Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 10(8): 18997–19017.
DOI: 10.3934/math.2025849
Received: 21 May 2025
Revised: 06 August 2025
Accepted: 13 August 2025
Published: 21 August 2025

Research article

Analytic approximations for foreign equity options under a stochastic volatility with fast mean reversion

Jaegi Jeon[1] **and Geonwoo Kim**[2,*]

[1] Graduate School of Data Science, Chonnam National University, Gwangju 61186, Republic of Korea

[2] School of Natural Sciences, Seoul National University of Science and Technology, Seoul 01811, Republic of Korea

* **Correspondence:** Email: geonwoo@seoultech.ac.kr; Tel: +82029706271.

Abstract: In this paper, we investigate analytical approximations for foreign equity options under a stochastic volatility model with fast mean reversion. Foreign equity options are financial derivatives whose payoffs are determined by the price of an underlying asset and the foreign exchange rate, and they involve two currencies where the currency of the actual payoff differs from the currency of the underlying asset. We consider a stochastic volatility model to capture key stylized facts observed in financial markets, such as volatility clustering and the volatility smile, by modeling the volatilities of both the foreign asset and the exchange rate as rapidly mean-reverting processes. Specifically, we employ asymptotic expansion techniques to derive analytical pricing formulas for two types of options: Option struck in foreign currency and option struck in the domestic currency. This methodology provides accurate price approximations without complex numerical methods. Additionally, we provide some examples to show the importance of model parameters in foreign equity option pricing.

Keywords: foreign equity option; stochastic volatility; fast mean reversion; asymptotic expansion
Mathematics Subject Classification: 91G20

1. Introduction

Since the growth of global financial markets and trade liberalization, the foreign exchange risk arising from foreign equity price fluctuations has become an important concern for investors. To manage both the foreign exchange risk and equity fluctuation risk, foreign equity options have been proposed by industry practitioners and academic researchers. A foreign equity option is a financial derivative whose payoff is determined by the price of an underlying asset and the foreign exchange rate. In other words, there are two kinds of currencies involved, and the currency of the actual payoff differs from the currency of the underlying asset of a foreign equity option.

The advantage of foreign equity options, which consist of a combination of a foreign asset and an exchange rate, is to provide various flexible ways for investors to manage multidimensional risk in international markets. These instruments allow investors to gain exposure to foreign assets without bearing the exchange rate uncertainty at maturity in some cases, or with specific exchange rate exposures tailored to their risk management needs. For foreign equity option pricing, the dynamics of the underlying asset and the exchange rate should be assumed. Early works considered the Black–Scholes model for the dynamics, assuming that underlying assets follow geometric Brownian motions [1, 2]. Under the Black–Scholes model, Kwok and Wong [3] studied some exotic foreign equity options including path-dependent options. However, it is well known that the classical Black–Scholes model does not depict some features of actual financial data. Specifically, the Black–Scholes model cannot explain the volatility smile or skew of actual data and the asymmetric leptokurtic features in asset pricing [4, 5]. Stochastic volatility (SV) models offer several crucial advantages over the classical Black–Scholes framework. First, SV models can reproduce the volatility smile effect observed in market option prices, where the implied volatilities vary across strike prices and maturities, a phenomenon that the constant volatility Black–Scholes model fails to capture. Second, empirical studies have shown that asset returns exhibit volatility clustering, where periods of high volatility tend to be followed by high volatility and vice versa, which SV models naturally accommodate through mean-reverting volatility processes. Third, SV models generate more realistic return distributions with heavier tails and excess kurtosis, better matching the empirical properties of financial data. Finally, the correlation between asset returns and volatility changes, known as the leverage effect, can be incorporated in SV models but is absent in the Black–Scholes framework. These advantages make SV models particularly suitable for pricing path-dependent and exotic options, including foreign equity options where accurate modeling of the volatility dynamics is essential for capturing the complex interactions between equity and currency risks.

To overcome these problems, there have been many extended studies for the valuation of foreign equity options with models such as SV models, jump-diffusion models, regime-switching models, and others. Huang and Hung [6] extended the Black–Scholes model with jumps and studied the valuation of foreign equity options when the underlying assets follow a multi-dimensional Lévy process. Xu et al. [7] dealt with foreign equity option pricing under SV with double jumps. Sun and Xu [8] considered the time-changed Lévy processes as SV model and used the characteristic function approach to obtain the foreign equity option prices. We also contribute to the development of foreign equity option pricing models with SV model.

More sophisticated SV models have been proposed to better capture market dynamics. The seminal work by Heston [9] introduced a tractable stochastic volatility framework with closed-form solutions for European options, including currency derivatives. Extending Heston's approach, several studies have applied SV models to foreign equity options, offering improvements in capturing market dynamics. For instance, Dimitroff et al. [4] analyzed foreign equity options within Heston's SV framework, although their model assumed constant correlation, thereby restricting flexibility. Subsequently, Lee [10] provided explicit pricing formulas under a double square-root SV setting, incorporating more realistic volatility dynamics. Recent advances in option pricing have extended beyond basic SV models to incorporate multiple sources of risk and market frictions. He et al. [11] developed a closed-form solution for European options under Heston's model that additionally accounts for credit and liquidity risks, demonstrating how market imperfections can significantly impact option values. The literature has also seen significant progress in regime-switching models. He et al. [12] present a closed-form formula for European options that combines SV with regime-switching and stochastic market liquidity. Their approach highlights the importance of capturing structural changes in market conditions, which can be especially pronounced in foreign exchange (FX) markets during periods of economic uncertainty.

In addition to volatility modeling, recent literature has emphasized the importance of modeling the dynamic correlation between the underlying asset and FX rates. Ma [13] and Teng et al. [14] introduced stochastic correlation into foreign equity option pricing, enhancing realism at the expense of requiring more complex numerical methods. Similarly, Kim et al. [15] considered fat-tailed distributions and asymmetric dependence, while Pellegrino [16] further developed stochastic correlation models within SV frameworks.

This pursuit of greater realism, however, comes with critical trade-off. The increased complexity of these models often results in numerical challenges and substantial computational costs. An elegant solution to this dilemma is the asymptotic expansion methodology for SV models with fast mean-reverting volatility, pioneered by Fouque et al. [17, 18]. This approach provides highly accurate analytical approximations for option prices, offering significant computational speed advantages over numerical methods. Despite its proven success for standard European options, its application to the two-factor pricing problem of foreign equity options, particularly under a framework that allows for correlation among all sources of risk, remains underexplored in the literature.

The motivation for employing a dual-factor, fast mean-reverting SV framework is strongly rooted in empirical evidence. High-frequency data studies report that both equity index and FX rate volatilities exhibit extremely rapid decay. For instance, [19] estimate a volatility half-life of only 2–5 trading days for the S&P 500, while [20] document a similarly short horizon for USD/JPY realized volatility. Such findings lend strong support to a fast mean-reverting specification (i.e., $\epsilon \ll 1$). Furthermore, the short-run interaction between equity and FX volatilities is known to materially affect the payoffs of quanto derivatives, making it essential to model them jointly within a correlated framework.

This paper advances this line of research by presenting a comprehensive and tractable solution. While the existing multiscale literature has provided foundational tools, its application to foreign equity options has often been limited to simple quanto drift adjustments or corrections within a single-factor Heston-type framework. Our primary contribution is to derive the first closed-form $O(\epsilon)$ pricing formula for a foreign equity option under a dual fast mean-reverting SV model that incorporates a full cross-correlation structure. We extend Fouque's single-factor expansion to this cross-correlated dual-SV setting, obtaining compact correction operators that systematically embed the complex interactions between the equity and FX volatilities. The result is a highly accurate, computationally efficient, and practitioner-oriented pricing tool that captures the essential dynamics of foreign equity options, thereby filling a gap in the literature.

This paper proceeds as follows. Section 2 introduces the mathematical framework for our two-factor fast mean-reverting SV model and provides the detailed derivation of our asymptotic approximation formula. Numerical results demonstrating the accuracy and efficiency of our method are presented in Section 3. Finally, Section 4 summarizes our findings and offers directions for future research.

2. Model

In this section, we present the dual-factor fast mean-reverting volatility model for foreign equity options. We adopt this framework, developed by Fouque et al. [21] using singular perturbation methods, as an extension to the classic geometric Brownian motion-based quanto specification of [3]. The central idea of this model is that volatility reverts rapidly to its mean level over the lifetime of a derivative contract. This property is widely recognized for its ability to capture key empirical market phenomena, such as the smile and skew of volatility, which simpler models cannot explain. To describe the foreign equity option, let S_t be the price of equity in foreign currency and let F_t be the foreign currency exchange rate, representing the value of one unit of foreign currency in terms of the domestic currency. Accordingly, the price of the equity in the domestic currency is given by $S_t^* = F_t S_t$. Following [21], we consider the dynamics of S_t and F_t, along with their volatility-driving processes Y_t^S and Y_t^F, respectively:

$$dS_t = \mu_s S_t dt + g_s\left(Y_t^S\right) S_t d\tilde{W}_t^S, \tag{2.1}$$

$$dF_t = \mu_f F_t dt + g_f\left(Y_t^F\right) F_t d\tilde{W}_t^F, \tag{2.2}$$

$$dY_t^S = \frac{k_s}{\epsilon}\left(m_s - Y_t^S\right) dt + \frac{\sqrt{2}\nu_s}{\sqrt{\epsilon}} d\tilde{Z}_t^S, \tag{2.3}$$

$$dY_t^F = \frac{k_f}{\epsilon}\left(m_f - Y_t^F\right) dt + \frac{\sqrt{2}\nu_f}{\sqrt{\epsilon}} d\tilde{Z}_t^F. \tag{2.4}$$

Here, μ_s and μ_f are the constant drift rates of S_t and F_t, respectively. The volatilities $g_s\left(Y_t^S\right)$ and $g_f\left(Y_t^F\right)$ are functions driven by the Ornstein–Uhlenbeck (OU) processes Y_t^S and Y_t^F, respectively. The small parameter $\epsilon > 0$ signifies the fast mean-reverting property of these volatility processes. For the OU processes Y_t^S and Y_t^F, k_s and k_f are the parameters governing the speed of mean reversion to their respective long-term means, m_s and m_f. The terms ν_s and ν_f represent the constant volatilities (or vol-of-vol) of these OU processes. The overall rates of mean reversion are thus k_s/ϵ and k_f/ϵ. While the equity and FX processes are modeled by stochastic differential equations with the same structure for analytical tractability, it is important to emphasize that their dynamics are driven by distinct sets of parameters. This allows the model to flexibly capture the unique characteristics of each market (such as different drift rates, long-term volatility levels, and mean-reversion speeds) through the separate parameterizations for the S_t and F_t processes. The Wiener processes $\tilde{W}_t^S$, $\tilde{W}_t^F$, $\tilde{Z}_t^S$, and $\tilde{Z}_t^F$ are assumed to have the following instantaneous correlation structure:

$$\left\langle d\tilde{W}_t^S, d\tilde{W}_t^F \right\rangle = \rho_{sf} dt, \quad \left\langle d\tilde{W}_t^S, d\tilde{Z}_t^S \right\rangle = \rho_s dt, \quad \left\langle d\tilde{W}_t^F, d\tilde{Z}_t^F \right\rangle = \rho_f dt.$$

All other pairwise correlations of $d\tilde{W}_t^S, d\tilde{W}_t^F, d\tilde{Z}_t^S$, and $d\tilde{Z}_t^F$ are assumed to be zero.

In this study, we consider two types of quanto options.

(1) Foreign equity call option struck in the foreign currency

$$\text{Terminal payoff function}: h^{(f)}(s, f) = f\left(s - K_f\right)^+. \tag{2.5}$$

(2) Foreign equity call options struck in the domestic currency

$$\text{Terminal payoff function}: h^{(d)}(s,f) = (fs - K_d)^+. \tag{2.6}$$

Here, K_f and K_d are the strike prices in the foreign and domestic currency, respectively. Option (1) involves exercising the foreign equity in foreign currency and then converting the proceeds to the domestic currency, whereas Option (2) involves converting the foreign equity to domestic currency first and then exercising the option. To deal with these two types of quanto options, we need to consider two risk-neutral measures: The domestic risk-neutral measure Q_d and the foreign risk-neutral measure Q_f, respectively.

First, let $\delta^d_{S^*}$, δ^d_S, and δ^d_F denote the drift rates of S^*_t, S_t, and F_t under Q_d, respectively. It is obvious that $\delta^d_{S^*} = r_d - q$ and $\delta^d_F = r_d - r_f$, where r_d and r_f are the domestic and foreign risk-free rate, respectively, and q is the dividend yield of the asset. Using the relation $S^*_t = F_t S_t$ and applying Ito's lemma, we obtain

$$\delta^d_{S^*} = \delta^d_F + \delta^d_S + \rho_{sf} g_s\left(Y^S_t\right) g_f\left(Y^F_t\right).$$

The drift rate of S_t is then given by

$$\delta^d_S = r_f - q - \rho_{sf} g_s\left(Y^S_t\right) g_f\left(Y^F_t\right).$$

Moreover, using the quanto–prewashing technique introduced by [3] and [22], we obtain the drift rates of S^*_t, S_t, and F_t under Q_f (see Appendix 4 for a derivation):

$$\begin{aligned}
\delta^f_{S^*} &= r_d - q + g_f^2\left(Y^F_t\right) + \rho_{sf} g_s\left(Y^S_t\right) g_f\left(Y^F_t\right),\\
\delta^f_S &= r_f - q,\\
\delta^f_F &= r_d - r_f + g_f^2\left(Y^F_t\right).
\end{aligned}$$

Therefore, the dynamics of S_t and F_t are defined differently under Q_d and Q_f. For example, the dynamics in (2.1)–(2.4) under Q_d are given by

$$dS_t = \delta^d_S S_t dt + g_s\left(Y^S_t\right) S_t dW^S_t, \tag{2.7}$$

$$dF_t = \delta^d_F F_t dt + g_f\left(Y^F_t\right) F_t dW^F_t, \tag{2.8}$$

$$dY^S_t = \left[\frac{k_s}{\epsilon}\left(m_s - Y^S_t\right) - \frac{\sqrt{2}\nu_s}{\sqrt{\epsilon}}\Lambda^d_S\left(Y^S_t\right)\right] dt + \frac{\sqrt{2}\nu_s}{\sqrt{\epsilon}} dZ^S_t, \tag{2.9}$$

$$dY^F_t = \left[\frac{k_f}{\epsilon}\left(m_f - Y^F_t\right) - \frac{\sqrt{2}\nu_f}{\sqrt{\epsilon}}\Lambda^d_F\left(Y^F_t\right)\right] dt + \frac{\sqrt{2}\nu_f}{\sqrt{\epsilon}} dZ^F_t. \tag{2.10}$$

Next, let P^ϵ be the price of a foreign equity option. By the fundamental theorem of asset pricing, P^ϵ is given by

$$P^\epsilon\left(t,s,f,y_s,y_f\right) = \mathrm{E}^{Q_i}\left[e^{-r_i(T-t)} h^{(i)}\left(S_T, F_T\right) \middle| S_t = s, F_t = f, Y^S_t = y_s, Y^F_t = y_f\right],$$

for $i \in \{f, d\}$, where for $i = f$, $(Q_f, r_f, h^{(f)})$ is used with $h^{(f)}$ from (2.5), and for $i = d$, $(Q_d, r_d, h^{(d)})$ is used with $h^{(d)}$ from (2.6).

From the Feynman–Kac theorem, P^ϵ satisfies the following partial differential equations (PDEs):

$$\begin{aligned}\mathcal{L}^\epsilon P^\epsilon\left(t,s,f,y_s,y_f\right)&=0,\\ P^\epsilon\left(T,s,f,y_s,y_f\right)&=h^{(i)}(s,f),\end{aligned}\tag{2.11}$$

where

$$\mathcal{L}^\epsilon=\frac{1}{\epsilon}\mathcal{L}_0+\frac{1}{\sqrt{\epsilon}}\mathcal{L}_1^{(i)}+\mathcal{L}_2^{(i)},$$

and

$$\begin{aligned}\mathcal{L}_0&=k_s(m_s-y_s)\frac{\partial}{\partial y_s}+k_f\left(m_f-y_f\right)\frac{\partial}{\partial y_f}+\nu_s^2\frac{\partial^2}{\partial y_s^2}+\nu_f^2\frac{\partial^2}{\partial y_f^2}\\ \mathcal{L}_1^{(i)}&=-\sqrt{2}\nu_s\Lambda_s^i(y_s)\frac{\partial}{\partial y_s}-\sqrt{2}\nu_f\Lambda_f^i\left(y_f\right)\frac{\partial}{\partial y_f}+\sqrt{2}\nu_s\rho_s g_s(y_s)\,s\frac{\partial^2}{\partial s\partial y_s}+\sqrt{2}\nu_f\rho_f g_f\left(y_f\right)f\frac{\partial^2}{\partial f\partial y_f}\\ \mathcal{L}_2^{(i)}&=\frac{\partial}{\partial t}+\delta_S^i s\frac{\partial}{\partial s}+\delta_F^i f\frac{\partial}{\partial f}+\frac{1}{2}g_s^2(y_s)\,s^2\frac{\partial^2}{\partial s^2}+\frac{1}{2}g_f^2\left(y_f\right)f^2\frac{\partial^2}{\partial f^2}+\rho_{sf}g_s(y_s)\,g_f\left(y_f\right)sf\frac{\partial^2}{\partial s\partial f}-r_i\mathcal{I}.\end{aligned}$$

Here, for each type $i\in\{f,d\}$, Λ_S^i and Λ_F^i are the market prices of risk, δ_S^i and δ_F^i are the drift rates of S_t and F_t under the risk neutral measure Q_i, respectively, and $\mathcal{I}$ is the identity operator.

We now expand P^ϵ in powers of $\sqrt{\epsilon}$

$$P^\epsilon=P_0+\sqrt{\epsilon}P_1+\epsilon P_2+\cdots,\tag{2.12}$$

where P_0 is the leading-order term and $\tilde{P}_1^\epsilon:=\sqrt{\epsilon}P_1$ is the first-order correction term. We utilize the asymptotic method developed in [21] to establish the following theorem regarding the accuracy of this approximation.

Theorem 1. *Assume that the partial derivatives of $P^\epsilon(t,s,f,y_s,y_f)$ with respect to y_s and y_f do not grow exponentially as y_s or y_f goes to infinity. Then, there is a positive constant C, independent of ϵ, such that for a sufficiently small $\epsilon>0$, we have*

$$\left|P^\epsilon\left(t,s,f,y_s,y_f\right)-\left(P_0(t,s,f)+\tilde{P}_1^\epsilon(t,s,f)\right)\right|\le C\epsilon\,|\ln\epsilon|.\tag{2.13}$$

Proof. The proof of this theorem, which relies on arguments similar to those in [17] and [18], will be provided at the end of Section 2. □

This theorem indicates that the option price P^ϵ can be effectively approximated by the sum of the leading-order term P_0 and the first-order correction term $\tilde{P}_1^\epsilon$, with an error of order $\epsilon\,|\ln\epsilon|$. Building upon this result, we proceed to explicitly calculate the leading-order term P_0 for the two types of quanto options considered. This calculation is crucial, as P_0 captures the primary contribution to the option price, with the correction term $\tilde{P}_1^\epsilon$ refining the approximation.

Theorem 2. *The leading order term P_0 in* (2.13) *for the two types of quanto options is given as follows.*

(1) For the option struck in the foreign currency (Type (f))

$$P_0(t,s,f) = f\left[se^{-q(T-t)}N(d_1) - K_f e^{-r_f(T-t)}N(d_2)\right],$$

where $N(\cdot)$ is the standard normal cumulative distribution function, and

$$\begin{aligned} d_1 &= \frac{\ln\frac{s}{K_f} + \left(r_f - q + \frac{1}{2}\bar{\sigma}_s^2\right)(T-t)}{\bar{\sigma}_s\sqrt{T-t}}, \\ d_2 &= d_1 - \bar{\sigma}_s\sqrt{T-t}. \end{aligned}$$

(2) For the option struck in the domestic currency (Type (d))

$$P_0(t,s,f) = sfe^{-q(T-t)}N(b_1) - K_d e^{-r_d(T-t)}N(b_2),$$

where

$$\begin{aligned} b_1 &= \frac{\ln\frac{sf}{K_d} + \left(r_d - q + \frac{1}{2}\bar{\sigma}_*^2\right)(T-t)}{\bar{\sigma}_*\sqrt{T-t}}, \\ b_2 &= b_1 - \bar{\sigma}_*\sqrt{T-t}. \end{aligned} \tag{2.14}$$

The averaged parameters used in the expressions for P_0 above are defined as:

$$\bar{\sigma}_s = \sqrt{\langle g_s^2(y_s)\rangle}, \tag{2.15}$$

$$\bar{\sigma}_f = \sqrt{\left\langle g_f^2\left(y_f\right)\right\rangle}, \tag{2.16}$$

$$\bar{\rho}_{sf} = \frac{\rho_{sf}\left\langle g_s(y_s)\, g_f\left(y_f\right)\right\rangle}{\bar{\sigma}_s\bar{\sigma}_f}, \tag{2.17}$$

$$\bar{\sigma}_*^2 = \bar{\sigma}_s^2 + 2\bar{\rho}_{sf}\bar{\sigma}_s\bar{\sigma}_f + \bar{\sigma}_f^2. \tag{2.18}$$

Here, $\langle\cdot\rangle$ denotes the expectation with respect to the invariant distribution(s) of the stochastic volatility process(es) Y_t^S (and Y_t^F).

Proof. Inserting the expansion (2.12) to (2.11), the following equation is obtained:

$$\frac{1}{\epsilon}\mathcal{L}_0P_0 + \frac{1}{\sqrt{\epsilon}}\left(\mathcal{L}_0P_1 + \mathcal{L}_1^{(i)}P_0\right) + \left(\mathcal{L}_0P_2 + \mathcal{L}_1^{(i)}P_1 + \mathcal{L}_2^{(i)}P_0\right) + \sqrt{\epsilon}\left(\mathcal{L}_0P_3 + \mathcal{L}_1^{(i)}P_2 + \mathcal{L}_2^{(i)}P_1\right) + \cdots = 0. \tag{2.19}$$

This substitution yields a series of equations by collecting terms of the same order in $\sqrt{\epsilon}$. Specifically, the term of order $O(\epsilon^{-1})$ gives the equation $\frac{1}{\epsilon}\mathcal{L}_0P_0 = 0$, which implies $\mathcal{L}_0P_0 = 0$. For this equation to have a non-trivial solution P_0 that does not grow exponentially in the variables y_s and y_f (which would be undesirable in a practical context, as it implies unbounded option prices), we must assume that P_0 is independent of y_s and y_f, i.e., $P_0 = P_0(t,s,f)$.

Proceeding to the terms of order $O(\epsilon^{-1/2})$ in (2.19), we obtain the equation $\mathcal{L}_0P_1 + \mathcal{L}_1^{(i)}P_0 = 0$. Since P_0 is independent of y_s and y_f, the operator $\mathcal{L}_1^{(i)}$, which contains derivatives with respect to y_s and y_f

acting on P_0, yields $\mathcal{L}_1^{(i)} P_0 = 0$. Thus, the equation reduces to $\mathcal{L}_0 P_1 = 0$. For the same reason as for P_0, we conclude that P_1 is also independent of y_s and y_f, i.e., $P_1 = P_1(t, s, f)$.

Next, collecting the terms of order $O(1)$ in (2.19) and noting that $\mathcal{L}_1^{(i)} P_1 = 0$, we have the following equation for P_2:

$$\mathcal{L}_0 P_2 + \mathcal{L}_2^{(i)} P_0 = 0. \tag{2.20}$$

This is a Poisson equation for P_2 with respect to the operator $\mathcal{L}_0$. For Eq (2.20) to have a solution, a solvability condition must be satisfied. This condition states that the expectation of $-\mathcal{L}_2^{(i)} P_0$ with respect to the invariant distribution $\Phi(y_s, y_f)$ of the two-dimensional process (Y_t^S, Y_t^F) is zero. That is, $\left\langle \mathcal{L}_2^{(i)} P_0 \right\rangle = 0$, where the angle brackets $\langle \cdot \rangle$ denote this expectation.

Since Y_t^S and Y_t^F are assumed to be independent OU processes, their joint invariant distribution $\Phi(y_s, y_f)$ is the product of their respective individual invariant distributions. Each individual invariant distribution for an OU process is a normal distribution. Therefore, the invariant distribution Φ is given by:

$$\Phi\left(y_s, y_f\right) = \prod_{j=s,f} \frac{1}{\sqrt{2\pi \tilde{v}_j^2}} \exp\left\{ -\frac{\left(y_j - m_j\right)^2}{2\tilde{v}_j^2} \right\},$$

where $\tilde{v}_j^2 = v_j^2 / k_j$.

Since P_0 does not depend on y_s and y_f, the solvability condition $\left\langle \mathcal{L}_2^{(i)} P_0 \right\rangle = 0$ becomes $\left\langle \mathcal{L}_2^{(i)} \right\rangle P_0(t, s, f) = 0$. This yields the following PDE for P_0:

$$\begin{aligned} \left\langle \mathcal{L}_2^{(i)} \right\rangle P_0\left(t, s, f\right) &= 0, \\ P_0\left(T, s, f\right) &= h^{(i)}\left(s, f\right), \end{aligned} \tag{2.21}$$

for each option type $i \in \{f, d\}$, where $h^{(i)}(s, f)$ is the terminal payoff function for the i-th type of option, as defined in Eqs (2.5) and (2.6). Here, the averaged operator $\left\langle \mathcal{L}_2^{(i)} \right\rangle$ for each of the two option types is as follows.

For $i = f$ (option struck in the foreign currency)

$$\begin{aligned} \left\langle \mathcal{L}_2^{(f)} \right\rangle = &\frac{\partial}{\partial t} + \left(r_f - q\right) s \frac{\partial}{\partial s} + \left(r_d - r_f + \bar{\sigma}_f^2\right) f \frac{\partial}{\partial f} \\ &+ \frac{1}{2}\bar{\sigma}_s^2 s^2 \frac{\partial^2}{\partial s^2} + \frac{1}{2}\bar{\sigma}_f^2 f^2 \frac{\partial^2}{\partial f^2} + \bar{\rho}_{sf}\bar{\sigma}_s\bar{\sigma}_f s f \frac{\partial^2}{\partial s \partial f} - r_f \mathcal{I}. \end{aligned}$$

For $i = d$ (option struck in the domestic currency)

$$\begin{aligned} \left\langle \mathcal{L}_2^{(d)} \right\rangle = &\frac{\partial}{\partial t} + \left(r_f - q - \bar{\rho}_{sf}\bar{\sigma}_s\bar{\sigma}_f\right) s \frac{\partial}{\partial s} + \left(r_d - r_f\right) f \frac{\partial}{\partial f} \\ &+ \frac{1}{2}\bar{\sigma}_s^2 s^2 \frac{\partial^2}{\partial s^2} + \frac{1}{2}\bar{\sigma}_f^2 f^2 \frac{\partial^2}{\partial f^2} + \bar{\rho}_{sf}\bar{\sigma}_s\bar{\sigma}_f s f \frac{\partial^2}{\partial s \partial f} - r_d \mathcal{I}. \end{aligned}$$

Here, $\bar{\sigma}_s$, $\bar{\sigma}_f$, and $\bar{\rho}_{sf}$ are the averaged volatility and correlation parameters defined in Eqs (2.15), (2.16), and (2.17), respectively.

Since the operator $\left\langle \mathcal{L}_2^{(i)} \right\rangle$ for each type i can be considered a two-dimensional Black–Scholes-type operator, and the pricing formulas for quanto options under such Black–Scholes frameworks are well–established, we can directly obtain the leading-order term P_0 for each type of option, as provided in Theorem 2. □

Having established the leading-order term P_0, we now turn our attention to the correction term $\tilde{P}_1^\epsilon$. This correction accounts for the effects of the SV that are not captured by the leading-order approximation.

Theorem 3. *The first-order correction term $\tilde{P}_1^\epsilon$ ($= \sqrt{\epsilon}P_1$) in (2.13) for the two types of quanto options ($i \in \{f, d\}$) is derived as an explicit expression as follows.*

(1) For the option struck in the foreign currency (Type (f))

$$\tilde{P}_1^\epsilon (t, s, f) = -(T-t)\,\mathcal{H}_{(f)}^\epsilon P_0 (t, s, f),$$

where $P_0(t, s, f)$ is the leading-order term for the Type (f) option (from Theorem 2), and

$$\begin{aligned}\mathcal{H}_{(f)}^\epsilon = &- V_s\mathcal{D}_s^3 - \left(2V_s + V_{sf} + W_s^{(f)}\right)\mathcal{D}_s^2 \\ &- \left(V_{sf} + V_{fs} + W_{sf}^{(f)} + W_{fs}^{(f)}\right)\mathcal{D}_s \\ &- 2\left(V_f + W_f^{(f)}\right)\mathcal{I}.\end{aligned}$$

(2) For the option struck in domestic currency (Type (d))

$$\tilde{P}_1^\epsilon (t, s, f) = -(T-t)\,\mathcal{H}_{(d)}^\epsilon \Gamma_{BS} (t, sf; K_d, r_d, \bar{\sigma}_*),$$

where Γ_{BS} is the Black–Scholes gamma for the corresponding underlying sf:

$$\Gamma_{BS} (t, sf; K_d, r_d, \bar{\sigma}_*) = e^{-q(T-t)} \frac{\varphi(b_1)}{sf\bar{\sigma}_* \sqrt{T-t}}, \tag{2.22}$$

where b_1 is defined as (2.14), and $\varphi(\cdot)$ is the standard normal probability density function. The operator $\mathcal{H}_{(d)}^\epsilon$ is defined as:

$$\begin{aligned}\mathcal{H}_{(d)}^\epsilon = &\left(-W_s^{(d)} - W_f^{(d)} + W_{sf}^{(d)} + W_{fs}^{(d)}\right)s^2 f^2 \mathcal{I} \\ &- \mathcal{D}_s\left(V_s - V_{sf}\right) - \mathcal{D}_f\left(V_f - V_{fs}\right).\end{aligned}$$

Here, the differential operators $\mathcal{D}_s^n$ and $\mathcal{D}_f^m$ are defined by

$$\begin{aligned}\mathcal{D}_s^n &= s^n \frac{\partial^n}{\partial s^n}, \\ \mathcal{D}_f^m &= f^m \frac{\partial^m}{\partial f^m}\end{aligned}$$

for $n, m \in \mathbb{N}$. The group parameters appearing in the operators $\mathcal{H}_{(f)}^\epsilon$ and $\mathcal{H}_{(d)}^\epsilon$ are given by

$$V_s = -\frac{\sqrt{\epsilon}}{\sqrt{2}}\nu_s\rho_s\left\langle g_s\left(y_s\right)\phi_s'\left(y_s\right)\right\rangle, \tag{2.23}$$

$$V_f = -\frac{\sqrt{\epsilon}}{\sqrt{2}}\nu_f\rho_f\left\langle g_f\left(y_f\right)\phi_f'\left(y_f\right)\right\rangle, \tag{2.24}$$

$$V_{sf} = -\sqrt{2\epsilon}\nu_s\rho_s\left\langle g_s\left(y_s\right)\frac{\partial}{\partial y_s}\phi_{sf}\left(y_s, y_f\right)\right\rangle, \tag{2.25}$$

$$V_{fs} = -\sqrt{2\epsilon}\nu_f\rho_f\left\langle g_f\left(y_f\right)\frac{\partial}{\partial y_f}\phi_{sf}\left(y_s, y_f\right)\right\rangle, \tag{2.26}$$

$$W_s^{(i)} = \frac{\sqrt{\epsilon}}{\sqrt{2}}\nu_s\left\langle\Lambda_s^{(i)}\left(y_s\right)\phi_s'\left(y_s\right)\right\rangle, \tag{2.27}$$

$$W_f^{(i)} = \frac{\sqrt{\epsilon}}{\sqrt{2}}\nu_f\left\langle\Lambda_f^{(i)}\left(y_f\right)\phi_f'\left(y_f\right)\right\rangle, \tag{2.28}$$

$$W_{sf}^{(i)} = \sqrt{2\epsilon}\nu_s\left\langle\Lambda_s^{(i)}\left(y_s\right)\frac{\partial}{\partial y_s}\phi_{sf}\left(y_s, y_f\right)\right\rangle, \tag{2.29}$$

$$W_{fs}^{(i)} = \sqrt{2\epsilon}\nu_f\left\langle\Lambda_f^{(i)}\left(y_f\right)\frac{\partial}{\partial y_f}\phi_{sf}\left(y_s, y_f\right)\right\rangle, \tag{2.30}$$

for $i \in \{f, d\}$. The functions $\phi_s(y_s)$, $\phi_f(y_f)$, and $\phi_{sf}(y_s, y_f)$ are solutions of the following Poisson equations, respectively:

$$\mathcal{L}_0\phi_s\left(y_s\right) = g_s^2\left(y_s\right) - \bar{\sigma}_s^2, \tag{2.31}$$

$$\mathcal{L}_0\phi_f\left(y_f\right) = g_f^2\left(y_f\right) - \bar{\sigma}_f^2, \tag{2.32}$$

$$\mathcal{L}_0\phi_{sf}\left(y_s, y_f\right) = \rho_{sf}g_s\left(y_s\right)g_f\left(y_f\right) - \bar{\rho}_{sf}\bar{\sigma}_s\bar{\sigma}_f. \tag{2.33}$$

Proof. The derivation of the first-order correction term $\tilde{P}_1^\epsilon$ begins with the terms of order $O(\sqrt{\epsilon})$ in the PDE expansion (2.19). This gives the following equation for P_3:

$$\mathcal{L}_0P_3 + \mathcal{L}_1^{(i)}P_2 + \mathcal{L}_2^{(i)}P_1 = 0.$$

For this equation to have a solution P_3, the following solvability condition must hold:

$$\left\langle\mathcal{L}_1^{(i)}P_2\right\rangle + \left\langle\mathcal{L}_2^{(i)}\right\rangle P_1 = 0. \tag{2.34}$$

Recall from (2.20) that $\mathcal{L}_0P_2 = -\mathcal{L}_2^{(i)}P_0$. Since $\left\langle\mathcal{L}_2^{(i)}\right\rangle P_0 = 0$, this can be written as $\mathcal{L}_0P_2 = -\left(\mathcal{L}_2^{(i)} - \left\langle\mathcal{L}_2^{(i)}\right\rangle\right)P_0$. The specific forms of P_2 for each option type $i \in \{f, d\}$, involving the functions ϕ_s, ϕ_f, and ϕ_{sf} (solutions to Poisson equations (2.31)–(2.33)) and the integration constants c_s, c_f, and c_{sf} (which depend on t, s, f but are independent of y_s, y_f), are given by

For $i = f$ (foreign currency strike)

$$P_2 = -\mathcal{L}_0^{-1}\left(\mathcal{L}_2^{(f)} - \left\langle\mathcal{L}_2^{(f)}\right\rangle\right)P_0$$

$$= -\mathcal{L}_0^{-1}\left\{\left(g_f^2\left(y_f\right) - \bar{\sigma}_f^2\right) f\frac{\partial}{\partial f} + \frac{1}{2}\left(g_s^2\left(y_s\right) - \bar{\sigma}_s^2\right) s^2\frac{\partial^2}{\partial s^2}\right.$$
$$\left. + \frac{1}{2}\left(g_f^2\left(y_f\right) - \bar{\sigma}_f^2\right) f^2\frac{\partial^2}{\partial f^2} + \left(\rho_{sf} g_s\left(y_s\right) g_f\left(y_f\right) - \bar{\rho}_{sf}\bar{\sigma}_s\bar{\sigma}_f\right) sf\frac{\partial^2}{\partial s\partial f}\right\} P_0$$
$$= -\frac{1}{2}\left(\phi_s\left(y_s\right) + c_s\left(t, s, f\right)\right) s^2\frac{\partial^2 P_0}{\partial s^2} - \frac{1}{2}\left(\phi_f\left(y_f\right) + c_f\left(t, s, f\right)\right)\left(2f\frac{\partial P_0}{\partial f} + f^2\frac{\partial^2}{\partial f^2}\right)$$
$$- \left(\phi_{sf}\left(y_s, y_f\right) + c_{sf}\left(t, s, f\right)\right) sf\frac{\partial^2 P_0}{\partial s\partial f}, \tag{2.35}$$

For $i = d$ (domestic currency strike)

$$P_2 = -\mathcal{L}_0^{-1}\left(\mathcal{L}_2^{(d)} - \left\langle\mathcal{L}_2^{(d)}\right\rangle\right) P_0$$
$$= -\mathcal{L}_0^{-1}\left\{\left(-\rho_{sf} g_s\left(y_s\right) g_f\left(y_f\right) + \bar{\rho}_{sf}\bar{\sigma}_s\bar{\sigma}_f\right) s\frac{\partial}{\partial s} + \frac{1}{2}\left(g_s^2\left(y_s\right) - \bar{\sigma}_s^2\right) s^2\frac{\partial^2}{\partial s^2}\right.$$
$$\left. + \frac{1}{2}\left(g_f^2\left(y_f\right) - \bar{\sigma}_f^2\right) f^2\frac{\partial^2}{\partial f^2} + \left(\rho_{sf} g_s\left(y_s\right) g_f\left(y_f\right) - \bar{\rho}_{sf}\bar{\sigma}_s\bar{\sigma}_f\right) sf\frac{\partial^2}{\partial s\partial f}\right\} P_0$$
$$= -\frac{1}{2}\left(\phi_s\left(y_s\right) + c_s\left(t, s, f\right)\right) s^2\frac{\partial^2 P_0}{\partial s^2} - \frac{1}{2}\left(\phi_f\left(y_f\right) + c_f\left(t, s, f\right)\right) f^2\frac{\partial^2 P_0}{\partial f^2}$$
$$- \left(\phi_{sf}\left(y_s, y_f\right) + c_{sf}\left(t, s, f\right)\right)\left(sf\frac{\partial^2 P_0}{\partial s\partial f} - s\frac{\partial P_0}{\partial s}\right), \tag{2.36}$$

The terms c_s, c_f, and c_{sf} in the expressions for P_2 (corresponding to each option type $i \in \{f, d\}$) are functions of integration that depend only on (t, s, f). Since the operator $\mathcal{L}_1^{(i)}$ involves derivatives with respect to y_s and y_f, these c terms, being independent of y_s and y_f, do not contribute to $\langle\mathcal{L}_1^{(i)} P_2^{(i)}\rangle$ after the averaging $\langle\cdot\rangle$ is performed; for this reason, the explicit type-dependent superscripts on c_s, c_f, and c_{sf} were omitted.

Note that ϕ_s and ϕ_f are functions of a single variable (y_s and y_f, respectively). This simplification arises from the spectral theory for the operator $\mathcal{L}_0$ and the assumed independence of the OU processes Y_t^S and Y_t^F. For a detailed argument, we refer to [23].

From the solvability condition (2.34), we have $\left\langle\mathcal{L}_2^{(i)}\right\rangle P_1 = -\left\langle\mathcal{L}_1^{(i)} P_2\right\rangle$. Multiplying by $\sqrt{\epsilon}$ and substituting P_2 in (2.35) and (2.36) leads to the following PDE for $\tilde{P}_1^{\epsilon}$:

$$\left\langle\mathcal{L}_2^{(i)}\right\rangle\tilde{P}_1^{\epsilon}\left(t, s, f\right) = \mathcal{H}_{(i)}^{\epsilon} P_0\left(t, s, f\right),$$
$$P_1^{\epsilon}\left(T, s, f\right) = 0,$$

where

$$\mathcal{H}_{(f)}^{\epsilon} = -\left(W_s^{(f)}\mathcal{D}_s^2 + W_{sf}^{(f)}\mathcal{D}_s\mathcal{D}_f\right)$$
$$- \left(W_f^{(f)}\left(2\mathcal{D}_f + \mathcal{D}_f^2\right) + W_{fs}^{(f)}\mathcal{D}_s\mathcal{D}_f\right)$$
$$- \mathcal{D}_s\left(V_s\mathcal{D}_s^2 + V_{sf}\mathcal{D}_s\mathcal{D}_f\right)$$
$$- \mathcal{D}_f\left(V_f\left(2\mathcal{D}_f + \mathcal{D}_f^2\right) + V_{fs}\mathcal{D}_s\mathcal{D}_f\right),$$

and

$$
\begin{aligned}
\mathcal{H}_{(d)}^{\epsilon} = &- \left(W_s^{(d)}\mathcal{D}_s^2 + W_{sf}^{(d)}\left(\mathcal{D}_s\mathcal{D}_f - \mathcal{D}_s\right)\right) \\
&- \left(W_f^{(d)}\mathcal{D}_f^2 + W_{fs}^{(d)}\left(\mathcal{D}_s\mathcal{D}_f - \mathcal{D}_s\right)\right) \\
&- \mathcal{D}_s\left(V_s\mathcal{D}_s^2 + V_{sf}\left(\mathcal{D}_s\mathcal{D}_f - \mathcal{D}_s\right)\right) \\
&- \mathcal{D}_f\left(V_f\mathcal{D}_f^2 + V_{fs}\left(\mathcal{D}_s\mathcal{D}_f - \mathcal{D}_s\right)\right).
\end{aligned}
$$

Since the operators $\mathcal{D}_s^n$ and $\mathcal{D}_f^m$ is commutative, the two-dimensional Black–Scholes operator $\left\langle \mathcal{L}_2^{(i)} \right\rangle$ and $\mathcal{H}_{(i)}^{\epsilon}$ are also commutative. Therefore, we can deduce the explicit form of $\tilde{P}_1^{\epsilon}$ given by

$$
\tilde{P}_1^{\epsilon}(t,s,f) = -(T-t)\,\mathcal{H}_{(i)}^{\epsilon}P_0(t,s,f),
$$

because $\tilde{P}(T,s,f) = 0$ and

$$
\begin{aligned}
\left\langle \mathcal{L}_2^{(i)} \right\rangle P_1^{\epsilon}(t,s,f) &= \left\langle \mathcal{L}_2^{(i)} \right\rangle \left(-(T-t)\,\mathcal{H}_{(i)}^{\epsilon}P_0(t,s,f)\right) \\
&= \mathcal{H}_{(i)}^{\epsilon}P_0(t,s,f) - (T-t)\left\langle \mathcal{L}_2^{(i)} \right\rangle \mathcal{H}_{(i)}^{\epsilon}P_0(t,s,f) \\
&= \mathcal{H}_{(i)}^{\epsilon}P_0(t,s,f) - (T-t)\,\mathcal{H}_{(i)}^{\epsilon}\left\langle \mathcal{L}_2^{(i)} \right\rangle P_0(t,s,f) \\
&= \mathcal{H}_{(i)}^{\epsilon}P_0(t,s,f).
\end{aligned}
$$

So, the explicit formula for the first-order correction term $\tilde{P}_1^{\epsilon}$ for each type of option ($i \in \{f, d\}$) is as follows:

(1) For the option of Type (f) with the payoff $h^{(f)}(S_T, F_T) = F_T\left(S_T - K_f\right)^+$: Since we can rewrite $P_0(t,s,f) = fC_{BS}\left(t, s; K_f, \bar{\sigma}_s, r_f, q\right)$, where $C_{BS}(t,s;K,\sigma,r,q)$ denotes the Black–Scholes call price with the strike K, the volatility σ, the interest rate r, and the dividend yield q, the following formulas hold:

$$
\mathcal{D}_f P_0 = P_0, \qquad \mathcal{D}_f^2 P_0 = 0.
$$

Therefore,

$$
\begin{aligned}
\mathcal{H}_{(f)}^{\epsilon} = &- V_s\mathcal{D}_s^3 - \left(2V_s + V_{sf} + W_s^{(f)}\right)\mathcal{D}_s^2 \\
&- \left(V_{sf} + V_{fs} + W_{sf}^{(f)} + W_{fs}^{(f)}\right)\mathcal{D}_s \\
&- 2\left(V_f + W_f^{(f)}\right)\mathcal{I}.
\end{aligned}
$$

(2) For the option of Type (d) with $h^{(d)}(S_T, F_T) = (F_T S_T - K_d)^+$: In this case, $P_0(t,s,f)$ is expressed simply as $C_{BS}(t, sf; K_d, \bar{\sigma}_*, r_d, q)$. Using the chain rule with $v = sf$, we have

$$
\begin{aligned}
\left(\mathcal{D}_s\mathcal{D}_f - \mathcal{D}_s\right)P_0 &= -s^2 f^2 \Gamma_{BS}(t, sf; K_d, r_d, \bar{\sigma}_*), \\
\mathcal{D}_s^2 P_0 = \mathcal{D}_f^2 P_0 &= s^2 f^2 \Gamma_{BS}(t, sf; K_d, r_d, \bar{\sigma}_*),
\end{aligned}
$$

where Γ_{BS} is the Black–Scholes Gamma in (2.22). Hence,

$$
P_1^{\epsilon}(t,s,f) = -(T-t)\,\mathcal{H}_{(d)}^{\epsilon}\Gamma_{BS}(t, sf; K_d, r_d, \bar{\sigma}_*),
$$

where

$$\mathcal{H}^{\epsilon}_{(d)} = \left(-W^{(d)}_{s} - W^{(d)}_{f} + W^{(d)}_{sf} + W^{(d)}_{fs}\right) s^2 f^2 \mathcal{I} \\ - \left(V_s - V_{sf}\right) \mathcal{D}_s - \left(V_f - V_{fs}\right) \mathcal{D}_f$$

□

Having derived the explicit formulas for both the leading-order term P_0 and the first-order correction term $\tilde{P}^{\epsilon}_{1}$, we now return to provide the proof for Theorem 1, which establishes the accuracy of our overall approximation $P_0 + \tilde{P}^{\epsilon}_{1}$.

Proof of Theorem 1. The argument for establishing the order of accuracy follows the general framework of asymptotic analysis, as detailed in, for example, [18] and [24]. Recall that the price P^{ϵ} for a quanto option solves the PDE (2.11) with the terminal payoff $h^{(i)}$ for each type $i \in \{f, d\}$, which is assumed to be continuous and piecewise smooth.

Let $\hat{P}^{\epsilon} = P_0 + \tilde{P}^{\epsilon}_{1} + \epsilon P_2 + \epsilon\sqrt{\epsilon} P_3$. The remainder term R^{ϵ} is then defined by:

$$\mathcal{R}^{\epsilon} = P^{\epsilon} - \hat{P}^{\epsilon}. \tag{2.37}$$

We assume that the payoff function $h^{(i)}(s, f)$ for P^{ϵ} and its derivatives are smooth and bounded. Substituting $P^{\epsilon} = \mathcal{R}^{\epsilon} + \hat{P}^{\epsilon}$ into $\mathcal{L}^{\epsilon} P^{\epsilon} = 0$ (from (2.11)) yields:

$$0 = \mathcal{L}^{\epsilon}\mathcal{R}^{\epsilon} + \frac{1}{\epsilon}\mathcal{L}_0 P_0 + \frac{1}{\sqrt{\epsilon}}(\mathcal{L}_0 P_1 + \mathcal{L}^{(i)}_{1} P_0) + (\mathcal{L}_0 P_2 + \mathcal{L}^{(i)}_{1} P_1 + \mathcal{L}^{(i)}_{2} P_0) \\ + \sqrt{\epsilon}(\mathcal{L}_0 P_3 + \mathcal{L}^{(i)}_{1} P_2 + \mathcal{L}^{(i)}_{2} P_1) + \epsilon(\mathcal{L}^{(i)}_{1} P_3 + \mathcal{L}^{(i)}_{2} P_2 + \sqrt{\epsilon}\mathcal{L}^{(i)}_{2} P_3). \tag{2.38}$$

By construction, P_0, P_1, P_2, and P_3 satisfy the following sequence of PDEs:

$$\begin{aligned} \mathcal{L}_0 P_0 &= 0, \\ \mathcal{L}_0 P_1 + \mathcal{L}^{(i)}_{1} P_0 &= 0, \\ \mathcal{L}_0 P_2 + \mathcal{L}^{(i)}_{1} P_1 + \mathcal{L}^{(i)}_{2} P_0 &= 0, \\ \mathcal{L}_0 P_3 + \mathcal{L}^{(i)}_{1} P_2 + \mathcal{L}^{(i)}_{2} P_1 &= 0. \end{aligned}$$

Using these relations, Eq (2.38) simplifies to:

$$\mathcal{L}^{\epsilon}\mathcal{R}^{\epsilon} + \epsilon\mathcal{R}^{\epsilon}_{1} = 0,$$

where $\mathcal{R}^{\epsilon}_{1} = \mathcal{L}^{(i)}_{1} P_3 + \mathcal{L}^{(i)}_{2} P_2 + \sqrt{\epsilon}\mathcal{L}^{(i)}_{2} P_3$.

The terminal conditions are $P_0(T) = h^{(i)}$ and $\tilde{P}^{\epsilon}_{1}(T) = 0$. Thus, the terminal condition for $\mathcal{R}^{\epsilon}$ is:

$$\begin{aligned} \mathcal{R}^{\epsilon}(T, s, f, y_s, y_f) &= P^{\epsilon}(T, s, f, y_s, y_f) - \hat{P}^{\epsilon}(T, s, f, y_s, y_f) \\ &= -\epsilon P_2(T, s, f, y_s, y_f) - \epsilon\sqrt{\epsilon} P_3(T, s, f, y_s, y_f). \end{aligned}$$

By the Feynman–Kac formula, $\mathcal{R}^{\epsilon}$ can be represented as:

$$\mathcal{R}^{\epsilon} = \epsilon\mathbb{E}^{Q_i}\Big[e^{-r_i(T-t)}(-P_2(T, S_T, F_T, Y^S_T, Y^F_T - \sqrt{\epsilon} P_3(T, S_T, F_T, Y^S_T, Y^F_T)$$

$$- \int_t^T e^{-r_i(\tau-t)}(-\epsilon\mathcal{R}_1^\epsilon(\tau, S_\tau, F_\tau, Y_\tau^S, Y_\tau^F))d\tau \,\Bigg|\, S_t, F_t, Y_t^S, Y_t^F\Bigg],$$

where Q_i is the appropriate risk-neutral measure and r_i is the corresponding risk-free rate of quanto option type $i \in \{f, d\}$.

Hence,

$$\begin{aligned}\left|P^\epsilon - (P_0 + \tilde{P}_1^\epsilon)\right| &\le |\mathcal{R}^\epsilon| + \left|\epsilon P_2 + \epsilon\sqrt{\epsilon}P_3\right| \\ &\le C_0\epsilon\end{aligned}$$

for some constant $C_0 > 0$. Thus, from this line of reasoning, $|P^\epsilon - (P_0 + \tilde{P}_1^\epsilon)|$ appears to be $O(\epsilon)$. However, the standard European option payoffs $h^{(i)}$, while continuous, are not globally smooth. For such piecewise smooth payoffs, as established in the singular perturbation literature (e.g., [17,21]), the interaction of the fast–scale volatility with the non–smoothness of the payoff typically leads to an error of order $\epsilon|\ln\epsilon|$.

Therefore, for the continuous and piecewise smooth payoffs considered for the quanto options, the approximation error is given by

$$\left|P^\epsilon\left(t, s, f, y_s, y_f\right) - \left(P_0\left(t, s, f\right) + \tilde{P}_1^\epsilon\left(t, s, f\right)\right)\right| \le C\epsilon\left|\ln\epsilon\right|,$$

for some constant $C > 0$ that is independent of ϵ. □

3. Numerical illustration: The exponential OU volatility model

For a numerical illustration of our derived pricing formulas, we now specify a concrete form for the volatility functions $g_s(Y_t^S)$ and $g_f(Y_t^F)$. We adopt an exponential OU (expOU) model, where the volatilities are given by:

$$g_s(Y_t^S) = e^{Y_t^S} \quad \text{and} \quad g_f(Y_t^F) = e^{Y_t^F}.$$

The dynamics of the underlying asset S_t, the exchange rate F_t, and their respective volatility-driving OU processes Y_t^S and Y_t^F are thus specified as

$$\begin{aligned}dS_t &= \mu_s S_t dt + e^{Y_t^S} S_t d\tilde{W}_t^S, \\ dF_t &= \mu_f F_t dt + e^{Y_t^F} F_t d\tilde{W}_t^F, \\ dY_t^S &= \frac{k_s}{\epsilon}\left(m_s - Y_t^S\right)dt + \frac{\sqrt{2}\nu_s}{\sqrt{\epsilon}} d\tilde{Z}_t^S, \\ dY_t^F &= \frac{k_f}{\epsilon}\left(m_f - Y_t^F\right)dt + \frac{\sqrt{2}\nu_f}{\sqrt{\epsilon}} d\tilde{Z}_t^F.\end{aligned}$$

These are the same dynamics as presented in (2.1)–(2.4), now with the explicit exponential form for the volatility functions. Furthermore, for simplicity in illustrating the core effects of stochastic volatility correlation, we assume that the market prices of volatility risk are zero, i.e., $\Lambda_S^{(i)}(y_s) = 0$ and $\Lambda_F^{(i)}(y_f) = 0$ for both option types $i \in \{f, d\}$. This assumption implies that all W-group parameters $(W_s^{(i)}, W_f^{(i)}, W_{sf}^{(i)}, W_{fs}^{(i)})$ defined in (2.23)–(2.30) become zero.

3.1. Leading-order price P_0 and averaged parameters

The leading-order price P_0 for both option types is given by the formulas in Theorem 2. These formulas depend on the averaged parameters $\bar{\sigma}_s$, $\bar{\sigma}_f$, and $\bar{\rho}_{sf}$. For the specified expOU model, these parameters are calculated (using the invariant normal distribution of Y_t^S and Y_t^F, where $\tilde{\nu}_j^2 = \nu_j^2/k_j$ for $j = s, f$) as follows:

$$\bar{\sigma}_s = \exp\left\{m_s + \tilde{\nu}_s^2\right\}, \tag{3.1}$$

$$\bar{\sigma}_f = \exp\left\{m_f + \tilde{\nu}_f^2\right\}, \tag{3.2}$$

$$\bar{\rho}_{sf} = \rho_{sf} \exp\left\{-\frac{1}{2}\left(\tilde{\nu}_s^2 + \tilde{\nu}_f^2\right)\right\}. \tag{3.3}$$

The term $\bar{\sigma}_*^2 = \bar{\sigma}_s^2 + 2\bar{\rho}_{sf}\bar{\sigma}_s\bar{\sigma}_f + \bar{\sigma}_f^2$ is then readily computed for the Type (d) option. The explicit formulas for $P_0(t, s, f)$ for Type (f) and Type (d) options remain as stated in Theorem 2, now using these specific averaged parameters.

3.2. First-order correction $\tilde{P}_1^\epsilon$ and simplified operators

The first-order correction term $\tilde{P}_1^\epsilon$ is given by the expressions in Theorem 3. With the assumption of zero market prices of volatility risk (all $W^{(i)}$ parameters are zero), the operators $\mathcal{H}_{(f)}^\epsilon$ and $\mathcal{H}_{(d)}^\epsilon$ from Theorem 3 simplify considerably.

(1) For the Type (f) option (foreign currency strike), the correction term is

$$\tilde{P}_1^\epsilon(t, s, f) = -(T-t)\,\mathcal{H}_{(f)}^\epsilon P_0(t, s, f),$$

where the simplified operator is

$$\mathcal{H}_{(f)}^\epsilon = -V_s\mathcal{D}_s^3 - \left(2V_s + V_{sf}\right)\mathcal{D}_s^2 - \left(V_{sf} + V_{fs}\right)\mathcal{D}_s - 2V_f\mathcal{I}.$$

(2) For the Type (d) option (domestic currency strike), the correction term is

$$\tilde{P}_1^\epsilon(t, s, f) = -(T-t)\,\mathcal{H}_{(d)}^\epsilon \Gamma_{BS}(t, sf; K_d, r_d, \bar{\sigma}_*),$$

where the simplified operator is

$$\mathcal{H}_{(d)}^\epsilon = -\mathcal{D}_s\left(V_s - V_{sf}\right) - \mathcal{D}_f\left(V_f - V_{fs}\right).$$

The remaining group parameters V_s, V_f, V_{sf}, and V_{fs} in these simplified operators, under the expOU model assumption, are derived from their general definitions in (2.23)–(2.30). These specific calculations yield:

$$V_s = -\frac{\sqrt{\epsilon}}{\sqrt{2}\nu_s}\rho_s \exp\left\{3m_s + \frac{5}{2}\tilde{\nu}_s^2\right\}\left(e^{2\tilde{\nu}_s^2} - 1\right), \tag{3.4}$$

$$V_f = -\frac{\sqrt{\epsilon}}{\sqrt{2}\nu_f}\rho_f \exp\left\{3m_f + \frac{5}{2}\tilde{\nu}_f^2\right\}\left(e^{2\tilde{\nu}_f^2} - 1\right), \tag{3.5}$$

$$V_{sf} = -\frac{\sqrt{2\epsilon}\rho_s\rho_{sf}}{\nu_s} \exp\left\{2m_s + m_f + \frac{1}{2}\left(2\tilde{\nu}_s^2 + \tilde{\nu}_f^2\right) + \tilde{\nu}_s^2\right\}\left(e^{\tilde{\nu}_s^2} - 1\right), \tag{3.6}$$

$$V_{fs} = -\frac{\sqrt{2\epsilon}\rho_f\rho_{sf}}{\nu_f} \exp\left\{m_s + 2m_f + \frac{1}{2}\left(\tilde{\nu}_s^2 + 2\tilde{\nu}_f^2\right) + \tilde{\nu}_f^2\right\}\left(e^{\tilde{\nu}_f^2} - 1\right). \tag{3.7}$$

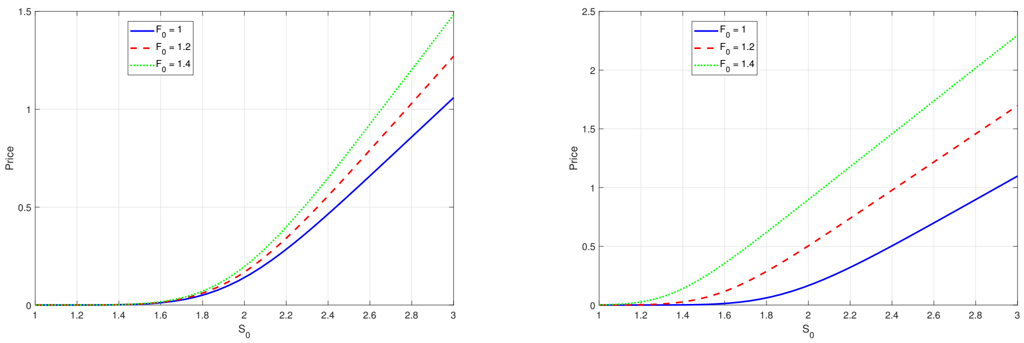
With these specific formulas for the averaged parameters and group parameters, numerical tests can be conducted. For the experiment, the baseline parameters are needed.

Table 1. Monte Carlo simulation results for Type (f) options.

Parameter	Value	Analytical	MC	Standard error	Relative error
	1	1.2294	1.2617	0.001	0.03
K_f	1.5	0.74414	0.75689	0.001	0.02
	2	0.28458	0.29865	0.0008	0.05
	1	0.28458	0.29783	0.0008	0.05
F_0	1.5	0.42687	0.44816	0.001	0.05
	2	0.56916	0.59309	0.002	0.04

Table 2. Monte Carlo simulation results for Type (d) options.

Parameter	Value	Analytical	MC	Standard error	Relative error
	1	1.2488	1.2498	0.001	0.001
K_d	1.5	0.77323	0.77204	0.001	0.001
	2	0.32027	0.33743	0.001	0.05
	1	0.32027	0.33673	0.001	0.05
F_0	1.5	1.3975	1.3974	0.001	0.001
	2	2.4975	2.4936	0.002	0.02



(a) Price of the Type (f) option varying with F_0.

(b) Price of the Type (d) option varying with F_0.

Figure 1. Option prices with respect to S_0 for different F_0.

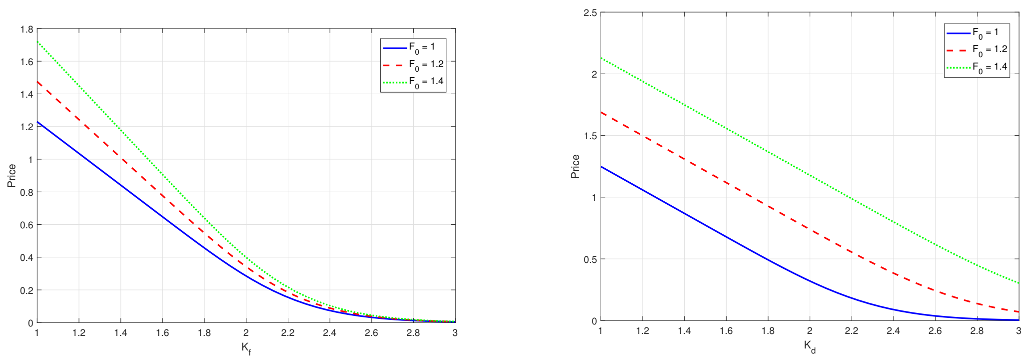
On the basis of the empirical results in Fouque et al. [21] and Giese [25], we choose the following baseline parameters $r_d = 0.05, r_f = 0.03, S_0 = 2.2, F_0 = 1, K_f = K_d = 2, q = 0, m_s = m_f =$

$-\ln 10, k_s = 1.5, k_f = 2, \nu_s = \nu_f = 0.7, \rho_s = \rho_{sf} = -0.6, \rho_f = 0.5, \epsilon = 0.01, T = 1$ unless otherwise stated. From these parameters, we can determine the the group parameters as follows.

$$V_s = 0.000149, \quad V_f = -0.0000688, \quad V_{sf} = -0.0000301, \quad V_{fs} = 0.0000164.$$

For the verification of the formulas, we carry out Monte Carlo (MC) simulations with the selected parameters using the Euler scheme. The MC methods in these implements are based on 100,000 sample paths and a time-step of 1/252. We implement all experiments using MATLAB. Table 1 presents the computational results for the foreign equity call option struck in the foreign currency. In Table 1, 'Analytical' is the values obtained by our formula and 'MC' is the benchmark values by MC simulations. Relative error (RE) in Table 1 means the absolute difference between 'Analytical' and 'MC' divided by the benchmark values. Similarly, Table 2 presents the computational results for the foreign equity call options struck in the domestic currency. These results show that our formulas are accurate and efficient.

Figure 1 demonstrates how option prices vary with respect to the underlying asset price S_0 for different foreign exchange rates F_0. Several important observations can be made. Both option types exhibit the characteristic convex relationship between the option price and the underlying asset price. As F_0 increases, option prices systematically increase for both types, with a more pronounced effect for the domestic currency-struck option (type d). This illustrates the multiplicative effect of the exchange rate in the pricing formula. The domestic currency option (type d) shows steeper price curves and greater sensitivity to both S_0 and F_0 compared with the foreign currency option (type f). This can be attributed to the fact that Type (d) options with the payoff $(fs - K_d)^+$ in (2.6) incorporate the exchange rate directly in the moneyness condition. In addition, we can find that the absolute price levels for Type (d) options are notably higher than those for Type (f), particularly at higher values of F_0, demonstrating the compounded effect of currency conversion on option valuation.



(a) Price of the Type (f) option varying with F_0.

(b) Price of the Type (d) option varying with F_0.

Figure 2. Option prices with respect to K_f and K_d for different F_0.

Figure 2 illustrates how option prices respond to changes in the strike prices K_f and K_d across different exchange rates. Expectably, both option types present the inverse relationship between the strike price and the option's value. The vertical separation between price curves for different F_0 values remains relatively constant across strikes for Type (f) options, whereas for Type (d) options, this separation increases at lower strike values. This indicates that the exchange rate effect interacts with the moneyness condition differently for each option type. Additionally, we can see that the rate of price decay as the strike increases appears to be more linear for Type (d) options, while Type (f) options demonstrate a more pronounced curvature, particularly in the region where strikes transition from in-the-money to out-of-the-money.

4. Concluding remarks

In this paper, we have developed analytical approximations for pricing foreign equity options under a stochastic volatility model with fast mean reversion. To overcome the limitations of traditional Black–Scholes models that fail to capture various features of actual financial data, we present a more accurate pricing methodology by incorporating stochastic volatility into option pricing.

Through asymptotic expansion techniques, we explicitly derived the leading-order term and first-order correction term for options struck in both foreign and domestic currencies. The leading-order approximation provides intuitive and computationally efficient pricing formulas resembling the classical Black–Scholes structure, while the correction term refines the approximation by incorporating higher-order effects of stochastic volatility. The theoretical error bound in Theorem 1 ensures the robustness of the approximation under regular conditions. In addition, through numerical examples, we find that options struck in the domestic currency exhibit greater sensitivity to exchange rate changes compared with those struck in the foreign currency, demonstrating the compound effect of currency conversion on option valuation.

This research can be extended in several promising directions. From a modeling perspective, the framework could be enhanced to capture complex market phenomena by incorporating jump processes, regime-switching behavior, or rough volatility dynamics. In addtion, future research directions include empirical validation of our analytical approximations using real market data for foreign equity options. Such empirical work would involve calibrating the model parameters to specific currency pairs and equity indices, and comparing the model prices with observed market prices. This would provide further validation of the practical applicability of our theoretical framework.

Authors contributions

Jaegi Jeon: Writing – original draft, Software, Methodology, Conceptualization; Geonwoo Kim: Writing – review & editing, Supervision, Funding acquisition, Conceptualization.

Use of Generative-AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

Acknowledgments

This research was supported by Global-Learning & Academic Research Institution for Master's and PhD students, and Postdocs (LAMP) Program of the National Research Foundation of Korea (NRF) grant funded by the Ministry of Education (No. RS-2024-00442775).

Conflict of interest

The authors declare that they have no conflict of interest.

References

1. E. Reiner, Quanto mechanics, *Risk*, **5** (1992), 59–63.
2. A. R. Dravid, M. Richardson, T.-s. Sun, Pricing foreign index contingent claims: An application to nikkei index warrants, *J. Deriv.*, **1** (1993), 33–51.
3. Y.-K. Kwok, H.-Y. Wong, Currency-translated foreign equity options with path dependent features and their multi-asset extensions, *Int. J. Theor. Appl. Finance*, **3** (2000), 257–278.
4. G. Dimitroff, A. Szimayer, A. Wagner, Quanto option pricing in the parsimonious Heston model, 2009. Available from: https://nbn-resolving.de/urn:nbn:de:hbz:386-kluedo-16317.
5. Y.-R. Lee, Pricing of quanto option under the hull and white stochastic volatility model, *Commun. Korean Math. Soc.*, **28** (2013), 489–496.
6. S.-C. Huang, M.-W. Hung, Pricing foreign equity options under lévy processes, *J. Futures Mark.: Futures, Options, Deriv. Prod.*, **25** (2005), 917–944. http://doi.org/10.1002/fut.20171
7. W. Xu, C. Wu, H. Li, Foreign equity option pricing under stochastic volatility model with double jumps, *Econ. Modell.*, **28** (2011), 1857–1863. http://doi.org/10.1016/j.econmod.2011.03.016
8. Q. Sun, W. Xu, Pricing foreign equity option with stochastic volatility, *Phys. A: Stat. Mech. Appl.*, **437** (2015), 89–100. http://doi.org/10.1016/j.physa.2015.05.059
9. S. L. Heston, A closed-form solution for options with stochastic volatility with applications to bond and currency options, *Rev. Financ. Stud.*, **6** (1993), 327–343. http://doi.org/10.1093/rfs/6.2.327
10. Y.-R. Lee, The pricing of quanto options in the double square root stochastic volatility model, *Commun. Korean Math. Soc.*, **29** (2014), 83–90.
11. X.-J. He, S.-D. Huang, S. Lin, A closed-form solution for pricing european-style options under the heston model with credit and liquidity risks, *Commun. Nonlinear Sci. Numer. Simul.*, **143** (2025), 108595. http://doi.org/10.1016/j.cnsns.2025.108595
12. X.-J. He, H. Chen, S. Lin, A closed-form formula for pricing european options with stochastic volatility, regime switching, and stochastic market liquidity, *J. Futures Mark.*, **45** (2025), 429–440. http://doi.org/10.1002/fut.22573
13. J. Ma, Pricing foreign equity options with stochastic correlation and volatility, *Ann. Econ. Finance*, **10** (2009), 343–362.
14. L. Teng, M. Ehrhardt, M. Günther, The pricing of quanto options under dynamic correlation, *J. Comput. Appl. Math.*, **275** (2015), 304–316.
15. Y. S. Kim, J. Lee, S. Mittnik, J. Park, Quanto option pricing in the presence of fat tails and asymmetric dependence, *J. Econometrics*, **187** (2015), 512–520. http://doi.org/10.1016/j.jeconom.2015.02.035
16. T. Pellegrino, Embedding stochastic correlation into the pricing of fx quanto options under stochastic volatility models, *J. Math. Finance*, **9** (2019), 345–362.

17. J.-P. Fouque, G. Papanicolaou, K. R. Sircar, *Derivatives in Financial Markets with Stochastic Volatility*, Cambridge University Press, 2000.
18. J.-P. Fouque, G. Papanicolaou, K. R. Sircar, K. Sølna, *Multiscale Stochastic Volatility for Equity, Interest Rate, and Credit Derivatives*, New York: Cambridge University Press, 2011.
19. T. G. Anderson, T. Bollerslev, F. X. Diebold, C. Vega, Micro effects of macro announcements: Real-time price discovery in foreign exchange, *Am. Econ. Rev.*, **93** (2003), 38–62. http://doi.org/10.1257/000282803321455151
20. T. G. Andersen, T. Bollerslev, F. X. Diebold, P. Labys, The distribution of realized exchange rate volatility, *J. Am. Stat. Assoc.*, **96** (2001), 42–55.
21. J.-P. Fouque, G. Papanicolaou, R. Sircar, K. Sølna, Singular perturbations in option pricing, *SIAM J. Appl. Math.*, **63** (2003), 1648–1665.
22. K. B. Toft, E. Reiner, Currency-translated foreign equity options: the american case, *Adv. Futures Options Res.*, **9** (1997), 233–264.
23. J. Jeon, J. Huh, G. Kim, An analytical approach to the pricing of an exchange option with default risk under a stochastic volatility model, *Adv. Contin. Discret. M.*, **2023** (2023), 37.
24. J. Huh, J. Jeon, J.-H. Kim, H. Park, A reduced pde method for european option pricing under multi-scale, multi-factor stochastic volatility, *Quant. Financ.*, **19** (2019), 155–175.
25. A. Giese, Quanto adjustments in the presence of stochastic volatility, *Risk*, **25** (2012), 67.

Supplementary

Quanto prewashing: Drift adjustments for the foreign risk-neutral measure

This appendix provides a self-contained derivation of the drift adjustments for the equity–FX system, transforming it from the domestic risk-neutral measure Q_d to the foreign risk-neutral measure Q_f. This procedure, known as quanto prewashing [3, 22], is instrumental in simplifying the pricing of derivatives on foreign assets. For notational convenience in this section, we let the stochastic volatility functions be represented by constants, such that $\sigma_S \equiv g_s(Y_t^S)$ and $\sigma_F \equiv g_f(Y_t^F)$.

For the reader's convenience, we restate the system's dynamics (2.7) and (2.8):

$$\frac{dS_t}{S_t} = \delta_S dt + \sigma_S dW_t^S$$
$$\frac{dF_t}{F_t} = \delta_F dt + \sigma_F dW_t^F,$$

with $\langle dW_t^S, dW_t^F\rangle = \rho_{sf}dt$. Here, S_t is the foreign asset price, F_t is the exchange rate (domestic/foreign), and δ_S, δ_F are the measure-dependent drift terms.

With the domestic measure Q_d, the standard no-arbitrage drifts are $\delta_F^d = r_d - r_f$ and $\delta_{S^*}^d = r_d - q$. As mentioned in the main text, applying Itô's lemma to the product $S_t^* = F_t S_t$ yields the drift for the foreign asset, which includes the essential covariance adjustment term:

$$\delta_S^d = r_f - q - \rho_{sf}\sigma_S\sigma_F.$$

This drift serves as the baseline before the change measure.

We now change the numéraire to the foreign money market account, thereby moving to the foreign risk-neutral measure Q_f. By definition of Q_f, the drift of the foreign asset S_t must be $\delta_S^f = r_f - q$. To find the drift of F_t, we consider the reciprocal rate $F_t' \equiv 1/F_t$. Its drift under Q_f must be $\delta_{F'}^f = r_f - r_d$. Applying Itô's lemma to $F_t = 1/F_t'$ yields $\delta_F^f = -\delta_{F'}^f + \sigma_{F'}^2$, which simplifies to:

$$\delta_F^f = -(r_f - r_d) + \sigma_F^2 = r_d - r_f + \sigma_F^2.$$

The term σ_F^2 arises purely from the Itô correction. Finally, the drift of the domestically-valued asset S_t^* under Q_f is recalculated using the new component drifts:

$$\begin{aligned}
\delta_{S^*}^f &= \delta_S^f + \delta_F^f + \rho_{sf}\sigma_S\sigma_F \\
&= (r_f - q) + (r_d - r_f + \sigma_F^2) + \rho_{sf}\sigma_S\sigma_F \\
&= r_d - q + \sigma_F^2 + \rho_{sf}\sigma_S\sigma_F.
\end{aligned}$$

In summary, the prewashed drift rates under Q_f, which are used for pricing in the main text, are:

$$\begin{aligned}
\delta_S^f &= r_f - q \\
\delta_F^f &= r_d - r_f + \sigma_F^2 \\
\delta_{S^*}^f &= r_d - q + \sigma_F^2 + \rho_{sf}\sigma_S\sigma_F
\end{aligned}$$