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Research article

Structured distance to normality of PDDT Toeplitz matrices

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Abstract: Investigating spectral properties and operator-space distance measurements, this research focused on tridiagonal Toeplitz matrices under perturbed Dirichlet boundary conditions (hereafter referred to as PDDT Toeplitz matrices). Explicit analytical expressions for eigenvalues and their associated eigenvectors were derived. These expressions emphasized their critical role in characterizing stability under perturbation conditions. Building on the structural features of PDDT Toeplitz matrices, we developed closed-form solutions to quantify normality distance and departure from normality. Additionally, these solutions analyzed ε-pseudospectra and eigenvalue sensitivity at both local and collective scales. By evaluating eigenvalue sensitivity through these parameters, our framework further enabled the evaluation of spectral sensitivity within PDDT Toeplitz environments. Through rigorous analysis, it has been shown that a dramatic increase in eigenvalue sensitivity depended exclusively on the magnitude ratio of lower to upper diagonal entries, demonstrating remarkable independence from diagonal terms or the complex phases of non-diagonal elements. The degree to which a matrix deviated from normality could be effectively characterized by examining the absolute values of its sub-diagonal and super-diagonal elements. To conclude, we explored an inverse eigenvalue problem embedded within a constrained optimization framework, which produced trapezoidal PDDT Toeplitz matrices as final optimal computational solutions.

Keywords: Toeplitz matrix; eigenvalue; condition number; normal matrix; distance; Frobenius norm; eigenvalue sensitivity; inverse eigenvalue problem
Mathematics Subject Classification: 15A18, 15A60

1. Introduction

When addressing wave equations via separation of variables in Cartesian coordinates, one frequently encounters eigenvalue problems of the form: find λ and a function $w(x)$ that satisfy [1]

$$\frac{d^2w}{dx^2} + \lambda w = 0, \tag{1.1}$$

subject to homogeneous Dirichlet boundary conditions (BCs) at τ and ψ

$$\begin{cases} D_\tau : & w(\tau) = 0, \\ D_\psi : & w(\psi) = 0. \end{cases} \tag{1.2}$$

When discretizing of Eq (1.1), let $x = \tau + ic$ and $w_i = w(\tau + ic)$, where c is the step size and i is an integer index $i = 1, \cdots, v$. A second-order finite difference discretization of Eq (1.1) yields:

$$\frac{w_{i-1} - 2w_i + w_{i+1}}{c^2} + \lambda_d w_i = 0. \tag{1.3}$$

This can be rewritten as

$$w_{i-1} + w_{i+1} = \left(2 - \lambda_d c^2\right) w_i, \tag{1.4}$$

where λ_d functions as the discrete representation of λ.

The translational invariance and reflection symmetry ($x \to -x$) of the operator $\frac{d^2}{dx^2}$ allow the BCs (1.2) to be satisfied by extending the function w. Both the left and right boundaries satisfy the Dirichlet BCs (odd-symmetry conditions), so w is extended as an odd function. By setting $\tau = c/2$ and $\psi = (v + 1)c$, w_i is extended using odd symmetry for all integers i. The difference scheme (1.3) is applied for $i = 1, \cdots, v$, and its matrix formulation is given by

$$\begin{pmatrix} 1 & 0 & 1 & & & & \\ & 1 & 0 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & 0 & 1 & \\ & & & & 1 & 0 & 1 \end{pmatrix}_{v\times(v+2)} \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ \vdots \\ w_{v-1} \\ w_v \\ w_{v+1} \end{pmatrix} = \varpi \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_{v-1} \\ w_v \end{pmatrix}, \tag{1.5}$$

where, according to Eq (1.4),

$$\varpi = 2 - \lambda_d c^2.$$

Observe that w_0 and w_{v+1} are positioned on the left-hand side of (1.5), and note that the matrix there is not square but has dimensions $v \times (v + 2)$. We now reformulate (1.5) by applying symmetry constraints at $c/2$ and $(v + 1)c$: the condition $w_0 = -w_1$ holds for $D_{c/2}$, while $w_{v+1} = 0$ applies to $D_{(v+1)c}$. This substitution removes w_0 and w_{v+1}, leaving w_1 and w_v as variables. Consequently, we derive the following eigenvalue problems for matrices

$$\begin{pmatrix} -1 & 1 & & & 0 \\ 1 & 0 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & 0 & 1 \\ 0 & & & 1 & 0 \end{pmatrix}_{v\times v} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_{v-1} \\ w_v \end{pmatrix} = \varpi \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_{v-1} \\ w_v \end{pmatrix}. \tag{1.6}$$

In (1.6), identical vectors are presented on the left and right sides, and the matrix is square. It is apparent that the diagonalization of the abovementioned coefficient matrix is possibly obtained via discrete sine transform type VII (DST7). The core of this study is to explore a wider set of matrix families diagonalizable via DST7.

Perturbed and standard tridiagonal Toeplitz matrices play an important role in various domains, such as in the potential solving of resistance network modeling [2, 3], quantum anomalous Hall effect analysis [4], and molecular orbital theory [5]. Also these matrices are useful to partial differential equations [6–9], time series decomposition [10], and regularization of discrete ill-posed problems via Tikhonov methods [11, 12]. Investigating the intrinsic computational characteristics of tridiagonal Toeplitz matrices is therefore of significant importance.

There are a variety of researches [13–17] that rigorously analyze the determinant properties and eigenvalue systems of bordered and periodic tridiagonal matrices. References [18–24] have systematically and in-depth discussed perturbed tridiagonal Toeplitz matrices, not only deriving explicit or approximate expressions for identities and inequalities related to their determinants, inverse matrices, and various norms, but also designing new numerical algorithms based on the above theoretical results. The analysis of two distinct regular matrix pairs has been systematically addressed in [25–27]. Matrix nearness problems have attracted significant interest among researchers, with notable contributions including [28–34] and subsequent investigations cited therein. The ε-pseudospectral separations related to banded Toeplitz matrices have been discussed in [35, 36], which provide foundational insights into their spectral behavior. Our focus on tridiagonal Toeplitz matrix operators originates from their unique capacity to compute expressions for critical metrics, coupled with their interdisciplinary relevance in applied mathematics. The investigation of multiple themes, notably inverse eigenvalue problems addressed in this work, has been pioneered by Biswa Datta in foundational studies [37–40].

This study focuses on PDDT Toeplitz matrices. It derives analytical expressions for eigenvalues and eigenvectors, constructs closed-form solutions for quantifying normality-related properties, analyzes ε-pseudospectra and eigenvalue sensitivity while establishing an evaluation framework, and explores inverse eigenvalue problems within the constrained optimization framework, resulting in several findings that can serve as references.

The paper is structured in the following manner. First, we introduce the symbols and definitions used throughout the work. Section 2 examines analytic expressions for the eigenvalues and eigenvectors of PDDT Toeplitz matrices. In Sections 3 and 4, the structured distances between PDDT Toeplitz matrices and the two algebraic varieties are explored. Eigenvalue sensitivity is explored in Section 5, and Section 6 presents numerical examples. An inverse eigenvalue problem is tackled in Section 7, where a minimization problem is formulated and the solution gives a trapezoidal PDDT Toeplitz matrix.

Concept and symbol

The Euclidean vector norm $\|\cdot\|_2$ and its induced matrix counterpart are employed in this work. In addition, the Frobenius norm $\|\cdot\|_F$, which has its own distinct applications, is used to quantify matrices and vectors. The following are some explanations of the symbols used in this paper.

(1) $\widetilde{D}$: A subspace of $\mathbf{C}^{\nu\times\nu}$ containing PDDT Toeplitz matrices.

(2) $\mathcal{A}$: An algebraic variety in $\mathbf{C}^{v\times v}$ consisting of normal matrices.

(3) $\mathcal{A}_{\widetilde{D}}$: $\mathcal{A}\bigcap\widetilde{D}$

(4) $\mathcal{B}$: The algebraic set within $\mathbf{C}^{v\times v}$ comprising matrices whose eigenvalues exhibit multiplicity.

(5) $\mathcal{B}_{\widetilde{D}}$: $\mathcal{B}\bigcap\widetilde{D}$

(6) $(\cdot)^T$: The symbol denotes the transpose operation.

(7) $(\cdot)^H$: The symbol indicates Hermitian adjoint.

Definition 1.

$$\mathbb{C}=\begin{bmatrix} \beta_0-\sqrt{\beta_1\beta_2} & \beta_1 & & & & O \\ \beta_2 & \beta_0 & \beta_1 & & & \\ & \beta_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & \beta_1 \\ O & & & & & \beta_2 & \beta_0 \end{bmatrix}\in\mathbf{C}^{v\times v}. \tag{1.7}$$

A square matrix of order v with Eq. (1.7) is abbreviated as the PDDT Toeplitz matrix and it is represented as $\mathbb{C}=(v;-\sqrt{\beta_1\beta_2},\beta_2,\beta_0,\beta_1)$. *Additionally, we define*

$$\omega_1=\arg\beta_2,\omega_2=\arg\beta_1,\omega_3=\arg\beta_0. \tag{1.8}$$

Specifically, for $\beta_0=0$, *the matrix has the symbolic form* $\mathbb{C}_0=(v;-\sqrt{\beta_1\beta_2},\beta_2,0,\beta_1)$.

Definition 2. *The Frobenius norm distance from a matrix* $\mathbb{E}\in\mathbf{C}^{v\times v}$ *to the set of normal matrices is defined as the minimum Frobenius norm of the difference between* $\mathbb{E}$ *and* $\mathbb{E}_{\mathcal{A}}\in\mathcal{A}$, *given by*

$$d_F(\mathbb{E},\mathcal{A})=\min_{\mathbb{E}_{\mathcal{A}}\in\mathcal{A}}\|\mathbb{E}-\mathbb{E}_{\mathcal{A}}\|_F. \tag{1.9}$$

Detailed discussions are provided in [28, 30, 41–44].

Definition 3. *We use the Frobenius norm to measure the deviation of the PDDT Toeplitz matrix* $\mathbb{C}$ *from the family* $\mathcal{A}_{\widetilde{D}}$. *This deviation is defined as*

$$d_F(\mathbb{C},\mathcal{A}_{\widetilde{D}})=\min_{\mathbb{C}_{\mathcal{A}}\in\mathcal{A}_{\widetilde{D}}}\|\mathbb{C}-\mathbb{C}_{\mathcal{A}}\|_F. \tag{1.10}$$

Definition 4. *For the PDDT Toeplitz matrix* $\mathbb{C}$, *its distance to the family* $\mathcal{B}_{\widetilde{D}}$ *under the Frobenius norm is defined as*

$$d_F(\mathbb{C},\mathcal{B}_{\widetilde{D}})=\min_{\mathbb{C}_{\mathcal{B}}\in\mathcal{B}_{\widetilde{D}}}\|\mathbb{C}-\mathbb{C}_{\mathcal{B}}\|_F. \tag{1.11}$$

2. Eigenvalues and their associated eigenvectors

This section employs similarity transformations to construct explicit representations of the eigenvalues and eigenvectors for the PDDT Toeplitz matrix $\mathbb{C}$, as introduced in (1.7). Suppose

$$\Delta_v = \text{diag}(1, \sqrt{\frac{\beta_1}{\beta_2}}, \cdots, (\sqrt{\frac{\beta_1}{\beta_2}})^{v-2}, (\sqrt{\frac{\beta_1}{\beta_2}})^{v-1}).$$

When $|\beta_1| = |\beta_2|$, it follows directly that Δ_v becomes a unitary matrix. From this property, it follows that $\Delta_v \Delta_v^H = I_v$, or equivalently $\Delta_v^{-1} = \Delta_v^H$. In this context, I_v is the identity matrix of dimension v, and Δ_v^H denotes the Hermitian adjoint of Δ_v.

Let

$$\mathbb{S}_v^{VII} = \left(\frac{2}{\sqrt{2v+1}} \sin \frac{(2k-1)j\pi}{2v+1} \right)_{k,j=1}^{v}.$$

The seventh discrete sine transform matrix $\mathbb{S}_v^{VII}$ is defined by its orthogonality property [45], and it automatically obeys

$$(\mathbb{S}_v^{VII})^{-1} = (\mathbb{S}_v^{VII})^T = \mathbb{S}_v^{VI}.$$

Here, $\mathbb{S}_v^{VI}$ [45] is the sixth discrete sine transform matrix.

The following equation is derived through diagonalization of $\mathbb{C}$ and a series of matrix multiplications and sophisticated algebraic manipulations.

$$(\mathbb{S}_v^{VII})^{-1} \Delta_v \mathbb{C} \Delta_v^{-1} (\mathbb{S}_v^{VII}) = \text{diag}(\lambda_1, \ldots, \lambda_v). \tag{2.1}$$

Eq (2.1) allows us to express $\mathbb{C}$ as

$$\mathbb{C} = \Delta_v^{-1} (\mathbb{S}_v^{VII}) \text{diag}(\lambda_1, \ldots, \lambda_v) (\mathbb{S}_v^{VII})^{-1} \Delta_v,$$

with λ_j defined by

$$\lambda_j = \beta_0 + 2\sqrt{\beta_1 \beta_2} \cos \frac{2j\pi}{2v+1}, \ j = 1, \ldots, v. \tag{2.2}$$

As explicitly shown in Eq (2.1), the PDDT Toeplitz matrix $\mathbb{C} = (\text{v}; -\sqrt{\beta_1\beta_2}, \beta_2, \beta_0, \beta_1)$ with these prescribed entries can be diagonalized to $\text{diag}(\lambda_1, \lambda_2, ..., \lambda_v)$, where each λ_j represents an eigenvalue of $\mathbb{C}$.

Left-multiplying Eq (2.1) by $\Delta_v^{-1} \mathbb{S}_v^{VII}$ yields

$$\mathbb{C} \Delta_v^{-1} \mathbb{S}_v^{VII} = \Delta_v^{-1} \mathbb{S}_v^{VII} \text{diag}(\lambda_1, \ldots, \lambda_v),$$

which can be rewritten as

$$\mathbb{C}(\xi^{(1)}, \xi^{(2)}, \cdots, \xi^{(v)}) = (\xi^{(1)}, \xi^{(2)}, \cdots, \xi^{(v)}) \text{diag}(\lambda_1, \ldots, \lambda_v), \tag{2.3}$$

where the vectors $\xi^{(j)} = (\xi_1^{(j)}, \ldots, \xi_v^{(j)})^T$ are defined by

$$\xi_k^{(j)} = \frac{2}{\sqrt{2v+1}}(\sqrt{\frac{\beta_2}{\beta_1}})^{k-1} \sin\frac{(2k-1)\,j\pi}{2v+1}, k = 1, \ldots, v, \quad j = 1, \ldots, v.$$

Eq (2.3) can be reformulated as

$$\mathbb{C}\xi^{(j)} = \lambda_j \xi^{(j)}, j = 1, 2, \ldots, v. \tag{2.4}$$

The right eigenvector $\xi^{(j)} = (\xi_1^{(j)}, \ldots, \xi_v^{(j)})^T$ corresponding to λ_j is derived directly from Eq. (2.4).

From Eqs (1.8) and (2.2), the eigenvalues of $\mathbb{C}$ are given by

$$\lambda_j(\mathbb{C}) = \beta_0 + 2\sqrt{|\beta_1\beta_2|}e^{\frac{(\omega_1+\omega_2)i}{2}} \cos\frac{2j\pi}{2v+1}, j = 1, \ldots, v. \tag{2.5}$$

When $\beta_1\beta_2 \neq 0$, the spectrum of $\mathbb{C}$ consists of v distinct eigenvalues. These lie on the closed segment

$$\mathcal{S}_{\lambda(\mathbb{C})} = \left\{\beta_0 + ae^{\frac{(\omega_1+\omega_2)i}{2}} : a \in \mathbf{R}, |a| \leq 2\sqrt{|\beta_1\beta_2|}\cos\frac{2\pi}{2v+1}\right\} \subset \mathbf{C}, \tag{2.6}$$

which is a subset of the complex plane. For the matrix $\mathbb{C}$, its spectral radius can be expressed as:

$$\rho(\mathbb{C}) = \max\left\{\left|\beta_0 + 2\sqrt{|\beta_1\beta_2|}e^{\frac{(\omega_1+\omega_2)i}{2}} \cos\frac{2\pi}{2v+1}\right|, \left|\beta_0 + 2\sqrt{|\beta_1\beta_2|}e^{\frac{(\omega_1+\omega_2)i}{2}} \cos\frac{2v\pi}{2v+1}\right|\right\}.$$

When $\mathbb{C}$ is nonsingular ($\lambda_j(\mathbb{C}) \neq 0$, $j = 1, \ldots, v$), the spectral radius of $\mathbb{C}^{-1}$ can be derived from Eq (2.5) as

$$\rho(\mathbb{C}^{-1}) = \max_{j=1,\ldots,v}\left|\beta_0 + 2\sqrt{|\beta_1\beta_2|}e^{\frac{(\omega_1+\omega_2)i}{2}} \cos\frac{2j\pi}{2v+1}\right|^{-1}.$$

If $\beta_1\beta_2 \neq 0$, the entries of the right eigenvector $\xi^{(j)} = [\xi_1^{(j)}, \cdots, \xi_v^{(j)}]^T$, associated with the eigenvalue $\lambda_j(\mathbb{C})$, are defined by

$$\xi_k^{(j)} = \frac{2}{\sqrt{2v+1}}(\sqrt{\frac{\beta_2}{\beta_1}})^{k-1} \sin\frac{(2k-1)\,j\pi}{2v+1}, k = 1, \ldots, v, j = 1, \ldots, v. \tag{2.7}$$

Similarly, the left eigenvector $\chi^{(j)} = [\chi_1^{(j)}, \cdots, \chi_v^{(j)}]^T$ is given by

$$\chi_k^{(j)} = \frac{2}{\sqrt{2v+1}}(\sqrt{\frac{\bar{\beta}_1}{\bar{\beta}_2}})^{k-1} \sin\frac{(2k-1)\,j\pi}{2v+1}, k = 1, \ldots, v, j = 1, \ldots, v, \tag{2.8}$$

where the overline denotes complex conjugation.

As demonstrated above, specifying the dimension and the $\frac{\beta_2}{\beta_1}$ ratio of the PDDT Toeplitz matrix $\mathbb{C}$ enables one to unambiguously determine all its left and right eigenvectors.

3. The structured distance between PDDT Toeplitz and the family of normal matrices

Focusing on a PDDT Toeplitz matrix, this section investigates both the distance from and the deviation of the family of normal matrices.

Theorem 1. *$\mathbb{C}$ defined by (1.7) is normal if, and only if, the equality*

$$|\beta_1| = |\beta_2| \tag{3.1}$$

holds.

Proof. The equality $\mathbb{C}^H\mathbb{C} = \mathbb{C}\mathbb{C}^H$ serves as an equivalent characterization of the condition (3.1). □

The above theorem reveals that a normal PDDT Toeplitz matrix can be represented in the following way:

$$\mathbb{C}' = (v; -\Gamma e^{\frac{(\omega_1'+\omega_2')i}{2}}, \Gamma e^{i\omega_1'}, \beta_0, \Gamma e^{i\omega_2'}) = \begin{bmatrix} \beta_0 - \Gamma e^{\frac{(\omega_1'+\omega_2')i}{2}} & \Gamma e^{i\omega_2'} & & & O \\ \Gamma e^{i\omega_1'} & \beta_0 & \Gamma e^{i\omega_2'} & & \\ & \Gamma e^{i\omega_1'} & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot \quad \cdot \\ & & & \cdot & \cdot \quad \Gamma e^{i\omega_2'} \\ O & & & \Gamma e^{i\omega_\Gamma'} & \beta_0 \end{bmatrix}, \tag{3.2}$$

where the parameters satisfy $\beta_0 \in \mathbf{C}$, $\Gamma \geq 0$, and ω_1', $\omega_2' \in \mathbf{R}$. Eq (2.5) provides an explicit characterization of the eigenvalues of the matrix $\mathbb{C}'$:

$$\lambda_j(\mathbb{C}') = \beta_0 + 2\Gamma e^{\frac{(\omega_1'+\omega_2')i}{2}} \cos\frac{2j\pi}{2v+1}, \; j = 1, \ldots, v.$$

The eigenvalues of $\mathbb{C}'$ lie on a closed line segment in the complex plane, parameterized as

$$\mathcal{S}_{\lambda(\mathbb{C}')} = \left\{\beta_0 + ae^{\frac{(\omega_1'+\omega_2')i}{2}} : a \in \mathbf{R}, |a| \leq 2\Gamma\cos\frac{2\pi}{2v+1}\right\} \subset \mathbf{C}.$$

Theorem 2. *When $\mathbb{C} = (v; -\sqrt{\beta_1\beta_2}, \beta_2, \beta_0, \beta_1)$ represents a PDDT Toeplitz matrix, the Frobenius distance minimization problem $\|\mathbb{C}_{\mathcal{A}} - \mathbb{C}\|_F$ admits a unique solution $\mathbb{C}^* = (v; -\sqrt{\beta_1^*\beta_2^*}, \beta_2^*, \beta_0^*, \beta_1^*) \in \mathcal{A}_{\widetilde{D}}$, whose entries are explicitly parameterized as*

$$\beta_0^* = \beta_0, \beta_1^* = \Gamma^* e^{i\omega_2}, \beta_2^* = \Gamma^* e^{i\omega_1}.$$

where ω_1 and ω_2 are specified by (1.8) *and* $\Gamma^* = \frac{|\beta_1|+|\beta_2|}{2} - \frac{(\sqrt{|\beta_1|}-\sqrt{|\beta_2|})^2}{2(2v-1)}$.

Proof. The Frobenius norm $\min\|\mathbb{C}-\mathbb{C}_{\mathcal{A}}\|_F$ is achieved by the matrix $\mathbb{C}^* = (v; -\sqrt{\beta_1^*\beta_2^*}, \beta_2^*, \beta_0^*, \beta_1^*) \in \mathcal{A}_{\widetilde{D}}$, with the equality $|\beta_1^*| = |\beta_2^*|$ prescribed by Theorem 1. The parameter assignments are necessarily chosen as

$$\beta_0^* = \beta_0, \beta_1^* = \Gamma^* e^{i\omega_2}, \beta_2^* = \Gamma^* e^{i\omega_1}.$$

Meanwhile, the parameter Γ^* is required to achieve the global minimum of the objective function $g(\Gamma) = (\Gamma - \sqrt{|\beta_1\beta_2|})^2 + (v-1)((\Gamma-\beta_1)^2 + (\Gamma-\beta_2)^2)$. The unique minimum is expressed as $\Gamma^* = \frac{|\beta_1|+|\beta_2|}{2} - \frac{(\sqrt{|\beta_1|}-\sqrt{|\beta_2|})^2}{2(2v-1)}$. Then, the theorem is proved. □

Corollary 1. *The eigenvalues of* $\mathbb{C}^* = (v; -\sqrt{\beta_1^*\beta_2^*}, \beta_2^*, \beta_0^*, \beta_1^*)$*, the closest normal PDDT Toeplitz matrix to* $\mathbb{C} = (v; -\sqrt{\beta_1\beta_2}, \beta_2, \beta_0, \beta_1)$*, are given by:*

$$\lambda_j(\mathbb{C}^*) = \beta_0 + 2\Gamma^* e^{\frac{(\omega_1+\omega_2)i}{2}} \cos\frac{2j\pi}{2v+1},\ j = 1, \dots, v, \tag{3.3}$$

where ω_1 *and* ω_2 *are defined by* (1.8)*. The eigenvalues are contained in the closed line segment*

$$\mathcal{S}_{\lambda(\mathbb{C}^*)} = \{\beta_0 + ae^{\frac{(\omega_1+\omega_2)i}{2}} : a \in \mathbf{R}, |a| \le 2\Gamma^* \cos\frac{2\pi}{2v+1}\}.$$

From the relation

$$|\beta_1| + |\beta_2| - \frac{(\sqrt{|\beta_1|} - \sqrt{|\beta_2|})^2}{2v-1} - 2\sqrt{|\beta_1\beta_2|} = \frac{2(v-1)}{2v-1}(\sqrt{|\beta_1|} - \sqrt{|\beta_2|})^2, \tag{3.4}$$

it follows that $\mathbb{C} \notin \mathcal{A}_{\widetilde{D}}$ if, and only if, the line segment of (2.6) is contained in this segment. The spectral radius of $\mathbb{C}^*$ is characterized by

$$\rho(\mathbb{C}^*) = \max\left\{\left|\beta_0 + 2\Gamma^* e^{\frac{(\omega_1+\omega_2)i}{2}} \cos\frac{2\pi}{2v+1}\right|, \left|\beta_0 + 2\Gamma^* e^{\frac{(\omega_1+\omega_2)i}{2}} \cos\frac{2v\pi}{2v+1}\right|\right\},$$

where $\Gamma^* = \frac{|\beta_1|+|\beta_2|}{2} - \frac{(\sqrt{|\beta_1|}-\sqrt{|\beta_2|})^2}{2(2v-1)}$.

For the PDDT Toeplitz matrix, the following result gives a concise expression for the distance to normality.

Theorem 3. *Define PDDT Toeplitz matrix* $\mathbb{C} = (v; -\sqrt{\beta_1\beta_2}, \beta_2, \beta_0, \beta_1)$*. The Frobenius distance from this matrix to the family* $\mathcal{A}_{\widetilde{D}}$ *is given by*

$$d_F(\mathbb{C}, \mathcal{A}_{\widetilde{D}}) = \sqrt{\frac{v-1}{2v-1}\left[v(|\beta_1| - |\beta_2|)^2 - 2\sqrt{|\beta_1\beta_2|}\left(\sqrt{|\beta_1|} - \sqrt{|\beta_2|}\right)^2\right]}, \tag{3.5}$$

where $\mathcal{A}_{\widetilde{D}}$ denotes the algebraic variety of normal PDDT Toeplitz matrices.

Proof. The proof is carried out by substituting Theorem 2 and Eq (3.4) into Eq (1.10)

$$\begin{aligned} \|\mathbb{C} - \mathbb{C}^*\|_F^2 &= (v-1)(|\beta_1 - \beta_1^*|^2 + |\beta_2 - \beta_2^*|^2) + |\sqrt{\beta_1\beta_2} - \sqrt{\beta_1^*\beta_2^*}|^2 \\ &= (v-1)(||\beta_1| - |\beta_1^*||^2 + ||\beta_2| - |\beta_2^*||^2) + |\sqrt{|\beta_1||\beta_2|} - \sqrt{|\beta_1^*\beta_2^*|}|^2 \\ &= (v-1)(||\beta_1| - l^*|^2 + ||\beta_2| - l^*|^2) + |\sqrt{|\beta_1\beta_2|} - l^*|^2 \\ &= (v-1)(|\beta_1|^2| + |\beta_2|^2) + |\beta_1||\beta_2| - (2v-1)(l^*)^2 \\ &= \tfrac{v(v-1)}{2v-1}(|\beta_1| - |\beta_2|)^2 - \tfrac{2(v-1)}{2v-1}\sqrt{|\beta_1\beta_2|}\left(\sqrt{|\beta_1|} - \sqrt{|\beta_2|}\right)^2. \end{aligned}$$

This completes the proof. □

Remark 1. *For the PDDT Toeplitz matrix* $\mathbb{C}$*, Theorem 3 shows independence of* $d_F(\mathbb{C}, \mathcal{A}_{\widetilde{D}})$ *from* β_0*. However, the normal PDDT Toeplitz matrix* $\mathbb{C}^*$ *that is closest to* $\mathbb{C}$ *will change with the variation of* β_0*. In other words, matrices varying only in* β_0 *maintain the same distance to* $\mathcal{A}_{\widetilde{D}}$*, yet exhibit distinct projections onto this algebraic variety. Furthermore, the pair* $\mathbb{C}_1 = (v, -\sqrt{\beta_1\beta_2}, \beta_2, \beta_0^1, \beta_1)$ *and* $\mathbb{C}_2 = (v, -\sqrt{\beta_1\beta_2}, \beta_2, \beta_0^2, \beta_1)$ *obeys the equality:*

$$\|\mathbb{C}_1^* - \mathbb{C}_2^*\|_F = \|\mathbb{C}_1 - \mathbb{C}_2\|_F = \sqrt{v}|\beta_0^1 - \beta_0^2|.$$

3.1. The distance from matrix $\mathbb{C}_0$ to matrix family $\mathcal{A}_{\widetilde{D}}$

The irregularity of a matrix $\mathbb{E}$ is quantified by the Frobenius norm deviation from the set of normal matrices, and is defined in [42] as

$$\Delta_F(\mathbb{E}) = (\|\mathbb{E}\|_F^2 - \sum_{j=1}^{v} |\lambda_j|^2)^{\frac{1}{2}}, \mathbb{E} \in \mathbf{C}^{v\times v}. \tag{3.6}$$

In order to calculate the formula, we have

$$\sum_{j=1}^{v} \cos^2 \frac{2j\pi}{2v+1} = \frac{2v-1}{4}. \tag{3.7}$$

Proof. The double angle formula transforms the original summation:

$$\sum_{j=1}^{v} \cos^2 \frac{2j\pi}{2v+1} = \frac{v}{2} + \frac{1}{2}\sum_{j=1}^{v} \cos \frac{4j\pi}{2v+1}.$$

Applying the product-to-sum formula, we simplify the summation

$$\sin\frac{2\pi}{2v+1}\sum_{j=1}^{v} \cos \frac{4j\pi}{2v+1} = \frac{1}{2}(\sin 2\pi - \sin\frac{2\pi}{2v+1}) = -\frac{1}{2}\sin\frac{2\pi}{2v+1},$$

which directly implies that

$$\sum_{j=1}^{v} \cos \frac{4j\pi}{2v+1} = \frac{-\frac{1}{2}\sin\frac{2\pi}{2v+1}}{\sin\frac{2\pi}{2v+1}} = -\frac{1}{2}.$$

Consequently,

$$\sum_{j=1}^{v} \cos^2 \frac{2j\pi}{2v+1} = \frac{v}{2} + \frac{1}{2}\sum_{j=1}^{v} \cos \frac{4j\pi}{2v+1} = \frac{v}{2} - \frac{1}{4} = \frac{2v-1}{4}.$$

□

Theorem 4. *For the PDDT Toeplitz matrix $\mathbb{C}_0$, its deviation $\Delta_F(\mathbb{C}_0)$ from the family $\mathcal{A}_{\widetilde{D}}$ relates to the distance $d_F(\mathbb{C}_0, \mathcal{A}_{\widetilde{D}})$ via:*

$$\Delta_F(\mathbb{C}_0) = \frac{\sqrt{(2v-1)}\,||\beta_1| - |\beta_2||}{\sqrt{v(|\beta_1| - |\beta_2|)^2 - 2\sqrt{|\beta_1\beta_2|}(\sqrt{|\beta_1|} - \sqrt{|\beta_2|})^2}} d_F(\mathbb{C}_0, \mathcal{A}_{\widetilde{D}}), \tag{3.8}$$

where the matrix family $\mathcal{A}_{\widetilde{D}}$ consists of the normal PDDT Toeplitz matrices.

Proof. Applying Eqs (3.6) and (3.7), the Frobenius deviation of the PDDT Toeplitz matrix $\mathbb{C}_0$ from the matrix family $\mathcal{A}_{\widetilde{D}}$ is computed as:

$$\Delta_F(\mathbb{C}_0) = [(\sqrt{|\beta_1\beta_2|})^2 + (v-1)(|\beta_1|^2 + |\beta_2|^2) - 4|\beta_1\beta_2| \sum_{j=1}^{v} \cos^2 \frac{2j\pi}{2v+1}]^{\frac{1}{2}}$$

$$= [(v-1)(|\beta_1|^2 + |\beta_2|^2 - 2|\beta_1||\beta_2|)]^{\frac{1}{2}}$$
$$= \sqrt{v-1}\,||\beta_1| - |\beta_2||\,. \tag{3.9}$$

The deviation can be expressed compactly using (3.5)

$$\Delta_F(\mathbb{C}_0) = \frac{\sqrt{(2v-1)}\,||\beta_1| - |\beta_2||}{\sqrt{v(|\beta_1| - |\beta_2|)^2 - 2\sqrt{|\beta_1\beta_2|}\left(\sqrt{|\beta_1|} - \sqrt{|\beta_2|}\right)^2}} d_F(\mathbb{C}_0, \mathcal{A}_{\widetilde{D}}).$$

□

Lemma 1. *[43] The Frobenius distance $d_F(\mathbb{E}, \mathcal{A})$ between a matrix $\mathbb{E} \in \mathbf{C}^{v\times v}$ and the normal matrix family $\mathcal{A}$ is bounded by:*

$$\frac{\Delta_F(\mathbb{E})}{\sqrt{v}} \leq d_F(\mathbb{E}, \mathcal{A}) \leq \Delta_F(\mathbb{E}). \tag{3.10}$$

It explicitly links the deviation $\Delta_F(\mathbb{C}_0)$ and the distance.

Corollary 2. *Let ø be defined by*

$$ø = \frac{\sqrt{(2v-1)}\,||\beta_1| - |\beta_2||}{\sqrt{v(|\beta_1| - |\beta_2|)^2 - 2\sqrt{|\beta_1\beta_2|}\left(\sqrt{|\beta_1|} - \sqrt{|\beta_2|}\right)^2}}, \tag{3.11}$$

Equation (3.8) can be rewritten as

$$\Delta_F(\mathbb{C}_0) = ø d_F(\mathbb{C}, \mathcal{A}_{\widetilde{D}}). \tag{3.12}$$

Inserting Eq (3.12) into Eq (3.10), we obtain

$$\frac{ø}{\sqrt{v}} d_F(\mathbb{C}_0, \mathcal{A}_{\widetilde{D}}) \leq d_F(\mathbb{C}_0, \mathcal{A}) \leq ø d_F(\mathbb{C}_0, \mathcal{A}_{\widetilde{D}}). \tag{3.13}$$

3.2. The distance of the spectra of $\mathbb{C}$ and $\mathbb{C}^*$

Theorem 5. *Let $\mathbb{C}^*$ be the nearest normal PDDT Toeplitz matrix to $\mathbb{C} = (v; -\sqrt{\beta_1\beta_2}, \beta_2, \beta_0, \beta_1)$. We define the eigenvalue vectors of $\mathbb{C}$ and $\mathbb{C}^*$ as follows:*

$$\vec{\delta} = [\lambda_1(\mathbb{C}), \lambda_2(\mathbb{C}), \cdots, \lambda_v(\mathbb{C})],$$

$$\vec{\delta}^* = [\lambda_1(\mathbb{C}^*), \lambda_2(\mathbb{C}^*), \cdots, \lambda_v(\mathbb{C}^*)].$$

The Euclidean distance between their eigenvalue vectors is

$$||\vec{\delta} - \vec{\delta}^*||_2 = \frac{v-1}{\sqrt{2v-1}}(\sqrt{|\beta_1|} - \sqrt{|\beta_2|})^2. \tag{3.14}$$

Also,

$$\lim_{\mathbb{C}\to\mathbb{C}^*} \frac{||\vec{\delta} - \vec{\delta}^*||_2}{d_F(\mathbb{C}, \mathcal{A}_{\widetilde{D}})} = 0, \tag{3.15}$$

where $d_F(\mathbb{C}, \mathcal{A}_{\widetilde{D}})$ is given by (3.5).

Proof. By combining the eigenvalue expressions given in Eqs (2.2) and (3.3) with the trigonometric summation identity in Eq (3.7), we derive the spectral distance as follows:

$$\begin{aligned}
\|\vec{\delta}-\vec{\delta}^{*}\|_2 &= \sqrt{\sum_{j=1}^{v}|\lambda_j(\mathbb{C})-\lambda_j(\mathbb{C}^{*})|^2}\\
&= \frac{2(v-1)}{2v-1}(\sqrt{|\beta_1|}-\sqrt{|\beta_2|})^2\sqrt{\sum_{j=1}^{v}\cos^2\frac{2j\pi}{2v+1}}\\
&= \frac{v-1}{\sqrt{2v-1}}(\sqrt{|\beta_1|}-\sqrt{|\beta_2|})^2.
\end{aligned}$$

Equation (3.15) is proved by substituting the expressions from Eqs (3.5) and (3.14) into the limit expression and then performing the necessary algebraic manipulations. □

Theorem 6. *When $\mathbb{C}\notin\mathcal{A}_{\widetilde{D}}$, we have*

$$\frac{\|\vec{\delta}-\vec{\delta}^{*}\|_2}{d_F(\mathbb{C},\mathcal{A}_{\widetilde{D}})}=\frac{\sqrt{v-1}(\sqrt{|\beta_1|}-\sqrt{|\beta_2|})}{\sqrt{v(\sqrt{|\beta_1|}+\sqrt{|\beta_2|})^2-2\sqrt{|\beta_1\beta_2|}}}. \tag{3.16}$$

Proof. Applying Eqs (3.5) and (3.14), we obtain:

$$\frac{\|\vec{\delta}-\vec{\delta}^{*}\|_2}{d_F(\mathbb{C},\mathcal{A}_{\widetilde{D}})}=\frac{\sqrt{v-1}(\sqrt{|\beta_1|}-\sqrt{|\beta_2|})}{\sqrt{v(\sqrt{|\beta_1|}+\sqrt{|\beta_2|})^2-2\sqrt{|\beta_1\beta_2|}}}.$$

□

3.3. Normalized distance of matrix $\mathbb{C}_0$ to the $\mathcal{A}_{\widetilde{D}}$ family

Theorem 7. *For a PDDT Toeplitz matrix $\mathbb{C}_0$ with $(\beta_1,\beta_2)\neq(0,0)$, the normalized Frobenius distance to the family of normal PDDT Toeplitz matrices $\mathcal{A}_{\widetilde{D}}$ is given by:*

$$\frac{d_F(\mathbb{C}_0,\mathcal{A}_{\widetilde{D}})}{\|\mathbb{C}_0\|_F}=\frac{\sqrt{(v-1)\left[v(\frac{|\beta_2|}{|\beta_1|}-1)^2-2\sqrt{\frac{|\beta_2|}{|\beta_1|}}(\sqrt{\frac{|\beta_2|}{|\beta_1|}}-1)^2\right]}}{\sqrt{(2v-1)\left[\frac{|\beta_2|}{|\beta_1|}+(v-1)\left(\left(\frac{|\beta_2|}{|\beta_1|}\right)^2+1\right)\right]}}. \tag{3.17}$$

It holds that

$$0\leq\frac{d_F(\mathbb{C}_0,\mathcal{A}_{\widetilde{D}})}{\|\mathbb{C}_0\|_F}<\sqrt{\frac{v}{2v-1}}.$$

Proof. In the case where $\beta_1\beta_2\neq 0$, we have the following equalities by Eq (3.5):

$$\frac{d_F(\mathbb{C}_0,\mathcal{A}_{\widetilde{D}})}{\|\mathbb{C}_0\|_F}=\frac{\sqrt{(v-1)\left[v(|\beta_1|-|\beta_2|)^2-2\sqrt{|\beta_1\beta_2|}(\sqrt{|\beta_1|}-\sqrt{|\beta_2|})^2\right]}}{\sqrt{(2v-1)\left[|\beta_1\beta_2|+(v-1)(|\beta_1|^2+|\beta_2|^2)\right]}}$$

$$= \frac{\sqrt{(v-1)\,[v(\frac{|\beta_1|}{|\beta_2|}-1)^2 - 2\sqrt{\frac{|\beta_1|}{|\beta_2|}}(\sqrt{\frac{|\beta_1|}{|\beta_2|}}-1)^2]}}{\sqrt{(2v-1)\,[\frac{|\beta_1|}{|\beta_2|}+(v-1)((\frac{|\beta_1|}{|\beta_2|})^2+1)]}}$$

$$= \frac{\sqrt{(v-1)\,[v(\frac{|\beta_2|}{|\beta_1|}-1)^2 - 2\sqrt{\frac{|\beta_2|}{|\beta_1|}}(\sqrt{\frac{|\beta_2|}{|\beta_1|}}-1)^2]}}{\sqrt{(2v-1)\,[\frac{|\beta_2|}{|\beta_1|}+(v-1)((\frac{|\beta_2|}{|\beta_1|})^2+1)]}}.$$

By Eq (3.17), we derive the following bounds for the normalized structured distance from $\mathbb{C}_0$ to normality $\mathcal{A}_{\widetilde{D}}$

$$0 \le \frac{d_F(\mathbb{C}_0, \mathcal{A}_{\widetilde{D}})}{||\mathbb{C}_0||_F} < \sqrt{\frac{v}{2v-1}}.$$

As either ratio $\frac{|\beta_1|}{|\beta_2|}$ or $\frac{|\beta_2|}{|\beta_1|}$ grows from 0 to 1, the normalized structured distance from $\mathbb{C}_0$ to normality $\mathcal{A}_{\widetilde{D}}$ decreases from $\sqrt{\frac{v}{2v-1}}$ to 0. □

Remark 2. *By Eq (3.17), the normalized distance from the matrix $\mathbb{C}_0$ to $\mathcal{A}_{\widetilde{D}}$ is*

$$\frac{d_F(\mathbb{C}_0, \mathcal{A}_{\widetilde{D}})}{||\mathbb{C}_0||_F} = \begin{cases} 0 & |\beta_1| = |\beta_2|, \\ \sqrt{\frac{v}{2v-1}} & \text{exactly one of } \beta_1 \text{ and } \beta_2 \text{ is zero.} \end{cases} \tag{3.18}$$

Remark 3. *The equality $d_F(\mathbb{C}, \mathcal{A}_{\widetilde{D}}) = d_F(\mathbb{C}_0, \mathcal{A}_{\widetilde{D}})$ clearly holds. Consequently, the normalized distance range for the PDDT Toeplitz matrix $\mathbb{C}$ to $\mathcal{A}_{\widetilde{D}}$ is bounded as shown in the following derivation:*

$$\begin{aligned} 0 \le \frac{d_F(\mathbb{C}, \mathcal{A}_{\widetilde{D}})}{||\mathbb{C}||_F} &= \frac{d_F(\mathbb{C}_0, \mathcal{A}_{\widetilde{D}})}{||\mathbb{C}_0||_F}\frac{||\mathbb{C}_0||_F}{||\mathbb{C}||_F} \\ &= \frac{d_F(\mathbb{C}_0, \mathcal{A}_{\widetilde{D}})}{||\mathbb{C}_0||_F}\sqrt{\frac{|\beta_1\beta_2| + (v-1)(|\beta_1|^2+|\beta_2|^2)}{|\beta_1\beta_2| + (v-1)(|\beta_1|^2+|\beta_2|^2) + v|\beta_0|^2 - 2|\beta_0|\sqrt{|\beta_1\beta_2|}}} \\ &\le \frac{d_F(\mathbb{C}_0, \mathcal{A}_{\widetilde{D}})}{||\mathbb{C}_0||_F} \le \sqrt{\frac{v}{2v-1}}. \end{aligned}$$

The upper bound is attained if, and only if, $\beta_0 = 0$, $\sqrt{\beta_1\beta_2} = 0$, and $\mathbb{C}$ is bi-diagonal.

3.4. Normalized distance of $\mathbb{C}_0$ from the normal matrix class $\mathcal{A}$

Theorem 8. *For $\mathbb{C}_0 = (v; -\sqrt{\beta_1\beta_2}, \beta_2, 0, \beta_1)$, the normalized distance $\frac{d_F(\mathbb{C}_0,\mathcal{A})}{||\mathbb{C}_0||_F}$ satisfies*

$$\frac{d_F(\mathbb{C}_0, \mathcal{A})}{||\mathbb{C}_0||_F} = \frac{1}{\sqrt{v}}$$

when either $\beta_2 = 0$, $\beta_1 \ne 0$ or $\beta_2 \ne 0$, $\beta_1 = 0$, with $\mathcal{A}$ as the family of normal matrices.

Proof. Equations (3.13) and (3.18) imply that $\frac{d_F(\mathbb{C}_0,\mathcal{A})}{||\mathbb{C}_0||_F} \ge \frac{1}{\sqrt{v}}$. To demonstrate equality, we exhibit a normal (circulant) matrix V whose normalized distance to $\mathbb{C}_0$ is $\frac{1}{\sqrt{v}}$.

For $\beta_2 \neq 0, \beta_1 = 0$, define

$$V = \frac{v-1}{v}(\mathbb{C}_0 + \beta_2 e_1 e_v^T).$$

For $\beta_2 = 0, \beta_1 \neq 0$, take

$$V = \frac{v-1}{v}(\mathbb{C}_0 + \beta_1 e_v e_1^T),$$

with e_j as the jth basis vector □

4. The distance between the PDDT Toeplitz matrix $\mathbb{C}$ and matrix family $\mathcal{B}_{\widetilde{D}}$

Remark 4. *Given the PDDT Toeplitz matrix* $\mathbb{C} = (v; -\sqrt{\beta_1\beta_2}, \beta_2, \beta_0, \beta_1)$*, its distance* $d_F(\mathbb{C}, \mathcal{B}_{\widetilde{D}})$ *is computed in the Frobenius norm. For the minimal distance matrix* $\mathbb{C}^+$ *in* $\mathcal{B}_{\widetilde{D}}$ *to* $\mathbb{C}$*, we have*

$$\mathbb{C}^+ = \begin{cases} (v; 0, \beta_2, \beta_0, 0) & |\beta_1| = \min\{|\beta_1|, |\beta_2|\}, \\ (v; 0, 0, \beta_0, \beta_1) & |\beta_2| = \min\{|\beta_1|, |\beta_2|\}. \end{cases} \tag{4.1}$$

$\mathcal{B}_{\widetilde{D}}$*, which has been mentioned in the previous text.*

Corollary 3. *For PDDT Toeplitz matrix* $\mathbb{C}$*, the Frobenius distance to the family* $\mathcal{B}_{\widetilde{D}}$ *is computed as follows:*

- *When* $\mathbb{C} \in \widetilde{D}$*,*

$$d_F(\mathbb{C}, \mathcal{B}_{\widetilde{D}}) = d_F(\mathbb{C}, \mathbb{C}^+) = \sqrt{(v-1)\min\{|\beta_1|, |\beta_2|\}^2 + |\beta_1\beta_2|}. \tag{4.2}$$

- *When* $\mathbb{C} \in \mathcal{A}_{\widetilde{D}}$*,*

$$d_F(\mathbb{C}, \mathcal{B}_{\widetilde{D}}) = d_F(\mathbb{C}, \mathbb{C}^+) = \sqrt{v}|\beta_1| = \sqrt{v}|\beta_2|.$$

- *When* $\mathbb{C} \notin \mathcal{A}_{\widetilde{D}}$*,*

$$d_F(\mathbb{C}^*, \mathcal{B}_{\widetilde{D}}) = d_F(\mathbb{C}^*, \mathbb{C}^+) = \sqrt{v}\Big(\frac{|\beta_1| + |\beta_2|}{2} - \frac{(\sqrt{|\beta_1|} - \sqrt{|\beta_2|})^2}{2(2v-1)}\Big), \tag{4.3}$$

where $\mathbb{C}^*$ *is the closest matrix in* $\mathcal{A}_{\widetilde{D}}$ *to* $\mathbb{C}$ *and* $\mathbb{C}^+$ *is mentioned in Eq. (4.1).*

Based on Eqs (4.2) and (4.3), we obtain

$$\begin{aligned} d_F(\mathbb{C}^*, \mathcal{B}_{\widetilde{D}}) - d_F(\mathbb{C}, \mathcal{B}_{\widetilde{D}}) &= d_F(\mathbb{C}^*, \mathbb{C}^+) - d_F(\mathbb{C}, \mathbb{C}^+) \\ &= \sqrt{v}(\frac{|\beta_1| + |\beta_2|}{2} - \frac{(\sqrt{|\beta_1|} - \sqrt{|\beta_2|})^2}{2(2v-1)}) - \sqrt{(v-1)\min\{|\beta_1|, |\beta_2|\}^2 + |\beta_1\beta_2|} \\ &= U d_F(\mathbb{C}, \mathcal{A}_{\widetilde{D}}), \end{aligned}$$

where $U = \dfrac{\frac{\sqrt{v(2v-1)}}{2}(|\beta_1|+\beta_2|) - \sqrt{\frac{v}{2v-1}}\left(\frac{(\sqrt{|\beta_1|}-\sqrt{|\beta_2|})^2}{2}\right) - \sqrt{(2v-1)\big(|\beta_1\beta_2|+(v-1)\min(|\beta_1|,|\beta_2|)^2\big)}}{\sqrt{(v(|\beta_1|-|\beta_2|)^2+2\sqrt{|\beta_1\beta_2|}(|\beta_2|-\sqrt{|\beta_1|})^2)(v-1)}}$.

We define the ratio m as

$$m = \frac{\min\{|\beta_1|, |\beta_2|\}}{\max\{|\beta_1|, |\beta_2|\}}, \tag{4.4}$$

which will be employed in analyzing the bounds of the normalized distance between $\mathbb{C}_0$ and $\mathcal{B}_{\widetilde{D}}$ within the following proof.

Theorem 9. *When $\beta_0 = 0$, the normalized Frobenius distance from a PDDT Toeplitz matrix $\mathbb{C}_0$ to the matrix family $\mathcal{B}_{\widetilde{D}}$ satisfies*

$$\frac{d_F(\mathbb{C}_0, \mathcal{B}_{\widetilde{D}})}{\|\mathbb{C}_0\|_F} \leq \sqrt{\frac{v}{2v-1}},$$

with equality holding true when $\mathbb{C}_0$ is normal.

Proof. Let $m = \frac{|\beta_1|}{|\beta_2|}$. According to Eqs (1.10) and (4.1), the following equation can be obtained:

$$\frac{d_F(\mathbb{C}_0, \mathcal{B}_{\widetilde{D}})}{\|\mathbb{C}_0\|_F} = \frac{d_F(\mathbb{C}_0, \mathbb{C}^+)}{\|\mathbb{C}_0\|_F} = \frac{\sqrt{(v-1)|\beta_1|^2 + |\beta_1\beta_2|}}{\sqrt{(v-1)(|\beta_1|^2 + |\beta_2|^2) + |\beta_1\beta_2|}} = \frac{\sqrt{(v-1)m^2 + m}}{\sqrt{(v-1)(m^2+1) + m}} \leq \sqrt{\frac{v}{2v-1}}.$$

As m declines from 1 to 0, the normalized structured distance transitions from $\sqrt{\frac{v}{2v-1}}$ to 0, the case where $m = \frac{|\beta_2|}{|\beta_1|}$ follows a symmetric argument. □

By considering what has been discussed previously, we can derive the subsequent conclusions:

(1) when $|\beta_1| = |\beta_2|$, $\frac{d_F(\mathbb{C}_0, \mathcal{B}_{\widetilde{D}})}{\|\mathbb{C}_0\|_F} = \sqrt{\frac{v}{2v-1}}$;

(2) when $\beta_2 \neq 0$, $\lim\limits_{|\beta_1| \to 0} \frac{d_F(\mathbb{C}_0, \mathcal{B}_{\widetilde{D}})}{\|\mathbb{C}_0\|_F} = 0$;

(3) when $\beta_1 \neq 0$, $\lim\limits_{|\beta_2| \to 0} \frac{d_F(\mathbb{C}_0, \mathcal{B}_{\widetilde{D}})}{\|\mathbb{C}_0\|_F} = 1$.

5. Eigenvalue sensitivity

Significant studies have explored multiple dimensions of eigenvalue analysis for general matrices, encompassing structural condition numbers, patterned conditioning parameters, and sensitivity properties, as demonstrated by references [46–52]. We analyze the eigenvalue sensitivity of matrices $\mathbb{C}_0$ and $\mathbb{C}$ from multiple perspectives.

To quantify the sensitivity of

$$\vec{\delta}(\mathbb{C}_0) = [\lambda_1(\mathbb{C}_0), \lambda_2(\mathbb{C}_0), \cdots, \lambda_v(\mathbb{C}_0)]$$

under perturbations to β_1 and β_2, we introduce a function

$$g : E \subset \mathbf{C}^2 \to g(E) \subset \mathbf{C}^v, E = \{(\beta_1, \beta_2) \in \mathbf{C}^2 : \beta_1\beta_2 \neq 0\} : \vec{\delta}(\mathbb{C}_0) = g(\beta_1, \beta_2).$$

The sensitivity of $\vec{\delta}(\mathbb{C}_0)$ to β_1 and β_2 perturbations is linked to the Jacobian matrix of g. The detailed analysis proceeds as follows.

Using (2.2), we obtain the Jacobian matrix of g as

$$J_g(\beta_1,\beta_2) = \begin{bmatrix} \sqrt{\frac{\beta_1}{\beta_2}}\cos\theta_1 & \sqrt{\frac{\beta_2}{\beta_1}}\cos\theta_1 \\ \sqrt{\frac{\beta_1}{\beta_2}}\cos\theta_2 & \sqrt{\frac{\beta_2}{\beta_1}}\cos\theta_2 \\ \vdots & \vdots \\ \sqrt{\frac{\beta_1}{\beta_2}}\cos\theta_v & \sqrt{\frac{\beta_2}{\beta_1}}\cos\theta_v \end{bmatrix} \in \mathbf{C}^{v\times 2}, \theta_j = \frac{2j\pi}{2v+1}, j = 1,\ldots,v. \tag{5.1}$$

Substituting Eqs (3.7) and (5.1) into the Frobenius norm, we obtain:

$$\|J_g(\beta_1,\beta_2)\|_F = \frac{\sqrt{2v-1}}{2}\sqrt{\frac{|\beta_2|}{|\beta_1|}+\frac{|\beta_1|}{|\beta_2|}} = \frac{\sqrt{2v-1}}{2}\sqrt{\frac{|\beta_1|^2+|\beta_2|^2}{|\beta_1\beta_2|}}. \tag{5.2}$$

When considering relative errors in β_1, β_2, and $\vec{\delta}(\mathbb{C}_0)$ instead, the counterpart to (5.1) becomes the $v\times 2$ matrix

$$\Omega_g(\beta_1,\beta_2) = \begin{bmatrix} \frac{\beta_2}{\vec{\delta}_1(\mathbb{C}_0)}(J_g(\beta_1,\beta_2))_{1,1} & \frac{\beta_1}{\vec{\delta}_1(\mathbb{C}_0)}(J_g(\beta_1,\beta_2))_{1,2} \\ \frac{\beta_2}{\vec{\delta}_2(\mathbb{C}_0)}(J_g(\beta_1,\beta_1))_{2,1} & \frac{\beta_1}{\vec{\delta}_2(\mathbb{C}_0)}(J_g(\beta_1,\beta_2))_{2,2} \\ \vdots & \vdots \\ \frac{\beta_2}{\vec{\delta}_v(\mathbb{C}_0)}(J_g(\beta_1,\beta_2))_{v,1} & \frac{\beta_1}{\vec{\delta}_v(\mathbb{C}_0)}(J_g(\beta_1,\beta_2))_{v,2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \\ \vdots & \vdots \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Calculations reveal that the matrix $\Omega_g(\beta_1,\beta_2)$ meets:

$$\Omega_g(\beta_1,\beta_2)^H\Omega_g(\beta_1,\beta_2) = \frac{v}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

while both its spectral norm and Frobenius norm yield

$$\|\Omega_g(\beta_1,\beta_2)\|_2 = \|\Omega_g(\beta_1,\beta_2)\|_F = \sqrt{\frac{v}{2}}.$$

Remark 5. *The norm of Ω_g remains unaffected by variations in β_1 and β_2, whereas the norm of J_g is influenced by m. If, and only if, $|\beta_1| = |\beta_2|$, which is equivalent to $\mathbb{C}$ being normal, the norm of J_g attains its minimum value $\sqrt{\frac{2v-1}{2}}$. As m decreases, the norm of J_g approaches $+\infty$.*

The sensitivity of eigenvalues can be quantified by the normalized distance between matrix $\mathbb{C}_0$ and the set $\mathcal{A}_{\widetilde{D}}$.

Theorem 10. *Under the condition $\beta_1\beta_2 \neq 0$, the following equality holds for $\left\|J_g(\beta_1,\beta_2)\right\|_F$:*

$$\left\|J_g(\beta_1,\beta_2)\right\|_F = \sqrt{\frac{(2v-1)\left(\frac{d_F{}^2(\mathbb{C}_0,\mathcal{A}_{\widetilde{D}})}{\|\mathbb{C}_0\|_F^2} - (2v^2-2v+2) + (2v-2)\left(\sqrt{\frac{|\beta_1|}{|\beta_2|}}+\sqrt{\frac{|\beta_2|}{|\beta_1|}}\right)\right)}{4(v-1)[3v-2-(2v-1)\frac{d_F{}^2(\mathbb{C}_0,\mathcal{A}_{\widetilde{D}})}{\|\mathbb{C}_0\|_F^2}]}}.$$

A relationship is established between the Frobenius norm of g's Jacobian matrix and the normalized distance of matrix $\mathbb{C}_0$ to $\mathcal{A}_{\widetilde{D}}$.

Proof. Under $\beta_1\beta_2 \neq 0$, we derive

$$\frac{{d_F}^2(\mathbb{C}_0, \mathcal{A}_{\widetilde{D}})}{\|\mathbb{C}_0\|_F^2} = \frac{\frac{(v-1)[v(|\beta_1|-|\beta_2|)^2-2\sqrt{|\beta_1\beta_2|}(\sqrt{|\beta_1|}-\sqrt{|\beta_2|})^2]}{2v-1}}{|\beta_1\beta_2| + (v-1)(|\beta_1|^2 + |\beta_2|^2)}$$

$$= \frac{(v-1)[v(\frac{|\beta_2|}{|\beta_1|} + \frac{|\beta_1|}{|\beta_2|} - 2) - 2(\sqrt{\frac{|\beta_2|}{|\beta_1|}} + \sqrt{\frac{|\beta_1|}{|\beta_2|}} - 2)]}{(2v-1)[1 + (v-1)(\frac{|\beta_2|}{|\beta_1|} + \frac{|\beta_1|}{|\beta_2|})]}.$$

From the final equation in (5.2), we derive

$$\frac{{d_F}^2(\mathbb{C}_0, \mathcal{A}_{\widetilde{D}})}{\|\mathbb{C}_0\|_F^2} = \frac{(v-1)\left[v\left(\frac{4\|J_g(\beta_1,\beta_2)\|_F^2}{2v-1} - 2\right) - 2\left(\sqrt{\frac{|\beta_1|}{|\beta_2|}} + \sqrt{\frac{|\beta_2|}{|\beta_1|}} - 2\right)\right]}{(2v-1)\left[1 + (v-1)\frac{4\|J_g(\beta_1,\beta_2)\|_F^2}{2v-1}\right]}.$$

By applying calculations, we can reach our intended objectives. □

5.1. Individual eigenvalue condition numbers

This section analyzes individual eigenvalue condition numbers. For $\beta_1\beta_2 \neq 0$, they are derived from (2.7) and (2.8) using the trigonometric identity

$$\sum_{k=1}^{v} \sin^2 \frac{(2k-1)\, j\pi}{2v+1} = \frac{2v+1}{4},\ j = 1, \ldots, v. \tag{5.3}$$

Specifically, for $j = 1, \ldots, v$,

$$\begin{aligned} \|\xi^{(j)}\|_2^2 &= \frac{4}{2v+1} \sum_{k=1}^{v} \left|\frac{\beta_2}{\beta_1}\right|^{k-1} \sin^2 \frac{(2k-1)j\pi}{2v+1}, \\ \|\chi^{(j)}\|_2^2 &= \frac{4}{2v+1} \sum_{k=1}^{v} \left|\frac{\beta_1}{\beta_2}\right|^{k-1} \sin^2 \frac{(2k-1)j\pi}{2v+1}, \end{aligned} \tag{5.4}$$

then,

$$|(\chi^{(j)})^H \xi^{(j)}| = \frac{4}{2v+1} \sum_{k=1}^{v} \sin^2 \frac{(2k-1)\, j\pi}{2v+1} = 1. \tag{5.5}$$

Substitute the above results into the individual eigenvalue condition number formula. By Eqs. (5.4) and (5.5), the condition number of the j-th eigenvalue is defined as

$$\kappa(\lambda_j(\mathbb{C})) = \frac{\|\xi^{(j)}\|_2\|\chi^{(j)}\|_2}{|(\chi^{(j)})^H\xi^{(j)}|} = \frac{4}{2v+1}\sqrt{\sum_{k=1}^{v}\left|\frac{\beta_2}{\beta_1}\right|^{k-1}\sin^2\left(\frac{(2k-1)j\pi}{2v+1}\right)\sum_{k=1}^{v}\left|\frac{\beta_1}{\beta_2}\right|^{k-1}\sin^2\left(\frac{(2k-1)j\pi}{2v+1}\right)}. \tag{5.6}$$

If $|\beta_1| = |\beta_2|$, the PDDT Toeplitz matrix $\mathbb{C}$ is normal, and the eigenvalue condition number reduces to

$$\kappa(\lambda_j(\mathbb{C})) = \frac{\|\xi^{(j)}\|_2\|\chi^{(j)}\|_2}{|\chi^{(j)})^H\xi^{(j)}|} = 1. \tag{5.7}$$

These results, namely, Eqs (5.4), (5.5), (5.6), and (5.7), are sourced from [53].

Theorem 11. *Let $\mathbb{C}$ be a PDDT Toeplitz matrix.*

If $|\beta_1| \neq |\beta_2|$, we mainly consider the case where $|\beta_1| > |\beta_2|$, and the other case is similar. Then, the condition number of the individual eigenvalue takes the form:

$$\kappa(\lambda_j(\mathbb{C})) = \frac{2}{(2v+1)m^{(v-1)/2}} \sqrt{B_{v,m}(j)B_{v,1/m}(j)},\ j = 1, \ldots, v, \tag{5.8}$$

and

$$B_{v,m}(j) = \frac{1-m^v}{1-m} - \frac{\cos p - m\cos p - m^v + m^{v+1}\cos 2p}{m^2 - 2m\cos 2p + 1},$$
$$B_{v,1/m}(j) = \frac{1-m^v}{1-m} - \frac{m^{v+1}\cos p - m^v\cos p - m + \cos 2p}{m^2 - 2m\cos 2p + 1},$$

where m is defined in (4.4) and $p = \frac{2j\pi}{2v+1}$.

Proof. Let $|\beta_1| > |\beta_2|$, then $m = \frac{|\beta_2|}{|\beta_1|}$. For an eigenvalue λ_j with $j = 1, \ldots, v$, define $p = \frac{2j\pi}{2v+1}$. Applying trigonometric half-angle identities, the summation simplifies to

$$\sum_{k=1}^{v} |\frac{\beta_2}{\beta_1}|^{k-1} \sin^2 \frac{(2k-1)j\pi}{2v+1} = \frac{1}{2}(\sum_{k=1}^{v} m^{k-1} - \sum_{k=0}^{v-1} m^k \cos(2k+1)p). \tag{5.9}$$

It follows from Euler's formula that

$$\begin{aligned}\sum_{k=0}^{v} n^k \sin kx &= \frac{n^{v+2}\sin vx - n^{v+1}\sin(v+1)x + n\sin x}{n^2 - 2n\cos x + 1},\\ \sum_{k=0}^{v} n^k \cos kx &= \frac{1 - n\cos x - n^{v+1}\cos(v+1)x + n^{v+2}\cos vx}{n^2 - 2n\cos x + 1}.\end{aligned} \tag{5.10}$$

It can be readily observed that

$$\begin{aligned}\sin(2(v-1)p) &= -\sin 3p, \quad \cos(2(v-1)p) = \cos 3p,\\ \sin(2vp) &= -\sin p, \qquad \cos(2vp) = \cos p.\end{aligned} \tag{5.11}$$

From Eqs (5.10) and (5.11), we derive:

$$\begin{aligned}\sum_{k=0}^{v-1} m^k \cos(2k+1)p &= \cos p \sum_{k=0}^{v-1} m^k \cos 2kp - \sin p \sum_{k=0}^{v-1} m^k \sin 2kp\\ &= \frac{\cos p - m\cos p - m^v + m^{v+1}\cos 2p}{m^2 - 2m\cos 2p + 1}.\end{aligned} \tag{5.12}$$

Hence, due to Eq (5.12), we have

$$\sum_{k=1}^{v} |\frac{\beta_2}{\beta_1}|^{k-1} \sin^2 \frac{(2k-1)j\pi}{(2v+1)} = \frac{1}{2}\Big(\frac{1-m^v}{1-m} - \frac{\cos p - m\cos p - m^v + m^{v+1}\cos 2p}{m^2 - 2m\cos 2p + 1}\Big),$$

and, analogously,

$$\sum_{k=1}^{v} |\frac{\beta_1}{\beta_2}|^{k-1} \sin^2 \frac{(2k-1)j\pi}{(2v+1)} = \frac{1}{2m^{v-1}}\Big(\frac{1-m^v}{1-m} - \frac{m^{v+1}\cos p - m^v\cos p - m + \cos 2p}{m^2 - 2m\cos 2p + 1}\Big).$$

Substituting these into (5.6) yields (5.8). □

Remark 6. *By algebraic manipulation, we can rewrite (5.8) in the following form:*

$$\kappa(\lambda_j(\mathbb{C})) = \frac{2}{(2v+1)m^{(v-1)/2}(1-m)(m^2-2m\cos\frac{4j\pi}{2v+1}+1)}K, \tag{5.13}$$

where $K=\sqrt{MN}$, $p=\frac{2j\pi}{2v+1}$ *with* $j=1,\dots,v$, *and* M, N *are defined by the polynomial combinations:*

$$M = \left(m^2+1\right)(1-\cos p) - 2m(\cos 2p - \cos p) - m^{v+1}(m+1)(1-\cos 2p),$$

$$N = -m^v\left(m^2+1\right)(1-\cos p) + 2m^{v+1}(\cos 2p - \cos p) + (m+1)(1-\cos 2p).$$

Based on theoretical analysis and numerical experiments, we obtain the following constraint:

$$0 \le \frac{K}{m^2-2m\cos\frac{4j\pi}{2v+1}+1} \le 1.513086. \tag{5.14}$$

For $0<m<1$ *defined as Eq. (4.4), the condition number* $\kappa(\lambda_j(\mathbb{C}))$ *exhibits exponential growth with* v.

5.2. The global eigenvalue condition number

The paper examines properties of the global condition number

$$\kappa_F(\lambda) = \sum_{k=1}^{v}\kappa(\lambda_j(\mathbb{C})). \tag{5.15}$$

It can be computed by summing up the individual condition numbers.

By substituting Eq (5.13) into Eq (5.15),

$$\kappa_F(\lambda) = \frac{2}{(2v+1)m^{(v-1)/2}(1-m)}\sum_{j=1}^{v}\frac{K}{m^2-2m\cos\frac{4j\pi}{2v+1}+1}$$

is obtained.

For any PDDT Toeplitz matrix $\mathbb{C}=(v;-\sqrt{\beta_1\beta_2},\beta_2,\beta_0,\beta_1)$ *with* $|\beta_1|=|\beta_2|$, *the bounds can be derived through Eqs (2.7) and (2.8).*

By applying (5.13) and (5.14), for PDDT Toeplitz matrix $\mathbb{C}=(v;-\sqrt{\beta_1\beta_2},\beta_2,\beta_0,\beta_1)$ *with* $|\beta_1|\neq|\beta_2|$, *we are able to find the bounds:*

$$0 \le \kappa_F(\lambda) \le 1.513086K_{v,m},$$

where

$$K_{v,m} = \frac{2v}{(2v+1)(1-m)m^{(v-1)/2}}, 0<m<1, \tag{5.16}$$

with m defined in (4.4).

5.3. The ε-pseudospectrum

Let $\varepsilon > 0$. Follow [54], and the ε-pseudospectrum of $\mathbb{E} \in \mathbf{C}^{v\times v}$ is the set

$$\Lambda_{\varepsilon}(\mathbb{E}) = \{w :\| (wI - \mathbb{E})^{-1} \|_2 \geq \varepsilon^{-1}\}.$$

An alternative definition to be employed in Section 7 is:

$$\Lambda_{\varepsilon}(\mathbb{E}) = \{w : \exists h \in \mathbf{C}^{v}, \| h \|_2 = 1, \text{ such that } \| (wI - \mathbb{E})h \|_2 \leq \varepsilon\}. \tag{5.17}$$

The vectors h in the above definition are termed ε-pseudoeigenvectors.

Define

$$f(w) = \beta_1 w + \beta_0 + \beta_2 w^{-1}.$$

The ellipse

$$f(S) = \{f(w) : w \in \mathbf{C}, | w |= 1\} \tag{5.18}$$

is the spectral boundary of $\mathbb{C}_{\infty} = (\infty; -\sqrt{\beta_1\beta_2}, \beta_2, \beta_0, \beta_1)$. For the ellipse $f(S)$, the major axis is

$$S_m = \{\beta_0 + ae^{i(\omega_1+\omega_2)/2}, a \in \mathbf{R}, |a| \leq |\beta_1| + |\beta_2| - \frac{(\sqrt{|\beta_1|} - \sqrt{|\beta_2|})^2}{2v - 1}\}. \tag{5.19}$$

The interval between the foci is similarly defined as

$$S_f = \{\beta_0 + ae^{i(\omega_1+\omega_2)/2}, a \in \mathbf{R}, |a| \leq 2\sqrt{|\beta_1\beta_2|}\}.$$

The spectral containment of $\mathbb{C} = (v; -\sqrt{\beta_1\beta_2}, \beta_2, \beta_0, \beta_1)$ is guaranteed by (2.6) to be within S_f for every finite $v \geq 1$, and no interval smaller than this one has such a property. Based on (3.3), the spectrum of the normal PDDT Toeplitz matrix $\mathbb{C}^*$ closest to $\mathbb{C}$ lies within interval (5.19).

5.4. Structured perturbations

The tridiagonal perturbation $E_{\zeta} = (v; \sqrt{\zeta\beta_1}, -\zeta, 0, 0)$ of $\mathbb{C} = (v; -\sqrt{\beta_1\beta_2}, \beta_2, \beta_0, \beta_1)$ is considered under the condition $|\beta_2| = \min\{|\beta_1|, |\beta_2|\}$. A family of diagonalizable matrices $\mathbb{C} + E_{\zeta}$ with simple eigenvalues is obtained for $\zeta = d\beta_2$ where $0 < d < 1$, emerging as the limit of $\mathbb{C} + E_{\zeta}$ when $d \longrightarrow 1$ is the defective matrix $\mathbb{C}^{+} = (v; 0, 0, \beta_0, \beta_1)$. The latter matrix has the unique eigenvalue β_0 of geometric multiplicity one. The structured perturbation

$$E_{\beta_2} = (v; \sqrt{\beta_1\beta_2}, -\beta_2, 0, 0), \| E_{\beta_2} \|_F = \sqrt{(v-1)|\beta_2|^2 + |\beta_1\beta_2|},$$

shifts all eigenvalues of matrix $\mathbb{C}$: the eigenvalues are all β_0.

Under the constraint $0 < |\beta_2| \leq |\beta_1|$, the rate of change of the jth eigenvalue of $\mathbb{C}$ satisfies

$$\frac{| \lambda_j(\mathbb{C} + E_{\beta_2}) - \lambda_j(\mathbb{C}) |}{\| E_{\beta_2} \|_F} = \frac{2\sqrt{| \beta_1\beta_2 |}\, | \cos \frac{2j\pi}{2v+1} |}{\sqrt{(v-1)|\beta_2|^2 + |\beta_1\beta_2|}} = \frac{2\, | \cos \frac{2j\pi}{2v+1} |}{\sqrt{(v-1)m + 1}}. \tag{5.20}$$

Define $E_{\zeta,\sigma} = (v; \sqrt{\zeta\sigma}, -\zeta, 0, -\sigma)$, with $\zeta = d\beta_2$ and $\sigma = d\beta_1$, where $0 < d < 1$. Subsequently,

$$\lim_{d\to 1}(\mathbb{C} + E_{\zeta,\sigma}) = \beta_0 I,$$

where I is the identity matrix. Thus, $\mathbb{C} + E_{\zeta,\sigma}$ is normal. This limit matrix is attained through the perturbation

$$E_{\beta_1,\beta_2} = (v;\ \sqrt{\beta_1\beta_2}, -\beta_2, 0, -\beta_1), \| E_{\beta_1,\beta_2} \|_F = \sqrt{(v-1)(|\beta_1|^2 + |\beta_2|^2) + |\beta_1\beta_2|}.$$

For indices $j = 1, 2, \cdots, v$, the eigenvalue variation rate under perturbation is

$$\frac{| \lambda_j(\mathbb{C} + E_{\beta_1,\beta_2}) - \lambda_j(\mathbb{C}) |}{\| E_{\beta_1,\beta_2} \|_F} = \frac{2\sqrt{|\beta_1\beta_2|}|\cos\frac{2j\pi}{2v+1}|}{\sqrt{(v-1)(|\beta_1|^2 + |\beta_2|^2) + |\beta_1\beta_2|}} = \frac{2\sqrt{2v-1}\left|\cos\frac{2j\pi}{2v+1}\right|}{\sqrt{4(v-1)\left\|J_g(\beta_1,\beta_2)\right\|_F^2 + 2v-1}}.$$

From the derived expressions, it can be clearly seen that an inverse proportionality is observed between the rate and the Frobenius norm of the Jacobian matrix (5.2). The maximal rate is attained when $\mathbb{C}$ is normal, as demonstrated in Remark 5. The sensitivity to structured perturbations increases with the distance between eigenvalues of $\mathbb{C}$ and β_0.

6. Examples of eigenvalue sensitivity

This section demonstrates computationally the properties of PDDT Toeplitz matrices and their eigenvalues, as previously analyzed.

To measure the relationship between the spectral distance between PDDT Toeplitz matrix $\mathbb{C}_{(m)}$ and its closest normal PDDT Toeplitz matrix $\mathbb{C}^*_{(m)}$, as well as the Frobenius distance, we use Eqs (3.5) and (3.14) obtain

$$\| \vec{\delta}(\mathbb{C}) - \vec{\delta}(\mathbb{C}^*) \|_2 = \frac{\sqrt{v-1}(\sqrt{|\beta_1|} - \sqrt{|\beta_2|})}{\sqrt{v(\sqrt{|\beta_1|} + \sqrt{|\beta_2|})^2 - 2\sqrt{|\beta_1\beta_2|}}} d_F(\mathbb{C}, \mathcal{A}_{\widetilde{D}}).$$

Table 1 lists numerical quantities corresponding to matrices of the form

$$\mathbb{C}_{(m)} = (50; -\sqrt{-5m(4+3i)}, (4+3i)m, 16-3i, -5), \tag{6.1}$$

where the parameter $0 < m < 1$ is defined as the ratio in (4.4). Observe that $\mathbb{C}_{(0)}$ is defective, whereas $\mathbb{C}_{(1)}$ is normal. Since $| 4 + 3i | = | -5 |$, the latter property holds. Computation of $d_F(\mathbb{C}_{(m)}, \mathcal{A}_{\widetilde{D}})$ employs the formula (3.5). The quantity $K_{50,m}$, defined in (5.16), serves as a measure of eigenvalue sensitivity.

The eigenvalues of matrices $\mathbb{C}_{(m)}$ and $\mathbb{C}^*_{(m)}$ are displayed in Figures 1, 2, 3, and 4. The eigenvalues are computed using (2.2) and (3.3). The figures illustrate the image of the unit circle under the matrices $\mathbb{C}_{(m)}$, as detailed in (5.18). Every ellipse shown constitutes the spectral boundary of $\mathbb{C}_\infty = (\infty; -\sqrt{-5(4+3i)m}, (4+3i)m, 16-3i, -5)$. In the process of eigenvalue computation, different methods can lead to different visual outcomes. For instance, if the QR algorithm (an iterative algorithm for solving eigenvalues and eigenvectors of matrices) is employed instead of Eq. (2.2) to compute the eigenvalues of $\mathbb{C}_{(0.1)}$, the visual representation in Figure 1 would differ markedly. Figure 5 illustrates this point. It shows the spectra of matrices $\mathbb{C}^T_{(0.1)}$ and $(\mathbb{C}^T_{(0.1)})^*$ calculated by the QR algorithm implemented with the MATLAB function eig. From Figures 1 and 5, one cannot easily observe that matrices $\mathbb{C}_{(0.1)}$ and $\mathbb{C}^T_{(0.1)}$ share the same eigenvalues. As depicted in Fig 5, when ε is set to the machine epsilon 2×10^{-16}, the spectrum of $\mathbb{C}^T_{(0.1)}$ is in close proximity to the boundary of the ε-pseudospectrum.

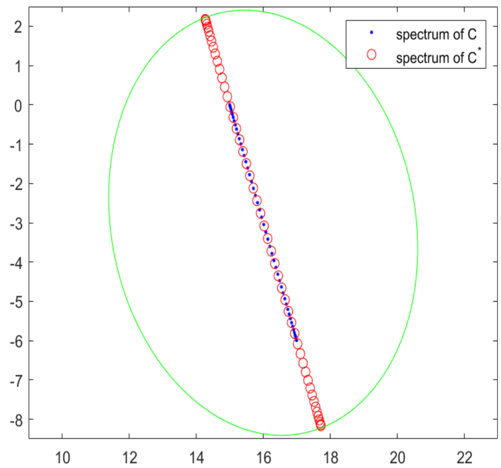

Figure 1. The spectra of matrix $\mathbb{C}_{(m)}$ and the closest normal PDDT Toeplitz matrix $\mathbb{C}^*_{(m)}$, along with the image of the unit circle under the symbol associated with $\mathbb{C}_{(m)}$ for $m = 0.1$. Horizontal and vertical axes, respectively, denote the real and imaginary parts of the eigenvalues.

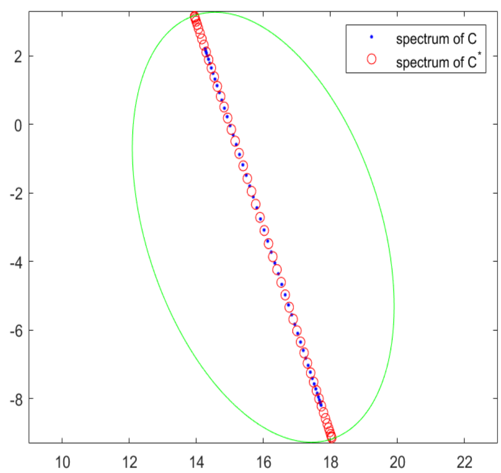

Figure 2. The spectra of matrix $\mathbb{C}_{(m)}$ and the closest normal PDDT Toeplitz matrix $\mathbb{C}^*_{(m)}$, along with the image of the unit circle under the symbol associated with $\mathbb{C}_{(m)}$ for $m = 0.3$. Horizontal and vertical axes, respectively, denote the real and imaginary parts of the eigenvalues.

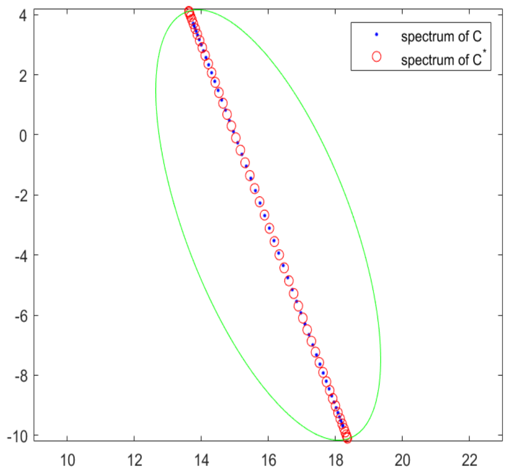

Figure 3. The spectra of matrix $\mathbb{C}_{(m)}$ and the closest normal PDDT Toeplitz matrix $\mathbb{C}^*_{(m)}$, along with the image of the unit circle under the symbol associated with $\mathbb{C}_{(m)}$ for $m = 0.5$. Horizontal and vertical axes, respectively, denote the real and imaginary parts of the eigenvalues.

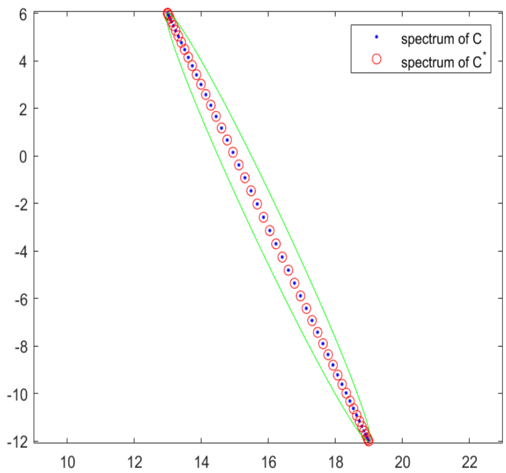

Figure 4. The spectra of matrix $\mathbb{C}_{(m)}$ and the closest normal PDDT Toeplitz matrix $\mathbb{C}^*_{(m)}$, along with the image of the unit circle under the symbol associated with $\mathbb{C}_{(m)}$ for $m = 0.9$. Horizontal and vertical axes, respectively, denote the real and imaginary parts of the eigenvalues.

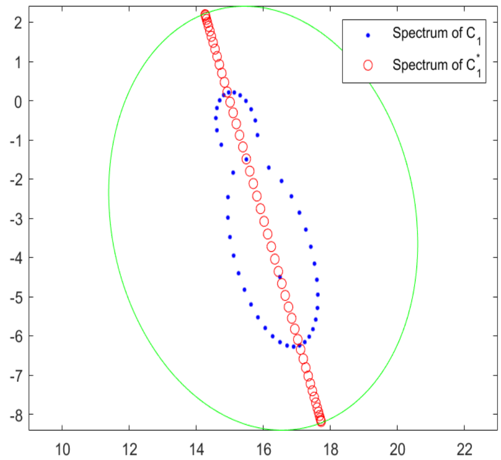

Figure 5. Using the eig function of MATLAB which employs the QR algorithm, we compute the spectra of $\mathbb{C}^T_{(0.1)}$ and $(\mathbb{C}^T_{(0.1)})^*$, denoted $\mathbb{C}_1$ and $\mathbb{C}_1^*$ in the legend. Horizontal and vertical axes, respectively, denote the real and imaginary parts of the eigenvalues.

Table 1. Parameters associated with $\mathbb{C}_{(m)}$ defined by (6.1) and the closest normal PDDT Toeplitz matrices $\mathbb{C}^*_{(m)}$.

m	$d_F(\mathbb{C}_{(m)}, \mathcal{A}_{\widetilde{D}})$	$K_{50,m}$	$\Vert \vec{\delta}(\mathbb{C}_{(m)}) - \vec{\delta}(\mathbb{C}^*_{(m)}) \Vert_2$
0.1	2.23042e+01	3.4789e+24	1.15125e+01
0.3	1.73316e+01	9.1434e+12	5.03680e+00
0.5	1.23762e+01	4.6983e+07	2.11240e+00
0.9	2.47490e+00	1.3084e+02	6.48000e–02

7. Inverse problems for $\mathbb{C}$

This section begins by addressing inverse eigenvalue problems for PDDT Toeplitz matrices, followed by an investigation of their inverse vector counterparts. A trapezoidal PDDT Toeplitz matrix is determined in the latter problem through minimization of the matrix-vector product norm with a specified vector.

Problem 1 : For two distinct complex numbers u_1, u_2 and a natural number v, construct a PDDT Toeplitz matrix $\mathbb{C} = (v; -\sqrt{\beta_1\beta_2}, \beta_2, \beta_0, \beta_1)$ such that its extremal eigenvalues are u_1 and u_2.

Sections 2–4 provide key insights into this problem. Although no unique solution exists for this problem, it should be noted that the eigenvalues of $\mathbb{C}$ are completely determined by the given data. Given the eigenvalues

$$\lambda_1 = u_1 = \beta_0 + 2\sqrt{\beta_1\beta_2}\cos\frac{2\pi}{2v+1}, \lambda_v = u_2 = \beta_0 + 2\sqrt{\beta_1\beta_2}\cos\frac{2v\pi}{2v+1},$$

the diagonal entry β_0 and the product $\beta_1\beta_2$ of super-diagonal entries are uniquely determined by

$$\sqrt{\beta_1\beta_2} = \frac{u_1 - u_2}{2(\cos\frac{2\pi}{2v+1} - \cos\frac{2v\pi}{2v+1})}, \beta_0 = \frac{u_1 \cos\frac{2v\pi}{2v+1} - u_2 \cos\frac{2\pi}{2v+1}}{\cos\frac{2v\pi}{2v+1} - \cos\frac{2\pi}{2v+1}}.$$

The data determines both the magnitude $|\beta_1\beta_2|$ and the angle $\arg(\beta_1) + \arg(\beta_2)$. We have the freedom to specify both the sub-diagonal or super-diagonal entry angles and the ratio m where $0 < m \leq 1$ as given in (4.4). The eigenvalues become more ill-conditioned as the parameter m approaches zero. When $m = 1$ is selected, which implies $|\beta_1| = |\beta_2|$, the resulting matrix becomes normal. Choosing different angles for the sub-diagonal or super-diagonal entry leads to distinct normal matrices. The uniqueness condition may be enforced through specifying either $\arg(\beta_1)$ or $\arg(\beta_2)$.

Problem 2 : For a given vector $\vec{\phi} \in \mathbf{C}^v$, construct a trapezoidal PDDT Toeplitz matrix

$$\begin{pmatrix} \beta_2 & 1-\sqrt{\beta_1\beta_2} & \beta_1 & & & & 0 \\ & \beta_2 & 1 & \beta_1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \beta_2 & 1 & \beta_1 & \\ 0 & & & & \beta_2 & 1 & \beta_1 \end{pmatrix} \in \mathbf{C}^{(v-2)\times v}$$

such that $\mathbb{C}$ achieves

$$\min_{\beta_2,\beta_1} \left\|\mathbb{C}\vec{\phi}\right\|_2. \tag{7.1}$$

Defining $\vec{\phi} = [\mu_1, \mu_2, ..., \mu_v]^T$, the problem (7.1) reduces to

$$\min_{\beta_1,\beta_2} \left\| \begin{bmatrix} \mu_1 & \mu_3 \\ \mu_2 & \mu_4 \\ . & . \\ . & . \\ \mu_{v-2} & \mu_v \end{bmatrix} \begin{bmatrix} \beta_2 \\ \beta_1 \end{bmatrix} + \begin{bmatrix} (1-\sqrt{\beta_1\beta_2})\mu_2 \\ \mu_3 \\ . \\ . \\ \mu_{v-1} \end{bmatrix} \right\|_2. \tag{7.2}$$

Provided that the columns of matrix (7.2) are linearly independent, this least-squares problem possesses a unique solution. The columns are linearly dependent if, and only if, there exists $\beta \in \mathbf{C}$ such that the components of $\vec{\phi}$ obey

$$\mu_{k+2} = \beta\mu_k, k = 1, \ldots, v-2.$$

Given the solution $\mathbb{C}$ of (7.1), an important question is to characterize unit vectors $\vec{\phi}$ for which $\left\|\mathbb{C}\vec{\phi}\right\|_2$ remains small. Let $\hat{\mathbb{C}} \in \mathbf{C}^{v\times v}$ be the PDDT Toeplitz matrix constructed by prepending and appending appropriately chosen rows to $\mathbb{C}$. As per definition of Eq (5.17), the set of ϵ-pseudoeigenvectors of $\hat{\mathbb{C}}$ corresponding to $w = 0$ is contained within

$$\{h : \|\mathbb{C}h\|_2 \leq \varepsilon, \|h\|_2 = 1\}.$$

Author contributions

Hongxiao Chu: Conceptualization, methodology, validation, writing-original draf; Ziwu Jiang: Investigation, supervision, writing-review and editing; Xiaoyu Jiang: Project administration, writing-review and editing; Yaru Fu: Project administration, writing-review and editing; Zhaolin Jiang: Conceptualization, writing-review and editing, validation. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Authors declare no conflicts of interest.

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