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Research article

Properties of monotonic solutions to half-linear second-order delay differential equations

Pakize Temtek[1,*] **and Yerzhan Turarov**[2]

[1] Department of Mathematics, Faculty of Sciences, Erciyes University, 38030 Kayseri, Turkey
[2] Department of Mathematics, Graduate School of Natural and Applied Science, Erciyes University, 38030 Kayseri, Turkey

* **Correspondence:** Email: temtek@erciyes.edu.tr.

Abstract: This paper establishes new monotonic properties of nonoscillatory solutions for second-order half-linear functional differential equations with delayed argument

$$(a(r)(u'(r))^m)' = b(r)u^m(\varphi(r))$$

where $m \in (0, 1)$. We develop several key monotonicity results for Kneser solutions and use these properties to derive criteria for the elimination of bounded nonoscillatory solutions. Our approach extends known techniques from linear differential equations to the half-linear case, providing new insights into the qualitative behavior of solutions.

Keywords: half-linear differential equations; delayed argument; monotonic properties; Kneser solutions; oscillation criteria
Mathematics Subject Classification: 34K25, 34K11, 34C10

1. Introduction

This paper investigates the qualitative behavior of solutions to the second-order half-linear differential equation with delayed argument:

$$(a(r)(u'(r))^m)' = b(r)u^m(\varphi(r)). \tag{1.1}$$

Throughout this paper, we assume the following:

(H1) $m \in (0, 1)$ is the ratio of two positive odd integers;

(H2) $a, b \in C([r_0, \infty))$ with $a(r) > 0$ and $b(r) > 0$ for all $r \geq r_0$;

(H3) $\varphi \in C^1([r_0,\infty))$ satisfies $\varphi(r) < r$, $\varphi'(r) > 0$, and $\lim_{r\to\infty} \varphi(r) = \infty$;

(H4) The integral $A(r) = \int_{r_0}^{r} \frac{1}{a^{1/m}(s)}\,ds$ diverges as $r \to \infty$, i.e., the equation is in canonical form.

A function u such that $u \in C^1([T_u,\infty))$ and $a(r)(u'(r))^m \in C^1([T_u,\infty))$ for some $T_u \geq r_0$ is called a *proper solution* of Eq (1.1) if $u(r)$ satisfies (1.1) for all $r \geq T_u$, with $\sup\{|u(r)| : r \geq T\} > 0$ for some $T \geq T_u$. Throughout this paper, we work under the assumption that such solutions exist for $r \geq r_0$. We classify a proper solution as *oscillatory* when it possesses zeros at arbitrarily large values in $[T_u,\infty)$, and *nonoscillatory* otherwise. We say Eq (1.1) is *oscillatory* if it admits only oscillatory proper solutions.

It is well known that every nonoscillatory solution $u(r)$ of (1.1) satisfies one of the following conditions:

(i) $u(r)u^{(i)}(r) > 0$,

(ii) $(-1)^i u(r)u^{(i)}(r) > 0$

for all sufficiently large r and integer i such that $0 \leq i \leq 2$.

A central object of interest in this paper is the class of nonoscillatory solutions $u(r)$ of (1.1) that satisfy condition (ii), where $(-1)^i u(r)u^{(i)}(r) > 0$ for $0 \leq i \leq 2$. These are commonly referred to as *Kneser solutions* in the literature. Due to the homogeneity property of Eq (1.1), if $u(r)$ is a solution, then $-u(r)$ is also a solution. This symmetry allows us to restrict our analysis to positive Kneser solutions without loss of generality.

The qualitative theory of differential equations with deviating arguments has attracted considerable research interest due to its theoretical importance and broad applications in various scientific fields. Within this field, second-order half-linear differential equations are of particular interest as they effectively generalize linear equations while preserving crucial structural properties, most notably the homogeneity of the solution space. Comprehensive research on the oscillation and asymptotic behavior of solutions to half-linear functional differential equations has been documented in numerous studies [1,6], building upon foundational work for linear equations.

Foundational contributions in this area include the results of Koplatadze and Chanturia [5], who proved that the equation

$$u''(r) = b(r)u(\varphi(r)), \quad \varphi(r) \leq r$$

has no Kneser solutions under the assumptions

$$\limsup_{r\to\infty} \int_{\varphi(r)}^{r} (s - \varphi(s))b(s)\,ds > 1.$$

Similar nonexistence results for Kneser-type solutions were established for nonlinear cases by Kusano and Lalli [7], who showed that the equation

$$(|u'(r)|^{m-1}u'(r))' = b(r)|u(\varphi(r))|^{m-1}u(\varphi(r)), \quad m > 0,$$

admits no such solutions provided

$$\limsup_{r\to\infty} \int_{\varphi(r)}^{r} (\varphi(r) - \varphi(s))^m b(s)\,ds > 1.$$

In parallel, researchers have developed various analytical techniques to study the qualitative behavior of these equations. A common approach involves transforming second-order nonlinear or neutral delay equations into first-order Riccati-type differential inequalities (see, e.g., [8–10]). Notably, Bohner et al. [4] presented a comparison theorem linking the oscillation of second-order neutral delay equations to that of associated first-order delay equations.

A significant recent advancement in this domain is due to Baculíková and Džurina [3], who derived new oscillation criteria for Eq (1.1). Their work has expanded the existing theory, offering sharper tools for analyzing solution behavior.

Most notably, Baculíková's recent work [2] improved upon the criteria of Koplatadze and Chanturia by introducing a novel approach to studying monotonic properties of nonoscillatory solutions. This innovative approach introduced a degree-based classification for solutions and offered more refined analytical tools to rule out bounded nonoscillatory solutions.

Motivated by the effectiveness of this monotonicity framework, our research extends and adapts these monotonicity-based techniques from the linear case to the more general half-linear delay equation context. We demonstrate that the concept of degree classification for nonoscillatory solutions and their monotonic behavior can be successfully generalized to the half-linear setting. This extension enables us to develop new oscillation criteria and enhance existing results in the field.

The paper is organized as follows: Section 2 presents essential preliminaries and foundational assumptions. Section 3 establishes our main theoretical results on monotonic properties of solutions. Section 4 provides illustrative examples that demonstrate the practical applicability of our theoretical findings.

2. Preliminary results

In this section, we present several mathematical inequalities that will be used in our main results.

Lemma 2.1 (Power inequality)**.** *Let* $b \geq 1$ *and* $A, B > 0$*. Then*

$$(A + B)^b \geq A^b + B^b. \tag{2.1}$$

Proof. We prove this inequality by considering the function $f(r) = (1 + r)^b - 1 - r^b$ for $r > 0$. Taking the derivative, we get

$$f'(r) = b(1 + r)^{b-1} - br^{b-1}.$$

Since $b \geq 1$ and $1 + r > r$ for all $r > 0$, we have $(1 + r)^{b-1} \geq r^{b-1}$ when $b \geq 1$. Therefore, $f'(r) \geq 0$ for all $r > 0$.

Since the derivative is non-negative, f is monotonically increasing on $(0, \infty)$. Given that $f(0) = 0$, we conclude that $f \geq 0$ for all points in $(0, \infty)$. This yields

$$(1 + r)^b \geq 1 + r^b \quad \text{for all } r > 0.$$

Setting $r = \frac{B}{A}$ and multiplying both sides by A^b, we obtain

$$(A + B)^b = A^b \left(1 + \frac{B}{A}\right)^b \geq A^b \left(1 + \left(\frac{B}{A}\right)^b\right) = A^b + B^b.$$

This completes the proof. □

The following lemma presents a specialized form of Hölder's inequality that will be crucial for our subsequent analysis.

Lemma 2.2. *Let $m \in (0,1)$ and f be a continuous function on $[a,b]$ for $0 \le a < b$. Then*

$$(b-a)^{1-1/m}\left(\int_a^b |f(r)|dr\right)^{1/m} \le \int_a^b |f(r)|^{1/m}dr. \tag{2.2}$$

Proof. Let $b = \frac{1}{m} > 1$ and $q = \frac{1}{1-m} > 1$, so that $\frac{1}{b} + \frac{1}{q} = 1$. By the standard Hölder's inequality, we have

$$\int_a^b |f(r)| \cdot 1dr \le \left(\int_a^b |f(r)|^b dr\right)^{1/b} \left(\int_a^b 1^q dr\right)^{1/q}.$$

Simplifying, we obtain

$$\int_a^b |f(r)|dr \le \left(\int_a^b |f(r)|^{1/m}dr\right)^m \left(\int_a^b dr\right)^{1-m}.$$

Since $\int_a^b dr = b - a$, we have

$$\int_a^b |f(r)|dr \le \left(\int_a^b |f(r)|^{1/m}dr\right)^m (b-a)^{1-m}.$$

Raising both sides to the power of $1/m$, we obtain

$$\left(\int_a^b |f(r)|dr\right)^{1/m} \le \left(\int_a^b |f(r)|^{1/m}dr\right)(b-a)^{(1-m)/m}.$$

Since $(1-m)/m = 1/m - 1$, we have

$$\left(\int_a^b |f(r)|dr\right)^{1/m} \le \left(\int_a^b |f(r)|^{1/m}dr\right)(b-a)^{1/m-1}.$$

Multiplying both sides by $(b-a)^{1-1/m}$, we get the desired inequality

$$(b-a)^{1-1/m}\left(\int_a^b |f(r)|dr\right)^{1/m} \le \int_a^b |f(r)|^{1/m}dr.$$

□

3. Main results

In this section, we establish our main results on monotonic properties of nonoscillatory solutions to Eq (1.1).

Lemma 3.1. *Let $u(r)$ be a positive Kneser solution of Eq* (1.1)*. Then*

$$a(r)(u'(r))^m < 0 \tag{3.1}$$

for all sufficiently large r. Moreover,

$$\lim_{r\to\infty} a(r)(u'(r))^m = 0. \tag{3.2}$$

Proof. Let $u(r)$ be a positive Kneser solution of Eq (1.1), so $u(r) > 0$ and $u'(r) < 0$ for $r \geq r_1 \geq r_0$. Since $u'(r) < 0$ for $r \geq r_1$ and (H1), we have $(u'(r))^m < 0$ for $r \geq r_1$. By condition (H2), we know that $a(r) > 0$ for $r \geq r_0$. Therefore,

$$a(r)(u'(r))^m < 0$$

for all $r \geq r_1$.

To prove the second part, note that from Eq (1.1), we have

$$\frac{d}{dr}\left[a(r)(u'(r))^m\right] = b(r)u(\varphi(r))^m.$$

Since $b(r) > 0$ and $u(\varphi(r)) > 0$ for r sufficiently large (due to $u(r) > 0$ and $\lim_{r\to\infty} \varphi(r) = \infty$), we have

$$\frac{d}{dr}\left[a(r)(u'(r))^m\right] > 0$$

for all sufficiently large r.

Since $a(r)(u'(r))^m < 0$ for all sufficiently large r (from the first part) and is increasing, it must have a limit as $r \to \infty$. Let

$$\lim_{r\to\infty} a(r)(u'(r))^m = -c$$

where $c \geq 0$.

We need to show that $c = 0$. Suppose, by contradiction, that $c > 0$. Then for sufficiently large r, we have

$$a(r)(u'(r))^m \leq -c < 0.$$

This implies

$$u'(r) \leq -c^{1/m}\left(\frac{1}{a(r)}\right)^{1/m}.$$

Integrating from r_2 to r, we obtain

$$u(r) - u(r_2) \leq -c^{1/m}\int_{r_2}^{r}\left(\frac{1}{a(s)}\right)^{1/m} ds.$$

Then as $r \to \infty$, the right-hand side tends to $-\infty$ by condition (H4), which contradicts the fact that $u(r) > 0$ for all $r \geq r_1$. Therefore, $c = 0$, and $\lim_{r\to\infty} a(r)(u'(r))^m = 0$. □

To simplify our notation, we define the following auxiliary functions:

$$\begin{aligned} \Lambda(r) &= \left(\frac{1}{a(r)}\int_{r}^{\varphi^{-1}(r)} b(s)ds\right)^{1/m} \\ \beta(r) &= \exp\left(\int_{r_1}^{r} \Lambda(s)ds\right) \end{aligned} \tag{3.3}$$

where $r_1 \geq r_0$ is an arbitrary but fixed constant.

Lemma 3.2. *Let $u(r)$ be a positive Kneser solution of Eq* (1.1). *Then*

$$\beta(r)u(r) \text{ is a decreasing function} \tag{3.4}$$

for $r \geq r_1$.

Proof. Let $u(r)$ be a positive Kneser solution of Eq (1.1). Then integrating Eq (1.1) from r to ∞ (note that φ^{-1} exists by (H3)), we have

$$\begin{aligned}-a(r)(u'(r))^m &= \int_r^\infty b(s)u(\varphi(s))^m ds\\ &\geq \int_r^{\varphi^{-1}(r)} b(s)u(\varphi(s))^m ds\\ &\geq u(r)^m \int_r^{\varphi^{-1}(r)} b(s)ds\end{aligned}$$

which in view of (3.3) we have

$$u'(r) + u(r)\Lambda(r) \leq 0.$$

Using the standard methods of calculus, it is easy to verify that for any $r_1 \geq r_0$

$$[u(r)\beta(r)]' = [u(r)e^{\int_{r_1}^r \Lambda(s)ds}]' = u'(r)e^{\int_{r_1}^r \Lambda(s)ds} + u(r)e^{\int_{r_1}^r \Lambda(s)ds}\Lambda(r) \leq 0$$

so we conclude that $\beta(r)u(r)$ is decreasing function for $r_1 \geq r_0$. □

Now we are ready to state our first theorem.

Theorem 3.3. *If*

$$\limsup_{r\to\infty} \beta(r)(r-\varphi(r))^{1-1/m}\left[\int_{\varphi(r)}^r \frac{b(s)}{\beta^m(\varphi(s))}\int_{\varphi(r)}^s \frac{1}{a(l)}dlds\right] > 1 \tag{3.5}$$

then Eq (1.1) *has no Kneser solutions.*

Proof. Assume, to the contrary, that there exists a positive Kneser solution $u(r)$ of Eq (1.1). By integrating Eq (1.1) once from l to r, we obtain

$$-u'(l) \geq \left(\frac{1}{a(l)}\int_l^r b(s)u^m(\varphi(s))ds\right)^{1/m},$$

where $r > l$.

Integrating this inequality once more and applying Lemma 2.2, we derive

$$\begin{aligned}u(l) &\geq \int_l^r \left(\frac{1}{a(k)}\int_k^r b(s)u^m(\varphi(s))ds\right)^{1/m} dk\\ &\geq (r-l)^{1-1/m}\left(\int_l^r \frac{1}{a(k)}\int_k^r b(s)u^m(\varphi(s))dsdk\right)^{1/m}.\end{aligned}$$

By changing the order of integration and using Lemma 3.2, we obtain

$$\begin{aligned}u(l) &\geq (r-l)^{1-1/m}\left(\int_l^r b(s)u^m(\varphi(s))\int_l^s \frac{dk}{a(k)}ds\right)^{1/m}\\ &\geq (r-l)^{1-1/m}u(\varphi(r))\beta(r)\left(\int_l^r \frac{b(s)}{\beta^m(\varphi(s))}\int_l^s \frac{dk}{a(k)}ds\right)^{1/m}.\end{aligned}$$

Setting $l = \varphi(r)$, we obtain

$$1 \geq (r - \varphi(r))^{1-1/m}\beta(r)\left(\int_{\varphi(r)}^{r} \frac{b(s)}{\beta^m(\varphi(s))} \int_{\varphi(r)}^{s} \frac{dk}{a(k)} ds\right)^{1/m}.$$

This contradicts our initial assumption, which completes the proof. □

For our next development, we need to establish the opposite monotonicity property for $u(r)$. By condition (H3), the inverse function $\varphi^{-1}(r)$ exists, which allows us to assume that there exists an invertible function $\psi \in C^1([r_0, \infty))$ satisfying

$$\psi(\psi(r)) = \varphi^{-1}(r), \tag{3.6}$$

and

$$\varphi(r) < \psi^{-1}(r) < r < \psi(r) < \varphi^{-1}(r) < \varphi^{-1}(\psi(r)) \quad \forall r. \tag{9a}$$

To facilitate our analysis, we introduce the following auxiliary functions:

$$\begin{aligned}
\Lambda_1(r) &= (\varphi^{-1}(r) - r)^{1-1/m}\beta(\psi^{-1}(r))\left(\int_{r}^{\psi(r)} \frac{b(s)}{\beta^m(\varphi(s))} \int_{r}^{s} \frac{dk}{a(k)} ds\right)^{1/m} \\
\Lambda_2(r) &= (\varphi^{-1}(r) - r)^{1-1/m}\beta(r)\left(\int_{\psi(r)}^{\varphi^{-1}(r)} \frac{b(s)}{\beta^m(\varphi(s))} \int_{r}^{s} \frac{dk}{a(k)} ds\right)^{1/m} \\
\Lambda_3(r) &= (\varphi^{-1}(r) - r)^{1-1/m}\beta(\psi(r))\left(\int_{\varphi^{-1}(r)}^{\varphi^{-1}(\psi(r))} \frac{b(s)}{\beta^m(\varphi(s))} \int_{r}^{s} \frac{dk}{a(k)} ds\right)^{1/m}.
\end{aligned} \tag{3.7}$$

When $\Lambda_2(r) < 1$, we define the normalized functions

$$\begin{aligned}
\Lambda_1^*(r) &= \frac{\Lambda_1(r)}{1 - \Lambda_2(r)} \\
\Lambda_3^*(r) &= \frac{\Lambda_3(r)}{1 - \Lambda_2(r)}.
\end{aligned} \tag{3.8}$$

Lemma 3.4. *Assume that there exists a function $\psi(r) \in C^1([r_0, \infty))$ satisfying* (3.6) *and $u(r)$ is a positive Kneser solution of Eq* (1.1). *Then*

$$u(\varphi(r)) \leq \frac{1 - \Lambda_3^*(\psi^{-1}(r))\Lambda_1^*(r) - \Lambda_1^*(\psi(r))\Lambda_3^*(r)}{\Lambda_1^*(r)\Lambda_1^*(\psi^{-1}(r))} u(r). \tag{3.9}$$

Proof. Let $u(r)$ be a positive Kneser solution of Eq (1.1). By integrating Eq (1.1) from r to l, we obtain

$$-u'(r) \geq \left(\frac{1}{a(r)} \int_{r}^{l} b(s)u^m(\varphi(s))ds\right)^{1/m}.$$

Integrating this inequality once more, applying Lemma 2.2, and changing the order of integration, we derive

$$
\begin{aligned}
u(r) &\geq (l-r)^{1-1/m}\left(\int_r^l \frac{1}{a(k)}\int_k^l b(s)u^m(\varphi(s))dsdk\right)^{1/m}\\
&= (l-r)^{1-1/m}\left(\int_r^l b(s)u^m(\varphi(s))\int_r^s \frac{dk}{a(k)}ds\right)^{1/m}.
\end{aligned}
$$

Using assumption (9a), we have

$$
\begin{aligned}
u(r) \geq & (\varphi^{-1}(\psi(r))-r)^{1-1/m}\left(\int_r^{\varphi^{-1}(\psi(r))} b(s)u^m(\varphi(s))\int_r^s \frac{dk}{a(k)}ds\right)^{1/m}\\
= & (\varphi^{-1}(\psi(r))-r)^{1-1/m}\left\{\int_r^{\psi(r)} b(s)u^m(\varphi(s))\int_r^s \frac{dk}{a(k)}ds\right.\\
& + \int_{\psi(r)}^{\varphi^{-1}(r)} b(s)u^m(\varphi(s))\int_r^s \frac{dk}{a(k)}ds\\
& \left. + \int_{\varphi^{-1}(r)}^{\varphi^{-1}(\psi(r))} b(s)u^m(\varphi(s))\int_r^s \frac{dk}{a(k)}ds\right\}^{1/m}.
\end{aligned}
$$

Using that $\beta(r)u(r)$ is decreasing and Lemma 3.2, we have

$$
\begin{aligned}
u(r) \geq & (\varphi^{-1}(\psi(r))-r)^{1-1/m}u(\psi^{-1}(r))\beta(\psi^{-1}(r))\left[\int_r^{\psi(r)} \frac{b(s)}{\beta^m(\varphi(s))}\int_r^s \frac{dk}{a(k)}ds\right]^{1/m}\\
& + (\varphi^{-1}(\psi(r))-r)^{1-1/m}u(r)\beta(r)\left[\int_{\psi(r)}^{\varphi^{-1}(r)} \frac{b(s)}{\beta^m(\varphi(s))}\int_r^s \frac{dk}{a(k)}ds\right]^{1/m}\\
& + (\varphi^{-1}(\psi(r))-r)^{1-1/m}u(\psi(r))\beta(\psi(r))\left[\int_{\varphi^{-1}(r)}^{\varphi^{-1}(\psi(r))} \frac{b(s)}{\beta^m(\varphi(s))}\int_r^s \frac{dk}{a(k)}ds\right]^{1/m}.
\end{aligned}
$$

Substituting the auxiliary functions defined earlier, we have

$$
u(r) \geq \Lambda_1(r)u(\psi^{-1}(r)) + \Lambda_2(r)u(r) + \Lambda_3(r)u(\psi(r)).
$$

When $\Lambda_2(r) < 1$, this inequality can be rearranged as

$$
u(r) \geq \Lambda_1^*(r)u(\psi^{-1}(r)) + \Lambda_3^*(r)u(\psi(r)). \tag{3.10}
$$

When we replace r by $\psi^{-1}(r)$ in Eq (3.10), we obtain

$$
u(\psi^{-1}(r)) \geq \Lambda_1^*(\psi^{-1}(r))u(\varphi(r)) + \Lambda_3^*(\psi^{-1}(r))u(r). \tag{3.11}
$$

Similarly, setting $r = \psi(r)$ yields

$$
u(\psi(r)) \geq \Lambda_1^*(\psi(r))u(r) + \Lambda_3^*(\psi(r))u(\varphi^{-1}(r)). \tag{3.12}
$$

Substituting Eqs (3.11) and (3.12) into (3.10), we derive

$$u(r) \geq \Lambda_1^*(r)\Lambda_1^*(\psi^{-1}(r))u(\varphi(r)) + \Lambda_1^*(r)\Lambda_3^*(\psi^{-1}(r))u(r) + \Lambda_3^*(r)\Lambda_1^*(\psi(r))u(r).$$

Rearranging terms, we conclude that

$$u(\varphi(r)) \leq \frac{1 - \Lambda_3^*(\psi^{-1}(r))\Lambda_1^*(r) - \Lambda_1^*(\psi(r))\Lambda_3^*(r)}{\Lambda_1^*(r)\Lambda_1^*(\psi^{-1}(r))}u(r).$$

The proof is complete. □

For the remainder of this paper, we shall assume that there exist positive constants Λ_i^*, $i = 1, 3$ such that

$$\Lambda_i^*(r) \geq \Lambda_i^*. \tag{3.13}$$

The following oscillation criteria are direct consequences of Lemma 3.4.

Corollary 3.5. *If* $\limsup_{r\to\infty} \Lambda_2(r) > 1$*, then Eq* (1.1) *has no Kneser solutions.*

Corollary 3.6. *Let condition* (3.13) *hold, and suppose there exists a function* $\psi(r) \in C^1([r_0, \infty))$ *satisfying* (3.6)*. If* $u(r)$ *is a positive Kneser solution of Eq* (1.1)*, then*

$$u(\varphi(r)) \leq \frac{1 - 2\Lambda_1^*\Lambda_3^*}{(\Lambda_1^*)^2}u(r). \tag{3.14}$$

We now establish the opposite monotonicity property for Kneser solutions. To accomplish this, we introduce the following auxiliary functions:

$$\begin{aligned} \Phi(r) &= \frac{1 - 2\Lambda_1^*\Lambda_3^*}{(\Lambda_1^*)^2}\frac{\beta(r)}{a^{1/m}(r)}\left(\int_r^\infty \frac{b(s)}{\beta^m(s)}ds\right)^{1/m} \\ \sigma(r) &= \exp\left(\int_{r_1}^r \Phi(s)ds\right) \end{aligned} \tag{3.15}$$

where $r_1 \geq r_0$ is an arbitrary constant and the improper integral is assumed to be convergent.

Lemma 3.7. *Let condition* (3.13) *hold, and suppose there exists a function* $\psi(r) \in C^1([r_0, \infty))$ *satisfying* (3.6)*. If* $u(r)$ *is a positive Kneser solution of Eq* (1.1)*, then*

$$\sigma(r)u(r) \text{ is an increasing function} \tag{3.16}$$

for $r \geq r_1$.

Proof. Let $u(r)$ be a positive Kneser solution of Eq (1.1). By integrating Eq (1.1) from r to ∞, we obtain

$$-a(r)(u'(r))^m = \int_r^\infty b(s)u(\varphi(s))^m ds \leq \int_r^\infty b(s)\left(\frac{1 - 2\Lambda_1^*\Lambda_3^*}{(\Lambda_1^*)^2}\right)^m u^m(s)ds.$$

Using Lemma 3.2, this inequality leads to

$$-u'(r) \leq \frac{1 - 2\Lambda_1^*\Lambda_3^*}{(\Lambda_1^*)^2}\frac{u(r)\beta(r)}{a^{1/m}(r)}\left(\int_r^\infty \frac{b(s)}{\beta^m(s)}ds\right)^{1/m} = \Phi(r)u(r).$$

From this differential inequality, we can conclude that $\sigma(r)u(r)$ is an increasing function for $r \geq r_1$, which completes the proof. □

We now present our second main result, which provides a more comprehensive criterion for the non-existence of Kneser solutions.

Theorem 3.8. *Let condition* (3.13) *hold, and suppose there exists a function* $\psi(r) \in C^1([r_0,\infty))$ *satisfying* (3.6). *If*

$$\limsup_{r\to\infty}\Bigg\{\left(r-\varphi(r)^{1-1/m}\right)\beta(\varphi(r))\left[\int_{\varphi(r)}^{r}\frac{b(s)}{\beta^m(\varphi(s))}\int_{\varphi(r)}^{s}\frac{dk}{a(k)}ds\right]^{1/m} + \left(r-\varphi(r)^{1-1/m}\right)\sigma(\varphi(r))\left[\int_{\varphi(r)}^{\infty}\frac{b(s)ds}{\sigma^m(\varphi(s))}\int_{\varphi(r)}^{r}\frac{dk}{a(k)}\right]^{1/m} + \sigma(\varphi(r))\int_{\varphi(r)}^{\infty}\frac{1}{a^{1/m}(k)}\left(\int_{k}^{\infty}\frac{b(s)}{\sigma^m(s)}ds\right)^{1/m}dk\Bigg\} > 1, \tag{3.17}$$

then Eq (1.1) *has no Kneser solutions.*

Proof. Assume, to the contrary, that $u(r)$ is a positive Kneser solution of Eq (1.1). By integrating Eq (1.1) from l to ∞, we obtain

$$-u'(l) \ge \left(\frac{1}{a(l)}\int_{l}^{\infty} b(s)u^m(\varphi(s))ds\right)^{1/m}. \tag{3.18}$$

Integrating this inequality once more, we derive

$$\begin{aligned} u(l) \ge& \int_{l}^{\infty}\left(\frac{1}{a(k)}\int_{k}^{\infty} b(s)u^m(\varphi(s))ds\right)^{1/m}dk \\ =& \int_{l}^{r}\left(\frac{1}{a(k)}\int_{k}^{\infty} b(s)u^m(\varphi(s))ds\right)^{1/m}dk + \int_{r}^{\infty}\left(\frac{1}{a(k)}\int_{k}^{\infty} b(s)u^m(\varphi(s))ds\right)^{1/m}dk. \end{aligned}$$

Applying Lemma 2.2, we obtain

$$\begin{aligned} u(l) \ge&(r-l)^{1-1/m}\left(\int_{l}^{r}\frac{1}{a(k)}\int_{k}^{\infty} b(s)u^m(\varphi(s))dsdk\right)^{1/m} \\ &+ \int_{r}^{\infty}\left(\frac{1}{a(k)}\int_{k}^{\infty} b(s)u^m(\varphi(s))ds\right)^{1/m}dk \\ u(l) \ge&(r-l)^{1-1/m}\left(\int_{l}^{r}\frac{1}{a(k)}\int_{k}^{r} b(s)u^m(\varphi(s))dsdk\right)^{1/m} \\ &+ (r-l)^{1-1/m}\left(\int_{l}^{r}\frac{1}{a(k)}\int_{r}^{\infty} b(s)u^m(\varphi(s))dsdk\right)^{1/m} \\ &+ \int_{r}^{\infty}\left(\frac{1}{a(k)}\int_{k}^{\infty} b(s)u^m(\varphi(s))ds\right)^{1/m}dk. \end{aligned}$$

By changing the order of integration and utilizing the fact that $\beta(r)u(r)$ is a decreasing function while $\sigma(r)u(r)$ is an increasing function, we derive

$$u(l) \ge(r-l)^{1-1/m}\left(\int_{l}^{r} b(s)u^m(\varphi(s))\int_{l}^{s}\frac{dk}{a(k)}ds\right)^{1/m}$$

$$+ (r-l)^{1-1/m}\left(\int_r^\infty b(s)u^m(\varphi(s))ds\int_l^r \frac{dk}{a(k)}\right)^{1/m}$$
$$+ \int_r^\infty \left(\frac{1}{a(k)}\int_k^\infty b(s)u^m(\varphi(s))ds\right)^{1/m} dk$$

$$\begin{aligned}
u(l) \geq &(r-l)^{1-1/m}u(\varphi(r))\beta(\varphi(r))\left(\int_l^r \frac{b(s)}{\beta^m(\varphi(s))}\int_l^s \frac{dk}{a(k)}ds\right)^{1/m}\\
&+ (r-l)^{1-1/m}\sigma(\varphi(r))u(\varphi(r))\left(\int_r^\infty \frac{b(s)ds}{\sigma^m(\varphi(s))}\int_l^r \frac{dk}{a(k)}\right)^{1/m}\\
&+ \sigma(\varphi(r))u(\varphi(r))\int_r^\infty \left(\frac{1}{a(k)}\int_k^\infty \frac{b(s)ds}{\sigma^m(s)}\right)^{1/m} dk.
\end{aligned}$$

When we replace l by $\varphi(r)$, we have

$$\begin{aligned}
1 \geq &(r-\varphi(r))^{1-1/m}\beta(\varphi(r))\left(\int_{\varphi(r)}^r \frac{b(s)}{\beta^m(\varphi(s))}\int_{\varphi(r)}^s \frac{dk}{a(k)}ds\right)^{1/m}\\
&+ (r-\varphi(r))^{1-1/m}\sigma(\varphi(r))\left(\int_r^\infty \frac{b(s)ds}{\sigma^m(\varphi(s))}\int_{\varphi(r)}^r \frac{dk}{a(k)}\right)^{1/m}\\
&+ \sigma(\varphi(r))\int_r^\infty \left(\frac{1}{a(k)}\int_k^\infty \frac{b(s)ds}{\sigma^m(s)}\right)^{1/m} dk.
\end{aligned}$$

This contradicts our initial assumption, which completes the proof. □

4. Example

To illustrate the applicability of our theoretical results, we consider a specific half-linear differential equation with delay. This example demonstrates how our criteria can be used to determine conditions under which all solutions are oscillatory.

Consider the half-linear differential equation with proportional delay

$$(r^\kappa (u'(r))^m)' = \frac{b_0}{r^{m-\kappa+1}}u^m(\lambda r) \tag{4.1}$$

where $b_0 > 0$ is a parameter and $\lambda \in (0, 1)$ is the delay coefficient.

This equation corresponds to our general form (1.1) with the following coefficient functions:

$$a(r) = r^\kappa, \quad b(r) = b_0 r^{-m-\kappa+1}, \quad \varphi(r) = \lambda r, \quad m \in (0, 1), \quad \kappa \in (-1, m+1).$$

To apply Theorem 3.8, we need to determine appropriate functions $\beta(r)$ and $\sigma(r)$. Based on the structure of our equation, we set

$$\Lambda(r) = \frac{A}{r}, \quad \beta(r) = r^A, \quad \Phi(r) = \frac{B}{r}, \quad \sigma(r) = r^B$$

where A and B are constants determined by

$$A = b_0^{1/m}\left[\frac{1-\lambda^{m-\kappa}}{m-\kappa}\right]^{1/m}$$
$$B = \frac{1-2\Lambda_1^*\Lambda_3^*}{(\Lambda_1^*)^2}\left[\frac{b_0}{m(A+1)-\kappa}\right]^{1/m}.$$

The auxiliary constants Λ_1, Λ_2, and Λ_3 are given by

$$\Lambda_1 = \frac{b_0^{1/m}\left(\lambda^{-3/2}-1\right)^{1-\frac{1}{m}}}{\lambda^{A/2}(\kappa-1)^{1/m}}\left[\frac{\lambda^{\frac{m(A+1)-1}{2}}-1}{m(A+1)-1}-\frac{\lambda^{\frac{m(A+1)-\kappa}{2}}-1}{m(A+1)-\kappa}\right]^{1/m}$$
$$\Lambda_2 = \frac{b_0^{1/m}\left(\lambda^{-3/2}-1\right)^{1-\frac{1}{m}}}{\lambda^{A}(\kappa-1)^{1/m}}\left[\frac{\lambda^{m(A+1)-1}-\lambda^{\frac{m(A+1)-1}{2}}}{m(A+1)-1}-\frac{\lambda^{m(A+1)-\kappa}-\lambda^{\frac{m(A+1)-\kappa}{2}}}{m(A+1)-\kappa}\right]^{1/m}$$
$$\Lambda_3 = \frac{b_0^{1/m}\left(\lambda^{-3/2}-1\right)^{1-\frac{1}{m}}}{\lambda^{3A/2}(\kappa-1)^{1/m}}\left[\frac{\lambda^{\frac{m(A+1)-1}{2/3}}-\lambda^{m(A+1)-1}}{m(A+1)-1}-\frac{\lambda^{\frac{m(A+1)-\kappa}{2/3}}-\lambda^{m(A+1)-\kappa}}{m(A+1)-\kappa}\right]^{1/m}.$$

By applying the expressions above to the criteria established in Theorems 3.3 and 3.8, we derive the following explicit conditions.

From Theorem 3.3, we obtain the criterion

$$\frac{b_0^{1/m}(1-\lambda)^{1-\frac{1}{m}}}{(\kappa-1)^{1/m}}\left[\frac{\lambda^{1-m(A+1)}-\lambda^{1-\kappa}}{m(A+1)-\kappa}+\frac{1-\lambda^{1-m(A+1)}}{m(A+1)-1}\right]^{1/m} > 1. \tag{4.2}$$

When we set $\lambda = 0.5$, $m = 0.9$, and $\kappa = 0.5$, numerical computation reveals that inequality (4.2) is satisfied when $b_0 \geq 2.8869$. According to Theorem 3.3, this ensures the absence of Kneser solutions to Eq (1.1) for these specific parameter values.

Correspondingly, the criterion derived from Theorem 3.8 is expressed as

$$\frac{b_0^{1/m}(1-\lambda)^{1-\frac{1}{m}}}{(\kappa-1)^{1/m}}\left[\frac{\lambda^{1-m(A+1)}-\lambda^{1-\kappa}}{m(A+1)-\kappa}+\frac{1-\lambda^{1-m(A+1)}}{m(A+1)-1}+\frac{\lambda^{1-\kappa}-1}{m(B+1)-\kappa}\right]^{1/m}+\frac{b_0^{1/m}}{B(m(B+1)-\kappa)^{1/m}} > 1. \tag{4.3}$$

Using the identical parameter configuration ($\lambda = 0.5, m = 0.9, \kappa = 0.5$), our numerical analysis demonstrates that inequality (4.3) holds when $b_0 \geq 2.6092$. This result, in conjunction with Theorem 3.8, confirms the non-existence of Kneser solutions to Eq (1.1) under these parameter constraints.

Our numerical comparison shows that Theorem 3.8 provides better results than Theorem 3.3, requiring less restrictive conditions. Specifically, Theorem 3.8 establishes the non-existence of Kneser solutions under a less restrictive threshold ($b_0 \geq 2.6092$) compared to the more stringent requirement of Theorem 3.3 ($b_0 \geq 2.8869$). This represents a significant enhancement in the oscillation criteria for half-linear differential equations with delay.

5. Conclusions

In this paper, we have investigated the qualitative behavior of solutions to second-order half-linear differential equations with deviating arguments. By extending and generalizing monotonicity-based techniques from the linear case, we developed new oscillation and nonoscillation criteria for half-linear equations with delay. Our approach accommodates a broader class of delay functions and nonlinearities, resulting in sharper and less restrictive conditions for the non-existence of Kneser solutions.

Through both theoretical analysis and examples, we demonstrated that our new criteria improve upon existing results in the literature, offering a more comprehensive understanding of the asymptotic properties of such equations. These findings not only unify and strengthen previous theorems but also open new avenues for further research in the qualitative theory of functional differential equations with delay.

Future work may explore the extension of these methods to more general classes of nonlinear equations, systems with multiple delays, or equations with variable exponents and more complex boundary conditions.

Author contributions

Pakize Temtek: Conceptualization, Methodology, Supervision, Project administration, Funding acquisition, Writing-review & editing; Yerzhan Turarov: Conceptualization, Methodology, Formal analysis, Investigation, Writing-original draft, Writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they used Artificial Intelligence (AI) tools to improve the language and clarity of the manuscript. AI tools used: ChatGPT (OpenAI). How were the AI tools used? To refine grammar, phrasing, and readability of the text. Where in the article is the information located? Language editing throughout the manuscript.

Conflict of interest

The authors declare that they have no conflicts of interest.

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