gamma_m|, 2\pi-|\gamma_n-\gamma_m|\} = 0.$$

Hence, $\{r_n\}$ *and* $\{\gamma_n\}(mod2\pi)$ *are Cauchy sequences in* $\mathbb{R}$*. Also, since* $(\mathbb{R}, |.|)$ *is complete, there exist* $1 \le r \in \mathbb{R}$ *and* $\gamma(mod2\pi) \in \mathbb{R}$*, such that* $r_n \to r$ *and* $\gamma_n \to \gamma$ *as* $n \to \infty$*. Hence, the Cauchy sequence is convergent in* $\eth$*, and so* $(\eth, \sigma)$ *is complete. Now, consider the subsets*

$$\mathfrak{R} = \left\{re^{i\gamma} : 3 \le r \le 4 \text{ and } -\frac{\pi}{2} \le \gamma \le \frac{\pi}{2}\right\}$$

and

$$\mathfrak{I} = \left\{re^{i\gamma} : 1 \leq r \leq 2 \text{ and } -\frac{\pi}{2} \leq \gamma \leq \frac{\pi}{2}\right\}.$$

Then, we have $\sigma(\mathfrak{R}, \mathfrak{I}) = 1$, *and it is clear that* $(\mathfrak{R}, \mathfrak{I})$ *has the weak* P_σ*-Property. Moreover, we get*

$$\mathfrak{R}_0 = \left\{3e^{i\gamma} : \ -\frac{\pi}{2} \leq \gamma \leq \frac{\pi}{2}\right\}$$

and

$$\mathfrak{I}_0 = \left\{2e^{i\gamma} : -\frac{\pi}{2} \leq \gamma \leq \frac{\pi}{2}\right\}.$$

Furthermore, it is obvious that $\mathfrak{R}$ *is closed. Define mappings* $\eta_1, \eta_2 : \mathfrak{R} \to \mathfrak{I}$ *as follows:*

$$\eta_1(re^{i\gamma}) = \frac{7-r}{2} e^{i\frac{|\gamma|}{2}}$$

and

$$\eta_2(re^{i\gamma}) = \frac{7-r}{2} e^{-i\frac{|\gamma|}{2}}.$$

It can be seen that f_i *is a continuous mapping satisfying* $\eta_i(\mathfrak{R}_0) \subseteq \mathfrak{I}_0$ *for* $i = 1, 2$. *Moreover, consider the functions* $\psi_{1,2} : [0, \infty) \to [0, \infty)$ *and* $\alpha : \mathfrak{R} \times \mathfrak{R} \to [0, \infty)$ *defined as* $\psi_{1,2}(t) = \frac{t}{2}$ *for all* $t \in [0, \infty)$ *and* $\alpha(\varkappa, \zeta) = 1$ *for all* $\varkappa, \zeta \in \mathfrak{R}$, *respectively. Further, for all* $i = 1, 2$, *there exist* $3e^{i0}$,$3e^{i0} \in \mathfrak{R}$, *such that* $\sigma(3e^{i0}, \eta_i 3e^{i0}) = 1$ *and* $\alpha(3e^{i0}, 3e^{i0}) \geq 1$. *Also,* η_i *is* α-ψ_i*-contraction mapping for all* $i = 1, 2$. *Indeed, for all* $u, v \in \mathfrak{R}$ *where* $u = r_1 e^{i\gamma_1}$ *and* $v = r_2 e^{i\gamma_2}$

$$\begin{aligned}
\alpha(u, v)\sigma(\eta_1 u, \eta_1 v) &= \left|\frac{7 - r_1 - 7 + r_2}{2}\right| + \min\left\{\frac{||\gamma_1| - |\gamma_2||}{2}, 2\pi - \frac{||\gamma_1| - |\gamma_2||}{2}\right\} \\
&= \left|\frac{r_1 - r_2}{2}\right| + \frac{||\gamma_1| - |\gamma_2||}{2} \\
&\leq = \left|\frac{r_1 - r_2}{2}\right| + \frac{|\gamma_1 - \gamma_2|}{2} \\
&= \psi_1\left(|r_1 - r_2| + |\gamma_1 - \gamma_2|\right) \\
&= \psi_1(\sigma(u, v)).
\end{aligned}$$

Similarly, η_2 *is* α-ψ_2*-contraction, too. Therefore, since all hypotheses of Theorem 6 are satisfied, the mapping* $S : K(\mathfrak{R}) \to K(\mathfrak{I})$ *defined as*

$$S(U) = \bigcup_{i=1}^{2} \eta_i(U)$$

has a best proximity point in $K(\mathfrak{R})$, *that is, the system*$\{\mathfrak{R}, \mathfrak{I}; \eta_1, \eta_2\}$ *has a best attractor. Let* $U_0 = \mathfrak{R}_0 = \{3e^{i\gamma} : \frac{\pi}{2} \leq \gamma \leq \frac{\pi}{2}\} \in (K(\mathfrak{R}))_0$. *We can construct the sequence* $\{U_n\}$ *as follow:*

$$U_1 = \left\{3e^{i\gamma} : \gamma \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]\right\},$$

$$U_2 = \left\{3e^{i\gamma} : \gamma \in \left[-\frac{\pi}{8}, \frac{\pi}{8}\right]\right\}$$

$$U_3 = \left\{3e^{i\gamma} : \gamma \in \left[-\frac{\pi}{16}, \frac{\pi}{16}\right]\right\}$$
$$U_4 = \left\{3e^{i\gamma} : \gamma \in \left[-\frac{\pi}{32}, \frac{\pi}{32}\right]\right\}$$
$$\vdots$$

Moreover, the following figure (Figure 1) shows a few steps of the sequence $\{U_n\}$.

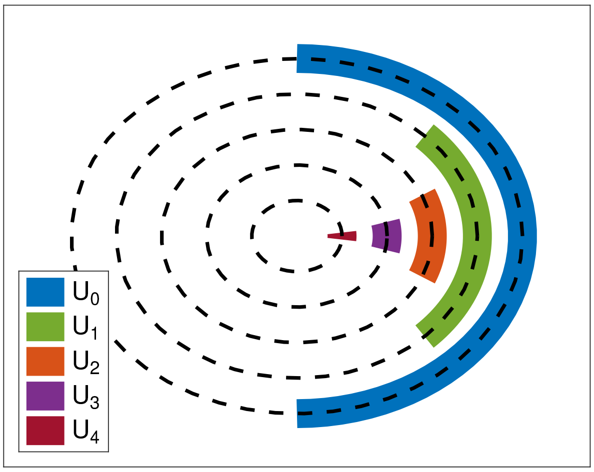

Figure 1. A few steps.

4. α-ψ-proximal iterated function systems

Now, we introduce the following definition:

Definition 10. *Let* $(\eth, \sigma)$ *be a metric space,* $\emptyset \neq \Re, \Im \subseteq \eth$ *and* $\eta_i : \Re \to \Im$ *be mappings for all* $i = 1, 2, \cdots, N$. *If the metric space* $(\eth, \sigma)$ *is complete, and* $\eta_i : \Re \to \Im$ *are continuous* α-ψ_i *-proximal contraction mappings for all* $i = 1, 2, \cdots, N$, *then the system* $\{\Re, \Im; \eta_i, i = 1, 2, \cdots, N\}$ *is called* α-ψ-*proximal-iterated function system, where* $\psi(\varkappa) = \max_{1 \leq i \leq N}\{\psi_i(\varkappa)\}$.

Theorem 7. *Let* $\{\Re, \Im; \eta_i, i = 1, 2, \cdots, N\}$ *be an* α-ψ-*proximal-iterated function system where* $\Re$ *is closed and* $\Im$ *is approximately compact with respect to* $\Re$. *Assume that for all* $1 \leq i \leq N$, *there exist* $\varkappa_0, \varkappa_1 \in \Re_0$, *such that* $\sigma(\varkappa_1, \eta_i \varkappa_0) = \sigma(\Re, \Im)$ *and* $\alpha(\varkappa_0, \varkappa_1) \geq 1$, $\Re_0 \neq \emptyset$ *and* $\eta_i(\Re_0) \subseteq \Im_0$ *for all* $1 \leq i \leq N$. *Then, the mapping* $S : K(\Re) \to K(\Im)$ *defined by*

$$S(C) = \bigcup_{i=1}^{\Im} \eta_i(C),$$

has a best proximity point C *in* $K(\Re)$.

Proof. Since $\Re$ is closed subsets of $\eth$, then $K(\Re)$ are closed subset of the complete metric space $(K(\eth), \hbar)$. From Lemma 4, Lemma 7, and Remark 1, we have $(K(\Re))_0 \neq \emptyset$ and $S((K(\Re))_0) \subseteq (K(\Im))_0$.

Further, from Lemma 8, $(K(\mathfrak{R})_0, \hbar)$ is complete. Moreover, if we choose $G_0 = \{\varkappa_0\}$, $G_1 = \{\varkappa_1\} \in K(\mathfrak{R}_0)$, then since for all $1 \leq i \leq N$, $\sigma(\varkappa_1, \eta_i \varkappa_0) = \sigma(\mathfrak{R}, \mathfrak{I})$ and $\alpha(\varkappa_0, \varkappa_1) \geq 1$, we get

$$\hbar(G_1, SG_0) = \sigma(\mathfrak{R}, \mathfrak{I}) = H(K(\mathfrak{R}), K(\mathfrak{I}))$$

and

$$\alpha(G_0, G_1) = \alpha(\varkappa_0, \varkappa_1) \geq 1.$$

Now, we want to show that S is α-ψ-proximal contraction, that is, the implication

$$\left.\begin{array}{l} \hbar(U_1, SV_1) = H\left(K(\mathfrak{R}), K(\mathfrak{I})\right) \\ \hbar(U_2, SV_2) = H\left(K(\mathfrak{R}), K(\mathfrak{I})\right) \end{array}\right\} \Rightarrow \alpha(U_1, U_2)\hbar(U_1, U_2) \leq \psi\left(\hbar(V_1, V_2)\right)$$

for all $U_1, U_2, V_1, V_2 \in K(\mathfrak{R})$ holds where $\psi(\varkappa) = \max_{1 \leq i \leq \mathfrak{I}}\{\psi_i(\varkappa)\}$. Hence, taking into account Lemma 2 it is enough to show the implication

$$\left.\begin{array}{l} \hbar(U_1, SV_1) = \sigma(\mathfrak{R}, \mathfrak{I}) \\ \hbar(U_2, SV_2) = \sigma(\mathfrak{R}, \mathfrak{I}) \end{array}\right\} \Rightarrow \alpha(U_1, U_2)\hbar(U_1, U_2) \leq \psi\left(\hbar(V_1, V_2)\right)$$

for all $U_1, U_2, V_1, V_2 \in K(\mathfrak{R})$. Let U_1, U_2, P_1, P_2 be arbitrary compact subsets of $\mathfrak{R}$, satisfying

$$\hbar(U_1, SV_1) = \sigma(\mathfrak{R}, \mathfrak{I}) \tag{4.1}$$

and

$$\hbar(U_2, SV_2) = \sigma(\mathfrak{R}, \mathfrak{I}). \tag{4.2}$$

Then, from (4.1) and (4.2), we have

$$D(U_1, SV_1) = \sigma(\mathfrak{R}, \mathfrak{I}) \tag{4.3}$$

and

$$D(SV_2, U_2) = \sigma(\mathfrak{R}, \mathfrak{I}). \tag{4.4}$$

Using the compactness of V_1 and U_1, from Eq (4.3) for each $u_1 \in U_1$, there exist $\varsigma_{u_1} \in V_1$ and $1 \leq i \leq N$, such that

$$\sigma(u_1, \eta_i \varsigma_{u_1}) = \sigma(\mathfrak{R}, \mathfrak{I}).$$

From the (4.4), we have

$$D(\eta_i P_2, U_2) = \sigma(\mathfrak{R}, \mathfrak{I}) \tag{4.5}$$

for all $1 \leq i \leq N$, and so using the compactness of P_2 and U_2, for each $v_2 \in V_2$ there exist $\varsigma_{u_2} \in U_1$ and $1 \leq j \leq N$, such that

$$\sigma(\varsigma_{v_2}, \eta_i v_2) = \sigma(\mathfrak{R}, \mathfrak{I}).$$

Now, for all $u_1 \in U_1$ and $v_2 \in V_2$, there exists $\varsigma_{u_1} \in V_1$ and $\varsigma_{u_2} \in U_2$, such that

$$\sigma(u_1, \eta_i \varsigma_{u_1}) = \sigma(\mathfrak{R}, \mathfrak{I}) \tag{4.6}$$

and

$$\sigma(\varsigma_{v_2}, \eta_i v_2) = \sigma(\mathfrak{R}, \mathfrak{I}). \tag{4.7}$$

Since (η_i, η_j) is α-$\psi_{\eta_i\eta_j}$-proximal contraction mappings, from (4.6) and (4.7), we have

$$\alpha(u_1, \varsigma_{v_2})\sigma(u_1, \varsigma_{v_2}) \leq \psi_i\left(\sigma(\varsigma_{u_1}, v_2)\right)$$

for all $u_1 \in U_1$ and $v_2 \in V_2$. Since U_1 is compact, there exist $u_1 \in U_1$, such that

$$D(U_1, U_2) = \min_{u_2 \in U_2} \sigma(u_1, u_2).$$

Then,

$$\begin{aligned}
\alpha_\hbar(U_1, U_2)D(U_1, U_2) &= \alpha_\hbar(U_1, U_2) \min_{u_2 \in U_2} \sigma(u_1, u_2) \\
&\leq \alpha(u_1, \varsigma_{v_2})\sigma(u_1, \varsigma_{v_2}) \\
&\leq \psi_i\left(\sigma(\varsigma_{u_1}, v_2)\right)
\end{aligned}$$

for all $v_2 \in V_2$. Then, let $b \in V_2$ satisfying $\min_{v_2 \in, V_2} \sigma(\varsigma_{u_1}, v_2) = \sigma(\varsigma_{u_1}, b)$.

$$\begin{aligned}
\alpha_\hbar(U_1, U_2)D(U_1, U_2) &\leq \psi_i\left(\sigma(\varsigma_{u_1}, b)\right) \\
&= \psi_i\left(\min_{v_2 \in, V_2} \sigma(\varsigma_{u_1}, v_2)\right) \\
&\leq \psi_i\left(\max_{u_1 \in U_1} \min_{u_2 \in U_2} \sigma(u_1, u_2)\right) \\
&= \psi_i(D(U_1, U_2)) \\
&\leq \psi(D(U_1, U_2)).
\end{aligned}$$

Also, we can find

$$\alpha_\hbar(U_1, U_2)D(U_2, U_1) \leq \psi(D(U_2, U_1)).$$

Hence, we get

$$\begin{aligned}
\alpha_\hbar(U_1, U_2)\hbar(U_1, U_2) &= \max\{\alpha_\hbar(U_1, U_2)D(U_1, U_2), \alpha_\hbar(U_1, U_2)D(U_2, U_1)\} \\
&\leq \max\{\psi(D(U_1, U_2)), \psi(D(U_2, U_1))\} \\
&\leq \psi(\max\{D(U_1, U_2), D(U_2, U_1)\}) \\
&= \psi(\hbar(U_1, U_2))
\end{aligned}$$

for all $U_1, U_2 \in K(\mathfrak{R})$. Hence, S is a α-ψ-proximal contraction mapping. Also, from Lemma 10, S is continuous. Hence, since the hypotheses of Theorem 3 hold, then S has a best proximity point in $K(\mathfrak{R})$, that is, the α-ψ-proximal iterated function system has a best attractor. □

Example 2. *Let* $(\mathbb{R}^2, \sigma)$ *be endowed with taxi-cub metric and*

$$\mathfrak{R} = \{(\varkappa, 1) : -1 \leq \varkappa \leq 1\}$$

and

$$\mathfrak{I} = \{(\varkappa, 2) : -1 \leq \varkappa \leq 1\}$$

be subsets of $\mathbb{R}^2$. *Then, it can be seen that* $\mathfrak{R}$ *is closed and* $\mathfrak{I}$ *is approximately compact with respect to* $\mathfrak{R}$. *Also, we have* $\mathfrak{R} = \mathfrak{R}_0$, $\mathfrak{I} = \mathfrak{I}_0$ *and* $\sigma(\mathfrak{R}, \mathfrak{I}) = 1$. *Assume that* $\alpha : \mathfrak{R} \times \mathfrak{R} \to [0, \infty)$ *is a function defined by*

$$\alpha(\varkappa, \zeta) = \begin{cases} 1 & , \quad \varkappa, \zeta \in \mathfrak{R}_0 \\ \\ 0 & , \quad otherwise \end{cases}.$$

Moreover, let $\eta_1 : \mathfrak{R} \to \mathfrak{I}$ *and* $\eta_2 : \mathfrak{R} \to \mathfrak{I}$ *be mappings defined by*

$$\eta_1(\varkappa, 1) = \left(\frac{2\varkappa^2}{5}, 2\right)$$

and

$$\eta_2(\varkappa, 1) = \left(-\frac{3\varkappa^2}{7}, 2\right)$$

for all $\varkappa \in \mathfrak{R}$, *respectively. Then, we have* $\eta_i(\mathfrak{R}_0) \subseteq \mathfrak{I}_0$ *for all* $i = 1, 2$. *Further, for all* $i = 1, 2$, *there exist* $(0, 1), (0, 1) \in \mathfrak{R}$, *such that* $\sigma((0, 1), \eta_i(0, 1)) = 1$ *and* $\alpha((0, 1), (0, 1)) \geq 1$. *Also,* η_i *is* α-ψ_i*-contraction mapping for all* $i = 1, 2$. *Assume that* $\psi : [0, \infty) \to [0, \infty)$ *is a function defined by* $\psi_i(t) = \left(\frac{2i+2}{2i+3}\right) t$ *for all* $t \in [0, \infty)$ *and* $i = 1, 2$. *We will show that* η_i *are* α-ψ_i*-proximal contraction mapping for* $i = 1, 2$. *Indeed, for all* $v_1 = (\varkappa_1, 1), v_2 = (\varkappa_2, 1) \in \mathfrak{R}$, *satisfying* $\sigma(u_1, \eta_1 v_1) = 1$ *and* $\sigma(u_2, \eta_1 v_2) = 1$ *we have* $u_1 = \left(\frac{2\varkappa_1^2}{5}, 1\right)$ *and* $u_2 = \left(\frac{2\varkappa_2^2}{5}, 1\right)$. *Hence, we get*

$$\begin{aligned} \alpha(u_1, u_2)\sigma(u_1, u_2) &= \frac{2\left|\varkappa_1^2 - \varkappa_2^2\right|}{5} \\ &= \frac{2\left|\varkappa_1 - \varkappa_2\right|\left|\varkappa_1 + \varkappa_2\right|}{5} \\ &\leq \frac{4\left|\varkappa_1 - \varkappa_2\right|}{5} \\ &= \psi_1(\left|\varkappa_1 - \varkappa_2\right|) \\ &= \psi_1(\sigma(v_1, v_2)), \end{aligned}$$

thus, η_1 *is* α-ψ_1*-proximal contraction mapping. Similarly,* η_2 *is* α-ψ_2*-proximal contraction mapping and so all hypotheses of Theorem 7 are satisfied. Therefore,* $S : K(\mathfrak{R}) \to K(\mathfrak{I})$ *defined as*

$$S(C) = \bigcup_{i=1}^{2} \eta_i(C)$$

has a unique best proximity point in $K(\mathfrak{R})$, *that is, the system has a unique best attractor. Now, we want to construct a few steps of the mentioned sequence* $\{C_n\}$ *with the initial set* $C_0 = \mathfrak{R} \in K(\mathfrak{R}_0)$. *In this case, we have*

$$C_1 = \left\{(\varkappa, 1) : \varkappa \in \left[-\frac{3}{7}, \frac{2}{5}\right]\right\},$$

$$C_2 = \left\{(\varkappa, 1) : \varkappa \in \left[-\frac{27}{343}, \frac{18}{245}\right]\right\},$$

$$C_3 = \left\{ (\varkappa, 1) : \varkappa \in \left[-\frac{2187}{823543}, \frac{1458}{588245} \right] \right\},$$

$$\vdots$$

Additionally, a few steps of the sequence $\{U_n\}$ can be seen in the Figure 2.

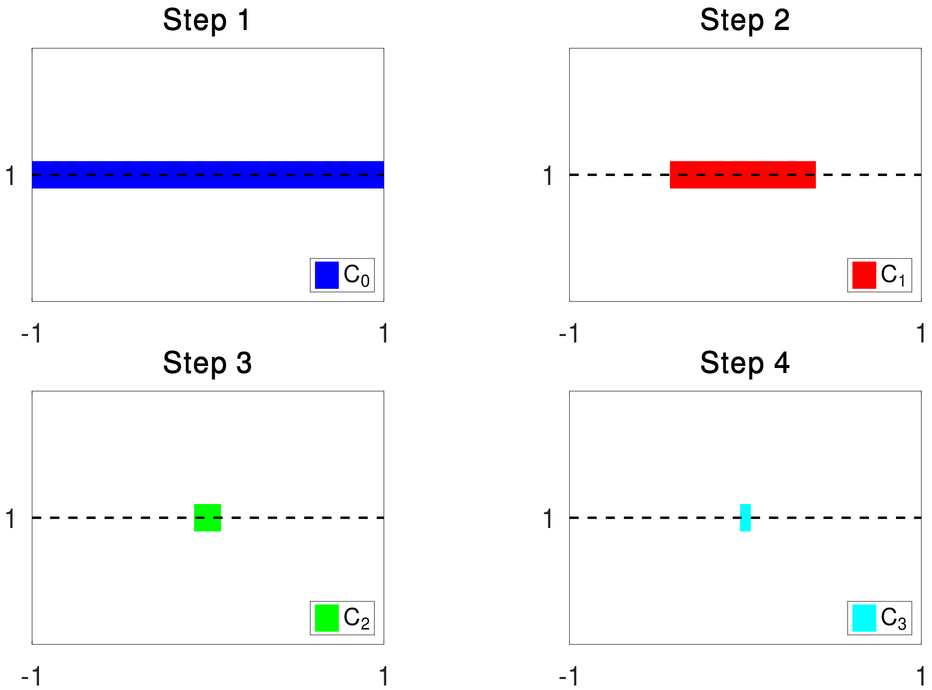

Figure 2. A few steps.

5. Conclusions

In this paper, we aim to introduce two new concepts: α-ψ-iterated function system and α-ψ-proximal iterated function system. Hence, as a result, we extend and generalize some previously published definitions and establish the relationship between the best proximity point and iterated function systems. Then, we derive some conclusions that determine the criteria required to obtain a fractal with an attractor in the aforementioned systems. Finally, we provide some intriguing instances of how to use our findings. In the future, more interesting results can be presented by studying new iterated function systems consisting of different types of contractions.

Author contributions

All authors contributed equally to all parts of the paper. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

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