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Research article

Some properties of Bazilevič harmonic functions

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Abstract: In the paper, a new class of Bazilevič harmonic functions is introduced. First, the sufficient and necessary conditions and integral expressions of the class are proved by the subordination relationship and the basic theory of harmonic functions. Then, the inclusion relation and radius problems are explored, yielding intriguing new findings.

Keywords: harmonic functions; Bazilevič functions; subordination; inclusion relation; radius; deviation
Mathematics Subject Classification: 30C45, 30C50, 30C62

1. Introduction

Let $\mathcal{A}$ signify the set of functions $h(z)$ that are analytic within the unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ and are expressed as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Let $\mathcal{S}$ be the subset of $\mathcal{A}$ consisting of univalent functions. Let $\mathcal{S}^*$ represent the subclass of starlike functions in $\mathcal{S}$. Define $\bar{\mathcal{P}}$ as the set of functions $p(z)$ that are analytic within $\mathcal{U}$ and satisfy the condition $p(0) = 1$. The symbol $\mathcal{P}$ represents the subclass of functions with positive real parts in $\bar{\mathcal{P}}$.

Assuming the functions $u(z)$ and $v(z)$ are analytic within the domain $\mathcal{U}$, $u(z)$ is defined to be subordinate to $v(z)$, denoted $u(z) \prec v(z)$, if there exists a function ω that is analytic satisfying $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathcal{U}$, such that $u(z) = v(\omega(z))$ for all $z \in \mathcal{U}$. Specifically, as stated in reference [1], when $v(z)$ is univalent in $\mathcal{U}$, the subordination $u(z) \prec v(z)$ holds if and only if $u(0) = v(0)$ and the image of u within $\mathcal{U}$ is a subset of the image of v within $\mathcal{U}$.

In 1955, Bazilevič introduced the function class $\mathcal{B}_g(\alpha,\beta)$ in [2]:

Definition 1.1. For $\alpha > 0, 0 \le \beta < 1$, and $h(z) \in \mathcal{A}$, the function $h(z)$ belongs to $\mathcal{B}_g(\alpha,\beta)$ if and only if there is a function $g(z) \in \mathcal{S}^*$ satisfying:

$$\operatorname{Re}\left\{\frac{zh'(z)}{h(z)}\left(\frac{h(z)}{g(z)}\right)^{\alpha}\right\} > \beta,$$

where the power takes the principal value.

Clearly, $\mathcal{B}_g(\alpha,\beta) \subset \mathcal{S}$ (for details, refer to the reference [2]). Let $g(z) = z$, we also obtain a subclass of $\mathcal{B}_g(\alpha,\beta)$:

$$\mathcal{B}(\alpha,\beta) = \left\{h \in \mathcal{A} : \operatorname{Re}\left(h'(z)\,(h(z)/z)^{\alpha-1}\right) > \beta\right\}. \tag{1.2}$$

Remark 1.2. The class $\mathcal{B}(\alpha,\beta)$ encompasses several notable special cases:

(1) $\mathcal{B}(\alpha, 0) = B_\alpha(\alpha > 0)$ (refer to [2]).

(2) $\mathcal{B}(\alpha,\beta) = B_{\alpha,\beta}(\alpha > 0, 0 \le \beta < 1)$ (refer to [3]).

(3) $\mathcal{B}(\alpha,\beta) = B_{\alpha,\beta}(\alpha > 0,\beta < 1)$ (refer to [4]).

(4) $\mathcal{B}(1,\beta) = \{h \in \mathcal{A} : \operatorname{Re}(h'(z)) > \beta\}$.

Given $\beta < 1$ and $\lambda \ge 0$, the class $\mathcal{R}(\lambda,\beta)$ is defined as the set of functions $p(z)$ in $\bar{\mathcal{P}}$ that satisfy the condition:

$$\operatorname{Re}\left(p(z) + \lambda z p'(z)\right) > \beta. \tag{1.3}$$

Specifically, when $\lambda = 0$, the class $\mathcal{R}(0,\beta)$ is denoted as $\mathcal{P}(\beta)$.

We define $\mathcal{H}$ as the set of complex harmonic functions $f(z) = h(z) + \overline{g(z)}$, where $h(z)$ and $g(z)$ are analytic in $\mathcal{U}$ with the initial conditions $h(0) = 0 = h'(0) - 1$ and $g(0) = 0$.

Let $\mathcal{S}_{\mathcal{H}}$ (a subclass of $\mathcal{H}$) represent the class of univalent harmonic functions of the form

$$f(z) = h(z) + \overline{g(z)}, \tag{1.4}$$

where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \ \ (|b_1| < 1). \tag{1.5}$$

A harmonic function $f(z) = h(z) + \overline{g(z)}$ is said to be locally univalent and sense-preserving if it satisfies the condition $|h'(z)| > |g'(z)|$ for $z \in \mathcal{U}$ (refer to [5, 6]).

The study of geometric properties of harmonic functions heavily relies on their analytic and conjugate analytic parts (see [7]). For $f(z) = h(z) + \overline{g(z)} \in \mathcal{S}_{\mathcal{H}}$ with $|b_1| = \sigma$ and $0 \le \sigma < 1$, the subclass $\bar{\mathcal{S}}^{\sigma}_{\mathcal{H}}$ of $\mathcal{S}_{\mathcal{H}}$, was explored in [8].

Throughout this paper, we assume all parameters adhere to the constraints: $\lambda > 0,\ \alpha > 0, \mu \ge 0, 0 \le \sigma < 1, 0 \le \beta < 1, |z| = r$ with $r \in (0, 1]$, and $\mathbb{N} = \{1, 2, \cdots\}$.

In the following, we define a class of Bazilevič harmonic functions, $\mathcal{B}^{\sigma}_{\mathcal{H}}(\alpha,\beta)$.

Definition 1.3. For $\alpha > 0, 0 \le \sigma < 1, 0 \le \beta < 1$, and $f(z) = h(z) + \overline{g(z)} \in \mathcal{S}_{\mathcal{H}}$ with $g'(z) = \omega_\sigma(z)h'(z)$, where $h(z) \in \mathcal{B}(\alpha,\beta)$, the function $f(z)$ is termed to be in the class $\mathcal{B}^{\sigma}_{\mathcal{H}}(\alpha,\beta)$. Here, $\omega_\sigma(z)$ is a Möbius self-transformation corresponding to the unit disk, defined as

$$\omega_\sigma(z) = \frac{z + \sigma}{1 + \sigma z}, 0 \le \sigma < 1. \tag{1.6}$$

In this paper, we introduce the class $\mathcal{B}_{\mathcal{H}}^{\sigma}(\alpha,\beta)$ of Bazilevič harmonic functions. Initially, we prove the necessary and sufficient conditions and integral representations for functions belonging to this class, utilizing subordination relationships and fundamental harmonic function theories. Building upon this foundation, we derive deviation theorems and coefficient inequalities for functions within this class. Lastly, we explore inclusion relationships and radius problems, yielding intriguing and novel findings.

2. Preliminary results

To delve into the primary findings, we require the subsequent lemmas.

It is easy to prove the identity

$$h'(z)(h(z)/z)^{\alpha-1} = H(z) + \frac{1}{\alpha}zH'(z), \alpha > 0$$

where $H(z) = (h(z)/z)^{\alpha}$.

Using this identity, it is not difficult to prove:

Lemma 2.1. For $\alpha > 0, \beta < 1$. The function $h(z)$ is in $\mathcal{B}(\alpha,\beta)$ if and only if $(h(z)/z)^{\alpha}$ belongs to $\mathcal{R}(\frac{1}{\alpha},\beta)$.

Lemma 2.2. (see [9]) Let $g(z)$ be analytic in $\mathcal{U}$ and $h(z)$ be convex in $\mathcal{U}$ satisfying $h(0) = 1$. If

$$g(z) + \frac{1}{\nu}zg'(z) \prec h(z),$$

where $\nu \neq 0$ and $\mathrm{Re}\nu \geq 0$, then

$$g(z) \prec \frac{\nu}{z^{\nu}}\int_0^z h(t)t^{\nu-1}dt \prec h(z).$$

Lemma 2.3. ([10] Avkhadiev-Wirths) Let $f(z) = h(z) + \overline{g(z)} \in \mathcal{H}$ satisfying $g'(z) = \omega_{\sigma}(z)h'(z)$, here $\omega_{\sigma}(z)$ is a Möbius transformation of $\mathcal{U}$ given by

$$\omega_{\sigma}(z) = \frac{z+\sigma}{1+\sigma z} = c_0 + \sum_{j=1}^{\infty} c_j z^j, \sigma \in (0,1), j \in \mathbb{N}.$$

Then the following conclusions hold:

(1) $c_0 = g'(0) = \sigma, |c_j| \leq 1 - |c_0|^2, j \in \mathbb{N}$,

(2) $\frac{|r-\sigma|}{1-\sigma r} \leq |\omega_{\sigma}(z)| \leq \frac{r+\sigma}{1+\sigma r}$,

(3) $|\omega'_{\sigma}(z)| \leq \frac{1-|\omega_{\sigma}(z)|^2}{1-r^2}$,

where $|z| = r \in [0,1)$.

Lemma 2.4. (see [11]) If $p(z) = 1 + \sum\limits_{n=k}^{\infty} p_n z^n (z \in \mathcal{U}, k \geq 1)$ belongs to the class $\mathcal{P}(\beta)$. For $|z| = r < 1$, we get

$$\left|\frac{zp'(z)}{p(z)}\right| \leq \frac{2k(1-\beta)r^k}{(1-r^k)\left[1+(1-2\beta)r^k\right]}.$$

The result is exact and optimal.

In 1986, Ma introduced the class $\mathcal{K}(\gamma,\delta)$ of $\gamma-$ convex functions of order δ (see [12]). The function $h(z)$ is in the class $\mathcal{K}(\gamma,\delta)$ if

$$\operatorname{Re}\left\{\gamma\left(1+\frac{zh''(z)}{h'(z)}\right)+(1-\gamma)\,\frac{zh'(z)}{h(z)}\right\}>\delta.$$

Especially, $\mathcal{K}(1,\delta)=\mathcal{K}(\delta)$ is the class of convex functions of order δ.

Lemma 2.5. Given $\alpha>0,\beta<1$, and $h(z)\in\mathcal{B}(\alpha,\beta)$, the function $h(z)$ is $\frac{1}{\alpha}$-convex of order 0 in disk $\mathcal{U}_{r_1}$, where $\mathcal{U}_{r_1}=\{z\in\mathbb{C}:|z|<r_1\}$. Here $r_1\in(0,1)$ is the smallest positive root of the following equation:

$$\alpha-2[\alpha\beta+k(1-\beta)]r^k-\alpha(1-2\beta)r^{2k}=0. \tag{2.1}$$

Proof. Given $h(z)\in\mathcal{B}(\alpha,\beta)$, we derive

$$h'(z)(h(z)/z)^{\alpha-1}=p(z), \tag{2.2}$$

where $p(z)\in\mathcal{P}(\beta)$.

Take the logarithmic derivative on both ends of the above equation and get

$$\left(1+\frac{zh''(z)}{h'(z)}\right)+(\alpha-1)\frac{zh'(z)}{h(z)}=\alpha+\frac{zp'(z)}{p(z)}.$$

By making use of Lemma 2.4, we have

$$\operatorname{Re}\left\{\left(1+\frac{zh''(z)}{h'(z)}\right)+(\alpha-1)\frac{zh'(z)}{h(z)}\right\}\geq\alpha-\left|\frac{zp'(z)}{p(z)}\right|\geq\frac{\alpha-2[\alpha\beta+k(1-\beta)]r^k-\alpha(1-2\beta)r^{2k}}{(1-r^k)\left[1+(1-2\beta)r^k\right]}.$$

Given the function $\varphi(r)=\alpha-2[\alpha\beta+k(1-\beta)]r^k-\alpha(1-2\beta)r^{2k}$, it is evident that $\varphi(r)$ is continuous on the interval $[0,1]$. With $\varphi(0)=\alpha>0$ and $\varphi(1)=-2k(1-\beta)<0$, we deduce that Eq (2.1) has a smallest positive root r_1 in $(0,1)$. For $|z|<r_1$, we have

$$\operatorname{Re}\left\{\frac{1}{\alpha}\left(1+\frac{zh''(z)}{h'(z)}\right)+\left(1-\frac{1}{\alpha}\right)\frac{zh'(z)}{h(z)}\right\}>0,$$

so, $h(z)$ is $\frac{1}{\alpha}$–convex of order 0 in disk $\mathcal{U}_{r_1}$. Thus the proof is finalized. □

Based on reference [12] and Lemma 2.5, the subsequent conclusions can readily be inferred:

Remark 2.6. $\mathcal{B}(\alpha,\beta)\subset\mathcal{K}\left(\frac{1}{\alpha},0\right)\subset\mathcal{S}^*(0),|z|<r_1$. Here, r_1 is given by Lemma 2.5.

Lemma 2.7. Given $\alpha>0,\beta<1$, and $h(z)\in\mathcal{B}(\alpha,\beta)$, the function $h(z)$ is a convex function of order $-\frac{1}{2}$ in the disk $\mathcal{U}_{r_0}$, where $r_0=\min\{r_1,r_2\}$; here, r_1 is given by Lemma 2.5, and

$$r_2=(\sqrt{4k^2+1}-2k)^{1/k}. \tag{2.3}$$

Proof. Given $h(z) \in \mathcal{B}(\alpha,\beta)$ for $|z| < r_1$, where r_1 is given by Lemma 2.5, there exists $p(z) \in \mathcal{P}(0)$ satisfying

$$zh'(z)/h(z) = p(z).$$

Take the logarithmic derivative at both ends of the above equation and get

$$\mathrm{Re}\left\{1+\frac{zh''(z)}{h'(z)}\right\} = \mathrm{Re}\left\{p(z)+\frac{zp'(z)}{p(z)}\right\} > -\left|\frac{zp'(z)}{p(z)}\right|.$$

By making use of Lemma 2.4, it can be deduced that

$$\mathrm{Re}\left\{1+\frac{zh''(z)}{h'(z)}\right\} \geq -\left|\frac{zp'(z)}{p(z)}\right| \geq -\frac{2kr^k}{(1-r^{2k})}.$$

Next, let us delve into the requirement for the right-hand side of the above inequality to exceed $-\frac{1}{2}$. In fact, if $-\frac{2kr^k}{(1-r^{2k})} > -\frac{1}{2}$, this is equivalent to

$$\frac{r^{2k}+4kr^k-1}{2\,(1-r^{2k})} < 0.$$

Let $\rho(r) = r^{2k} + 4kr^k - 1$; then $\rho(r)$ is continuous on $[0,1]$ such that $\rho(0) = -1 < 0$ and $\rho(1) = 4k > 0$. According to the continuity of the function $\rho(r)$, there exists $r_2 \in (0,1)$, if $r < r_2$, then $\rho(r) = -1 + 4kr^k + r^{2k} < 0$. That is, $r_2 = (\sqrt{4k^2+1} - 2k)^{1/k}$ is the minimum positive root of the equation $r^{2k} + 4kr^k - 1 = 0$. Let $r_0 = \min\{r_1, r_2\}$. For $|z| < r_0$, we have $h(z)$ is convex function of order $-\frac{1}{2}$ in disk $|z| < r_0$. Thus the proof is complete. □

Lemma 2.7 leads to the following conclusions:

Remark 2.8. $\mathcal{B}(\alpha,\beta) \subset \mathcal{K}\left(-\frac{1}{2}\right) \subset \mathcal{S}(|z| \leq r_0)$. Here, r_0 is given by Lemma 2.7.

Lemma 2.9. Given $q(z) = 1 + q_k z^k + \cdots \in \mathcal{P}(\beta)$ for $z \in \mathcal{U}$ and $k \geq 1$, and considering $z = re^{i\theta}$, $0 \leq r < 1$ and $0 \leq \theta < 2\pi$, it follows that

$$\frac{1}{2\pi}\int_0^{2\pi} |q(re^{i\theta})|^2 d\theta \leq \frac{1+(4\beta^2-8\beta+3)r^2}{1-r^2}.$$

This inequality is exact.

Proof. Since $q(z) = 1 + q_k z^k + \cdots \in \mathcal{P}(\beta), |q_k| \leq 2(1-\beta)$, then we have

$$\frac{1}{2\pi}\int_0^{2\pi} |q(re^{i\theta})|^2 d\theta \leq \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{k=0}^{\infty}|q_k|^2 r^{2k}\right)d\theta \leq 1+4(1-\beta)^2\sum_{k=1}^{\infty} r^{2k} = \frac{1+(4\beta^2-8\beta+3)r^2}{1-r^2}.$$

Thus the proof is complete. □

3. Results

Theorem 3.1. Provided that $f(z)$ is in the class $\mathcal{B}_{\mathcal{H}}^{\sigma}(\alpha,\beta)$, it can be represented as

$$f(z) = h(z) + \overline{\int_0^z \frac{u+\sigma}{1+\sigma u} h'(u)du}, \tag{3.1}$$

where

$$h(z) = z\left\{\alpha \int_0^1 \frac{1+(1-2\beta)\varpi(zt)}{1-\varpi(zt)} t^{\alpha-1} dt\right\}^{1/\alpha}, \tag{3.2}$$

and the power is taken as its principal value. Here, $\varpi(z)$ denotes a Schwarz function that satisfies $\varpi(0) = 0$ and $|\varpi(z)| < 1$ for all z in the unit disk $\mathcal{U}$.

Proof. Let $f(z) = h(z) + \overline{g(z)} \in \mathcal{B}_{H}^{\sigma}(\alpha,\beta)$, we have $h(z) \in \mathcal{B}(\alpha,\beta)$, from (1.2), that is,

$$h'(z)(\frac{h(z)}{z})^{\alpha-1} \prec \frac{1+(1-2\beta)z}{1-z} \quad (z \in \mathcal{U}). \tag{3.3}$$

Utilizing the concept of subordination among analytic functions in conjunction with (3.3), we deduce that:

$$h'(z)(\frac{h(z)}{z})^{\alpha-1} = \frac{1+(1-2\beta)\varpi(z)}{1-\varpi(z)} \quad (z \in \mathcal{U}), \tag{3.4}$$

where $\varpi(z)$ is a Schwarz function satisfying $\varpi(0) = 0$ and $|\varpi(z)| < 1$ for $z \in \mathcal{U}$.

From (3.4), we can further derive:

$$H(z) + \frac{1}{\alpha} zH'(z) = \frac{1+(1-2\beta)\varpi(z)}{1-\varpi(z)}, \tag{3.5}$$

where $H(z) = (h(z)/z)^{\alpha}$.

After simple calculations, we can obtain that

$$(z^{\alpha} H(z))' = \alpha z^{\alpha-1} \frac{1+(1-2\beta)\varpi(z)}{1-\varpi(z)}.$$

By integrating both sides of the preceding equation, we obtain:

$$h(z) = z\left\{\alpha \int_0^1 \frac{1+(1-2\beta)\varpi(zt)}{1-\varpi(zt)} t^{\alpha-1} dt\right\}^{1/\alpha}. \tag{3.6}$$

By Definition 1.3 and Lemma 2.3, we have

$$g(z) = \int_0^z \frac{u+\sigma}{1+\sigma u} h'(u)du. \tag{3.7}$$

By integrating (3.6) and (3.7), we confirm that Theorem 3.1's conclusion stands. Thus, the proof is established. □

In particular, by setting $\alpha = 1, \beta = \frac{1}{2}$ in Theorem 3.1, the following corollary can be derived.

Corollary 3.2. Provided that f belongs to the class $\mathcal{B}_H^{\sigma}(1, \frac{1}{2})$, the function $f(z)$ can be expressed as:

$$f(z) = \int_0^z \frac{d\xi}{1 - \varpi(\xi)} + \overline{\int_0^z \frac{u + \sigma}{(1 + \sigma u)(1 - \varpi(u))} du},$$

where the power takes the principal value and ϖ is a Schwarz function satisfying $\varpi(0) = 0$ and $|\varpi(z)| < 1$ for all z in the disk $\mathcal{U}$.

In particular, by setting $\sigma = \frac{1}{2}$, we can get the extreme functions as follows.

$$f_1(z) = \log \frac{1}{1 - z} - \overline{\log(1 + \frac{3z}{2} + \frac{z^2}{2})}$$

or

$$f_2(z) = \frac{1}{2} \log\left(\frac{1 + z}{1 - z}\right) + \frac{1}{2}\overline{\log \frac{(1 + \frac{1}{2}z)^2}{(1 - z^2)}}.$$

Additionally, we illustrate the graphs of the extreme functions $f_1(z)$ and $f_2(z)$ in Figures 1 and 2, respectively. In these figures, the horizontal axis depicts the real part of the function, while the vertical axis represents its imaginary part.

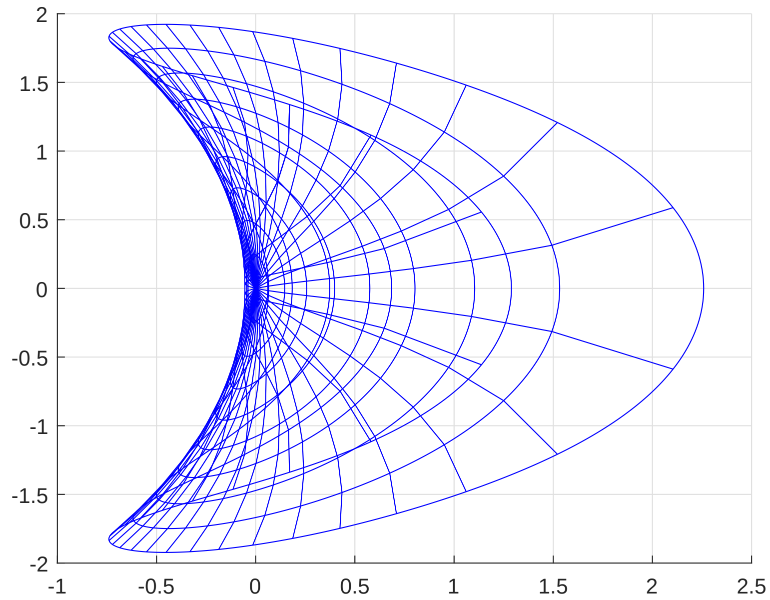

Figure 1. Image of $f_1(z)$.

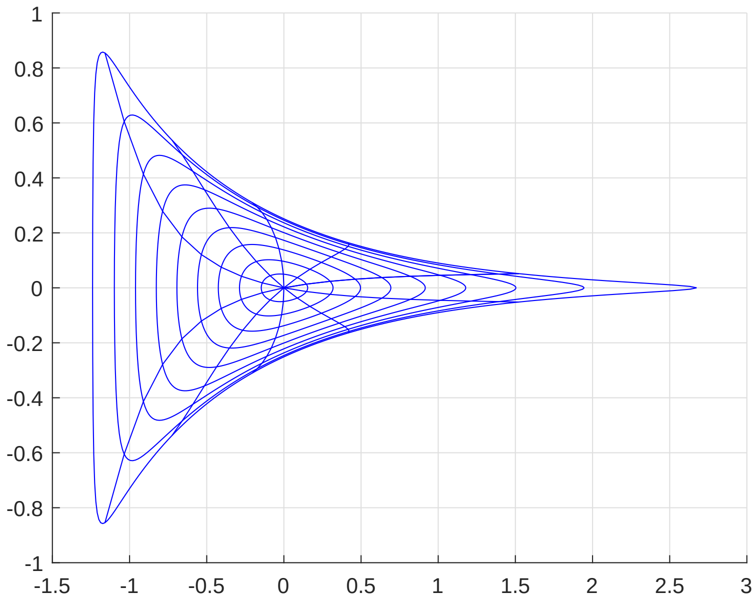

Figure 2. Image of $f_2(z)$.

Theorem 3.3. Given that $f(z) = h(z) + \overline{g(z)} \in \mathcal{B}_H^{\sigma}(\alpha,\beta)$ and $|z| = r < 1$, the following bounds hold:

(1) $rK(\alpha,\beta;-r) \le |h(z)| \le rK(\alpha,\beta;r)$, where $\alpha > 0$,

$$K(\alpha,\beta;r) = \left(\alpha \int_0^1 \mathcal{F}(\beta, rt)t^{\alpha-1}\,\mathrm{d}t\right)^{\frac{1}{\alpha}}, \tag{3.8}$$

and $\mathcal{F}(\beta,r) = \frac{1+(1-2\beta)r}{1-rt}$.

(2) If $0 < \alpha \le 1$, then

$$\mathcal{F}(\beta,-r)K^{1-\alpha}(\alpha,\beta;-r) \le |h'(z)| \le \mathcal{F}(\beta,r)K^{1-\alpha}(\alpha,\beta;r).$$

If $\alpha > 1$, then

$$\mathcal{F}(\beta,-r)K^{1-\alpha}(\alpha,\beta;r) \le |h'(z)| \le \mathcal{F}(\beta,r)K^{1-\alpha}(\alpha,\beta;-r).$$

(3) If $0 < \alpha \le 1$, then

$$\frac{|r-\sigma|}{(1-\sigma r)}\mathcal{F}(\beta,-r)K^{1-\alpha}(\alpha,\beta;-r) \le |g'(z)| \le \frac{(r+\sigma)}{(1+\sigma r)}\mathcal{F}(\beta,r)K^{1-\alpha}(\alpha,\beta;r).$$

If $\alpha > 1$, then

$$\frac{|r-\sigma|}{(1-\sigma r)}\mathcal{F}(\beta,-r)K^{1-\alpha}(\alpha,\beta;r) \le |g'(z)| \le \frac{(r+\sigma)}{(1+\sigma r)}\mathcal{F}(\beta,r)K^{1-\alpha}(\alpha,\beta;-r).$$

(4) If $0 < \alpha \le 1, |z| = r < r_0$, then

$$\int_0^r \frac{|s-\sigma|}{(1-\sigma s)}\mathcal{F}(\beta,-s)K^{1-\alpha}(\alpha,\beta;-s)ds \le |g(z)| \le \int_0^r \frac{(s+\sigma)}{(1+\sigma s)}\mathcal{F}(\beta,s)K^{1-\alpha}(\alpha,\beta;s)ds.$$

The above power takes the principal value.

Proof. According to Theorem 3.1 and subordination relationship, we can get

$$(h(z)/z)^{\alpha} = \alpha \int_0^1 \frac{1 + (1 - 2\beta)\varpi(z)t}{1 - \varpi(z)t} t^{\alpha-1} \mathrm{d}t,$$

where $\varpi(z) = c_1 z + c_2 z^2 + \cdots$ is analytic in $\mathcal{U}$ and $|\varpi(z)| \leq |z| = r < 1$,

$$|h(z)/z|^{\alpha} \leq \alpha \int_0^1 \frac{1 + (1 - 2\beta)|\varpi(z)|t}{1 - |\varpi(z)|t} t^{\alpha-1}\ \mathrm{d}t \leq \alpha \int_0^1 \frac{1 + (1 - 2\beta)rt}{1 - rt} t^{\alpha-1} \mathrm{d}t,$$

and

$$|h(z)/z|^{\alpha} \geq \alpha \int_0^1 \min_{z\in\mathcal{U}} \mathrm{Re}\left(\frac{1 + (1 - 2\beta)zt}{1 - zt}\right) t^{\alpha-1} \mathrm{d}t \geq \alpha \int_0^1 \frac{1 - (1 - 2\beta)rt}{1 + rt} t^{\alpha-1} \mathrm{d}t.$$

Therefore, we have

$$r\left(\alpha \int_0^1 \frac{1 - (1 - 2\beta)rt}{1 + rt} t^{\alpha-1}\ \mathrm{d}t\right)^{\frac{1}{\alpha}} \leq |h(z)| \leq r\left(\alpha \int_0^1 \frac{1 + (1 - 2\beta)rt}{1 - rt} t^{\alpha-1}\ \mathrm{d}t\right)^{\frac{1}{\alpha}}. \tag{3.9}$$

On the other hand, due to

$$h'(z)(\frac{h(z)}{z})^{\alpha-1} \prec \frac{1 + (1 - 2\beta)z}{1 - z} \quad (z \in \mathcal{U}).$$

According to the subordinate relationship, for $|z| = r < 1$, we have

$$\frac{1 - (1 - 2\beta)r}{1 + r} \leq \left|h'(z)(\frac{h(z)}{z})^{\alpha-1}\right| \leq \frac{1 + (1 - 2\beta)r}{1 - r}. \tag{3.10}$$

By utilizing both (3.9) and (3.10), we can derive an estimate for $h'(z)$ in the manner described below. If $0 < \alpha \leq 1$, then

$$|h'(z)| \leq \left(\alpha \int_0^1 \frac{1 + (1 - 2\beta)rt}{1 - rt} t^{\alpha-1}\ \mathrm{d}t\right)^{\frac{1}{\alpha}-1} \left(\frac{1 + (1 - 2\beta)r}{1 - r}\right) \tag{3.11}$$

and

$$|h'(z)| \geq \left(\alpha \int_0^1 \frac{1 - (1 - 2\beta)rt}{1 + rt} t^{\alpha-1}\ \mathrm{d}t\right)^{\frac{1}{\alpha}-1} \left(\frac{1 - (1 - 2\beta)r}{1 + r}\right). \tag{3.12}$$

If $\alpha > 1$, then

$$|h'(z)| \leq \left(\alpha \int_0^1 \frac{1 - (1 - 2\beta)rt}{1 + rt} t^{\alpha-1}\ \mathrm{d}t\right)^{\frac{1}{\alpha}-1} \left(\frac{1 + (1 - 2\beta)r}{1 - r}\right) \tag{3.13}$$

and

$$|h'(z)| \geq \left(\alpha \int_0^1 \frac{1 + (1 - 2\beta)rt}{1 - rt} t^{\alpha-1}\ \mathrm{d}t\right)^{\frac{1}{\alpha}-1} \left(\frac{1 - (1 - 2\beta)r}{1 + r}\right). \tag{3.14}$$

Utilizing the relationship $g'(z) = w(z)h'(z)$ from Lemma 2.3, we derive the inequalities:

$$\frac{|r - \sigma|}{(1 - \sigma r)}|h'(z)| \leq |g'(z)| \leq \frac{(r + \sigma)}{(1 + \sigma r)}|h'(z)|.$$

Thus, we obtained an estimate of (2) and (3) in Theorem 3.3.

Next, we continue to consider conclusion (4). Taking the logarithmic derivative of equation $g'(z) = w(z)h'(z)$, we obtain:

$$\frac{zg''(z)}{g'(z)} = \frac{zh''(z)}{h'(z)} + \frac{zw'(z)}{w(z)}, \quad z \in \mathcal{U}. \tag{3.15}$$

For $h \in \mathcal{B}(\alpha, \beta)$, applying Lemma 2.7, we have

$$\mathrm{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}, \quad |z| < r_0, \tag{3.16}$$

where r_0 is determined by Lemma 2.7.

Therefore, by using Lemma 2.3, (3.15), and (3.16), we deduce

$$\mathrm{Re}\left(1 + \frac{zg''(z)}{g'(z)}\right) > \frac{1-\alpha}{1+\alpha} - \frac{1}{2} > -\frac{1}{2}.$$

This demonstrates that the function g is univalent (refer to [13]). According to [14], if the function ψ is univalent and satisfies:

$$M'(r) \leq |\chi'(z)| \leq N'(r),$$

then

$$\int_0^r M'(t)\mathrm{d}t \leq |\chi(z)| \leq \int_0^r N'(t)\mathrm{d}t.$$

Applying the estimate of g' in (3), we can get the estimate of g in (4). Thus the proof is complete. □

Specifically, by assigning $\alpha = 1, \beta = \frac{1}{2}$ in Theorem 3.3, the subsequent corollary can be obtained.

Corollary 3.4. Suppose $f(z) = h(z) + \overline{g(z)} \in \mathcal{B}^{\sigma}_{\mathcal{H}}(1, \frac{1}{2})$, then

(1) $\log(1+r) \leq |h(z)| \leq \log \frac{1}{1-r}$.

(2) $\frac{1}{1+r} \leq |h'(z)| \leq \frac{1}{1-r}$.

(3) $\frac{(|r-\sigma|)}{(1-\sigma r)(1+r)} \leq |g'(z)| \leq \frac{(r+\sigma)}{(1+\sigma r)(1-r)}$.

(4) $\max\left\{\log\left[(1+r)(1-\sigma r)^{\frac{1-\sigma}{\sigma}}\right], \log\left[(\frac{1-2\sigma^2+\sigma^4}{1-\sigma r})^{\frac{1-\sigma}{\sigma}}\left(\frac{1+2\sigma+\sigma^2}{1+r}\right)\right]\right\} \leq |g(z)| \leq \log \frac{(1+\sigma r)^{\frac{\sigma-1}{\sigma}}}{1-r}$.

Next, we study the coefficient estimation of Bazilevič function.

Let C_r represent the closed curve obtained by transforming the circle $|z| = r < 1$ under the function $w = f(z)$, and let $L_r(f(z))$ denote the length of this curve. Here is the resulting conclusion.

Theorem 3.5. Given $f(z) = h(z) + \overline{g(z)} \in \mathcal{B}^{\sigma}_{H}(\alpha, \beta)$, the length $L_r(g(z))$ can be formulated as:

$$L_r(g(z)) = \int_0^{2\pi} |\omega_\sigma(z) z^\alpha h^{1-\alpha}(z) q(z)| d\theta.$$

This integral is subject to the following bound:

$$L_r(g(z)) \leq \pi\left(\frac{r+\sigma}{1+\sigma r}\right)\chi(\alpha, \beta; r) \times \begin{cases} (rK(\alpha, \beta; r))^{1-\alpha}, & 0 \leq \alpha < 1, \\ (rK(\alpha, \beta; -r))^{1-\alpha}, & \alpha > 1, \end{cases}$$

where $K(\alpha,\beta;r)$ is defined by Eq (3.8) and

$$\chi(\alpha,\beta;r) = \frac{\alpha}{\alpha+1}r^{\alpha+1} + \frac{\alpha}{\alpha-1}r^{\alpha-1} + \frac{4(1-\beta)(1+\alpha-\alpha\beta)}{\alpha+1}r^{\alpha+1}{}_2F_1(1,\frac{\alpha+1}{2},\frac{\alpha+3}{2};r^2). \tag{3.17}$$

Proof. For any function $f(z) = h(z)+\overline{g(z)}$ in the class $\mathcal{B}_H^{\sigma}(\alpha,\beta)$, it is implied that $h(z)$ belongs to $\mathcal{B}(\alpha,\beta)$ and fulfills the condition:

$$zh'(z) = h^{1-\alpha}(z)z^{\alpha}q(z), \tag{3.18}$$

where $q(z)$ is a member of $\mathcal{P}(\beta)$. Additionally, the derivative of $g(z)$ is related to $h'(z)$ by

$$g'(z) = \omega_{\sigma}(z)h'(z), \tag{3.19}$$

with ω_{σ} defined as in Eq (1.6).

From these relationships, we derive

$$zg'(z) = \omega_{\sigma}(z)h^{1-\alpha}(z)z^{\alpha}q(z).$$

For $z = re^{i\theta}$, where $0 < r < 1$ and $0 < \alpha \leq 1$, the length $L_r(g(z))$ can be calculated:

$$\begin{aligned} L_r(g(z)) &= \int_0^{2\pi} |zg'(z)|d\theta = \int_0^{2\pi} |\omega_{\sigma}(z)z^{\alpha}h^{1-\alpha}(z)q(z)|d\theta \\ &\leq \left(\frac{r+\sigma}{1+\sigma r}\right)(rK(\alpha,\beta;r))^{1-\alpha}\int_0^{2\pi}\int_0^r |\alpha z^{\alpha-1}q(z) + z^{\alpha}q'(z)|dsd\theta \\ &\leq \left(\frac{r+\sigma}{1+\sigma r}\right)(rK(\alpha,\beta;r))^{1-\alpha}I(r,\alpha), \end{aligned} \tag{3.20}$$

where $K(\alpha,\beta;r)$ is given by (3.8) and

$$I(r,\alpha) = \int_0^{2\pi}\int_0^r \alpha s^{\alpha-1}|q(z)|dsd\theta + \int_0^{2\pi}\int_0^r s^{\alpha-1}|zq'(z)|dsd\theta. \tag{3.21}$$

By using the Cauchy–Schwarz inequality, we obtain

$$I(r,\alpha) \leq \int_0^r s^{\alpha-1}\left\{\alpha\sqrt{\int_0^{2\pi} 1^2 d\theta}\sqrt{\int_0^{2\pi} |q(z)|^2 d\theta} + \int_0^{2\pi} |zq'(z)|d\theta\right\}ds. \tag{3.22}$$

According to Lemma 2.9 for $\mathcal{P}(\beta)(0 \leq \beta < 1)$, along with the result

$$\int_0^{2\pi} |zq'(z)|d\theta \leq \frac{4\pi(1-\beta)r}{1-r^2},$$

for $q(z) \in \mathcal{P}(\beta)(0 \leq \beta < 1)$(see [15]), we can write

$$I(r,\alpha) \leq 2\pi\int_0^r s^{\alpha-1}\left\{\alpha\sqrt{\frac{1+(4\beta^2-8\beta+3)s^2}{1-s^2}} + \frac{2(1-\beta)s}{1-s^2}\right\}ds. \tag{3.23}$$

Using inequality

$$s^2 + \frac{1+(4\beta^2-8\beta+3)s^2}{1-s^2} \geq 2s\sqrt{\frac{1+(4\beta^2-8\beta+3)s^2}{1-s^2}},$$

we obtain

$$I(r,\alpha) \leq 2\pi \int_0^r s^{\alpha-2}\left\{\frac{\alpha}{2}(s^2 + \frac{1+(4\beta^2-8\beta+3)s^2}{1-s^2}) + \frac{2(1-\beta)s^2}{1-s^2}\right\}ds. \tag{3.24}$$

From (3.24), we get

$$\begin{aligned} I(r,\alpha) &\leq \pi \int_0^r \alpha s^\alpha + s^{\alpha-2}\left\{\alpha\left(\frac{1+(4\beta^2-8\beta+3)s^2}{1-s^2}\right) + \frac{4(1-\beta)s^2}{1-s^2}\right\}ds \\ &= \pi \int_0^r \alpha s^\alpha + \alpha s^{\alpha-2} + \frac{4(1-\beta)(1+\alpha-\alpha\beta)}{1-s^2}s^\alpha ds \\ &= \frac{\pi\alpha}{\alpha+1}r^{\alpha+1} + \frac{\pi\alpha}{\alpha-1}r^{\alpha-1} + \frac{4\pi(1-\beta)(1+\alpha-\alpha\beta)}{\alpha+1}r^{\alpha+1}{}_2F_1(1,\frac{\alpha+1}{2},\frac{\alpha+3}{2};r^2) \\ &:= \pi\chi(\alpha,\beta;r). \end{aligned}$$

Thus, we have

$$L_r(g(z)) \leq \pi\chi(\alpha,\beta;r)\left(\frac{r+\sigma}{1+\sigma r}\right)(rK(\alpha,\beta;r))^{1-\alpha}, \quad 0<\alpha\leq 1.$$

When $\alpha > 1$, we can prove similarly as above to get

$$L_r(g(z)) \leq \pi\chi(\alpha,\beta;r)\left(\frac{r+\sigma}{1+\sigma r}\right)(rK(\alpha,\beta;-r))^{1-\alpha}.$$

Thus the proof is complete. □

Theorem 3.6. Given $f(z) = h(z) + \overline{g(z)} \in \mathcal{B}^\sigma_{\mathcal{H}}(\alpha,\beta)$, the coefficient b_n is bounded by:

$$|b_n| \leq \begin{cases} \dfrac{1}{2n}\lim\limits_{r\to 1^-}\chi(\alpha,\beta;r)\,(K(\alpha,\beta;r))^{1-\alpha}, & 0\leq\alpha<1, \\ \dfrac{1}{2n}\lim\limits_{r\to 1^-}\chi(\alpha,\beta;r)\,(K(\alpha,\beta;-r))^{1-\alpha}, & \alpha>1, \end{cases}$$

where $K(\alpha,\beta;r)$ is given by (3.8) and $\chi(\alpha,\beta;r)$ is given by (3.17).

Proof. Utilizing Cauchy's theorem, for $z = re^{i\theta}$ where $n \geq 2$, we derive:

$$nb_n = \frac{1}{2\pi r^n}\int_0^{2\pi} zg'(z)e^{-in\theta}d\theta.$$

Thus,

$$n\,|b_n| \leq \frac{1}{2\pi r^n}\int_0^{2\pi} |zg'(z)|\,d\theta = \frac{1}{2\pi r^n}L_r(g(z)).$$

For the case $0<\alpha<1$, using Theorem 3.5, we obtain

$$n\,|b_n| \leq \frac{1}{2r^n}(rK(\alpha,\beta;r))^{1-\alpha}\left(\frac{r+\sigma}{1+\sigma r}\right)\chi(\alpha,\beta;r).$$

When r approaches 1^-, we have

$$|b_n| \leq \frac{1}{2n} \lim_{r \to 1^-} (K(\alpha,\beta;r))^{1-\alpha} \chi(\alpha,\beta;r).$$

For $\alpha > 1$, we have

$$|b_n| \leq \frac{1}{2n} \lim_{r \to 1^-} (K(\alpha,\beta;-r))^{1-\alpha} \chi(\alpha,\beta;r).$$

□

Corollary 3.7. Given $f(z) = h(z) + \overline{g(z)} \in \mathcal{B}^{\sigma}_{\mathcal{H}}(\frac{3}{2},\beta)$, the coefficients b_n is bounded by:

$$|b_n| \leq \frac{1}{2n} \lim_{r \to 1^-} \frac{\chi(\frac{3}{2},\beta;r)}{(K(\frac{3}{2},\beta;-r))^{\frac{1}{2}}},$$

where $K(\frac{3}{2},\beta;-r)$ and $\chi(\frac{3}{2},\beta;r)$ are specified in Eqs (3.8) and (3.17) respectively.

4. Conclusions

This paper introduces a new class of Bazilevič harmonic functions, $\mathcal{B}^{\sigma}_{\mathcal{H}}(\alpha,\beta)$. Using subordination relationships and basic harmonic function theories, it establishes the necessary and sufficient conditions and integral expressions for this class. Further exploration of inclusion relations and radius problems yields new findings.

Author contributions

Shuhai Li: Conceptualization, methodology, investigation, writing—original draft preparation, project administration, funding acquisition; Lina Ma: Validation, writing—review and editing; Xiaomeng Niu: Software, data curation, visualization; Huo Tang: Supervision, formal analysis; Jingyu Yang: Data curation, resources. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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