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Research article

Fuzzy topology with $\mathcal{I}$-convergence

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Abstract: In this paper, a new approach to specifying the closure of subsets in J^E is proposed, based on the concept of $\mathcal{I}$-convergence of sequences. The notion of fuzzy $\mathcal{I}$-closure is defined using this convergence, and from it, the fuzzy $\mathcal{I}$-topology is introduced. Moreover, the concepts of continuity and $\mathcal{I}$-continuity in fuzzy topological spaces are examined. A comparison between the proposed $\mathcal{I}$-topology and the classical fuzzy topology is presented, highlighting certain fundamental properties and structural differences. This work contributes to the generalization of fuzzy topological structures by extending the role of sequence convergence through the use of $\mathcal{I}$-convergence. Furthermore, a practical example illustrating temperature control in an industrial furnace is provided through the application of $\mathcal{I}$-convergence of fuzzy sequences.

Keywords: fuzzy $\mathcal{I}$-closure; fuzzy $\mathcal{I}$-interior; fuzzy $\mathcal{I}$-topology; fuzzy $\mathcal{I}$-continuity; asymptotic density; t-norm
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1. Introduction

The idea of uncertainty is one of the many paradigm shifts that have occurred in science and mathematics. An alternative viewpoint that embraces uncertainty, considering it essential to scientific inquiry, has gradually replaced the old position in science, which asserts that vagueness is unwanted in science and needs to be eliminated whenever possible. In light of characterizing fuzzy sets through functions that map a set to the interval $[0, 1]$ on the real line, Zadeh [1] presented the fundamental notion of fuzzy sets in the year 1965. With the aim of characterizing groups of things for which there are no clear membership criteria, fuzzy sets were proposed. These kinds of collections have ambiguous or "fuzz" boundaries; it is impossible to tell whether an object is part of the collection or not. Although fuzzy sets are sometimes mistaken for probability theory, their specialization is capturing the concept of partial membership [2]; therefore, they can be thought of as a generalization of the traditional sets. Chang [3] was the first to define fuzzy topology, and later, Lowen [4], Hutton and Reilly [5], redefined it in somewhat different ways. The framework of fuzzy sets and its various applications have been extensively developed by a number of writers [6, 7]. Chang initially examined fuzzy continuity, and numerous studies have been conducted in this area, as seen in [8,9]. Different forms of fuzzy continuity have been defined using different methodologies. Notable contributions were made by K. K. Azad [10], S. Saha [11], and R. H. Warren [12].

In topology, a large number of researchers have looked into how interior and closure serve as alternative methods to describe open sets. In [13, 14], Y. S. Eom and S. J. Lee introduced the notions of fuzzy θ-closure and fuzzy δ-closure in fuzzy topological spaces. Furthermore, the notion of strong δ-continuity was explored in [15], and the concept of δ-continuity on function spaces was examinaded in [16]. The notion of fuzzy weak continuity was examined in [17] by Tripathy and Ray. They also investigated fuzzy δ-I-continuity in [18].

One of the key concepts in analysis is the notion of convergence. The idea of convergence for fuzzy filters was proposed and explored by Lowen [19], published in 1979. The findings were then used to characterize fuzzy continuity and fuzzy compactness. The concept of convergence in fuzzy topological spaces was further developed by several authors, including C. M. Hu [20], R. Lowen [21], and B. Y. Lee et al. [22], who extended the theory in different directions. Different types of difference sequence spaces were developed by Tripathy and Borgohain in [23,24]. They also studied the classes of extended difference bounded, convergent, and null sequences of fuzzy real numbers, which are defined using an Orlicz function.

In the context of general topological spaces, more general nets or filters must be utilized, as sequences alone are insufficient to fully describe the topology, and sequence convergence does not adequately capture the topological structure. A technique for creating a new topology τ^s that is naturally related to the original one is provided by sequential closure with respect to a given topology τ. We refer to this topology as sequential topology, and it is created by taking open sets and combining them with sets whose complements are finer and progressively closed in relation to the original topology [25].

The notion of $\mathcal{I}$-convergence, introduced in [26], represents a significant generalization of statistical convergence by replacing asymptotic density with an abstract ideal $\mathcal{I}$ of subsets of the natural numbers. While statistical convergence focuses on the density of index sets where a sequence behaves regularly, it may fail to capture more refined patterns of convergence, especially in the presence of structured or rare but influential perturbations. In contrast, $\mathcal{I}$-convergence offers greater flexibility by allowing the selection of different ideals—such as the ideal of sets of density zero (recovering statistical convergence) or other more specialized ideals (e.g., analytic or summable ideals)—to tailor the notion of "negligibl" index sets to specific applications. This ideal-based framework not only unifies various convergence methods but also enables a more nuanced analysis of sequences in complex, uncertain, or non-uniform environments. Since its introduction, $\mathcal{I}$-convergence has been extensively studied. Šalát et al. established its fundamental properties [27] and later investigated the $\mathcal{I}$-convergence field [28]. Das et al. extended the concept to double sequences by introducing $\mathcal{I}-$ and $\mathcal{I}^*$-convergence [29].

In this paper, we will apply the $\mathcal{I}$-convergence of sequences of fuzzy points to investigate properties related to the concept of $\mathcal{I}$-closure. We will introduce and develop a new fuzzy topology, which we will call the fuzzy $\mathcal{I}$-topology, and show that it is finer than the original fuzzy topology. We will prove that every fuzzy convergent sequence is also $\mathcal{I}$-convergent and demonstrate that every fuzzy continuous function is necessarily fuzzy $\mathcal{I}$-continuous. Moreover, we will illustrate the practical relevance of fuzzy $\mathcal{I}$-convergence through an application to temperature regulation in an industrial furnace, where the system is affected by noisy sensor data. The article is organized into five sections. The first section provides an introductory overview and review of relevant literature. In the second section, we present the fundamental theoretical background necessary for understanding fuzzy sets. The third section contains the main contributions of this work, where we introduce the concept of $\mathcal{I}$-convergence and establish several new results. In the fourth section, we illustrate the applicability of $\mathcal{I}$-convergence and fuzzy theory through a practical example. Finally, the last section offers concluding remarks along with suggestions for future research directions.

2. Preliminaries

In this work, E is a set, and J is the closed interval $[0, 1]$ on the real line.

Definition 2.1. [30, Definition 1.1] A fuzzy set in E is a member of J^E, a mapping from E into J.

Definition 2.2. [30, Definition 1.3] Assuming $\theta, \kappa \in J^E$, we state that θ includes κ (written $\kappa \subset \theta$) if $\kappa \leqslant \theta$ (i.e., for all $e \in E$, $\kappa(e) \leqslant \theta(e)$).

Definition 2.3. [31, 32] Let $q(e_0, s)$ be a fuzzy set characterized by the membership function θ_q determined by, $\theta_q(e) = \begin{cases} s \text{ if } e = e_0, \\ 0 \text{ if not.} \end{cases}$

In this situation, this fuzzy set is called a fuzzy point in E, where s is referred to as the value of q and e_0 its support. Likewise, q belongs to a fuzzy set θ (or, $q \in \theta$) if and only if $\theta(e_0) > \theta_q(e_0)$. Therefore, q is not an element of θ if and only if $\theta(e_0) \leqslant \theta_q(e_0)$. The set of all fuzzy points in E is indicated as $\mathcal{Q}(E)$.

Definition 2.4. [30, Definition 1.4] For all $\theta, \kappa \in J^E$:

(1) $\theta \wedge \kappa$, is described as $(\theta \wedge \kappa)(e) = \min\{\theta(e), \kappa(e)\}$, $\forall e \in E$.

(2) $\theta \vee \kappa \in J^E$, is framed as $(\theta \vee \kappa)(e) = max\{\theta(e), \kappa(e)\}$, $\forall e \in E$.

(3) $\theta^c \in J^E$, is presented as $\theta^c(e) = 1 - \theta(e)$, $\forall e \in E$.

(4) $q' \in \mathcal{Q}(E)$, is illustrated as $q'(e, s') = q(e, 1 - s)$.

(5) $c_s \in J^E$, is described as $c_s(e) = s$, $\forall e \in E$ and $s \in J$.

Proposition 2.5. *[33] For all $\theta, \kappa \in J^E$ and $(\theta_i)_{i \in N}$ a family of fuzzy subsets in J^E. The following features hold true:*

(1) $c_1 = (c_0)^c$ *and* $c_0 = (c_1)^c$.

(2) $\theta = (\theta^c)^c$.

(3) $\theta \leqslant \kappa \implies \kappa^c \leqslant \theta^c$.

(4) $\underset{i \in \mathcal{N}}{\wedge} \theta_i^c = (\underset{i \in \mathcal{N}}{\vee} \theta_i)^c$.

(5) $\underset{i \in \mathcal{N}}{\vee} \theta_i^c = (\underset{i \in \mathcal{N}}{\wedge} \theta_i)^c$.

(6) $\theta(e) \vee \theta^c(e) \geq \frac{1}{2}, \forall e \in E$.

(7) $\theta(e) \wedge \theta^c(e) \leqslant \frac{1}{2}, \forall e \in E$.

Remark 2.6. The fuzzy set θ has a complement θ^c, which is known as the Zadeh's complement [1]. However, it is not considered a complement in the order-theoretic sense due to $\theta \vee (1 - \theta) \neq c_1$ and $\theta \wedge (1 - \theta) \neq c_0$. For this purpose, we can examine the constant fuzzy set $\theta : E \to J : E \to \frac{1}{2}$, We obtain $\theta^c = \theta$, $\theta \vee (1 - \theta) \neq c_1$ and $\theta \wedge (1 - \theta) \neq c_0$. From the perspective of classical set theory, this is a highly unusual situation, so caution is necessary when interpreting this pseudocomplement [33].

Definition 2.7. [30, Definition 1.5] Let $\sigma \subset J^E$ fulfill the following conditions:

(1) $c_0, c_1 \in \sigma$.

(2) Let θ_1 and θ_2 be contained in σ; consequently $\theta_1 \wedge \theta_2$ is included in σ.

(3) Let $\{\theta_i : i \in \mathcal{N}\}$ be a family that is a subset of σ; hence, $\bigvee_{i \in \mathcal{N}} \{\theta_i\} \in \sigma$.

σ is referred to as a fuzzy topology on E, and (E, σ) a fuzzy topological space (shortly, f.t.s). The pair (E, σ) is called a fuzzy topological space. The elements belonging to σ are called fuzzy open sets in E.

Definition 2.8. [30, Definition 1.6] The set of all fuzzy neighborhoods corresponding to a fuzzy point q in an f.t.s. (E, σ) is indicated by $\mathcal{V}_q^\sigma$. A collection $\mathcal{D}_q^\sigma$ of subsets of $\mathcal{V}_q^\sigma$ forms a local base at q if every $\kappa \in \mathcal{V}_q^\sigma$, there exists some $\theta \in \mathcal{D}_q^\sigma$ such that $\theta \subset \kappa$.

Theorem 2.9. *[32, Theorem 2.4] A fuzzy set in (E, σ) is considered fuzzy open if every fuzzy point within it possesses a fuzzy neighborhood.*

Definition 2.10. [30, Definition 1.7] Let (E, σ) and (F, ϖ) be f.t.s, and let φ be a mapping from E into F. φ is considered a continuous fuzzy mapping if, for every $q \in \mathcal{Q}(E)$ and every $\kappa \in \mathcal{V}_{\varphi(q)}^\gamma$, there exists some $\theta \in \mathcal{V}_q^\sigma$ such that $\varphi(\theta) \subset \kappa$.

Theorem 2.11. *[30, Theorem 1.1 (5)] Let (E, σ) and (F, ϖ) be f.t.s. and let φ be a map from E into F. Then φ is fuzzy continuous if and only if for every $\kappa \in \varpi$, the inverse image $\varphi^{-1}(\kappa) \in \sigma$.*

Definition 2.12. [30, Definition 1.8] Let (E, σ) be an f.t.s., $q(x, t) \in \mathcal{Q}(E)$, and $\theta \in J^E$. The fuzzy point q is described as an adherence value of θ if for every $\kappa \in \mathcal{V}_{q'}^\sigma$, it holds that $\lambda \not\subset \theta^c$.

Proposition 2.13. *[34, Proposition 1.2] $\bigvee\{p : p$ is an adherence value of $\theta\} = \overline{\theta}$.*

Definition 2.14. [35, Definition 3.1]. A fuzzy topological space (E, σ) is said to be separated (or T_2) if and only if for each pair $q_1(e_1, s_1)$, $q_2(e_2, s_2)$ of fuzzy points in E, with $e_1 \neq e_2$, there exist fuzzy open sets θ and κ in E such that $q_1 \in \theta$, $q_2 \in \kappa$, and $(\theta \wedge \kappa)(z) = 0$ for every $z \in E$.

Definition 2.15. [36] Let E and F be any pair of sets, neither of which is empty, $\varphi : E \to F$ be a map, and θ be a fuzzy subset in J^E. Then $\varphi(\theta)$ is a fuzzy subset of F given by:

$$\varphi(\theta)(f) = \begin{cases} \sup_{\varphi \in \varphi^{-1}(f)} \theta(e) \text{ if } \varphi^{-1}(f) \neq 0, \\ 0 \qquad\qquad\qquad\qquad \text{else,} \end{cases}$$

for all $f \in F$, where $\varphi^{-1}(f) = \{e : \varphi(e) = f\}$. If κ is a fuzzy subset of F, the fuzzy subset $\varphi^{-1}(\kappa)$ of E is described by $\varphi^{-1}(\kappa)(e) = \kappa(\varphi(e))$ for all $e \in E$.

3. Main results

In what follows, (E, σ) is viewed as a fuzzy topological space.

Definition 3.1. [3, Definition 3.1] Let $(q_n(e_n, s_n))$ (briefly, (q_n)) be a sequence of fuzzy points (referred to as a fuzzy sequence) of E having supports (e_n) and corresponding membership value (s_n). Let $q(e, s)$ (in brief, q) be an element of E with support e and membership level s. Then q_n is claimed to converge to q, written $q_n \underset{n\to+\infty}{\longrightarrow} q$, if and only if for every member θ of σ such that $q \in \theta$, There is an integer m where $q_n \in \theta$ for any $n > m$. In this situation, the fuzzy point $q(e, s)$ is represented as $\lim(q_n)$.

Define $\varepsilon(\Omega)$ as the asymptotic density of $\Omega \subset \mathbb{N}$, represented by $\varepsilon(\Omega)$. The value of $\varepsilon(\Omega)$ for each subset Ω is calculated using the formula

$$\varepsilon(\Omega) = \lim_{n\to+\infty} \frac{1}{m} |\{n \in \Omega : m \geqslant n\}|,$$

if it can be found, lies within the interval $[0, 1]$, with $|\,|$ is the cardinality (number of elements). We must also recall that for any Ω, $1 - \varepsilon(\Omega) = \varepsilon(\mathbb{N} \setminus \Omega)$ [37].

A fuzzy sequence $(q_n)_n$ in E is said to converge statistically to a fuzzy point q in E if, for every fuzzy neighborhood θ of q in E, $\varepsilon(\{n \in \mathbb{N} : q_n \notin \theta\}) = 0$ [38], that is, $\varepsilon(\{n \in \mathbb{N} : q_n \in \theta\}) = 1$, which is indicated by $statist - \lim_{n\to\infty} q_n = p$ or $q_n \overset{statist}{\to} p$.

Let $\Omega_\theta(q_n) = \{n \in \mathbb{N} : q_n \notin \theta\}$, which is represented by Ω_θ if no confusion arises, where $(q_n)_n$ is a fuzzy sequence and θ a fuzzy subset of E. It is simple to verify that a fuzzy sequence $(q_n)_n$ statistically converges to $q \in E$ holds exactly when, for any fuzzy neighborhood θ of q in E, we obtain $\varepsilon(\Omega_\theta) = 0$ [39].

Assume that the collection of all possible subsets of $\mathbb{N}$ is $\mathcal{Q}(\mathbb{N})$. An ideal $\mathcal{I}$ that is part of $\mathbb{N}$ is a collection of subsets of $\mathbb{N}$ fulfilling the following requirements:

(1) Additivity: For each $\mathcal{N}, \mathcal{M} \in \mathcal{I}$, we obtain $\mathcal{N} \cup \mathcal{M} \in \mathcal{I}$.

(2) Hereditary: For each $\mathcal{M} \subset \mathbb{N}$ and for any $\mathcal{N} \in \mathcal{I}$, whenever $\mathcal{M} \subseteq \mathcal{N}$ it implies that $\mathcal{M} \in \mathcal{I}$.

We call an ideal $\mathcal{I}$ of $\mathbb{N}$ non-trivial when it satisfies $\mathcal{I}$ does not contain $\mathbb{N}$, and $\mathcal{I}$ is non-empty. $\mathcal{I}$ is described as admissible whenever $\mathcal{I}$ is non-trivial and it includes all finite subsets of $\mathbb{N}$. This means that every singleton subset $\{n\}$ is included in $\mathcal{I}$.

Consider $\mathcal{I}_f$ to be the collection formed by taking all finite subsets of $\mathbb{N}$. The family of subsets $\mathcal{N} \subseteq \mathbb{N}$ with $\varepsilon(\mathcal{N}) = 0$ is represented by $\mathcal{I}_\varepsilon$, i.e, $\mathcal{I}_\varepsilon = \{\mathcal{N} \subseteq \mathbb{N} : \varepsilon(\theta) = 0\}$. Its dual filter is $F_{\mathcal{I}_\varepsilon} = \{\mathcal{N} \subseteq \mathbb{N} : \varepsilon(\theta) = 1\}$ [27]. Then, $\mathcal{I}_\varepsilon$ is an admissible ideal.

For the purposes of this study, we take $\mathcal{I}$ to be a non-trivial admissible ideal of $\mathbb{N}$.

Definition 3.2. [40] A fuzzy sequence $(q_n)_n$ in E is said to converge with respect to the ideal $\mathcal{I}$ (alternatively, $\mathcal{I}$-convergent) toward a fuzzy point $q \in E$ if for each fuzzy neighborhood θ of q, the set Ω_θ belongs to $\mathcal{I}$. This is written as

$$\mathcal{I}\text{-}\lim_{n\to\infty} q_n = q \text{ or } q_n \xrightarrow{\mathcal{I}} q,$$

q is identified as the $\mathcal{I}$-limit of the fuzzy sequence $(q_n)_n$.

Example. Let $\mathbb{R}_\mathcal{F}$ denote the set of all fuzzy numbers defined on $\mathbb{R}$. Each fuzzy number $u \in \mathbb{R}_\mathcal{F}$ can be represented by its α-cuts, denoted:

$$[u]_\alpha = [u^-(\alpha), u^+(\alpha)], \quad \alpha \in [0,1],$$

where:

$$\begin{cases} u^-(\alpha) \text{ is the lower bound of the } \alpha\text{-cut}, \\ u^+(\alpha) \text{ is the upper bound of the } \alpha\text{-cut}, \\ u^-(\alpha) \text{ is non-decreasing with respect to } \alpha, \\ u^+(\alpha) \text{ is non-increasing with respect to } \alpha. \end{cases}$$

We consider the fuzzy topology on $\mathbb{R}_\mathcal{F}$ induced by the fuzzy metric, for all $u, v \in \mathbb{R}_\mathcal{F}$:

$$d(u,v) = \sup_{\alpha\in[0,1]} \max\left(\left|u^-(\alpha) - v^-(\alpha)\right|, \left|u^+(\alpha) - v^+(\alpha)\right|\right),$$

Let $q \in \mathbb{R}_\mathcal{F}$ be the triangular fuzzy number centered at 0, defined by the membership function

$$\mu_q(x) = \max(1 - |x|, 0), \quad x \in \mathbb{R}.$$

Define a sequence (q_n) of fuzzy numbers in $\mathbb{R}_\mathcal{F}$ by:

$$q_n(x) = \begin{cases} \max(1 - |x|, 0), & \text{if } n \text{ is not a power of 2}, \\ \max(1 - |x-1|, 0), & \text{if } n = 2^k \text{ for some } k \in \mathbb{N}. \end{cases}$$

Thus, q_n is centered at 0 for most n but centered at 1 when $n = 2, 4, 8, 16, \ldots$

Now, define an ideal $\mathcal{I}$ on $\mathbb{N}$ by:

$$\mathcal{I} = \left\{A \subseteq \mathbb{N} : \sum_{n\in A} \frac{1}{n} < \infty\right\}.$$

This is a proper ideal (since $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$), known as a *summable ideal*. Let θ be any fuzzy neighborhood of q in $\mathbb{R}_\mathcal{F}$. Since the distance between a fuzzy number centered at 0 and one at 1 is at least 1. Thus, the set

$$\Omega_\theta = \{n \in \mathbb{N} : q_n \notin \theta\} \subseteq \{2^k : k \in \mathbb{N}\}.$$

Now compute:

$$\sum_{n\in\{2^k\}} \frac{1}{n} = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1 < \infty,$$

so $\{2^k : k \in \mathbb{N}\} \in \mathcal{I}$, and hence $\Omega_\theta \in \mathcal{I}$. Since this holds for every fuzzy neighborhood θ of q, we conclude that $\mathcal{I}\text{-}\lim_{n\to\infty} q_n = q$. Note that Ω_θ is infinite, so (q_n) does not converge classically to q, and the convergence is genuinely generalized via the ideal $\mathcal{I}$.

The following lemma is obvious.

Lemma 3.3. *Consider two ideals $\mathcal{J}$ and $\mathcal{I}$ of $\mathbb{N}$ satisfying $\mathcal{J} \supseteq \mathcal{I}$. If a fuzzy sequence $(q_n)_n$ in E converges to a fuzzy point q with respect to the ideal $\mathcal{I}$ (i.e., $q_n \xrightarrow{\mathcal{I}} q$) then $q_n \xrightarrow{\mathcal{J}} q$.*

Proposition 3.4. *A fuzzy sequence $(q_n)_n$ in E that converges to q is necessarily $\mathcal{I}$-convergent with limit q.*

Proof. Under the assumption that $\mathcal{I}$ is admissible, it follows that $\mathcal{I} \supseteq \mathcal{I}_f$, and hence the result is established as a consequence of the preceding lemma. □

The example given below demonstrates that the above proposition's converse is untrue.

Example. Assume that E is a fuzzy separated topological space that contains at least two distinct fuzzy points $q(e,s)$ and $q(f,r)$. Let $(q_n(e_n,s_n))$ be a fuzzy sequence in E defined by

$$q_n(e_n,s_n) = \begin{cases} q(e,s), & \text{if } n \text{ is a square,} \\ q(f,t), & \text{if not.} \end{cases}$$

By Definition 3.2 $(q_n(e_n,s_n))$ is $\mathcal{I}_\varepsilon$-convegent to $q(f,t)$; in fact, given any fuzzy neighborhood θ of q, it holds that $\Omega_\theta \subset \mathcal{N}$, where $\mathcal{N} = \{m \in \mathbb{N} / \exists k \in \mathbb{N}, m = k^2\}$ and moreover, $\mathcal{N} \in \mathcal{I}_\varepsilon$ (For the re,ason that $\varepsilon(\mathcal{N}) = 0$). However, $(q_n(e_n,s_n))$ fails to converge to $q(e,s)$.

Definition 3.5. [30, Definition 1.4]

(1) Let $q(e,s)$ represent a fuzzy point in E. We define q' by setting $q'(e,s) = q(e,1-s)$.

(2) A fuzzy subset $c_s \in J^E$, is described as $c_s(e) = s$, $\forall e \in E$, and s is an element of $J = [0,1]$.

Remark 3.6. The complement of c_0 is c_1, and conversely, the complement of c_1 is c_0, i.e.,

$$c_1 = (c_0)^c, \text{ and } c_0 = (c_1)^c.$$

Remark 3.7. Zorn's lemma can be used to demonstrate that we can find a maximal ideal (in terms of inclusion) among all admissible ideals $\mathcal{I}$ of $\mathbb{N}$. For any $\mathcal{N} \subset \mathbb{N}$ and $\mathcal{I}$ ideal of $\mathbb{N}$, whenever $\mathcal{J}$ is a maximum ideal, we have either $\mathcal{N} \in \mathcal{J}$ or $\mathcal{N} \in \mathbb{N} \setminus \mathcal{J}$.

The most well-known axioms of convergence are as follows (see [41]):

(1) Any constant fuzzy sequence (q_n), where $q_n = q$ for every $n \in \mathbb{N}$, converges to q.

(2) If the limit of a fuzzy sequence $(q_n)_n$ is q, all of its subsequences converge to q as well.

(3) Whenever every subsequence of (q_n) has a further subsequence that converges to q, it implies that (q_n) also converges to q.

(4) It is not possible for a fuzzy sequence to converge to two different fuzzy points that have distinct supports.

The validity of certain properties under $\mathcal{I}$-convergence of fuzzy sequences is demonstrated in the following theorem.

Theorem 3.8. *The $\mathcal{I}$-convergence of fuzzy sequences in E meets the required conditions* (1)–(3). *Moreover, if E is separated, the additional property* (4) *is also satisfied.*

Proof. There is no difficulty in proving (1) and (2).

(3) Suppose, with the aim of reaching a contradiction, that the sequence (q_n) is not $\mathcal{I}$-convergent to q. Consequently, there is $\theta \in \sigma$, such that $\Omega_\theta = \{n \in \mathbb{N} : q_n \notin \theta\} \notin \mathcal{I}$, then Ω_θ is infinite. We are able to write $\Omega'_\theta = \{m_k \in \Omega_\theta : m_1 < m_2 < ... < m_k < ...\}$. The fuzzy sequence (q_{m_k}) is a subsequence of (q_n). Let $(q_{m_{k_{k'}}})$ be a subsequence of (q_{m_k}). Based on the hypothesis, $(q_{m_{k_{k'}}})$ is expected to be $\mathcal{I}$-convergent to q i.e., $\Omega''_\theta = \{m_{k_{k'}} \in \Omega'_\theta : q_{m_{k_{k'}}} \notin \theta\} \in \mathcal{I}$. Since $\Omega_\theta \notin \mathcal{I}$ and $\Omega''_\theta \subset \Omega'_\theta \subset \Omega_\theta$, then $\Omega_\theta \in \mathbb{N} \setminus \mathcal{I}$. Consequently, $\Omega''_\theta \in \mathbb{N} \setminus \mathcal{I}$, so $\Omega''_\theta \notin \mathcal{I}$, which is not true.

(4) Assume that E is separated. Suppose, for contradiction, that the fuzzy sequence (q_n) is convergent to two different points $q_1(e_1, s_1)$ and $q_2(e_2, s_2)$ $(e_1 \neq e_2)$. Since $q_n \xrightarrow{\mathcal{I}} q_1$ and $q_n \xrightarrow{\mathcal{I}} q_2$, then $\forall\theta \in \sigma$ with $q_1 \in \theta, \Omega_\theta \in \mathcal{I}$ and $\forall\kappa \in \sigma$ with $q_2 \in \kappa, \Omega_\kappa \in \mathcal{I}$, i.e., $\Omega_\theta \cup \Omega_\kappa \in \mathcal{I}$, therefore

$$\begin{aligned}\Omega_\kappa \cup \Omega_\theta &= \{m \in \mathbb{N} : q_m \text{ does not belong to } \kappa\} \cup \{m \in \mathbb{N} : q_m \text{ does not}\\ &\quad\ \text{belong to } \theta\}\\ &= \{m \in \mathbb{N} : q_m \in \kappa\} \cap \{m \in \mathbb{N} : q_m \in \theta\}\\ &= \{m \in \mathbb{N} : q_m \in \kappa \wedge \theta\}.\end{aligned}$$

Given that E is a fuzzy separated space, this implies that $\theta \wedge \kappa = c_0$. Therefore $\{m \in \mathbb{N} : q_m \in \theta \wedge \kappa\} = \emptyset \in \mathcal{I}$, This is not true because $\mathcal{I}$ is not trivial.

□

Definition 3.9. (1) For any fuzzy subset θ in J^E, we define the fuzzy $\mathcal{I}$-closure of θ by:

$$\overline{\theta}^{\mathcal{I}} = \{q \in E : \exists(q_n) \subset \theta \text{ and } q'_n \xrightarrow{\mathcal{I}} q'\}.$$

We note $\overline{\theta}^{\mathcal{I}} = \overline{\kappa}^{\mathcal{I}}$ where $\kappa = \theta$ for all θ in J^E.

(2) A subset $\theta \in J^E$ is termed fuzzy provided that it is $\mathcal{I}$-closed if $\overline{\theta}^{\mathcal{I}} = \theta$.

(3) In the lattice J^E, a fuzzy subset κ is $\mathcal{I}$-open precisely when its negation lies in the class of $\mathcal{I}$-closed sets.

Example. Let $E = \mathbb{R}_{\mathcal{F}}$ denote the space of fuzzy real numbers, equipped with the Hausdorff fuzzy metric on α-cuts:

$$d(u, v) = \sup_{\alpha\in[0,1]} \max\left\{|u_\alpha^- - v_\alpha^-|, |u_\alpha^+ - v_\alpha^+|\right\}.$$

Let $\theta \subset \mathbb{R}_{\mathcal{F}}$ be the set of triangular fuzzy numbers defined by:

$$\theta = \left\{q_n \in \mathbb{R}_{\mathcal{F}} : \mu_{q_n}(x) = \max\left(1 - \left|x - \frac{1}{n}\right|, 0\right),\ n \in \mathbb{N}\right\}.$$

Each q_n is a triangular fuzzy number centered at $\frac{1}{n}$ with support of width 2. Let $\mathcal{I}$ be the ideal of subsets of $\mathbb{N}$ with asymptotic density zero:

$$\mathcal{I} = \left\{A \subseteq \mathbb{N} : \lim_{n\to\infty} \frac{|A \cap \{1, 2, \ldots, n\}|}{n} = 0\right\}.$$

This is the ideal associated with statistical convergence. Now, consider the sequence $(q_n) \subset \theta$. We have:

$$\mu_{q_n'}(x) = 1 - \mu_{q_n}(x) = 1 - \max\left(1 - \left|x - \frac{1}{n}\right|, 0\right).$$

This simplifies to:

$$\mu_{q_n'}(x) = \begin{cases} \left|x - \frac{1}{n}\right|, & \text{if } \left|x - \frac{1}{n}\right| \leq 1, \\ 1, & \text{otherwise.} \end{cases}$$

Let $q_0 \in \mathbb{R}_{\mathcal{F}}$ be the triangular fuzzy number centered at 0:

$$\mu_{q_0}(x) = \max(1 - |x|, 0).$$

We claim that $q_n' \xrightarrow{\mathcal{I}} q_0$. To verify this, let γ be any fuzzy neighborhood of q_0 in the fuzzy metric topology. Since $\mu_{q_n'}(x) \to \mu_{q_0}(x)$ pointwise as $n \to \infty$, and uniformly on compact sets, we have $d(q_n', q_0) \to 0$. Therefore, the set $\Omega_\gamma = \{n \in \mathbb{N} : q_n' \notin \gamma\}$ is finite, so $\Omega_\gamma \in \mathcal{I}$.

Thus, $q_n' \xrightarrow{\mathcal{I}} q_0$, i.e., $q_n' \xrightarrow{\mathcal{I}} q'$ with $q' = q_0$. Therefore, $q_0 \in \overline{\theta}^{\mathcal{I}}$. This shows that even though $q_0 \notin \theta$ (since no q_n is centered exactly at 0), it belongs to the fuzzy $\mathcal{I}$-closure due to the $\mathcal{I}$-convergence of (q_n') to q'.

In the theorem that follows, we compare the closure and the $\mathcal{I}$-closure of a fuzzy subsets.

Theorem 3.10. *Consider a fuzzy subset θ to be a fuzzy subset in J^E; it follows that:*

(1) $\theta \leqslant \overline{\theta}^{\mathcal{I}} \leqslant \overline{\theta}$.

(2) $\overline{c_0}^{\mathcal{I}} = c_0$ *and* $\overline{c_1}^{\mathcal{I}} = c_1$.

Proof. (1) Let $q(e, s) \in \theta$ and let $q_n(e_n, s_n) = q(e, s)$, for all n contained in $\mathbb{N}$. Evidently, $q_n'(e_n, 1-s_n) = q'(x, 1-s)$, given that $q_n' \xrightarrow{\mathcal{I}} q'$ then $q \in \overline{\theta}^{\mathcal{I}}$. Now, if $q \in \overline{\theta}^{\mathcal{I}}$, then a fuzzy sequence $(q_n)_n$ in θ exists for which $q_n' \xrightarrow{\mathcal{I}} q'$, i.e., for any fuzzy neighborhood κ of q', $\Omega_\kappa \in \mathcal{I}$. Therefore, for any $n \in \mathbb{N}$, $s_n < \theta(e_n)$, and for every $n \notin \Omega_\kappa$, $1 - s_n < \kappa(e_n)$. Hence, for all $n \notin \Omega_\kappa$, $\kappa(e_n) > 1 - \theta(e_n)$ $(\kappa \not\subset \theta^c)$ i.e., $q \in \overline{\theta}$ ([30, Definition 1.8]).

(2) It is so simple.

□

Additionally, we possess the properties listed below:

Theorem 3.11. *Let θ and κ be fuzzy subsets in J^E, and let $(\theta_i)_{i\in\mathcal{N}}$ be a collection of fuzzy subsets within J^E. We have the following relationships:*

(1) $\kappa \geqslant \theta \Rightarrow \overline{\kappa}^{\mathcal{I}} \geqslant \overline{\theta}^{\mathcal{I}}$.

(2) $\bigwedge_{i\in\mathcal{N}} \overline{\theta_i}^{\mathcal{I}} \geqslant \overline{\bigwedge_{i\in\mathcal{N}} \theta_i}^{\mathcal{I}}$.

(3) $\overline{\bigvee_{i\in\mathcal{N}} \theta_i}^{\mathcal{I}} \geqslant \bigvee_{i\in\mathcal{N}} \overline{\theta_i}^{\mathcal{I}}$.

(4) $\overline{\theta}^{\mathcal{I}} \vee \overline{\kappa}^{\mathcal{I}} = \overline{\theta \vee \kappa}^{\mathcal{I}}$.

Proof. (1) Consider q a fuzzy point in $\overline{\theta}^{\mathcal{I}}$. Consequently, we can find a fuzzy sequence (q_n) in θ for which $q'_n \xrightarrow{\mathcal{I}} q'$, That is, for each member ζ of σ such that $q' \in \zeta$, $\Omega_\zeta \in \mathcal{I}$. Since $\theta \leqslant \kappa$ this implies that for all possible n lying in $\mathbb{N}$, q_n belongs to κ. Thus, q is in $\overline{\kappa}^{\mathcal{I}}$.

(2) and (3) are straightforward by (1).

(4) $\overline{\theta \vee \kappa}^{\mathcal{I}} \geqslant \overline{\theta}^{\mathcal{I}} \vee \overline{\kappa}^{\mathcal{I}}$ is easily shown by (3). Conversely, let q be a fuzzy point in $\overline{\theta \vee \kappa}^{\mathcal{I}}$. Thus, there exists a fuzzy sequence (q_n) in $\theta \vee \kappa$ where $q'_n \xrightarrow{\mathcal{I}} q'$. Assume, in the direction of a contradiction, that $\{n$ is a member of $\mathbb{N} : q_n \in \kappa\}$ and $\{n$ is a member of $\mathbb{N} : q_n \in \theta\}$ are finite. It follows that there exists n_0 lies in $\mathbb{N}$ satisfying $q_n \notin \theta \vee \kappa$ is not finite for every $n > n_0$, which option is untrue (because (q_n) is in $\theta \vee \kappa$). Speculate that $\mathcal{M} = \{m \in \mathbb{N} : q_m \in \theta\}$ and define the set $\mathcal{M}' = \{m'_1 < m'_2 < ... < m'_k < ...$ for which $m'_k \in \mathcal{M}\}$. As a result, $q_{m'_k}$ is a subsequence of q_n. Based on Theorem 3.8, $q \in \overline{\theta}^{\mathcal{I}}$. Hence, $q \in \overline{\theta}^{\mathcal{I}} \vee \overline{\kappa}^{\mathcal{I}}$.

□

Definition 3.12. Consider θ as a fuzzy subset within J^E. The fuzzy $\mathcal{I}$-interior of θ can be expressed as

$$\overset{\circ \mathcal{I}}{\theta} = \theta \wedge (1 - \overline{1-\theta}^{\mathcal{I}}).$$

Example. Let $E = \mathbb{R}_{\mathcal{F}}$ denote the space of fuzzy real numbers. A triangular fuzzy number is defined by a triple (a, b, c) with $a < b < c$, and its membership function is:

$$\mu(x) = \begin{cases} 0, & x \leq a, \\ \frac{x-a}{b-a}, & a < x \leq b, \\ \frac{c-x}{c-b}, & b < x \leq c, \\ 0, & x > c. \end{cases}$$

This represents a fuzzy number with a peak at b and support $[a, c]$.

Let (ξ_n) be a sequence in $\mathbb{R}_{\mathcal{F}}$ defined by: $\xi_n = \left(1 - \frac{1}{n},\ 1,\ 1 + \frac{1}{n}\right), \quad n \in \mathbb{N}$. So: $\xi_1 = (0, 1, 2)$, $\xi_2 = (0.5, 1, 1.5)$, $\xi_3 = \left(\frac{2}{3}, 1, \frac{4}{3}\right)$, As $n \to \infty$, the support $[1 - \frac{1}{n}, 1 + \frac{1}{n}]$ converges to the singleton $\{1\}$.

This sequence converges to the crisp number 1, represented as the degenerate triangular fuzzy number: $\xi = (1, 1, 1)$.

Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be the ideal of finite sets: $\mathcal{I} = \{A \subseteq \mathbb{N} : A \text{ is finite}\}$. A sequence (ζ_n) of fuzzy numbers $\mathcal{I}$-converges to ζ if for every fuzzy neighborhood U of ζ, $\{n \in \mathbb{N} : \zeta_n \notin U\} \in \mathcal{I}$. Since $\mathcal{I}$ contains only finite sets, this means $\zeta_n \in U$ for all but finitely many n. Thus, $\mathcal{I}$-convergence coincides with usual convergence in fuzzy topology. Therefore, $\xi_n \xrightarrow{\mathcal{I}} \xi$.

Let $\theta : \mathbb{R}_{\mathcal{F}} \to [0, 1]$ be a fuzzy subset defined by:

$$\theta(\zeta) = \begin{cases} 1, & \zeta = \xi_n \text{ for some } n \in \mathbb{N}, \\ \frac{1}{2}, & \zeta = \xi, \\ 0, & \text{otherwise.} \end{cases}$$

The complement is: $(1 - \theta)(\zeta) = 1 - \theta(\zeta)$, so:

$$(1 - \theta)(\zeta) = \begin{cases} 0, & \zeta = \xi_n \text{ for some } n \in \mathbb{N}, \\ 1 - \frac{1}{2} = \frac{1}{2}, & \zeta = \xi, \\ 1, & \text{otherwise.} \end{cases}$$

Thus, $1 - \theta$ is 1 on a dense set of fuzzy numbers (since only countably many are excluded). The fuzzy $\mathcal{I}$-closure of $1-\theta$ includes all fuzzy numbers that are $\mathcal{I}$-limits of sequences in the support of $1-\theta$. Since $(1 - \theta)(\zeta) = 1$ for all ζ except $\xi_1, \xi_2, \dots$ and ξ, and this set is dense in $\mathbb{R}_{\mathcal{F}}$ under standard fuzzy metrics (for example, d_∞), we can find, for any $\zeta_0 \in \mathbb{R}_{\mathcal{F}}$, a sequence (ζ_k) such that: $(1-\theta)(\zeta_k) = 1$, $\zeta_k \to \zeta_0$ (hence $\zeta_k \xrightarrow{\mathcal{I}} \zeta_0$). Therefore, every ζ_0 is an $\mathcal{I}$-cluster point of $1 - \theta$, so: $\overline{1-\theta}^{\mathcal{I}}(\zeta) = 1 \quad$ for all $\zeta \in \mathbb{R}_{\mathcal{F}}$ and $(1 - \overline{1-\theta}^{\mathcal{I}})(\zeta) = 1 - 1 = 0$ for all ζ.

The fuzzy $\mathcal{I}$-interior of θ is defined by: $\overset{\circ}{\theta}^{\mathcal{I}} = \theta \wedge (1 - \overline{1-\theta}^{\mathcal{I}})$, where $\wedge$ denotes pointwise minimum (fuzzy intersection). Thus:

$$\begin{cases} \overset{\circ}{\theta}^{\mathcal{I}}(\xi_n) & = \min(\theta(\xi_n),\ 0) = \min(1, 0) = 0, \\ \overset{\circ}{\theta}^{\mathcal{I}}(\xi) & = \min\left(\frac{1}{2},\ 0\right) = 0, \\ \overset{\circ}{\theta}^{\mathcal{I}}(\zeta) & = \min(0, 0) = 0 \text{ for all other } \zeta. \end{cases}$$

Hence, $\overset{\circ}{\theta}^{\mathcal{I}}(\zeta) = 0$ for all $\zeta \in \mathbb{R}_{\mathcal{F}}$. We conclude that: $\theta(\xi_1) = 1$, $\theta(\xi_2) = 1$, $\theta(\xi) = \frac{1}{2}$. However, $\overset{\circ}{\theta}^{\mathcal{I}}(\zeta) = 0$ for all $\zeta \in \mathbb{R}_{\mathcal{F}}$. Therefore, $\overset{\circ}{\theta}^{\mathcal{I}} \neq \theta$.

In the following, we present several relationships by utilizing the fuzzy $\mathcal{I}$-interior of subsets.

Theorem 3.13. *Consider θ and κ as fuzzy subsets of J^E and $(\theta_i)_{i \in N}$ as a collection of fuzzy subsets of J^E. Our relationships are as follows:*

(1) $\overset{\circ}{\kappa}^{\mathcal{I}} \geqslant \overset{\circ}{\theta}^{\mathcal{I}} \Leftarrow \kappa \geqslant \theta$.

(2) $\overset{\circ}{\theta \vee \kappa}^{\mathcal{I}} \geqslant \overset{\circ}{\theta}^{\mathcal{I}} \vee \overset{\circ}{\kappa}^{\mathcal{I}}$.

(3) $\theta \overset{\circ\mathcal{I}}{\wedge} \kappa = \overset{\circ\mathcal{I}}{\theta} \wedge \overset{\circ\mathcal{I}}{\kappa}$.

Proof. (1) Because of this $\kappa \geqslant \theta$, therefore $\overline{1-\theta}^{\mathcal{I}} \geqslant \overline{1-\kappa}^{\mathcal{I}}$, that is $1 - \overline{1-\kappa}^{\mathcal{I}} \geqslant 1 - \overline{1-\theta}^{\mathcal{I}}$. It follows that, $\min(\kappa(e), 1 - \overline{1-\kappa}^{\mathcal{I}}(e)) \geqslant \min(\theta(e), 1 - \overline{1-\theta}^{\mathcal{I}}(e))$ for any $e \in E$, i.e., $\kappa \wedge (1 - \overline{1-\kappa}^{\mathcal{I}}) \geqslant \theta \wedge (1 - \overline{1-\theta}^{\mathcal{I}})$. Hence, $\overset{\circ\mathcal{I}}{\kappa} \geqslant \overset{\circ\mathcal{I}}{\theta}$.

(2) Consider $\theta \overset{\circ\mathcal{I}}{\vee} \kappa \geqslant \overset{\circ\mathcal{I}}{\theta}$ and $\theta \overset{\circ\mathcal{I}}{\vee} \kappa \geqslant \overset{\circ\mathcal{I}}{\kappa}$. Consequently, we have that $\theta \overset{\circ\mathcal{I}}{\vee} \kappa \geqslant \overset{\circ\mathcal{I}}{\theta} \vee \overset{\circ\mathcal{I}}{\kappa}$.

(3) As a result of this $(1-\kappa) \vee (1-\theta) = 1 - (\kappa \wedge \theta)$ we have $\overline{(1-\kappa) \vee (1-\theta)}^{\mathcal{I}} = \overline{(1-\kappa \wedge \theta)}^{\mathcal{I}}$. By Theorem 3.11 $\overline{1-\kappa}^{\mathcal{I}} \vee \overline{1-\theta}^{\mathcal{I}} = \overline{1-(\kappa \wedge \theta)}^{\mathcal{I}}$. Therefore $1 - (\overline{1-\kappa}^{\mathcal{I}} \vee \overline{1-\theta}^{\mathcal{I}}) = 1 - (\overline{1-\kappa \wedge \theta}^{\mathcal{I}})$, i.e., $(1 - \overline{1-\kappa}^{\mathcal{I}}) \wedge (1 - \overline{1-\theta}^{\mathcal{I}}) = 1 - (\overline{1-\kappa \wedge \theta}^{\mathcal{I}})$. Hence,

$$
\begin{aligned}
(\kappa \wedge \theta) \wedge [1 - (\overline{1-\kappa \wedge \theta}^{\mathcal{I}})] &= (\kappa \wedge \theta) \wedge [(1 - \overline{1-\kappa}^{\mathcal{I}}) \wedge (1 - \overline{1-\theta}^{\mathcal{I}})] \\
&= [\kappa \wedge (1 - \overline{1-\kappa}^{\mathcal{I}})] \wedge [\theta \wedge (1 - \overline{1-\theta}^{\mathcal{I}})].
\end{aligned}
$$

Thus, $\theta \overset{\circ\mathcal{I}}{\wedge} \kappa = \overset{\circ\mathcal{I}}{\theta} \wedge \overset{\circ\mathcal{I}}{\kappa}$.

□

Definition 3.14. Consider κ as a subset of J^E. Suppose there exists a fuzzy $\mathcal{I}$-open subset θ, for which $q \in \theta$ and $\theta \leqslant \kappa$; then κ is described as a fuzzy $\mathcal{I}$-neighborhood of $q(e, s)$.

Next, we'll show that fuzzy closedness implies fuzzy $\mathcal{I}$-closedness, and we'll explore the characterization of the $\mathcal{I}$-interior.

Theorem 3.15. *(1) Any set that is fuzzy closed is also classified as fuzzy $\mathcal{I}$-closed. In the same way, any set that is fuzzy open is also classified as fuzzy $\mathcal{I}$-open.*

(2) The $\mathcal{I}$-interior of θ equals θ itself, precisely when θ is fuzzy $\mathcal{I}$-open.

Proof. (1)

$$
\begin{aligned}
\theta \text{ is a fuzzy closed subset of } J^E &\Rightarrow \overline{\theta} = \theta \\
&\Rightarrow \overline{\theta}^{\mathcal{I}} = \theta \text{ (based on Theorem 3.10)} \\
&\Rightarrow \theta \text{ is } \mathcal{I}\text{-closed (*)}.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\theta \text{ is a fuzzy open subset of } J^E &\Rightarrow 1-\theta \text{ is fuzzy closed} \\
&\Rightarrow 1-\theta \text{ is fuzzy } \mathcal{I}\text{-closed (by (*))} \\
&\Rightarrow \theta \text{ is } \mathcal{I}\text{-open}.
\end{aligned}
$$

(2)

$$
\theta \text{ is fuzzy } \mathcal{I}\text{-open} \Rightarrow 1-\theta \text{ is fuzzy } \mathcal{I}\text{-closed}
$$

$$\Rightarrow \overset{\circ\mathcal{I}}{\theta} = \theta \wedge (1 - (1 - \theta))$$
$$\Rightarrow \overset{\circ\mathcal{I}}{\theta} = \theta.$$

In contrast, let $\theta = (1 - \overline{1-\theta}^{\mathcal{I}}) \wedge \theta = \overset{\circ\mathcal{I}}{\theta}$. It follows that,

$$\begin{aligned} 1 - \theta &= 1 - [(1 - \overline{1-\theta}^{\mathcal{I}}) \wedge \theta] \\ &= (1 - \theta) \vee [1 - (1 - \overline{1-\theta}^{\mathcal{I}})] \\ &= \overline{1-\theta}^{\mathcal{I}} \vee (1 - \theta). \end{aligned}$$

As a result, $1 - \theta \geqslant \overline{1-\theta}^{\mathcal{I}}$. Thus, $1 - \theta = \overline{1-\theta}^{\mathcal{I}}$, i.e., θ is fuzzy $\mathcal{I}$-open. □

Remark 3.16. A fuzzy subset κ of E is fuzzy $\mathcal{I}$-open exactly when κ is a fuzzy $\mathcal{I}$-neighborhood for all its points.

We now establish the order of the interior, $\mathcal{I}$-interior, closure, and $\mathcal{I}$-closure of a fuzzy set in the next theorem.

Theorem 3.17. *Consider θ as a fuzzy subset in J^E. As a result, we have the following relationships of inclusion:*

$$\overline{\theta} \geqslant \overline{\theta}^{\mathcal{I}} \geqslant \theta \geqslant \overset{\circ\mathcal{I}}{\theta} \geqslant \mathring{\theta}.$$

Proof. Refer to Theorem 3.10 for $\overline{\theta} \geqslant \overline{\theta}^{\mathcal{I}} \geqslant \theta$. By [3, Definition 2.4] $\theta \geqslant \mathring{\theta}$. Then, $\overline{1-\mathring{\theta}}^{\mathcal{I}} \geqslant \overline{1-\theta}^{\mathcal{I}}$. Therefore, $1 - \overline{1-\theta}^{\mathcal{I}} \geqslant 1 - \overline{1-\mathring{\theta}}^{\mathcal{I}}$. Hence, $(1 - \overline{1-\theta}^{\mathcal{I}}) \wedge \theta \geqslant (1 - \overline{1-\mathring{\theta}}^{\mathcal{I}}) \wedge \mathring{\theta}$. Thus, $\overset{\circ\mathcal{I}}{\theta} \geqslant \mathring{\theta}$. Moreover, by definition, $\theta \geqslant \overset{\circ\mathcal{I}}{\theta}$. Accordingly, $\overline{\theta} \geqslant \overline{\theta}^{\mathcal{I}} \geqslant \theta \geqslant \overset{\circ\mathcal{I}}{\theta} \geqslant \mathring{\theta}$. □

A characterization of $\mathcal{I}$-closed subsets is given below.

Theorem 3.18. *Consider the fuzzy subset θ in J^E.*

A fuzzy set θ is $\mathcal{I}$-closed $\Leftrightarrow \theta = \wedge\{\kappa/\kappa$ is described as $\mathcal{I}$-closed and $\theta \leqslant \kappa\}$.

Proof. It is straightforward that if θ is $\mathcal{I}$-closed, then the equality is true. Inversely assume that $\theta = \wedge\{\kappa/\kappa$ is described as $\mathcal{I}$-closed and $\theta \leqslant \kappa\}$. Let us demonstrate that θ is fuzzy $\mathcal{I}$-closed. We only need to establish that $\overline{\theta}^{\mathcal{I}} \leqslant \theta$. Let $q(e, s)$ be a fuzzy point outside of θ. Consequently, we can find a fuzzy $\mathcal{I}$-closed subset κ where $q \notin \kappa$ and $\kappa \geqslant \theta$. Suppose, to derive a contradiction, that $q \in \overline{\theta}^{\mathcal{I}}$. This implies that a fuzzy sequence (q_n) of fuzzy points exists in θ satisfying $q'_n \xrightarrow{\mathcal{I}} q'$. As $\theta \leqslant \kappa$ it follows that (q_n) is also a member of κ. Therefore, $q \in \kappa$, which isn't true. So, $q \in \overline{\theta}^{\mathcal{I}}$. □

A characterization of the $\mathcal{I}$-interior of subsets follows directly from the preceding theorem.

Corollary 3.19. *Consider the fuzzy subset θ in J^E.*

A fuzzy set θ is $\mathcal{I}$-open $\Leftrightarrow \theta = \vee\{\zeta/\zeta$ is described as $\mathcal{I}$-open and $\zeta \leqslant \theta\}$.

Proof. It is straightforward that if θ is $\mathcal{I}$-open, then the equality is true. Conversely,

$$\theta = \vee\Big\{\zeta/\zeta \text{ is } \mathcal{I}\text{-open and } \zeta \leqslant \theta\Big\} \Rightarrow 1-\theta = \wedge\Big\{1-\zeta/1-\zeta \text{ is } \mathcal{I}\text{-closed and } 1-\theta \leqslant 1-\zeta\Big\}$$
$$\Rightarrow 1-\theta \text{ is } \mathcal{I}\text{-closed (by earlier theorem)}$$
$$\Rightarrow \theta \text{ is } \mathcal{I}\text{-open.}$$

□

With the help of the earlier result, we can now introduce a new topology called the fuzzy topology with $\mathcal{I}$-convergence, which is presented below.

Theorem 3.20. *Take $\sigma^{\mathcal{I}}$ to be the subfamily of J^E described by:*

$$\sigma^{\mathcal{I}} = \Big\{\theta \in J^E / \textit{ for each fuzzy sequence } (q_n)_n \textit{ in } 1-\theta \textit{ for which } q'_n \xrightarrow{\mathcal{I}} q' \textit{ then } q \textit{ lies in } 1-\theta\Big\},$$

that is a fuzzy topology on E. It is finer than (or equal to) σ, denoted by $\sigma^{\mathcal{I}} \geqslant \sigma$. We refer to it as the fuzzy topology with $\mathcal{I}$-convergence (or, in short, the fuzzy $\mathcal{I}$-topology).

Proof. Let $\theta, \kappa \in \sigma^{\mathcal{I}}$, and let $(\theta_j)_{j\in I}$ be a family in $\sigma^{\mathcal{I}}$. It is clear that $c_0, c_1 \in \sigma^{\mathcal{I}}$. Since $(\theta \wedge \kappa)^c = \theta^c \vee \kappa^c$, then by Theorems 3.11 and 3.18, $\theta^c \vee \kappa^c$ is fuzzy $\mathcal{I}$-closed, so $\theta \wedge \kappa \in \sigma^{\mathcal{I}}$. However, $(\underset{j\in I}{\vee}\theta_j)^c = \underset{j\in I}{\wedge}\theta_j^c$. Using the same theorems, it follows that $\underset{j\in I}{\wedge}\theta_j^c$ is fuzzy $\mathcal{I}$-closed. In this case, $\underset{j\in I}{\vee}\theta_j \in \sigma^{\mathcal{I}}$. Consequently, $\sigma^{\mathcal{I}}$ forms a fuzzy topology on E. Now, consider $\theta \in \sigma$, It follows that θ^c is fuzzy closed in E. Thus, according to Theorem 3.15, it is fuzzy $\mathcal{I}$-closed that is $\theta \in \sigma^{\mathcal{I}}$. Therefore $\sigma^{\mathcal{I}} \geqslant \sigma$. □

Example. Let $E = \mathbb{R}_{\mathcal{F}}$, the space of all fuzzy real numbers. Each u is represented by its α-cuts:

$$[u]_\alpha = [u^-(\alpha), u^+(\alpha)], \quad \alpha \in [0,1].$$

Equip E with the standard fuzzy topology σ induced by the metric:

$$d(u,v) = \sup_{\alpha\in[0,1]} \max\left(|u^-(\alpha) - v^-(\alpha)|, |u^+(\alpha) - v^+(\alpha)|\right).$$

The topology σ is generated by open balls:

$$B_r(u) = \{v \in E : d(u,v) < r\}, \quad r > 0.$$

Now, let $\mathcal{I}$ be the ideal of sets with asymptotic density zero:

$$\mathcal{I} = \left\{A \subseteq \mathbb{N} : \lim_{n\to\infty} \frac{1}{n}\Big|A \cap \{1,2,\ldots,n\}\Big| = 0\right\}.$$

Then, a sequence (u_n) in E is said to be $\mathcal{I}$-convergent to u, written $u_n \xrightarrow{\mathcal{I}} u$, if for every $\varepsilon > 0$,

$$\{n \in \mathbb{N} : d(u_n, u) \geq \varepsilon\} \in \mathcal{I}.$$

Define the fuzzy $\mathcal{I}$-topology $\sigma^{\mathcal{I}}$ on E as the family of all fuzzy sets $\theta \in J^E$ such that: For every sequence (u_n) in $1-\theta$, if $u_n \xrightarrow{\mathcal{I}} u$, then $u \in 1-\theta$. In other words, $1-\theta$ must be closed under $\mathcal{I}$-limits.

We now construct a fuzzy set $\theta \in \sigma^{\mathcal{I}}$ such that $\theta \notin \sigma$, proving that $\sigma^{\mathcal{I}} \supsetneq \sigma$. Define a sequence (q_n) of triangular fuzzy numbers. For each $n \in \mathbb{N}$, define q_n as the triangular fuzzy number centered at $\frac{1}{n}$ with support width 2:

$$\mu_{q_n}(x) = \max\left(1 - \left|x - \frac{1}{n}\right|, 0\right), \quad x \in \mathbb{R}.$$

Then $[q_n]_\alpha = \left[\frac{1}{n} - (1-\alpha), \frac{1}{n} + (1-\alpha)\right]$. Let q be the triangular fuzzy number centered at 0: $\mu_q(x) = \max(1-|x|, 0)$. As $n \to \infty$, $\frac{1}{n} \to 0$, and one can verify that:

$$d(q_n, q) = \sup_{\alpha\in[0,1]} \max\left(\left|\left(\frac{1}{n} - (1-\alpha)\right) - (-(1-\alpha))\right|, \left|\left(\frac{1}{n} + (1-\alpha)\right) - (1-\alpha)\right|\right) = \frac{1}{n} \to 0.$$

So $q_n \to q$ in the metric d, hence also $q_n \xrightarrow{\mathcal{I}} q$ (since ordinary convergence implies $\mathcal{I}$-convergence).

Define $\theta \in J^E$ by:

$$\theta(u) = \begin{cases} 1 & \text{if } u = q_n \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

So θ is the fuzzy characteristic function of the sequence (q_n).

Suppose $\theta \in \sigma$. Then for each q_n, there exists $\varepsilon > 0$ such that the open ball $B_\varepsilon(q_n) \subseteq \{u : \theta(u) > 0\}$. But $B_\varepsilon(q_n)$ contains fuzzy numbers arbitrarily close to q_n that are not equal to any q_m, so $\theta(u) = 0$ for such u. Hence, no such ε-ball is contained in θ, so θ is not open in σ. Thus, $\theta \notin \sigma$.

Now, we must show that if a sequence (u_k) satisfies $\theta(u_k) = 0$ (i.e., $u_k \neq q_n$ for all n) and $u_k \xrightarrow{\mathcal{I}} u$, then $\theta(u) = 0$ (i.e., $u \neq q_n$ for all n). But suppose $u = q_N$ for some N. Then $u_k \xrightarrow{\mathcal{I}} q_N$. However, since the q_n are isolated points in the metric space (they are at a positive distance from each other), there exists $\delta > 0$ such that $d(u_k, q_N) < \delta$ only for finitely many k, unless $u_k = q_N$ for infinitely many k. But $\theta(u_k) = 0$, so $u_k \neq q_N$, and thus $d(u_k, q_N) \geq \delta$ for all but finitely many k. So the set $\{k : d(u_k, q_N) \geq \delta\}$ is cofinite, hence not in $\mathcal{I}$, which is a contradiction.

Therefore, $u \notin \{q_n\}$, so $\theta(u) = 0$. Hence, $1-\theta$ is $\mathcal{I}$-sequentially closed, so $\theta \in \sigma^{\mathcal{I}}$. We have constructed a fuzzy set θ such that: $\theta \in \sigma^{\mathcal{I}}$, but $\theta \notin \sigma$. Therefore, $\sigma^{\mathcal{I}}$ proving that the fuzzy $\mathcal{I}$-topology is strictly finer than the original fuzzy topology.

Theorem 3.21. *Consider a fuzzy sequence (q_n) in E, and take q as a fuzzy point that lies in E. Whenever the sequence (q_n) is $\sigma^{\mathcal{I}}$-convergent to q, it is necessarily σ-convergent to q as well.*

Proof. The result is correct since $\sigma^{\mathcal{I}} \succcurlyeq \sigma$. □

As will be shown below, every fuzzy continuous mapping is fuzzy $\mathcal{I}$-continuous.

Theorem 3.22. *Let $\varphi : E \to F$ be a function, for which (E, σ) and (F, ρ) are fuzzy topological spaces. Whenever φ is fuzzy continuous, it necessarily follows that φ is fuzzy $\mathcal{I}$-continuous.*

Proof. Let us suppose that φ is fuzzy continuous. Consider a fuzzy sequence (q_n) in E and a fuzzy point q lies in E for which $q_n \xrightarrow{\mathcal{I}} q$. We must show that $\varphi(q_n) \xrightarrow{\mathcal{I}} \varphi(q)$ in (F, ρ). Let $\theta \in \rho$ where

$\varphi(q) \in \theta$. Through the fuzzy continuity of φ, $\varphi^{-1}(\theta) \in \sigma$, we denote $\varphi^{-1}(\theta)$ by κ. Consequently, $q \in \kappa$. However, for all $f \in F$:

$$\begin{aligned}\varphi(\kappa)(f) &= \sup_{e\in\varphi^{-1}(f)} \kappa(e) \quad ([36, \text{Definition } 2.1])\\ &= \sup_{e\in\varphi^{-1}(f)} \varphi^{-1}(\theta)(e)\\ &= \sup_{e\in\varphi^{-1}(f)} \theta(\varphi(e))\\ &= \begin{cases}\theta(f), & \text{if } f = \varphi(e),\\ 0, & \text{if not.}\end{cases}\end{aligned}$$

Then, $\varphi(\kappa)(f) \leqslant \theta(f)$, for any $f \in F$. Therefore, $\varphi(\kappa) \leqslant \theta$. By assumption, $q_n \xrightarrow{\mathcal{I}} q$. Which means that $\Omega_\kappa = \{n \in \mathbb{N} : q_n \notin \kappa\} \in \mathcal{I}$; in the other way, for all possible $n \in \mathbb{N} \setminus \mathcal{I}$, $q_n \in \kappa$. Additionally, we obtain $\varphi(\kappa)(f) = \sup_{e\in\varphi^{-1}(f)} \kappa(e)$. Then, for every $n \in \mathbb{N} \setminus \mathcal{I}$, $\varphi(\kappa)(\varphi(e_n)) = \sup_{z\in\varphi^{-1}\circ\varphi(e_n)} \kappa(z)$. Consequently, for every $n \in \mathbb{N} \setminus \mathcal{I}$, $\varphi(\kappa)(\varphi(e_n)) > 0$. Hence, for each $n \in \mathbb{N} \setminus \mathcal{I}$, $\varphi(q_n) \in \varphi(\kappa) \leqslant \theta$, i.e, $\varphi(q_n) \in \theta$, for any $n \in \mathbb{N} \setminus \mathcal{I}$. Hence, $\Omega_\kappa = \{n \in \mathbb{N} : \varphi(q_n) \notin \theta\} \in \mathcal{I}$. Thus, $\varphi(q_n) \xrightarrow{\mathcal{I}} \varphi(q)$ in (E, σ). □

4. Example in fuzzy analysis using $\mathcal{I}$-convergence

Consider a fuzzy logic-based temperature control system for an industrial furnace. The system uses multiple sensors to monitor internal temperatures and adjusts heating elements accordingly. However, due to sensor drift or environmental interference, some readings may be inaccurate:

$$x_n = \{200.1,\ 200.3,\ 200.5,\ 999.9,\ 200.6,\ 200.4,\ -100.0,\ 200.7,\ \ldots\}.$$

Here, values like 999.9 and −100.0 are outliers caused by sensor malfunction. We aim to estimate the true underlying temperature trend despite these sparse but large deviations using *fuzzy $\mathcal{I}$-convergence*, which allows us to ignore negligible subsets (e.g., outliers) while preserving convergence behavior. Let X denote the space of all bounded sequences representing temperature readings. Define a *fuzzy norm N* on X as:

$$N(x,t) = \begin{cases}\frac{t}{t+\|x\|}, & \text{if } t > 0,\\ 0, & \text{otherwise.}\end{cases}$$

where $\|x\| = \sup_n |x_n|$. This makes $(X, N, *)$ a *fuzzy normed linear space*, with $*$ being a continuous t-norm (e.g., minimum or product). Let $\mathcal{I}$ be the ideal of subsets of $\mathbb{N}$ with asymptotic density zero:

$$\mathcal{I} = \left\{A \subseteq \mathbb{N} : \lim_{n\to\infty} \frac{|A(n)|}{n} = 0\right\}.$$

This ideal represents sets of indices corresponding to negligible events—such as sensor errors.

A sequence x_n in X is said to be *fuzzy $\mathcal{I}$-convergent* to $L \in \mathbb{R}$ if for every $\varepsilon > 0$ and $t > 0$, the set:

$$\{n \in \mathbb{N} : N(x_n - L, t) < 1 - \varepsilon\} \in \mathcal{I}.$$

In simpler terms, only a negligible number of indices deviate significantly from the limit L in the fuzzy sense.

Using robust statistical techniques (e.g., median filtering), we estimate the central tendency of the sequence x_n. Suppose we estimate the true temperature to be approximately $L = 200.5$.

Now check the set:

$$E_{\varepsilon,t} = \{n : N(x_n - 200.5, t) < 1 - \varepsilon\}$$

for small $\varepsilon = 0.1$, $t = 1$. If the proportion of such indices grows slower than linearly (i.e., has asymptotic density zero), then:

$$E_{\varepsilon,t} \in \mathcal{I},$$

and hence $x_n \xrightarrow{\mathcal{I}} 200.5$ in the fuzzy sense.

5. Conclusions

In this study, we investigated the concept of fuzzy $\mathcal{I}$-convergence and its relationship with standard fuzzy convergence. We proved that every fuzzy convergent sequence is also $\mathcal{I}$-convergent; however, the converse does not hold in general, as illustrated by a counterexample provided in this paper. Furthermore, we introduced a new closure operator based on $\mathcal{I}$-convergence, referred to as the $\mathcal{I}$-closure, and derived the corresponding notion of the $\mathcal{I}$-interior. These constructions give rise to a new topology known as the topology of $\mathcal{I}$-convergence which turns out to be finer than the usual fuzzy topology on J^E. We also showed that every fuzzy continuous function is necessarily fuzzy $\mathcal{I}$-continuous, thereby extending classical results from ideal topological spaces to the fuzzy setting. Our characterization of the topology induced by $\mathcal{I}$-convergence offers a more general framework for studying subsets compared to those based solely on ordinary fuzzy convergence. Moreover, we utilized the concept of fuzzy $\mathcal{I}$-convergence in a practical example involving temperature regulation in an industrial furnace, where sensor data is affected by noise. By integrating tools from fuzzy normed linear spaces and ideal theory, we demonstrated how abstract mathematical concepts such as asymptotic density, t-norms, and $\mathcal{I}$-convergence can be effectively applied to solve practical engineering problems involving uncertainty, noise, and outliers. We showed that by defining a suitable fuzzy norm and choosing an appropriate ideal $\mathcal{I}$ (such as subsets of natural numbers with asymptotic density zero), we can filter out sparse but large deviations in the data while preserving the convergence behavior of the majority of the sequence. This example demonstrated how fuzzy $\mathcal{I}$-convergence, rooted in advanced mathematical theory, provided a practical framework for handling uncertainty and noise in real-world data sets, particularly in fuzzy control systems and signal processing applications.

Future research could explore the development of analogs to classical functional analysis results—such as the Hahn–Banach theorem—within the framework of this newly considered topology. Another promising direction involves investigating the conditions under which fuzzy $\mathcal{I}$-limits of sequences are uniquely determined, especially in connection with the fuzzy $\mathcal{I}$-closedness of sequentially $\mathcal{I}$-compact subsets. The study of $\mathcal{I}$-convergence in fuzzy settings is deeply rooted in robust mathematical theory, drawing from fundamental concepts such as asymptotic density, ideal theory, and fuzzy norms. These tools provide a rigorous foundation for analyzing $\mathcal{I}$-convergence in environments characterized by uncertainty and imprecision. By appropriately selecting ideals, the method can be tailored to accommodate various types of noise and disturbances, offering a highly adaptable framework that can be fine-tuned for specific application areas. This flexibility positions fuzzy $\mathcal{I}$-convergence as a powerful tool for addressing complex problems across diverse real-world domains, including sensor networks, financial forecasting, medical diagnostics, and cybersecurity. Fuzzy $\mathcal{I}$-convergence proves particularly effective in scenarios where classical convergence methods face limitations, especially in the presence of rare but significant disturbances such as sensor failures or cyber intrusions. By focusing on the dominant trend in the data while efficiently filtering out sporadic anomalies, it maintains the accuracy and reliability of the analytical process, even under adverse or uncertain conditions.

It is worth noting, however, that as a mathematical framework rooted in fuzzy topology and generalized convergence, the applicability of the proposed method extends beyond the applications explicitly mentioned. In particular, it holds significant promise for fuzzy control systems—a major domain where uncertainty, imprecision, and dynamic adaptability are central challenges. For instance, recent advances in structural topology optimization have demonstrated the effectiveness of integrating fuzzy logic into control mechanisms, such as in the parameterized level set method (PLSM) based on reaction-diffusion equations and fuzzy PID control [42]. In such approaches, fuzzy control is employed to manage volume constraints robustly, ensuring stable convergence while enabling topological changes like hole generation with minimal sensitivity to initial designs. The synergy between fuzzy logic and evolutionary systems—where convergence behavior must be both flexible and resilient—suggests that fuzzy $\mathcal{I}$-convergence could provide a rigorous theoretical foundation for analyzing and enhancing such control processes, particularly in complex, high-dimensional, or uncertain environments.

Author Contributions

Abdelhak Razouki: Conceptualization, supervision; Omar El Ogri: Methodology, visualization of the application; Jaouad EL-Mekkaoui: Investigation, proof development; Zamir Ahmad Ansari: Formal analysis, validation; Naglaa F. Soliman: Resources, writing-review and editing; Abeer D. Algarni: Writing-original draft, investigation. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

There are no competing interests.

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