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Research article

On a parameterized inversion-free iterative algorithm for solving the nonlinear matrix equation $X + \sum_{i=1}^{m} A_i^* X^{-p_i} A_i = I$

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Abstract: In this paper, we propose a parameterized inversion-free iterative algorithm to compute the positive definite solution of the nonlinear matrix equation $X + \sum_{i=1}^{m} A_i^* X^{-p_i} A_i = I$. Then, we prove that the proposed algorithm converges. Finally, in comparison with three well-known existing algorithms, the accuracy and effectiveness of the proposed algorithm are demonstrated by some numerical examples.

Keywords: nonlinear matrix equations; positive definite solution; inversion-free iteration methods; convergence analysis
Mathematics Subject Classification: 15A24, 47H10, 65H05

1. Introduction

In this paper, we study the following nonlinear matrix equation:

$$X + \sum_{i=1}^{m} A_i^* X^{-p_i} A_i = I, \tag{1.1}$$

where $A_i \in \mathbb{C}^{n\times n}$ are given matrices, $0 < p_i \leq 1$, $i = 1, 2, \ldots, m$, A^* denotes the conjugate transpose of matrix A, I is the identity matrix, and X is an unknown $n \times n$ matrix to be determined.

This type of Eq (1.1) has wide applications in ladder networks [1], dynamic programming [2], control theory [3], stochastic filtering [4], and Nano research [5]. To solve some stochastic linear quadratic control problems [6], it is necessary to solve the stochastic algebraic Riccati equation. Given the widespread application of Eq (1.1) in natural science and engineering problems, researchers have conducted extensive research on Eq (1.1).

For the case where $m = 1$ and $p_i = 1$ in Eq (1.1), this equation has been extensively and deeply studied. The research results on this equation mainly include the following aspects: Sufficient conditions [7] for the existence of positive definite solutions of this equation; and iterative algorithms to solve positive definite solutions, such as basic fixed point iteration (BFPI) algorithms, Newton's iteration method (NIM) [8], etc. Different algorithms have their advantages and disadvantages, and fixed point iteration algorithms are relatively simple and easy to program. NIM generally has a quadratic convergence rate, thoug the computational complexity is high. Both BFPI and NIM involve matrix inversion operations. In practical problems, matrix inversion often has large errors and a high complexity. Given this, researchers have proposed a new iterative algorithm, the inversion-free iterative (IFI) algorithm, which does not involve matrix inversion operations in the iterative format. The inversion-free iterative algorithm to compute the maximum positive definite solution of this equation first appeared in Zhang's work [9]. Meanwhile, this algorithm was proven to be numerically stable and convergent. El Sayed and Al Dbiban proposed a new inversion-free iterative algorithm [10] to solve this equation and provided the convergence conditions and sufficient conditions for the existence of positive definite solutions. Additionally, Monsalve and Raydan [11] proposed a new inversion-free method to solve the equation, provided a convergence analysis, and discussed the stability properties. Zhang and Ding [12] proposed an inversion-free iterative algorithm to solve this equation based on the hierarchical identification principle and convergence factor.

Peng et al. [13] proposed a new inversion-free iterative algorithm to solve the minimal positive definite solution of equations $X + A^*X^{-\alpha}A = Q$, where $\alpha \geq 1$. Zhang [14] presented an inversion-free iterative method based on the damped Newton's method to solve $X + A^*X^{-\alpha}A = Q$, where $\alpha \geq 1$. The numerical results show that the proposed method can improve the numerical performance compared to the method proposed in [13].

For the case where $m = 2$ and $p_i = 1$ in Eq (1.1), Long and Hu [15] proposed an inversion-free iterative algorithm to solve this equation and proved that the algorithm converges. Hasanov and Ali [16] provided a new convergence result for the inversion-free iterative algorithm proposed by Long et al. [15]. Sayevand et al. [17] presented another inversion-free iterative algorithm to compute the maximal positive definite solution of this equation.

For the case where $m \geq 1$, $p_i = q$, and $q < 1$ in Eq (1.1), Yin et al. [18] proposed an inversion-free iterative algorithm to compute the maximal positive definite solution of the equation, alongside establishing necessary and sufficient conditions for the existence of positive definite solutions for this equation. For the case where $m \geq 1$ and $p_i = 1$, Huang and Ma [19] proposed some inversion-free iterative algorithms to solve this equation. In addition, Liu and Chen [20] presented a sufficient condition for the existence of a unique positive definite solution of the equation $X^s + \sum_{i=1}^{m} A_i^* X^{-t_i} A_i = Q$, where $0 < t_i \leq 1$, and designed an algorithm to solve this equation. Li and Zhang [21] provided a sufficient and necessary condition for the unique positive definite solution of Eq (1.1) using the integral representation of matrix functions and the properties of the Kronecker product.

In this paper, motivated by the iterative algorithm proposed in [22, 23], we propose a parameterized inversion-free iterative algorithm to solve Eq (1.1). This paper's contributions are outlined as follows:

- A parameterized inversion-free iterative algorithm is proposed to solve Eq (1.1);
- both the convergence analysis and the determination of convergence order are implemented for

the proposed algorithm;
- numerical experiments verify the superiority of the proposed algorithm compared with the BFPI, IFI, and NIM algorithms in some respects.

Throughout this paper, a positive definite solution means a symmetric positive definite solution. Let $\mathbb{C}^{n\times n}$, $\mathbb{H}^{n\times n}$, and $\mathbb{P}(n)$ denote the set of all $n \times n$ complex matrices, all $n \times n$ Hermitian matrices, and $n \times n$ Hermitian positive definite matrices, respectively. For $A \in \mathbb{C}^{n\times n}$, we use $\|A\|$ to denote the spectral norm of A (i.e., $\|A\| = \sqrt{\lambda_1(AA^*)}$), where $\lambda_1(AA^*)$ represents the maximum eigenvalue of the matrix AA^*. For $X, Y \in \mathbb{H}^{n\times n}, X \geq Y$ means $X - Y$ is positive semidefinite.

The rest of the paper is structured as follows: In Section 2, we propose a parameterized inversion-free iterative algorithm to solve Eq (1.1); in Section 3, we prove that the proposed iterative algorithm converges; in Section 4, we compare three well-known existing algorithms with the proposed iterative algorithm in some examples; and in Section 5, we conclude the paper.

2. A parameterized inversion-free iterative algorithm

In this section, we propose a parameterized inversion-free iterative algorithm to solve Eq (1.1). For this purpose, let $Y = X^{-1}$; then, Eq (1.1) is equivalent to the following:

$$Y^{-1} + \sum_{i=1}^{m} A_i^* Y^{p_i} A_i = I. \tag{2.1}$$

Then, by pre-multiplying and post-multiplying the two sides of Eq (2.1) by Y, we obtain the following equation:

$$Y - Y(I - \sum_{i=1}^{m} A_i^* Y^{p_i} A_i)Y = 0. \tag{2.2}$$

Thus, we write Eq (2.2) as follows:

$$Y = -Y(I - \sum_{i=1}^{m} A_i^* Y^{p_i} A_i)Y + 2Y. \tag{2.3}$$

Remark 2.1. *Evidently Y is the positive definite solution of Eq* (2.3) *if and only if Y is the positive definite solution of Eq* (2.1).

Therefore, in order to solve Eq (2.1), it is necessary to solve Eq (2.3). Based on the structure of Eq (2.3), the following iterative algorithm

$$\begin{cases} Z_n = I - \sum_{i=1}^{m} A_i^* Y_n^{p_i} A_i, \\ Y_{n+1} = (1+t)Y_n - tY_n Z_n Y_n, \ 0 < t \leq 1, \end{cases} \tag{2.4}$$

has been designed to solve Eq (2.3). The calculation process of the iterative algorithm (2.4) can be found in Algorithm 1.

Lemma 2.1. *[9] Let C and P be Hermitian matrices of the same order and $P > 0$. Then,*

$$CPC + P^{-1} \geq 2C. \tag{2.5}$$

Algorithm 1 An inversion-free iterative algorithm for solving Eq (2.1)

1: Input matrices $A_i \in \mathbb{C}^{n\times n}$, $i = 1, 2, \ldots m$.
2: Input initial guess $Y_0 = I$ and tolerance, tol > 0.
3: $n \Leftarrow 0$.
4: Calculate Y_{n+1} by iterative scheme (2.4).
5: If $\|Y_{n+1} - Y_n\| \leq$ tol, then stop and output Y_{n+1}. Otherwise, $n \Leftarrow n + 1$, continue to perform Step 4.

Lemma 2.2. *If Eq* (2.1) *has a positive definite solution Y, then $I \leq Y$.*

Proof. Suppose that Eq (2.1) has a positive definite solution $Y > 0$. According to Eq (2.1), we have the following:

$$\sum_{i=1}^{m} A_i^* Y^{p_i} A_i \geq 0.$$

Thus,

$$Y^{-1} = I - \sum_{i=1}^{m} A_i^* Y^{p_i} A_i \leq I,$$

which implies that $I \leq Y$. □

Lemma 2.3. *If Eq* (2.1) *has a positive definite solution Y, then $\sum_{i=1}^{m} A_i^* A_i < I$.*

Proof. Let Y is a positive definite solution of Eq (2.1), which yields $Y^{-1} \leq I$ and $I \leq Y$. Then, we have the following:

$$\sum_{i=1}^{m} A_i^* A_i \leq \sum_{i=1}^{m} A_i^* Y^{p_i} A_i = I - Y^{-1} < I.$$

The proof is completed. □

3. Convergence analysis

In this section, the following theorem indicates that the sequence $\{Y_n\}_{n\geq 0}$ generated by Algorithm (2.4) converges to the minimal positive definite solution of Eq (2.1).

Theorem 3.1. *Assume Eq* (2.1) *has a positive definite solution. Then, the sequence $\{Y_n\}_{n\geq 0}$ defined by Algorithm* (2.4) *is monotone increasing and converges to the minimal positive definite solution $\hat{Y}$ of Eq* (2.1).

Proof. Let Y be any positive definite solution of Eq (2.1). We claim that $0 < Y_n \leq Y_{n+1} \leq Y$, $\forall n \in \mathbb{N}$. For $n = 0$, we have $Y_0 = I > 0$. Using (2.4), we obtain the following:

$$\begin{aligned} Y_1 &= (1 + t)Y_0 - tY_0(I - \sum_{i=1}^{m} A_i^* Y_0^{p_i} A_i)Y_0 \\ &= (1 + t)I - t(I - \sum_{i=1}^{m} A_i^* A_i) \end{aligned}$$

$$
\begin{aligned}
&= I + t\sum_{i=1}^{m} A_i^* A_i \\
&> I = Y_0.
\end{aligned}
$$

According to Lemma 2.3, we immediately obtain $I - \sum_{i=1}^{m} A_i^* A_i > 0$; then, using Lemma 2.1, we have the following:

$$
\begin{aligned}
Y_1 &= (1+t)Y_0 - tY_0(I - \sum_{i=1}^{m} A_i^* Y_0^{p_i} A_i)Y_0 \\
&= (1-t)Y_0 + t\left[2Y_0 - Y_0(I - \sum_{i=1}^{m} A_i^* Y_0^{p_i} A_i)Y_0\right] \\
&\leq (1-t)I + t(I - \sum_{i=1}^{m} A_i^* Y^{p_i} A_i)^{-1} \\
&\leq (1-t)Y + tY \\
&\leq Y.
\end{aligned} \tag{3.1}
$$

From Lemma 2.2, the second inequality in (3.1) holds. Thus, $0 < Y_0 \leq Y_1 \leq Y$.

Next, we assume that $0 < Y_n \leq Y_{n+1} \leq Y$ holds for $n = k \geq 0$, because $I - \sum_{i=1}^{m} A_i^* Y_{k+1}^{p_i} A_i \geq I - \sum_{i=1}^{m} A_i^* Y^{p_i} A_i = Y^{-1} > 0$. According to Lemma 2.1, we have the following:

$$
\begin{aligned}
Y_{k+2} &= (1+t)Y_{k+1} - tY_{k+1}(I - \sum_{i=1}^{m} A_i^* Y_{k+1}^{p_i} A_i)Y_{k+1} \\
&= (1-t)Y_{k+1} + 2tY_{k+1} - tY_{k+1}(I - \sum_{i=1}^{m} A_i^* Y_{k+1}^{p_i} A_i)Y_{k+1} \\
&= (1-t)Y_{k+1} + t\left[2Y_{k+1} - Y_{k+1}(I - \sum_{i=1}^{m} A_i^* Y_{k+1}^{p_i} A_i)Y_{k+1}\right] \\
&\leq (1-t)Y_{k+1} + t(I - \sum_{i=1}^{m} A_i^* Y_{k+1}^{p_i} A_i)^{-1} \\
&\leq (1-t)Y + t(I - \sum_{i=1}^{m} A_i^* Y^{p_i} A_i)^{-1} \\
&= (1-t)Y + tY \\
&= Y.
\end{aligned} \tag{3.2}
$$

Therefore, $Y_{k+2} \leq Y$.

According to the assumption, we obtain $I - \sum_{i=1}^{m} A_i^* Y_k^{p_i} A_i \geq I - \sum_{i=1}^{m} A_i^* Y_{k+1}^{p_i} A_i \geq I - \sum_{i=1}^{m} A_i^* Y^{p_i} A_i = Y^{-1} > 0$. Using Lemma 2.1 and $Y_k > 0$, we have the following:

$$
\begin{aligned}
Y_{k+1} &= (1+t)Y_k - tY_k(I - \sum_{i=1}^{m} A_i^* Y_k^{p_i} A_i)Y_k \\
&= (1-t)Y_k + t\left[2Y_k - Y_k(I - \sum_{i=1}^{m} A_i^* Y_k^{p_i} A_i)Y_k\right] \\
&\leq (1-t)Y_k + t(I - \sum_{i=1}^{m} A_i^* Y_k^{p_i} A_i)^{-1} \\
&\leq (1-t)Y_k + t(I - \sum_{i=1}^{m} A_i^* Y_{k+1}^{p_i} A_i)^{-1}.
\end{aligned}
\tag{3.3}
$$

Thus,

$$
Y_{k+1}^{-1} \geq I - \sum_{i=1}^{m} A_i^* Y_{k+1}^{p_i} A_i. \tag{3.4}
$$

Next, we prove $Y_{k+1} \leq Y_{k+2}$. By (2.4), we have the following:

$$
\begin{aligned}
Y_{k+2} - Y_{k+1} &= (1+t)Y_{k+1} - tY_{k+1}(I - \sum_{i=1}^{m} A_i^* Y_{k+1}^{p_i} A_i)Y_{k+1} - Y_{k+1} \\
&= tY_{k+1} - tY_{k+1}(I - \sum_{i=1}^{m} A_i^* Y_{k+1}^{p_i} A_i)Y_{k+1} \\
&= Y_{k+1}\left[tY_{k+1}^{-1} - t(I - \sum_{i=1}^{m} A_i^* Y_{k+1}^{p_i} A_i)\right]Y_{k+1}.
\end{aligned}
$$

According to (3.4), we obtain $Y_{k+2} \geq Y_{k+1}$. Thus, we complete the proof which claims $0 < Y_n \leq Y_{n+1} \leq Y$ holds for all $n \in \mathbb{N}$. The sequence $\{Y_n\}$ is monotone increasing and bounded from above by Y; thus, we know the sequence exist limit $\hat{Y}$ and $\hat{Y} \leq Y$.

Taking the limit on both sides of the equality

$$
Y_{n+1} = (1+t)Y_n - tY_n(I - \sum_{i=1}^{m} A_i^* Y_n^{p_i} A_i)Y_n, \tag{3.5}
$$

we conclude that $\hat{Y}$ is the positive definite solution of Eq (2.1). Consequently, the proof of Theorem 3.1 is completed. □

Lemma 3.1. *[24, p. 305] Let $A, B \in \mathbb{P}^{n\times n}$ that are bounded below by a; i.e., $A \geq aI$ and $B \geq aI$ for the positive number a. Then for every unitarily invariant norm, we have*

$$
\|A^r - B^r\| \leq ra^{r-1}\|A - B\|,\ 0 < r < 1.
$$

The following theorem shows that Algorithm (2.4) linearly converges when the parameter t is set to 1.

Theorem 3.2. *If Y is the positive definite solution of Eq* (2.1)*, then with any $\epsilon > 0$, we have*

$$
\|Y - Y_{n+1}\| \leq \sum_{i=1}^{m} p_i(\|A_i Y\| + \epsilon)^2 \|Y - Y_n\|, \tag{3.6}
$$

$$\|Z_n - Y^{-1}\| \leq \sum_{i=1}^{m} p_i \|A_i\|^2 \|Y - Y_n\|, \tag{3.7}$$

for all n large enough.

Proof. Let Y be the solution of Eq (2.1) computed by Algorithm (2.4) when the parameter $t = 1$. Thus, we have the following:

$$\begin{aligned} Y_{n+1} &= 2Y_n - Y_n(I - \sum_{i=1}^{m} A_i^* Y_n^{p_i} A_i) Y_n \\ &= 2Y_n - Y_n[I - \sum_{i=1}^{m} A_i^*(Y^{p_i} + Y_n^{p_i} - Y^{p_i}) A_i] Y_n \\ &= 2Y_n - Y_n[I - \sum_{i=1}^{m} A_i^* Y^{p_i} A_i + \sum_{i=1}^{m} A_i^*(Y^{p_i} - Y_n^{p_i}) A_i] Y_n \\ &= 2Y_n - Y_n Y^{-1} Y_n - Y_n \left(\sum_{i=1}^{m} A_i^*(Y^{p_i} - Y_n^{p_i}) A_i \right) Y_n. \end{aligned} \tag{3.8}$$

Then,

$$\begin{aligned} Y - Y_{n+1} &= Y - 2Y_n + Y_n Y^{-1} Y_n + Y_n \left(\sum_{i=1}^{m} A_i^*(Y^{p_i} - Y_n^{p_i}) A_i \right) Y_n \\ &= (Y - Y_n) Y^{-1} (Y - Y_n) + Y_n \left(\sum_{i=1}^{m} A_i^*(Y^{p_i} - Y_n^{p_i}) A_i \right) Y_n. \end{aligned} \tag{3.9}$$

Because $Y_n \to Y$, as $n \to \infty$, we have the following:

$$\begin{aligned} \|Y - Y_{n+1}\| &\leq \|(Y - Y_n) Y^{-1} (Y - Y_n)\| + \|Y_n \left(\sum_{i=1}^{m} A_i^*(Y^{p_i} - Y_n^{p_i}) A_i \right) Y_n\| \\ &\leq \|(Y - Y_n) Y^{-1}\| \|Y - Y_n\| + \sum_{i=1}^{m} \|A_i Y_n\|^2 \|Y^{p_i} - Y_n^{p_i}\| \\ &\leq p_1 \left[\frac{1}{p_1} \|(Y - Y_n) Y^{-1}\| + \|A_1(Y_n - Y_s) + A_1 Y_s\|^2 \right] \|Y - Y_n\| \\ &+ \sum_{i=2}^{m} p_i \|A_i Y_n\|^2 \|Y - Y_n\| \\ &\leq p_1 \left[\frac{1}{p_1} \|(Y - Y_n) Y^{-1}\| + (\|A_1(Y_n - Y_s)\| + \|A_1 Y_s\|)^2 \right] \|Y - Y_n\| \\ &+ \sum_{i=2}^{m} p_i \|A_i Y_n\|^2 \|Y - Y_n\| \\ &\leq p_1 \left(\sqrt{\frac{1}{p_1} \|(Y - Y_n) Y^{-1}\|} + \|A_1(Y_n - Y_s)\| + \|A_1 Y_s\| \right)^2 \|Y - Y_n\| \end{aligned}$$

$$+\sum_{i=2}^{m} p_i\|A_iY_n\|^2\|Y-Y_n\|$$
$$\leq p_1(\|A_1Y\|+\epsilon)^2\|Y-Y_n\|+\sum_{i=2}^{m} p_i(\|A_iY\|+\epsilon)^2\|Y-Y_n\|.$$

Since

$$\begin{aligned} Z_n - Y^{-1} &= I-\sum_{i=1}^{m} A_i^*Y_n^{p_i}A_i-(I-\sum_{i=1}^{m} A_i^*Y^{p_i}A_i) \\ &= \sum_{i=1}^{m} A_i^*(Y^{p_i}-Y_n^{p_i})A_i, \end{aligned}$$

according to Lemma 3.1, we have the following:

$$\begin{aligned} \|Z_n - Y^{-1}\| &= \|\sum_{i=1}^{m} A_i^*(Y^{p_i}-Y_n^{p_i})A_i\| \\ &\leq \sum_{i=1}^{m} \|A_i^*(Y^{p_i}-Y_n^{p_i})A_i\| \\ &\leq \sum_{i=1}^{m} p_i\|A_i\|^2\|Y-Y_n\|. \end{aligned}$$

Thus, the proof is completed. □

4. Numerical experiments

In this section, we demonstrate the feasibility of Algorithm (2.4) to solve Eq (1.1) using some numerical examples. All programs were implemented in MATLAB, R2018a, on a Windows 10 computer equipped with an Intel i5 Processor (1.6 GHz) and 8 GB of RAM. Furthermore, we compare the performance of the proposed Algorithm (2.4) with the BFPI algorithm, the IFI algorithm, and the NIM for several numerical examples. We stop the iterations when $\|Y_{k+1}-Y_k\|_\infty \leq 1\times 10^{-8}$, and the output $X_{k+1}=Y_{k+1}^{-1}$. The infinite norm of the residue is as follows:

$$\mathtt{Res}(X_{k+1})=\|X_{k+1}+\sum_{i=1}^{m} A_i^*X_{k+1}^{-p_i}A_i-I\|_\infty.$$

Example 4.1. *We consider Eq* (1.1) *with* $m=2,\ p_1=p_2=0.5$*, along with the coefficient matrices*

$$A_1=\begin{pmatrix} 0.0165 & 0.0417 & 0.0508 \\ 0.0025 & 0.0400 & 0.0515 \\ 0.0620 & 0.0754 & 0.0716 \end{pmatrix},$$

$$A_2=\begin{pmatrix} 0.0403 & 0.0120 & 0.0245 \\ 0.0288 & 0.0443 & 0.0084 \\ 0.0091 & 0.0014 & 0.0489 \end{pmatrix}.$$

Using the iterative Algorithm (2.4)*, we take parameter* $t = \frac{1}{2}$ *and obtain the positive definite solution of Eq* (1.1) *after 22 iterations, as follows:*

$$X_{22} = Y_{22}^{-1} = \begin{pmatrix} 0.9933 & -0.0073 & -0.0072 \\ -0.0073 & 0.9887 & -0.0105 \\ -0.0072 & -0.0105 & 0.9864 \end{pmatrix},$$

where $\mathtt{Res}(X_{22}) = 5.2080 \times 10^{-9}$.

The numerical results with different parameter t for Example 4.1 are presented in Table 1. From Table 1, it becomes evident that the parameter t has a substantial impact on the performance of Algorithm (2.4). In some cases, selecting a value of t that exceeds 1 can enhance the efficiency of Algorithm (2.4). Although this paper solely establishes the convergence of Algorithm (2.4) when $0 < t \leq 1$, it is feasible to run Algorithm (2.4) with $t > 1$ in practical computations. This flexibility offers a broadened range of options to address practical problems.

Table 1. Numerical results of Algorithm (2.4) for Example 4.1.

parameter t	number of iteration	$\mathrm{Res}(X_{k+1})$
0.2	61	3.7854×10^{-8}
0.25	49	2.2267×10^{-8}
0.33	36	1.4813×10^{-8}
0.5	22	5.2080×10^{-9}
1	4	4.7044×10^{-11}
1.01	3	7.9185×10^{-11}
1.02	4	8.8916×10^{-12}
1.03	4	6.2927×10^{-11}

Example 4.2. *We consider Eq* (1.1) *with* $m = 2,\ p_1 = 0.5,\ p_2 = 0.2$*, along with the coefficient matrices*

$$A_1 = \begin{pmatrix} 0.0628 & 0.0444 & 0.0085 & 0.0588 & 0.0654 \\ 0.0278 & 0.0359 & 0.0666 & 0.0446 & 0.0104 \\ 0.0655 & 0.0465 & 0.0114 & 0.0127 & 0.0570 \\ 0.0201 & 0.0444 & 0.0022 & 0.0246 & 0.0430 \\ 0.0467 & 0.0119 & 0.0374 & 0.0307 & 0.0251 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0.0076 & 0.0090 & 0.0247 & 0.0138 & 0.0327 \\ 0.0171 & 0.0154 & 0.0106 & 0.0234 & 0.0104 \\ 0.0193 & 0.0233 & 0.0330 & 0.0043 & 0.0238 \\ 0.0048 & 0.0101 & 0.0393 & 0.0363 & 0.0009 \\ 0.0236 & 0.0116 & 0.0292 & 0.0352 & 0.0170 \end{pmatrix}.$$

Using the iterative Algorithm (2.4)*, we take parameter* $t = \frac{1}{2}$ *and obtain the positive definite solution* *of Eq* (1.1) *after 23 iterations, as follows:*

$$X_{23} = Y_{23}^{-1} = \begin{pmatrix} 0.9869 & -0.0096 & -0.0069 & -0.0095 & -0.0117 \\ -0.0096 & 0.9912 & -0.0058 & -0.0078 & -0.0095 \\ -0.0069 & -0.0058 & 0.9896 & -0.0081 & -0.0053 \\ -0.0095 & -0.0078 & -0.0081 & 0.9894 & -0.0085 \\ -0.0117 & -0.0095 & -0.0053 & -0.0085 & 0.9876 \end{pmatrix},$$

where $\texttt{Res}(X_{23}) = 4.9820 \times 10^{-9}$.

The numerical results with different parameter t for Example 4.2 are presented in Table 2.

Table 2. Numerical results of Algorithm (2.4) for Example 4.2.

parameter t	number of iteration	$\mathrm{Res}(X_{k+1})$
0.2	64	3.4968×10^{-8}
0.25	51	2.2742×10^{-8}
0.33	37	1.8162×10^{-8}
0.5	23	4.9820×10^{-9}
1	5	1.0069×10^{-11}

Based on the numerical results presented in Tables 1 and 2, as well as Figures 1 and 2, we conclude that the choice of parameter t significantly influences both the number of iterations and the computational accuracy. Specifically, as the value of parameter t approaches 1, the iterative Algorithm (2.4) experiences a reduction in the number of iterations, which is accompanied by an enhancement in accuracy.

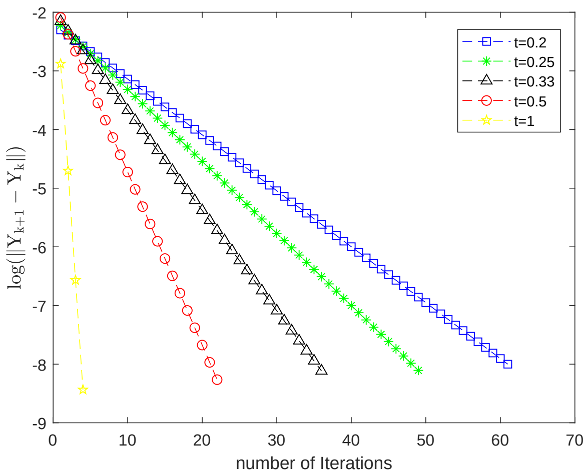

Figure 1. Numerical comparison of the parameterized inversion-free iterative Algorithm (2.4) with different parameter t for Example 4.1.

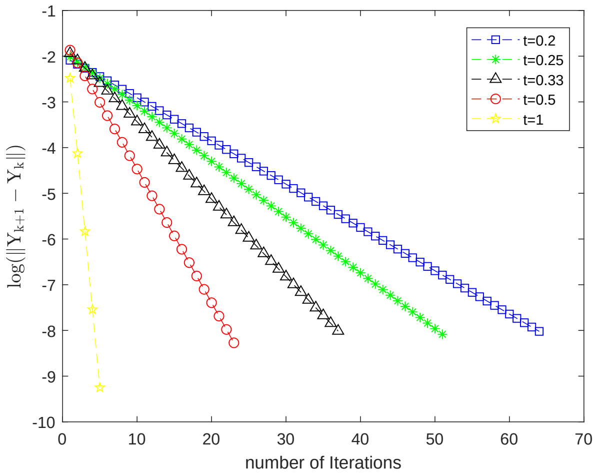

Figure 2. Numerical comparison of the parameterized inversion-free iterative Algorithm (2.4) with different parameter t for Example 4.2.

Example 4.3. *We consider Eq* (1.1) *with* $m = 2$, $p_1 = p_2 = 1$, *where the coefficient matrices are the matrices in Example 4.2. Using the iterative Algorithm* (2.4), *we take parameter* $t = 1$ *and obtain the positive definite solution of Eq* (1.1) *after 6 iterations, as follows:*

$$X_6 = Y_6^{-1} = \begin{pmatrix} 0.9866 & -0.0098 & -0.0072 & -0.0097 & -0.0119 \\ -0.0098 & 0.9910 & -0.0060 & -0.0080 & -0.0098 \\ -0.0072 & -0.0060 & 0.9894 & -0.0084 & -0.0055 \\ -0.0097 & -0.0080 & -0.0084 & 0.9891 & -0.0087 \\ -0.0119 & -0.0098 & -0.0055 & -0.0087 & 0.9874 \end{pmatrix},$$

where $\mathtt{Res}(X_6) = 4.2342 \times 10^{-11}$.

We report the number of iterations, computing time (denoted by time, measured in seconds), and residue in Table 3.

Table 3. Numerical comparison of the four algorithms for Example 4.3.

Algorithm	number of iterations	time	$\text{Res}(X_{k+1})$
BFPI	8	0.004318	9.6479×10^{-9}
IFI	10	0.004845	9.6547×10^{-9}
NIM	4	0.024453	8.8177×10^{-10}
Algorithm (2.4)	6	0.003268	4.2342×10^{-11}

Table 3 and Figure 3 demonstrate that the proposed Algorithm (2.4) to solve Eq (1.1) requires fewer iterations, a shorter computing time, and a higher computational accuracy than the BFPI and IFI algorithms. Algorithm (2.4) has a higher computational accuracy and a shorter computing time compared to NIM, but converges in more iterations. In summary, Algorithm (2.4) has advantages compared with the BFPI algorithm, the IFI algorithm, and NIM in some aspects.

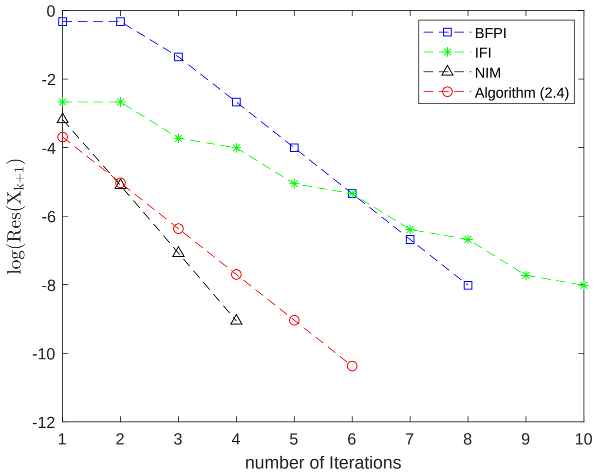

Figure 3. The comparison of convergence rates of the four algorithms for Example 4.3.

Example 4.4. *We consider Eq* (1.1) *with* $m = 2$, $p_1 = 0.5, p_2 = 0.3$, *along with two relatively larger coefficient matrices*

$$A_1 = \begin{pmatrix} 0.0464 & 0.0421 & 0.0167 & 0.0209 & 0.0655 & 0.0196 & 0.0495 & 0.0305 & 0.0066 & 0.0361 \\ 0.0084 & 0.0084 & 0.0595 & 0.0111 & 0.0268 & 0.0354 & 0.0283 & 0.0160 & 0.0549 & 0.0453 \\ 0.0087 & 0.0090 & 0.0469 & 0.0415 & 0.0414 & 0.0555 & 0.0286 & 0.0509 & 0.0117 & 0.0024 \\ 0.0062 & 0.0066 & 0.0370 & 0.0659 & 0.0103 & 0.0398 & 0.0083 & 0.0506 & 0.0109 & 0.0539 \\ 0.0005 & 0.0095 & 0.0123 & 0.0114 & 0.0254 & 0.0224 & 0.0016 & 0.0494 & 0.0444 & 0.0499 \\ 0.0282 & 0.0112 & 0.0141 & 0.0172 & 0.0107 & 0.0199 & 0.0193 & 0.0496 & 0.0596 & 0.0080 \\ 0.0437 & 0.0131 & 0.0052 & 0.0265 & 0.0505 & 0.0302 & 0.0212 & 0.0071 & 0.0344 & 0.0350 \\ 0.0482 & 0.0212 & 0.0609 & 0.0049 & 0.0581 & 0.0282 & 0.0436 & 0.0454 & 0.0468 & 0.0217 \\ 0.0354 & 0.0211 & 0.0471 & 0.0456 & 0.0234 & 0.0240 & 0.0638 & 0.0309 & 0.0102 & 0.0364 \\ 0.0073 & 0.0145 & 0.0372 & 0.0268 & 0.0457 & 0.0372 & 0.0624 & 0.0141 & 0.0636 & 0.0266 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0.0166 & 0.0231 & 0.0168 & 0.0178 & 0.0057 & 0.0383 & 0.0139 & 0.0247 & 0.0214 & 0.0354 \\ 0.0072 & 0.0176 & 0.0156 & 0.0024 & 0.0010 & 0.0106 & 0.0264 & 0.0230 & 0.0035 & 0.0360 \\ 0.0102 & 0.0103 & 0.0326 & 0.0347 & 0.0168 & 0.0370 & 0.0154 & 0.0212 & 0.0321 & 0.0250 \\ 0.0008 & 0.0301 & 0.0127 & 0.0252 & 0.0074 & 0.0090 & 0.0251 & 0.0110 & 0.0396 & 0.0055 \\ 0.0369 & 0.0091 & 0.0326 & 0.0142 & 0.0290 & 0.0149 & 0.0009 & 0.0099 & 0.0027 & 0.0087 \\ 0.0261 & 0.0026 & 0.0316 & 0.0399 & 0.0148 & 0.0035 & 0.0364 & 0.0181 & 0.0376 & 0.0073 \\ 0.0373 & 0.0307 & 0.0341 & 0.009 & 0.0337 & 0.0256 & 0.0320 & 0.0091 & 0.0007 & 0.0017 \\ 0.0065 & 0.0268 & 0.0202 & 0.0261 & 0.0294 & 0.0072 & 0.0298 & 0.0322 & 0.0274 & 0.0043 \\ 0.0368 & 0.0286 & 0.0254 & 0.0242 & 0.0228 & 0.0018 & 0.0325 & 0.0394 & 0.0313 & 0.0247 \\ 0.0318 & 0.0257 & 0.0380 & 0.0155 & 0.0071 & 0.0289 & 0.0153 & 0.0012 & 0.0214 & 0.0376 \end{pmatrix}.$$

Using the iterative Algorithm (2.4), *we take parameter* $t = 1$ *and obtain the positive definite solution* *of Eq* (1.1) *after 7 iterations, as follows:*

$$X_7 = Y_7^{-1} = \begin{pmatrix} 1.0176 & 0.0111 & 0.0172 & 0.0124 & 0.0177 & 0.0129 & 0.0171 & 0.0143 & 0.0143 & 0.0134 \\ 0.0111 & 1.0101 & 0.0128 & 0.0099 & 0.0128 & 0.0102 & 0.0138 & 0.0115 & 0.0115 & 0.0109 \\ 0.0172 & 0.0128 & 1.0274 & 0.0187 & 0.0213 & 0.0203 & 0.0229 & 0.0207 & 0.0215 & 0.0191 \\ 0.0124 & 0.0099 & 0.0187 & 1.0192 & 0.0156 & 0.0160 & 0.0174 & 0.0176 & 0.0162 & 0.0156 \\ 0.0177 & 0.0128 & 0.0213 & 0.0156 & 1.0244 & 0.0176 & 0.0216 & 0.0186 & 0.0193 & 0.0165 \\ 0.0129 & 0.0102 & 0.0203 & 0.0160 & 0.0176 & 1.0190 & 0.0171 & 0.0174 & 0.0175 & 0.0168 \\ 0.0171 & 0.0138 & 0.0229 & 0.0174 & 0.0216 & 0.0171 & 1.0255 & 0.0188 & 0.0206 & 0.0170 \\ 0.0143 & 0.0115 & 0.0207 & 0.0176 & 0.0186 & 0.0174 & 0.0188 & 1.0233 & 0.0196 & 0.0176 \\ 0.0143 & 0.0115 & 0.0215 & 0.0162 & 0.0193 & 0.0175 & 0.0206 & 0.0196 & 1.0273 & 0.0179 \\ 0.0134 & 0.0109 & 0.0191 & 0.0156 & 0.0165 & 0.0168 & 0.0170 & 0.0176 & 0.0179 & 1.0213 \end{pmatrix},$$

where $\mathtt{Res}(X_7) = 4.1333 \times 10^{-10}$.

5. Conclusions

In this paper, the proposed Algorithm (2.4) to solve Eq (1.1) is efficient in terms of the computational accuracy. Additionally, this algorithm delivers accurate results with fewer iterations and a shorter computing time compared with the BFPI and IFI algorithms. Although Algorithm (2.4) converges in more iterations than NIM, it has a higher computational accuracy and a shorter computing time than NIM.

Author contributions

Changzhou Li: Conceptualization, writing original draft, funding acquisition; Boyu Wang: Methodology; Chao Yuan: Software; Shiliang Chen: Validation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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