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Research article

A simplified approach to solve hesitant fuzzy linear programming problem with hesitant decision variables and right-hand-side values

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Abstract: An approach (a generalization of the classical simplex algorithm) is proposed to solve hesitant fuzzy linear programming problems (HFLPPs). In this paper, we pointed out that much computational effort is required to solve HFLPPs by the existing approach. Moreover, to reduce the computational efforts, an alternative approach is proposed to solve HFLPPs. Furthermore, some other advantages of the proposed alternative approach (PrAlApp) over the existing approach are discussed. Finally, an existing HFLPP is solved by the PrAlApp.

Keywords: simplex algorithm; hesitant fuzzy linear programming problem (HFLPP); trapezoidal hesitant fuzzy number (THFN)
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1. Introduction

Torra [1] proposed the concept of hesitant fuzzy set (HFS) by generalizing the concept of fuzzy set [2]. Ranjbar et al. [3] pointed out that in a HFS, each element is represented by a non-negative real number belonging to the closed interval 0 to 1. However, it is more realistic to represent each element of a HFS by a fuzzy number instead of a real number belonging to the closed interval 0 to 1. Hence, Ranjbar et al. [3] proposed a new type of HFS (named as THFN) by representing each element of a HFS with a trapezoidal fuzzy number. Ranjbar et al. [3] also proposed arithmetic operations of trapezoidal hesitant fuzzy numbers (THFNs) by generalizing the existing arithmetic operations of hesitant fuzzy sets [1] with the help of existing arithmetic operations of trapezoidal fuzzy numbers [4]. Furthermore, Ranjbar et al. [3] proposed an approach to solve the HFLPP (P_1).

Problem (P_1)

$Maximize \left(\sum_{j=1}^{n} \tilde{\tilde{c}}_j x_j\right)$

$Subject\ to$

$$\sum_{j=1}^{n} \tilde{\tilde{a}}_{ij} x_j \preccurlyeq \tilde{\tilde{b}}_i, i = 1, \dots, m,$$

$x_j \succcurlyeq 0, j = 1,2, \dots, n,$

where, $\tilde{\tilde{c}}_j, \tilde{\tilde{b}}_i, \tilde{\tilde{a}}_{ij}$ $(j = 1,2, \dots, n, i = 1,2, \dots, m)$ are THFNs and x_j is a real number.

Ranjbar et al. [5] pointed out that much computational effort is required to apply the existing arithmetic operations of THFNs [4]. Moreover, to reduce the computational efforts, Ranjbar et al. [5] proposed new arithmetic operations of THFNs. Furthermore, Ranjbar et al. [5] suggested to use their proposed arithmetic operations to evaluate the value of the objective function of the HFLPP (P_1) corresponding to the existing optimal solution (OS) [3] of the HFLPP (P_1).

Ranjbar et al. [6] claimed that no method is proposed in the literature to solve the HFLPP (P_2).

Problem (P_2)

$Maximize \left(\sum_{j=1}^{n} \left(\tilde{\tilde{c}}_j \tilde{\tilde{x}}_j\right)\right)$

$Subject\ to$

$\sum_{j=1}^{n} \left(\tilde{\tilde{a}}_{ij} \tilde{\tilde{x}}_j\right) \preccurlyeq or = or \geq \tilde{\tilde{b}}_i, i = 1, \dots, m,$

$\tilde{\tilde{x}}_j \succcurlyeq \tilde{\tilde{0}}, j = 1,2, \dots, n,$

where, $\tilde{\tilde{c}}_j, \tilde{\tilde{x}}_j, \tilde{\tilde{a}}_{ij}, \tilde{\tilde{b}}_i$ and $\tilde{\tilde{0}}$ are THFNs.

Using the existing arithmetic operations of THFNs [5], Saghi et al. [7] proposed two approaches to solve the HFLPP (P_3).

Problem (P_3)

$Maximize \left(\sum_{j=1}^{n} \left(\tilde{\tilde{c}}_j x_j\right)\right)$

$Subject\ to$

$\sum_{j=1}^{n} a_{ij} x_j \preccurlyeq or = or \geq b_i, i = 1,2, \dots, m;$

$x_j \succcurlyeq 0, j = 1,2, \dots, n,$

where, $\tilde{\tilde{c}}_j$ is a THFN and b_i, a_{ij} are real numbers.

Ahuja and Kumar [8] pointed out that if in the HFLPP (P_3), the cardinality of all the hesitant fuzzy numbers, $\tilde{\tilde{c}}_j, j = 1,2, \dots, n$ is not the same. Then, Saghi et al.'s [7] approach fails to find the correct THFN (representing the optimal value of a HFLPP). Ahuja and Kumar [8] also discussed the reasons for the failure of Saghi et al.'s [7] approach. Furthermore, to overcome this limitation, Ahuja and Kumar [8] proposed a new approach (named as Mehar approach) to solve the HFLPP (P_3).

Saghi et al. [9] proposed an approach to solve the HFLPP (P_4).

Problem (P_4)

$Maximize\left(\sum_{j=1}^{n}\left(c_j\tilde{\tilde{x}}_j\right)\right)$
$Subject\ to$
$\sum_{j=1}^{n} a_{ij}\tilde{\tilde{x}}_j \preccurlyeq \tilde{\tilde{b}}_i, i = 1,2,\ldots,m;$
$\tilde{\tilde{x}}_j \succcurlyeq \tilde{\tilde{0}}, j = 1,2,\ldots,n,$
where, $\tilde{\tilde{x}}_j$, $\tilde{\tilde{b}}_i$ are THFNs and c_j, a_{ij} are real numbers.

In this paper, it is pointed out that much computational is required to solve the HFLPP (P_4) by Saghi et al.'s [9] approach. Also, to reduce computational efforts, an alternative approach is proposed to solve the HFLPP (P_4). Furthermore, to illustrate the PrAlApp, an existing HFLPP is solved by the PrAlApp.

This paper is organized as follows. In section 2, some basic definitions are discussed. In Section 3, an existing method, used in Saghi et al.'s [9] approach to compare THFNs, is discussed. In Section 4, a method, used in Saghi et al.'s [9] approach to evaluate $k_1\tilde{\tilde{A}}_1 + k_2\tilde{\tilde{A}}_2$, where k_1, k_2 are real numbers and $\tilde{\tilde{A}}_1, \tilde{\tilde{A}}_2$ are THFNs, is discussed. In Section 5, Saghi et al.'s [9] approach is discussed. In Section 6, an alternative approach is proposed to solve the HFLPP (P_4). In Section 7, the origin of the PrAlApp is discussed. In Section 8, an existing HFLPP is solved by the PrAlApp. In Section 9, advantages of the PrAlApp over Saghi et al.'s [9] approach are discussed. In Section 10, we conclude the paper.

2. Preliminaries

Some basic definitions are presented this section.

Definition 2.1. [1] The set $\tilde{\tilde{A}} = \{(x, \{\mu_{\tilde{A}^1}(x), \mu_{\tilde{A}^2}(x), \ldots, \mu_{\tilde{A}^n}(x)\}): x \in X, 0 \le \mu_{\tilde{A}^1}(x) \le 1, 0 \le \mu_{\tilde{A}^2}(x) \le 1, \ldots, 0 \le \mu_{\tilde{A}^n}(x) \le 1\}$, defined on the universal set X, is said to be a HFS, where $\mu_{\tilde{A}^i}(x), i = 1,2,\ldots,n$ represents the degree of satisfaction of the i^{th} expert for $x \in \tilde{\tilde{A}}$. If the universal set X is a discrete set, then the HFS $\tilde{\tilde{A}}$ is said to be a discrete HFS. While, if the universal set X is a continuous set, then the HFS $\tilde{\tilde{A}}$ is said to be a continuous HFS.

For example, the statement "real numbers close to 3" may be represented by the HFS, $\tilde{\tilde{A}} = \{(x, \{\mu_{\tilde{A}^1}(x), \mu_{\tilde{A}^2}(x)\}): x \in \mathbb{R}\}$, where $\mu_{\tilde{A}^1}(x)$ and $\mu_{\tilde{A}^2}(x)$ represents the degree of satisfaction of the first and second decision-maker, respectively, for $x \in \tilde{\tilde{A}}$ and are defined as

$$\mu_{\tilde{A}^1}(x) = \begin{cases} \frac{x-1}{2}, 1 \le x < 3 \\ 1, \quad x = 3 \\ \frac{5-x}{2}, 3 < x \le 5 \\ 0, \ elsewhere \end{cases} \quad \text{and} \quad \mu_{\tilde{A}^2}(x) = \begin{cases} \frac{x-1}{1.5}, 1 \le x < 2.5 \\ 1, \ 2.5 \le x \le 3.5 \\ \frac{5-x}{1.5}, \ 3.5 < x \le 5 \\ 0, \ elsewhere \end{cases}.$$

Definition 2.2. [3] A continuous HFS $\tilde{\tilde{A}} = \{(x, \{\mu_{\tilde{A}^1}(x), \mu_{\tilde{A}^2}(x), \ldots, \mu_{\tilde{A}^p}(x)\}): x \in X\}$ is said to be a HFN if the following properties are satisfied.

(i) $\tilde{A}^1, \tilde{A}^2, \ldots, \tilde{A}^p$ are fuzzy numbers.

(ii) $\cap_{j=1}^{p}\left\{x \in X: \mu_{\tilde{A}^j}(x) \ge 1\right\} \ne \emptyset$.

Furthermore, if $\tilde{A}^1, \tilde{A}^2, \ldots, \tilde{A}^p$ are trapezoidal fuzzy numbers, then $\tilde{\tilde{A}} = \{(x, \{\mu_{\tilde{A}^1}(x), \mu_{\tilde{A}^2}(x), \ldots, \mu_{\tilde{A}^p}(x)\}): x \in X\}$ is said to be a THFN.

3. Existing method for comparing THFNs

Our aim of this paper is to propose an alternative approach for solving the HFLPP (P_4). To achieve this aim, there is a need to discuss the method used in Saghi et al.'s [9]approach for comparing THFNs. Thus, the same is discussed in this section.

Let $\tilde{\tilde{A}}_1 = \{(a_{111}, a_{112}, a_{113}, a_{114}), (a_{121}, a_{122}, a_{123}, a_{124}), \dots, (a_{1p1}, a_{1p2}, a_{1p3}, a_{1p4})\}$ and $\tilde{\tilde{A}}_2 = \{(a_{211}, a_{212}, a_{213}, a_{214}), (a_{221}, a_{222}, a_{223}, a_{224}), \dots, (a_{2q1}, a_{2q2}, a_{2q3}, a_{2q4})\}$ be two THFNs with cardinality p and q respectively. Then, $\tilde{\tilde{A}}_1$ and $\tilde{\tilde{A}}_2$ can be compared as follows.

Step 1: Evaluate $\Re\left(\tilde{\tilde{A}}_1\right) = \frac{\sum_{j=1}^{p}(a_{1j1}+a_{1j2}+a_{1j3}+a_{1j4})}{4p}$ and $\Re\left(\tilde{\tilde{A}}_2\right) = \frac{\sum_{j=1}^{q}(a_{2j1}+a_{2j2}+a_{2j3}+a_{2j4})}{4q}$.

Step 2: Check that $\Re\left(\tilde{\tilde{A}}_1\right) > \Re\left(\tilde{\tilde{A}}_2\right)$ or $\Re\left(\tilde{\tilde{A}}_1\right) < \Re\left(\tilde{\tilde{A}}_2\right)$ or $\Re\left(\tilde{\tilde{A}}_1\right) = \Re\left(\tilde{\tilde{A}}_2\right)$.

Case (i): If $\Re\left(\tilde{\tilde{A}}_1\right) > \Re\left(\tilde{\tilde{A}}_2\right)$, then $\tilde{\tilde{A}}_1 \succ \tilde{\tilde{A}}_2$.

Case (ii): If $\Re\left(\tilde{\tilde{A}}_1\right) < \Re\left(\tilde{\tilde{A}}_2\right)$, then $\tilde{\tilde{A}}_1 \prec \tilde{\tilde{A}}_2$.

Case (iii): If $\Re\left(\tilde{\tilde{A}}_1\right) = \Re\left(\tilde{\tilde{A}}_2\right)$, then $\tilde{\tilde{A}}_1 \approx \tilde{\tilde{A}}_2$.

4. Saghi et al.'s expressions to evaluate $k_1\tilde{\tilde{A}}_1 + k_2\tilde{\tilde{A}}_2$

To apply Saghi et al.'s [9] approach, there is a need to evaluate $k_1\tilde{\tilde{A}}_1 + k_2\tilde{\tilde{A}}_2$, where $\tilde{\tilde{A}}_1 = \{(a_{111}, a_{112}, a_{113}, a_{114}), (a_{121}, a_{122}, a_{123}, a_{124}), \dots, (a_{1p1}, a_{1p2}, a_{1p3}, a_{1p4})\}$, $\tilde{\tilde{A}}_2 = \{(a_{211}, a_{212}, a_{213}, a_{214}), (a_{221}, a_{222}, a_{223}, a_{224}), \dots, (a_{2p1}, a_{2p2}, a_{2p3}, a_{2p4})\}$ are two THFNs and k_1, k_2 are two real numbers.

Saghi et al. [9] have used expressions (1)–(4) to evaluate $k_1\tilde{\tilde{A}}_1 + k_2\tilde{\tilde{A}}_2$.

$$k_1\tilde{\tilde{A}}_1 + k_2\tilde{\tilde{A}}_2 = \{(k_1a_{111} + k_2a_{211}, k_1a_{112} + k_2a_{212}, k_1a_{113} + k_2a_{213}, k_1a_{114} + k_2a_{214}), (k_1a_{121} + k_2a_{221}, k_1a_{122} + k_2a_{222}, k_1a_{123} + k_2a_{223} k_1a_{124} + k_2a_{224}), \dots, (k_1a_{1p1} + k_2a_{2p1}, k_1a_{1p2} + k_2a_{2p2}, k_1a_{1p3} + k_2a_{2p3}, k_1a_{1p4} + k_2a_{2p4})\} \text{ if } k_1 \geq 0, k_2 \geq 0. \quad (1)$$

$$k_1\tilde{\tilde{A}}_1 + k_2\tilde{\tilde{A}}_2 = \{(k_1a_{111} + k_2a_{214}, k_1a_{112} + k_2a_{213}, k_1a_{113} + k_2a_{212}, k_1a_{114} + k_2a_{211}), (k_1a_{121} + k_2a_{224}, k_1a_{122} + k_2a_{223}, k_1a_{123} + k_2a_{222}, k_1a_{124} + k_2a_{221}), \dots, (k_1a_{1p1} + k_2a_{2p4} k_1a_{1p2} + k_2a_{2p3}, k_1a_{1p3} + k_2a_{2p2}, k_1a_{1p4} + k_2a_{2p1})\} \text{if } k_1 \geq 0, k_2 \leq 0. \quad (2)$$

$$k_1\tilde{\tilde{A}}_1 + k_2\tilde{\tilde{A}}_2 = \{(k_1a_{114} + k_2a_{211}, k_1a_{113} + k_2a_{212}, k_1a_{112} + k_2a_{213}, k_1a_{111} + k_2a_{214}), (k_1a_{124} + k_2a_{221}, k_1a_{123} + k_1a_{222}, k_1a_{122} + k_2a_{223}, k_1a_{121} + k_2a_{224}), \dots, (k_1a_{1p4} + k_2a_{2p1}, k_1a_{1p3} + k_2a_{2p2}, k_1a_{1p2} + k_2a_{2p3}, k_1a_{1p1} + k_2a_{2p4})\} \text{ if } k_1 \leq 0, k_2 \geq 0. \quad (3)$$

$$k_1\tilde{\tilde{A}}_1 + k_2\tilde{\tilde{A}}_2 = \{(k_1a_{114} + k_2a_{214}, k_1a_{113} + k_2a_{213}, k_1a_{112} + k_2a_{212}, k_1a_{111} + k_2a_{211}), (k_1a_{124} + k_2a_{224}, k_1a_{123} + k_1a_{223}, k_1a_{122} + k_2a_{222}, k_1a_{121} + k_2a_{221}), \dots, (k_1a_{1p4} + k_2a_{2p4}, k_1a_{1p3} + k_2a_{2p3}, k_1a_{1p2} + k_2a_{2p2}, k_1a_{1p1} + k_2a_{2p1})\} \text{ if } k_1 \leq 0, k_2 \leq 0. \quad (4)$$

Saghi et al. [9] pointed out that expressions (1)–(4) can be used only if the cardinality of $\tilde{\tilde{A}}_1$ is equal to the cardinality of $\tilde{\tilde{A}}_2$. Therefore, if the cardinality of $\tilde{\tilde{A}}_1$ is not equal to the cardinality of $\tilde{\tilde{A}}_2$, then, to evaluate $k_1\tilde{\tilde{A}}_1 + k_2\tilde{\tilde{A}}_2$, there is a need to change the cardinality of $\tilde{\tilde{A}}_1$ from s to t if $s < t$ or there is a need to change the cardinality of $\tilde{\tilde{A}}_2$ from t to s if $t < s$, where s and t represent the cardinality of $\tilde{\tilde{A}}_1$ and $\tilde{\tilde{A}}_2$, respectively.

The following method is used in Saghi et al.'s [9] approach to achieve the same.

Evaluate $maximum\,\{s, t\}$.

Case (i): If $s > t$. Then, evaluate $maximum\,\{(a_{211}+a_{212}+a_{213}+a_{214}),(a_{221}+a_{222}+a_{223}+a_{224}),\ldots,(a_{2t1}+a_{2t2}+a_{2t3}+a_{2t4})\}$.
Let $maximum\,\{(a_{211}+a_{212}+a_{213}+a_{214}),(a_{221}+a_{222}+a_{223}+a_{224}),\ldots,(a_{2t1}+a_{2t2}+a_{2t3}+a_{2t4})\}=(a_{2k1}+a_{2k2}+a_{2k3}+a_{2k4})$. Then, increase the cardinality of $\tilde{\tilde{A}}_2$ by repeating $(a_{2k1},a_{2k2},a_{2k3},a_{2k4})$ $(s-t)$ times in $\tilde{\tilde{A}}_2$ i.e., $\tilde{\tilde{A}}_2=\{(a_{211},a_{212},a_{213},a_{214}),(a_{221},a_{222},a_{223},a_{224}),\ldots,(a_{2t1},a_{2t2},a_{2t3},a_{2t4}),(a_{2k1},a_{2k2},a_{2k3},a_{2k4}),(a_{2k1},a_{2k2},a_{2k3},a_{2k4}),\ldots,(s-t)\ times,\ldots,(a_{2k1},a_{2k2},a_{2k3},a_{2k4})\}$.
Case (ii): If $s < t$. Then, evaluate $maximum\,\{(a_{111}+a_{112}+a_{113}+a_{114}),(a_{121}+a_{122}+a_{123}+a_{124}),\ldots,(a_{1s1}+a_{1s2}+a_{1s3}+a_{1s4})\}$.
Let $maximum\,\{(a_{111}+a_{112}+a_{113}+a_{114}),(a_{121}+a_{122}+a_{123}+a_{124}),\ldots,(a_{1s1}+a_{1s2}+a_{1s3}+a_{1s4})\}=(a_{1k1}+a_{1k2}+a_{1k3}+a_{1k4})$. Then, increase the cardinality of $\tilde{\tilde{A}}_1$ by repeating $(a_{1k1},a_{1k2},a_{1k3},a_{1k4})$ $(t-s)$ times in $\tilde{\tilde{A}}_1$ i.e., $\tilde{\tilde{A}}_1=\{(a_{111},a_{112},a_{113},a_{114}),(a_{121},a_{122},a_{123},a_{124}),\ldots,(a_{1s1},a_{1s2},a_{1s3},a_{1s4}),(a_{1k1},a_{1k2},a_{1k3},a_{1k4}),(a_{1k1},a_{1k2},a_{1k3}\ a_{1k4}),\ldots,(s-t)\ times,\ldots,(a_{1k1},a_{1k2},a_{1k3},a_{1k4})\}$.

5. Saghi et al.'s approach

Saghi et al. [9] proposed an approach for solving HFLPP (P_4) by incorporating the following modification in the classical simplex algorithm.
(i) Use the method discussed in Section 4 to evaluate the value of $k_1\tilde{\tilde{A}}_1+k_2\tilde{\tilde{A}}_2$.
(ii) Use the method discussed in Section 3 to compare THFNs.
To illustrate Saghi et al.'s [9] approach, the existing HFLPP (P_5) [6, Section 5, Example 1] is solved by Saghi et al.'s [9] approach.

Problem $(\boldsymbol{P_5})$

$Maximize\,\left(4\tilde{\tilde{x}}_1+5\tilde{\tilde{x}}_2+9\tilde{\tilde{x}}_3+11\tilde{\tilde{x}}_4\right)$
$Subject\ to$
$\tilde{\tilde{x}}_1+\tilde{\tilde{x}}_2+\tilde{\tilde{x}}_3+\tilde{\tilde{x}}_4\preccurlyeq\{(10,14,16,20),(12,15,15,20),(8,13,18,24)\}$,
$7\tilde{\tilde{x}}_1+5\tilde{\tilde{x}}_2+3\tilde{\tilde{x}}_3+2\tilde{\tilde{x}}_4\preccurlyeq\{(35,70,90,120),(40,80,90,120)\}$,
$3\tilde{\tilde{x}}_1+4\tilde{\tilde{x}}_2+10\tilde{\tilde{x}}_3+15\tilde{\tilde{x}}_4\preccurlyeq\{(70,100,100,130),(80,95,110,140)\}$,
$\tilde{\tilde{x}}_1,\tilde{\tilde{x}}_2,\tilde{\tilde{x}}_3,\tilde{\tilde{x}}_4\succcurlyeq\tilde{\tilde{0}}$.
Using Saghi et al.'s [9] approach, the existing HFLPP (P_5) can be solved as follows.
Upon introducing slack variables $\tilde{\tilde{x}}_5,\tilde{\tilde{x}}_6$, and $\tilde{\tilde{x}}_7$, solving the HFLPP (P_5) is equivalent to solving the HFLPP (P_6).

Problem $(\boldsymbol{P_6})$

$Maximize\,\left(4\tilde{\tilde{x}}_1+5\tilde{\tilde{x}}_2+9\tilde{\tilde{x}}_3+11\tilde{\tilde{x}}_4+0\tilde{\tilde{x}}_5+0\tilde{\tilde{x}}_6+0\tilde{\tilde{x}}_7\right)$
$Subject\ to$
$\tilde{\tilde{x}}_1+\tilde{\tilde{x}}_2+\tilde{\tilde{x}}_3+\tilde{\tilde{x}}_4+\tilde{\tilde{x}}_5\approx\{(10,14,16,20),(12,15,15,20),(8,13,18,24)\}$,
$7\tilde{\tilde{x}}_1+5\tilde{\tilde{x}}_2+3\tilde{\tilde{x}}_3+2\tilde{\tilde{x}}_4+\tilde{\tilde{x}}_6\approx\{(35,70,90,120),(40,80,90,120)\}$,
$3\tilde{\tilde{x}}_1+4\tilde{\tilde{x}}_2+10\tilde{\tilde{x}}_3+15\tilde{\tilde{x}}_4+\tilde{\tilde{x}}_7\approx\{(70,100,100,130),(80,95,110,140)\}$,
$\tilde{\tilde{x}}_1,\tilde{\tilde{x}}_2,\tilde{\tilde{x}}_3,\tilde{\tilde{x}}_4,\tilde{\tilde{x}}_5,\tilde{\tilde{x}}_6,\tilde{\tilde{x}}_7\succcurlyeq\tilde{\tilde{0}}$.
Upon considering $\tilde{\tilde{x}}_5,\tilde{\tilde{x}}_6$, and $\tilde{\tilde{x}}_7$ as basic variables, the initial simplex table (Table 1) is obtained.

Table 1. Initial simplex table [6, Section 5, Table 3].

Basic variables	$\tilde{\tilde{x}}_1$	$\tilde{\tilde{x}}_2$	$\tilde{\tilde{x}}_3$	$\tilde{\tilde{x}}_4$	$\tilde{\tilde{x}}_5$	$\tilde{\tilde{x}}_6$	$\tilde{\tilde{x}}_7$	RHS
$\tilde{\tilde{z}}_j - \tilde{\tilde{c}}_j$	-4	-5	-9	-11	0	0	0	
$\tilde{\tilde{x}}_5$	1	1	1	1	1	0	0	$\{(10,14,16,20),(12,15,15,20),(8,13,18,24)\}$
$\tilde{\tilde{x}}_6$	7	5	3	2	0	1	0	$\{(35,70,90,120),(40,80,90,120)\}$
$\tilde{\tilde{x}}_7$	3	4	10	15	0	0	1	$\{(70,100,100,130),(80,95,110,140)\}$

Since, $minimum\{z_1 - c_1, z_2 - c_2, z_3 - c_3, z_4 - c_4\} = z_4 - c_4 = -11$ is not a non-negative real number, the obtained solution is not an OS, and variable $\tilde{\tilde{x}}_4$ is the entering variable.

Furthermore, as $minimum\left\{\frac{\Re\left\{\begin{matrix}(10,14,16,20),(12,15,15,20),\\(8,13,18,24)\end{matrix}\right\}}{1}, \frac{\Re\{(35,70,90,120),(40,80,90,120)\}}{2}, \frac{\Re\{(70,100,100,130),(80,95,110,140)\}}{15}\right\} = minimum\left\{\frac{185}{12}, \frac{645}{16}, \frac{55}{8}\right\} = \frac{55}{8}$, which is corresponding to variable $\tilde{\tilde{x}}_7$. Thus, $\tilde{\tilde{x}}_7$ is the leaving variable.

Upon considering $\tilde{\tilde{x}}_5, \tilde{\tilde{x}}_6$, and $\tilde{\tilde{x}}_4$ as basic variables, and using the row operations, the simplex table (Table 2) is obtained.

The values of the last column of the simplex table (Table 2) are obtained as follows.

(i) $\frac{\{(70,100,100,130),(80,95,110,140)\}}{15} = \left\{\left(\frac{70}{15}, \frac{100}{15}, \frac{100}{15}, \frac{130}{15}\right), \left(\frac{80}{15}, \frac{95}{15}, \frac{110}{15}, \frac{140}{15}\right)\right\} = \left\{\left(\frac{14}{3}, \frac{20}{3}, \frac{20}{3}, \frac{26}{3}\right), \left(\frac{16}{3}, \frac{19}{3}, \frac{22}{3}, \frac{28}{3}\right)\right\}$.

(ii) $\{(35,70,90,120),(40,80,90,120)\} - 2 \times \left\{\left(\frac{14}{3}, \frac{20}{3}, \frac{20}{3}, \frac{26}{3}\right), \left(\frac{16}{3}, \frac{19}{3}, \frac{22}{3}, \frac{28}{3}\right)\right\} = \{(35,70,90,120),(40,80,90,120)\} - \left\{\left(\frac{28}{3}, \frac{40}{3}, \frac{40}{3}, \frac{52}{3}\right), \left(\frac{32}{3}, \frac{38}{3}, \frac{44}{3}, \frac{56}{3}\right)\right\} = \left\{\left(\frac{53}{3}, \frac{170}{3}, \frac{230}{3}, \frac{332}{3}\right), \left(\frac{64}{3}, \frac{196}{3}, \frac{232}{3}, \frac{328}{3}\right)\right\}$.

(iii) $\{(10,14,16,20),(12,15,15,20),(8,13,18,24)\} - \left\{\left(\frac{14}{3}, \frac{20}{3}, \frac{20}{3}, \frac{26}{3}\right), \left(\frac{16}{3}, \frac{19}{3}, \frac{22}{3}, \frac{28}{3}\right)\right\} = \{(10,14,16,20),(12,15,15,20),(8,13,18,24)\} - \left\{\left(\frac{14}{3}, \frac{20}{3}, \frac{20}{3}, \frac{26}{3}\right), \left(\frac{16}{3}, \frac{19}{3}, \frac{22}{3}, \frac{28}{3}\right), \left(\frac{16}{3}, \frac{19}{3}, \frac{22}{3}, \frac{28}{3}\right)\right\} = \left\{\left(\frac{4}{3}, \frac{22}{3}, \frac{28}{3}, \frac{46}{3}\right), \left(\frac{8}{3}, \frac{23}{3}, \frac{26}{3}, \frac{44}{3}\right), \left(-\frac{4}{3}, \frac{17}{3}, \frac{35}{3}, \frac{56}{3}\right)\right\}$.

Table 2. Simplex table after first iteration.

Basic variables	$\tilde{\tilde{x}}_1$	$\tilde{\tilde{x}}_2$	$\tilde{\tilde{x}}_3$	$\tilde{\tilde{x}}_4$	$\tilde{\tilde{x}}_5$	$\tilde{\tilde{x}}_6$	$\tilde{\tilde{x}}_7$	RHS
$\tilde{\tilde{z}}_j - \tilde{\tilde{c}}_j$	$-\frac{9}{5}$	$-\frac{31}{15}$	$-\frac{5}{3}$	0	0	0	$\frac{11}{15}$	
$\tilde{\tilde{x}}_5$	$\frac{4}{5}$	$\frac{11}{15}$	$\frac{1}{3}$	0	1	0	$-\frac{1}{15}$	$\left\{\left(\frac{4}{3},\frac{22}{3},\frac{28}{3},\frac{46}{3}\right),\left(\frac{8}{3},\frac{23}{3},\frac{26}{3},\frac{44}{3}\right),\left(-\frac{4}{3},\frac{17}{3},\frac{35}{3},\frac{56}{3}\right)\right\}$
$\tilde{\tilde{x}}_6$	$\frac{33}{5}$	$\frac{67}{15}$	$\frac{5}{3}$	0	0	1	$-\frac{2}{15}$	$\left\{\left(\frac{53}{3},\frac{170}{3},\frac{230}{3},\frac{332}{3}\right),\left(\frac{64}{3},\frac{196}{3},\frac{232}{3},\frac{328}{3}\right)\right\}$
$\tilde{\tilde{x}}_4$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{10}{15}$	1	0	0	$\frac{1}{15}$	$\left\{\left(\frac{14}{3},\frac{20}{3},\frac{20}{3},\frac{26}{3}\right),\left(\frac{16}{3},\frac{19}{3},\frac{22}{3},\frac{28}{3}\right)\right\}$

Since, $minimum\{z_1 - c_1, z_2 - c_2, z_3 - c_3\} = z_2 - c_2 = -\frac{31}{15}$ is not a non-negative real number, the obtained solution is not an OS, and variable $\tilde{\tilde{x}}_2$ is the entering variable.

Furthermore, as $minimum\left\{\frac{\Re\left\{\left(\frac{4}{3},\frac{22}{3},\frac{28}{3},\frac{46}{3}\right),\left(\frac{8}{3},\frac{23}{3},\frac{26}{3},\frac{44}{3}\right),\left(-\frac{4}{3},\frac{17}{3},\frac{35}{3},\frac{56}{3}\right)\right\}}{11/15}, \frac{\Re\left\{\left(\frac{53}{3},\frac{170}{3},\frac{230}{3},\frac{332}{3}\right),\left(\frac{64}{3},\frac{196}{3},\frac{232}{3},\frac{328}{3}\right)\right\}}{67/15},\right.$

$\left.\frac{\Re\left\{\left(\frac{14}{3},\frac{20}{3},\frac{20}{3},\frac{26}{3}\right),\left(\frac{16}{3},\frac{19}{3},\frac{22}{3},\frac{28}{3}\right)\right\}}{4/15}\right\} = minimum\left\{\frac{1525}{132},\frac{8025}{536},\frac{825}{32}\right\} = \frac{1525}{132}$, which is corresponding to variable $\tilde{\tilde{x}}_5$. Thus, $\tilde{\tilde{x}}_5$ is the leaving variable.

Upon considering $\tilde{\tilde{x}}_2, \tilde{\tilde{x}}_6$, and $\tilde{\tilde{x}}_4$ as basic variables and using the row operations, the simplex table (Table 3) is obtained.

The values of the last column of the simplex table (Table 3) are obtained as follows.

(i) $\frac{\left\{\left(\frac{4}{3},\frac{22}{3},\frac{28}{3},\frac{46}{3}\right),\left(\frac{8}{3},\frac{23}{3},\frac{26}{3},\frac{44}{3}\right),\left(-\frac{4}{3},\frac{17}{3},\frac{35}{3},\frac{56}{3}\right)\right\}}{\frac{11}{15}} = \left\{\left(\frac{20}{11},10,\frac{140}{11},\frac{230}{11}\right),\left(\frac{40}{11},\frac{115}{11},\frac{130}{11},20\right),\left(-\frac{20}{11},\frac{85}{11},\frac{175}{11},\frac{280}{11}\right)\right\}$.

(ii) $\left\{\left(\frac{53}{3},\frac{170}{3},\frac{230}{3},\frac{332}{3}\right),\left(\frac{64}{3},\frac{196}{3},\frac{232}{3},\frac{328}{3}\right)\right\} - \frac{67}{15} \times \left\{\left(\frac{20}{11},10,\frac{140}{11},\frac{230}{11}\right),\left(\frac{40}{11},\frac{115}{11},\frac{130}{11},20\right),\left(-\frac{20}{11},\frac{85}{11},\frac{175}{11},\frac{280}{11}\right)\right\} = \left\{\left(\frac{53}{3},\frac{170}{3},\frac{230}{3},\frac{332}{3}\right),\left(\frac{64}{3},\frac{196}{3},\frac{232}{3},\frac{328}{3}\right)\right\} - \left\{\left(\frac{268}{33},\frac{134}{3},\frac{1876}{33},\frac{3082}{33}\right),\left(\frac{536}{33},\frac{1541}{33},\frac{1742}{33},\frac{268}{3}\right),\left(-\frac{268}{33},\frac{1139}{33},\frac{2345}{33},\frac{3752}{33}\right)\right\} = \left\{\left(-\frac{833}{11},-\frac{2}{11},32,\frac{1128}{11}\right),\left(-68,\frac{138}{11},\frac{337}{11},\frac{1024}{11}\right),\left(-\frac{1016}{11},-\frac{63}{11},\frac{471}{11},\frac{1292}{11}\right)\right\}$.

(iii) $\left\{\left(\frac{14}{3},\frac{20}{3},\frac{20}{3},\frac{26}{3}\right),\left(\frac{16}{3},\frac{19}{3},\frac{22}{3},\frac{28}{3}\right)\right\} - \frac{4}{15} \times \left\{\left(\frac{20}{11},10,\frac{140}{11},\frac{230}{11}\right),\left(\frac{40}{11},\frac{115}{11},\frac{130}{11},20\right),\left(-\frac{20}{11},\frac{85}{11},\frac{175}{11},\frac{280}{11}\right)\right\} = \left\{\left(\frac{14}{3},\frac{20}{3},\frac{20}{3},\frac{26}{3}\right),\left(\frac{16}{3},\frac{19}{3},\frac{22}{3},\frac{28}{3}\right)\right\} - \left\{\left(\frac{16}{33},\frac{8}{3},\frac{112}{33},\frac{184}{33}\right),\left(\frac{32}{33},\frac{92}{33},\frac{104}{33},\frac{16}{3}\right),\left(-\frac{16}{33},\frac{68}{33},\frac{140}{33},\frac{224}{33}\right)\right\} = \left\{\left(-\frac{10}{11},\frac{36}{11},4,\frac{90}{11}\right),\left(0,\frac{35}{11},\frac{50}{11},\frac{92}{11}\right),\left(-\frac{16}{11},\frac{23}{11},\frac{58}{11},\frac{108}{11}\right)\right\}$.

Table 3. Simplex table after second iteration.

Basic Variable	$\tilde{\tilde{x}}_1$	$\tilde{\tilde{x}}_2$	$\tilde{\tilde{x}}_3$	$\tilde{\tilde{x}}_4$	$\tilde{\tilde{x}}_5$	$\tilde{\tilde{x}}_6$	$\tilde{\tilde{x}}_7$	RHS
$\tilde{\tilde{z}}_j - \tilde{\tilde{c}}_j$	$\frac{5}{11}$	0	$-\frac{8}{11}$	0	$\frac{31}{11}$	0	$\frac{6}{11}$	
$\tilde{\tilde{x}}_2$	$\frac{12}{11}$	1	$\frac{5}{11}$	0	$\frac{15}{11}$	0	$-\frac{1}{11}$	$\left\{\left(\frac{20}{11},10,\frac{140}{11},\frac{230}{11}\right),\left(\frac{40}{11},\frac{115}{11},\frac{130}{11},20\right) \left(-\frac{20}{11},\frac{85}{11},\frac{175}{11},\frac{280}{11}\right)\right\}$
$\tilde{\tilde{x}}_6$	$\frac{19}{11}$	0	$-\frac{4}{11}$	0	$-\frac{67}{11}$	1	$\frac{3}{11}$	$\left\{\left(-\frac{833}{11},-\frac{2}{11},32,\frac{1128}{11}\right),\left(-68,\frac{138}{11},\frac{337}{11},\frac{1024}{11}\right) \left(-\frac{1016}{11},-\frac{63}{11},\frac{471}{11},\frac{1292}{11}\right)\right\}$
$\tilde{\tilde{x}}_4$	$-\frac{1}{11}$	0	$\frac{6}{11}$	1	$-\frac{4}{11}$	0	$\frac{1}{11}$	$\left\{\left(-\frac{10}{11},\frac{36}{11},4,\frac{90}{11}\right),\left(0,\frac{35}{11},\frac{50}{11},\frac{92}{11}\right) \left(-\frac{16}{11},\frac{23}{11},\frac{58}{11},\frac{108}{11}\right)\right\}$

Since, $minimum\{z_1 - c_1, z_4 - c_4\} = z_3 - c_3 = -\frac{8}{11}$ is not a non-negative real number, the obtained solution is not an OS, and variable $\tilde{\tilde{x}}_3$ is the entering variable.

Furthermore, as $minimum\left\{\frac{\Re\left\{\left(\frac{20}{11},10,\frac{140}{11},\frac{230}{11}\right),\left(\frac{40}{11},\frac{115}{11},\frac{130}{11},20\right) \left(-\frac{20}{11},\frac{85}{11},\frac{175}{11},\frac{280}{11}\right)\right\}}{5/11}, \frac{\Re\left\{\left(-\frac{10}{11},\frac{36}{11},4,\frac{90}{11}\right),\left(0,\frac{35}{11},\frac{50}{11},\frac{92}{11}\right) \left(-\frac{16}{11},\frac{23}{11},\frac{58}{11},\frac{108}{11}\right)\right\}}{6/11}\right\} =$

$\left\{\frac{305}{12},\frac{85}{12}\right\} = \frac{85}{12}$, which is corresponding to variable $\tilde{\tilde{x}}_4$, then $\tilde{\tilde{x}}_4$ is the leaving variable.
Upon considering $\tilde{\tilde{x}}_2, \tilde{\tilde{x}}_6$, and $\tilde{\tilde{x}}_3$ as basic variables and using the row operations, the simplex table (Table 4) is obtained.

Table 4. Simplex table after third iteration [6, Section 5, Table 4].

Basic Variables	$\tilde{\tilde{x}}_1$	$\tilde{\tilde{x}}_2$	$\tilde{\tilde{x}}_3$	$\tilde{\tilde{x}}_4$	$\tilde{\tilde{x}}_5$	$\tilde{\tilde{x}}_6$	$\tilde{\tilde{x}}_7$	RHS
$\tilde{\tilde{z}}_j - \tilde{\tilde{c}}_j$	$\frac{1}{3}$	0	0	$\frac{4}{3}$	$\frac{7}{3}$	0	$\frac{2}{3}$	
$\tilde{\tilde{x}}_2$	$\frac{7}{6}$	1	0	$-\frac{5}{6}$	$\frac{5}{3}$	0	$-\frac{1}{6}$	$\left\{\left(-5,\frac{20}{3},10,\frac{65}{3}\right),\left(-\frac{10}{3},\frac{20}{3},\frac{55}{6},20\right) \left(-10,\frac{10}{3},\frac{85}{6},\frac{80}{3}\right)\right\}$
$\tilde{\tilde{x}}_6$	$\frac{5}{3}$	0	0	$\frac{2}{3}$	$-\frac{19}{3}$	1	$\frac{1}{3}$	$\left\{\left(-\frac{229}{3},2,\frac{104}{3},108\right),\left(-68,\frac{44}{3},\frac{101}{3},\frac{296}{3}\right) \left(-\frac{280}{3},-\frac{13}{3},\frac{139}{3},124\right)\right\}$
$\tilde{\tilde{x}}_3$	$-\frac{1}{6}$	0	1	$\frac{11}{6}$	$-\frac{2}{3}$	0	$\frac{1}{6}$	$\left\{\left(-\frac{5}{3},6,\frac{22}{3},15\right),\left(0,\frac{35}{6},\frac{25}{3},\frac{46}{3}\right) \left(-\frac{8}{3},\frac{23}{6},\frac{29}{3},18\right)\right\}$

The values of the last column of the simplex table (Table 4) are obtained as follows.

(i) $\frac{\left\{\left(-\frac{10}{11},\frac{36}{11},4,\frac{90}{11}\right),\left(0,\frac{35}{11},\frac{50}{11},\frac{92}{11}\right),\left(-\frac{16}{11},\frac{23}{11},\frac{58}{11},\frac{108}{11}\right)\right\}}{6/_{11}}=\left\{\left(-\frac{5}{3},6,\frac{22}{3},15\right),\left(0,\frac{35}{6},\frac{25}{3},\frac{46}{3}\right),\left(-\frac{8}{3},\frac{23}{6},\frac{29}{3},18\right)\right\}.$

(ii) $\left\{\left(-\frac{833}{11},-\frac{2}{11},32,\frac{1128}{11}\right),\left(-68,\frac{138}{11},\frac{337}{11},\frac{1024}{11}\right),\left(-\frac{1016}{11},-\frac{63}{11},\frac{471}{11},\frac{1292}{11}\right)\right\}+\frac{4}{11}\times\left\{\left(-\frac{5}{3},6,\frac{22}{3},15\right),\left(0,\frac{35}{6},\frac{25}{3},\frac{46}{3}\right),\left(-\frac{8}{3},\frac{23}{6},\frac{29}{3},18\right)\right\}=\left\{\left(-\frac{833}{11},-\frac{2}{11},32,\frac{1128}{11}\right),\left(-68,\frac{138}{11},\frac{337}{11},\frac{1024}{11}\right),\left(-\frac{1016}{11},-\frac{63}{11},\frac{471}{11},\frac{1292}{11}\right)\right\}+\left\{\left(-\frac{20}{33},\frac{24}{11},\frac{8}{3},\frac{60}{11}\right),\left(0,\frac{70}{33},\frac{100}{33},\frac{184}{33}\right),\left(-\frac{32}{33},\frac{46}{33},\frac{116}{33},\frac{72}{11}\right)\right\}=\left\{\left(-\frac{229}{3},2,\frac{104}{3},108\right),\left(-68,\frac{44}{3},\frac{101}{3},\frac{296}{3}\right),\left(-\frac{280}{3},-\frac{13}{3},\frac{139}{3},124\right)\right\}.$

(iii) $\left\{\left(\frac{20}{11},10,\frac{140}{11},\frac{230}{11}\right),\left(\frac{40}{11},\frac{115}{11},\frac{130}{11},20\right),\left(-\frac{20}{11},\frac{85}{11},\frac{175}{11},\frac{280}{11}\right)\right\}-\frac{5}{11}\left\{\left(-\frac{5}{3},6,\frac{22}{3},15\right),\left(0,\frac{35}{6},\frac{25}{3},\frac{46}{3}\right),\left(-\frac{8}{3},\frac{23}{6},\frac{29}{3},18\right)\right\}=\left\{\left(\frac{20}{11},10,\frac{140}{11},\frac{230}{11}\right),\left(\frac{40}{11},\frac{115}{11},\frac{130}{11},20\right),\left(-\frac{20}{11},\frac{85}{11},\frac{175}{11},\frac{280}{11}\right)\right\}-\left\{\left(-\frac{25}{11},\frac{30}{11},\frac{10}{3},\frac{75}{11}\right),\left(0,\frac{175}{66},\frac{125}{33},\frac{230}{33}\right),\left(-\frac{40}{33},\frac{115}{66},\frac{145}{33},\frac{90}{11}\right)\right\}=\left\{\left(-5,\frac{20}{3},10,\frac{65}{3}\right),\left(-\frac{10}{3},\frac{20}{3},\frac{55}{6},20\right),\left(-10,\frac{10}{3},\frac{85}{6},\frac{80}{3}\right)\right\}.$

Since, in Table 4, the value of z_j-c_j corresponding to each non-basic variable $\tilde{\tilde{x}}_j$ is a non-negative real number, the obtained OS is $\tilde{\tilde{x}}_1=\{(0,0,0,0),(0,0,0,0),(0,0,0,0)\}$, $\tilde{\tilde{x}}_2=\left\{\left(-5,\frac{20}{3},10,\frac{65}{3}\right),\left(-\frac{10}{3},\frac{20}{3},\frac{55}{6},20\right),\left(-10,\frac{10}{3},\frac{85}{6},\frac{80}{3}\right)\right\}$, $\tilde{\tilde{x}}_3=\left\{\left(-\frac{5}{3},6,\frac{22}{3},15\right),\left(0,\frac{35}{6},\frac{25}{3},\frac{46}{3}\right)\left(-\frac{8}{3},\frac{23}{6},\frac{29}{3},18\right)\right\}$, $\tilde{\tilde{x}}_4=\{(0,0,0,0),(0,0,0,0),(0,0,0,0)\}$, $\tilde{\tilde{x}}_5=\{(0,0,0,0),(0,0,0,0),(0,0,0,0)\}$, $\tilde{\tilde{x}}_6=\left\{\left(-\frac{229}{3},2,\frac{104}{3},108\right),\left(-68,\frac{44}{3},\frac{101}{3},\frac{296}{3}\right),\left(-\frac{280}{3},-\frac{13}{3},\frac{139}{3},124\right)\right\}$ and $\tilde{\tilde{x}}_7=\{(0,0,0,0),(0,0,0,0),(0,0,0,0)\}$.

6. PrAlApp

From Section 5, to solve the HFLPP (P_4) by Saghi et al.'s [9] approach, there is a need to use arithmetic operations of THFNs. Moreover, since much computational effort is required to apply arithmetic operations of THFNs, it is difficult to solve the HFLPP (P_4) corresponding to large-scaled real-life problems. To reduce the computational efforts, an alternative approach is proposed to solve the HFLPP (P_4).

The steps of the PrAlApp are as follows.

Step 1: Evaluate $\underset{1\le i\le m}{maximum}\left\{cardinality\ of\ \tilde{\tilde{b}}_i\right\}$ and go to Step 2.

Step 2: If $\underset{1\le i\le m}{maximum}\left\{cardinality\ of\ \tilde{\tilde{b}}_i\right\}=p$, then check that the cardinality of each $\tilde{\tilde{b}}_i$ is p or not.

Case (i): If the answer is yes, then find an OS x_j, $j=1,2,\ldots,n$ of the crisp linear programming problem (CLPP) (P_7).

Problem (P_7)

$Maximize\left(\sum_{j=1}^{n}\left(c_jx_j\right)\right)$

$Subject\ to$

$\sum_{j=1}^{n}a_{ij}x_j\le\Re\left(\tilde{\tilde{b}}_i\right),i=1,2,\ldots,m;$

$x_j\ge0,j=1,2,\ldots,n.$

Case (ii): If the answer is no, then find an OS x_j, $j=1,2,\ldots,n$ of the CLPP (P_8).

Problem $(\boldsymbol{P_8})$

$Maximize\left(\sum_{j=1}^{n}\left(c_j x_j\right)\right)$
$Subject\ to$
$\sum_{j=1}^{n} a_{ij}x_j \leq \Re\left(\tilde{\tilde{b}}_i{}'\right), i = 1,2,\dots,m;$
$x_j \geq 0, j = 1,2,\dots,n,$
where,

$$\tilde{\tilde{b}}_i' = \begin{cases} \tilde{\tilde{b}}_i, if\ cardinality\ of \tilde{\tilde{b}}_i\ is\ p, \\ \tilde{\tilde{b}}_j, if\ cardinality\ of \tilde{\tilde{b}}_i\ is\ q\ (less\ than\ p) \end{cases}$$

The trapezoidal number $\tilde{\tilde{b}}_j$ is obtained by increasing the cardinality of $\tilde{\tilde{b}}_i$ from q to p by the method discussed in Section 4.

Step 3: If $x_j = \gamma_j, j = 1,2,\dots,n$, then evaluate a THFN $\tilde{\tilde{x}}_j = \{(x_{i11}, x_{i12}, x_{i13}, x_{i14}), (x_{i21}, x_{i22}$ $x_{i23}, x_{i24}), \dots, (x_{ip1}, x_{ip2}, x_{ip3}, x_{ip4})\}$ having cardinality p such that $\Re(\tilde{\tilde{x}}_j) = \gamma_j$. The evaluated values of $\tilde{\tilde{x}}_j, j = 1,2,\dots,n$ represents an OS of the HFLPP (P_4).

Step 4: Use the following steps to find unique optimal solution of the HFLPP (P_4).

Step 4(a): Construct the optimal basis matrix B.

Step 4(b): Evaluate the multiplicative inverse of the matrix B i.e., B^{-1}.

Step 4(c): Evaluate the optimal solution matrix $B^{-1} \times \tilde{\tilde{b}}_i'$.

7. Origin of the PrAlApp

The origin of the PrAlApp is as follows.

Step 1: To find an OS of the HFLPP (P_4) is equivalent to finding an OS of the HFLPP (P_9).

Problem $(\boldsymbol{P_9})$

$Maximize\ \left(\sum_{j=1}^{n}\left(c_j \tilde{\tilde{x}}_j\right)\right)$
$Subject\ to$
$\sum_{j=1}^{n} a_{ij}\tilde{\tilde{x}}_j - \tilde{\tilde{b}}_i \preccurlyeq \tilde{\tilde{b}}_i - \tilde{\tilde{b}}_i, i = 1,2,\dots,m;$
$\tilde{\tilde{x}}_j \succcurlyeq \tilde{\tilde{0}}, j = 1,2,\dots,n.$

Step 2: If the cardinality of $\tilde{\tilde{b}}_i$ is θ_i, then, as discussed in Section 4, $\tilde{\tilde{b}}_i - \tilde{\tilde{b}}_i = \tilde{\tilde{0}}$, where $\tilde{\tilde{0}} = \{(0,0,0,0), (0,0,0,0), \dots, \theta_i\ times, \dots, (0,0,0,0)\}$. Thus, to find an OS of the HFLPP (P_9) is equivalent to finding an OS of the HFLPP (P_{10}).

Problem $(\boldsymbol{P_{10}})$

$Maximize\ \left(\sum_{j=1}^{n}\left(c_j \tilde{\tilde{x}}_j\right)\right)$
$Subject\ to$
$\sum_{j=1}^{n} a_{ij}\tilde{\tilde{x}}_j - \tilde{\tilde{b}}_i \preccurlyeq \tilde{\tilde{0}}, i = 1,2,\dots,m;$
$\tilde{\tilde{x}}_j \succcurlyeq \tilde{\tilde{0}}, j = 1,2,\dots,n.$

Step 3: As discussed in Section 4, $\sum_{j=1}^{n} a_{ij}\tilde{\tilde{x}}_j - \tilde{\tilde{b}}_i$ can be evaluated only if the cardinality of each $\tilde{\tilde{x}}_j$ and each $\tilde{\tilde{b}}_i$ is the same. Also, as the cardinality of each $\tilde{\tilde{x}}_j$ is unknown, the cardinality of each $\tilde{\tilde{x}}_j$ may be considered $\underset{1\leq i\leq m}{maximum}\left\{cardinality\ of\ \tilde{\tilde{b}}_i\right\}$. Furthermore, if the cardinality of each $\tilde{\tilde{b}}_i$ is not equal to $\underset{1\leq i\leq m}{maximum}\left\{cardinality\ of\ \tilde{\tilde{b}}_i\right\}$, then each $\tilde{\tilde{b}}_i$ can be replaced by $\tilde{\tilde{b}}_i{}'$, where $\tilde{\tilde{b}}_i{}'$ is defined in Problem (P_8). Hence, to find an OS of the HFLPP (P_{10}) is equivalent to finding an OS of the HFLPP (P_{11}).

Problem (P_{11})

$Maximize\ \left(\sum_{j=1}^{n}\left(c_j\tilde{\tilde{x}}_j\right)\right)$
$Subject\ to$
$\sum_{j=1}^{n} a_{ij}\tilde{\tilde{x}}_j - \tilde{\tilde{b}}_i{}' \preccurlyeq \tilde{\tilde{0}}, i = 1,2,\dots,m;$
$\tilde{\tilde{x}}_j \succcurlyeq \tilde{\tilde{0}}, j = 1,2,\dots,n.$

Step 4: Using the comparing method, discussed in Section 3, to find an OS of the HFLPP (P_{11}) is equivalent to finding an OS of the CLPP (P_{12}).

Problem (P_{12})

$Maximize\ \left(\Re\left(\sum_{j=1}^{n}\left(c_j\tilde{\tilde{x}}_j\right)\right)\right)$
$Subject\ to$
$\Re\left(\sum_{j=1}^{n} a_{ij}\tilde{\tilde{x}}_j - \tilde{\tilde{b}}_i{}'\right) \leq \Re\left(\tilde{\tilde{0}}\right), i = 1,2,\dots,m;$
$\Re\left(\tilde{\tilde{x}}_j\right) \geq \Re\left(\tilde{\tilde{0}}\right), j = 1,2,\dots,n.$

Step 5: Using the relations $\left(\Re\left(\sum_{j=1}^{n}\left(c_j\tilde{\tilde{x}}_j\right)\right)\right) = \sum_{j=1}^{n}\left(c_j \times \Re\left(\tilde{\tilde{x}}_j\right)\right), \Re\left(\sum_{j=1}^{n} a_{ij}\tilde{\tilde{x}}_j - \tilde{\tilde{b}}_i{}'\right) = \sum_{j=1}^{n} a_{ij}\Re\left(\tilde{\tilde{x}}_j\right) - \Re\left(\tilde{\tilde{b}}_i{}'\right)$, and $\Re\left(\tilde{\tilde{0}}\right) = 0$ to find an OS of the CLPP (P_{12}) is equivalent to finding an OS of the CLPP (P_{13}).

Problem (P_{13})

$Maximize\ \left(\sum_{j=1}^{n}\left(c_j \times \Re\left(\tilde{\tilde{x}}_j\right)\right)\right)$
$Subject\ to$
$\sum_{j=1}^{n} a_{ij}\Re\left(\tilde{\tilde{x}}_j\right) - \Re\left(\tilde{\tilde{b}}_i{}'\right) \leq 0, i = 1,2,\dots,m;$
$\Re\left(\tilde{\tilde{x}}_j\right) \geq 0, j = 1,2,\dots,n.$

Step 6: Assuming $\Re\left(\tilde{\tilde{x}}_j\right) = x_j$, to find an OS of the CLPP (P_{13}) is equivalent to finding an OS of the CLPP (P_8).

8. Illustrative example

Saghi et al. [9] solved the HFLPP (P_5) to illustrate their proposed approach. In this section, the same HFLPP is solved by the PrAlApp.

Using the PrAlApp, optimal solutions of the HFLPP (P_5) can be obtained as follows.

Step 1: Since, the cardinality of $\tilde{\tilde{b}}_1 = \{(10,14,16,20),(12,15,15,20),(8,13,18,24)\}$, $\tilde{\tilde{b}}_2 = \{(35,70,90,120),(40,80,90,120)\}$, and $\tilde{\tilde{b}}_3 = \{(70,100,100,130),(80,95,110,140)\}$ is $3, 2$ and 2 respectively, then $\underset{1\leq i\leq 3}{maximum}\left\{cardinality\ of\ \tilde{\tilde{b}}_i\right\} = maximum\{3,2,2\} = 3.$

Step 2: Since, the cardinality of each $\tilde{\tilde{b}}_i$ is not 3, then, according to Case (ii) of Step 2 of the PrAlApp, there is a need to find an OS of the CLPP (P_{15}), which is obtained by substituting $\Re\{(10,14,16,20),(12,15,15,20),(8,13,18,24)\} = \frac{185}{12}, \Re\{(35,70,90,120),(40,80,90,120),(40,80,90,120)\} = \frac{975}{12}$, and $\Re\{(70,100,100,130),(80,95,110,140),(80,95,110,140)\} = \frac{1250}{12}$ in the CLPP (P_{14}).

Problem $(\boldsymbol{P_{14}})$

$$
\begin{aligned}
&Maximize\ \ (4x_1 + 5x_2 + 9x_3 + 11x_4)\\
&Subject\ to\\
&x_1 + x_2 + x_3 + x_4 \leq \Re\{(10,14,16,20),(12,15,15,20),(8,13,18,24)\},\\
&7x_1 + 5x_2 + 3x_3 + 2x_4 \leq \Re\{(35,70,90,120),(40,80,90,120),(40,80,90,120)\},\\
&3x_1 + 4x_2 + 10x_3 + 15x_4 \leq \Re\{(70,100,100,130),(80,95,110,140),(80,95,110,140)\},\\
&x_1, x_2, x_3, x_4 \geq 0,
\end{aligned}
$$

where, $\{(35,70,90,120),(40,80,90,120),(40,80,90,120)\}$ and $\{(70,100,100,130),(80,95,110,140),(80,95,110,140)\}$ is obtained by increasing the cardinality of $\{(35,70,90,120),(40,80,90,120)\}$, and $\{(70,100,100,130),(80,95,110,140)\}$, respectively, from 2 to 3 with the method discussed in Section 4.

Problem $(\boldsymbol{P_{15}})$

$$
\begin{aligned}
&Maximize\ \ (4x_1 + 5x_2 + 9x_3 + 11x_4)\\
&Subject\ to\\
&x_1 + x_2 + x_3 + x_4 \leq \frac{185}{12},\\
&7x_1 + 5x_2 + 3x_3 + 2x_4 \leq \frac{975}{12},\\
&3x_1 + 4x_2 + 10x_3 + 15x_4 \leq \frac{1250}{12},\\
&x_1, x_2, x_3, x_4 \geq 0.
\end{aligned}
$$

It can be verified that $x_1 = 0, x_2 = 100, x_3 = 85, x_4 = 0, x_5 = 0, x_6 = 220$, and $x_7 = 0$ are OS's of the CLPP (P_{15}), where x_5, x_6, and x_7 are slack variables corresponding to the first, second, and third constraint, respectively, of the CLPP (P_{15}).

Step 3: Since $x_1 = 0, x_2 = 100, x_3 = 85, x_4 = 0, x_5 = 0, x_6 = 220, x_7 = 0$, and $\underset{1 \leq i \leq 3}{maximum}\left\{cardinality\ of\ \tilde{\tilde{b}}_i\right\} = maximum\{3,2,2\} = 3$, then, according to Step 3 of the PrAlApp, there is a need to find THFNs $\tilde{\tilde{x}}_1, \tilde{\tilde{x}}_2, \tilde{\tilde{x}}_3, \tilde{\tilde{x}}_4, \tilde{\tilde{x}}_5, \tilde{\tilde{x}}_6$, and $\tilde{\tilde{x}}_7$ having cardinality 3, such that $\Re\left(\tilde{\tilde{x}}_1\right) = 0, \Re\left(\tilde{\tilde{x}}_2\right) = \frac{100}{12}, \Re\left(\tilde{\tilde{x}}_3\right) = \frac{85}{12}, \Re\left(\tilde{\tilde{x}}_4\right) = 0, \Re\left(\tilde{\tilde{x}}_5\right) = 0, \Re\left(\tilde{\tilde{x}}_6\right) = \frac{220}{12}$, and $\Re\left(\tilde{\tilde{x}}_7\right) = 0$.

It is pertinent to mention that infinite numbers of such $\tilde{\tilde{x}}_1, \tilde{\tilde{x}}_2, \tilde{\tilde{x}}_3, \tilde{\tilde{x}}_4, \tilde{\tilde{x}}_5, \tilde{\tilde{x}}_6$, and $\tilde{\tilde{x}}_7$ can be obtained. Some are as follows:

(i) $\tilde{\tilde{x}}_1 = \{(0,0,0,0),(0,0,0,0),(0,0,0,0)\}, \tilde{\tilde{x}}_2 = \left\{\left(-\frac{10}{3},\frac{20}{3},\frac{55}{6},20\right),\left(-5,\frac{20}{3},10,\frac{65}{3}\right),\left(-10,\frac{10}{3},\frac{85}{6},\frac{80}{3}\right)\right\}, \tilde{\tilde{x}}_3 = \left\{\left(-\frac{5}{3},6,\frac{22}{3},15\right),\left(-\frac{8}{3},\frac{23}{6},\frac{29}{3},18\right),\left(0,\frac{35}{6},\frac{25}{3},\frac{46}{3}\right)\right\} \tilde{\tilde{x}}_4 = \{(0,0,0,0),(0,0,0,0),(0,0,0,0)\}, \tilde{\tilde{x}}_5 = \{(0,0,0,0),(0,0,0,0),(0,0,0,0)\}, \tilde{\tilde{x}}_6 = \left\{\left(-\frac{229}{3}\ 2,\frac{104}{3},108\right),\left(-\frac{280}{3},-\frac{13}{3},\frac{139}{3},124\right),\left(-68,\frac{44}{3},\frac{101}{3},\frac{296}{3}\right)\right\}$ and $\tilde{\tilde{x}}_7 = \{(0,0,0,0),(0,0,0,0),(0,0,0,0)\}$.

(ii) $\tilde{\tilde{x}}_1 = \{(0,0,0,0),(0,0,0,0),(0,0,0,0)\}, \tilde{\tilde{x}}_2 = \{(2,4,6,7),(4,10,12,14),(4,10,12,15)\}, \tilde{\tilde{x}}_3 = \{(2,3,4,8),(4,5,9,10),(7,10,11,12)\}, \tilde{\tilde{x}}_4 = \{(0,0,0,0),(0,0,0,0),(0,0,0,0)\}, \tilde{\tilde{x}}_5 = \{(0,0,0,0),(0,0,0,0),(0,0,0,0)\}, \tilde{\tilde{x}}_6 = \{(5,15,19,24),(8,17,20,24),(10,20,25,33)\}$ $\{(0,0,0,0),(0,0,0,0),(0,0,0,0)\}$ and $\tilde{\tilde{x}}_7 = \{(0,0,0,0),(0,0,0,0),(0,0,0,0)\}$.

(iii) $\tilde{\tilde{x}}_1 = \{(0,0,0,0),(0,0,0,0),(0,0,0,0)\}, \tilde{\tilde{x}}_2 = \{(2,4,9,10),(3,7,10,14),(6,10,12,13)\}$,

$\tilde{\tilde{x}}_3 = \{(3,5,6,7),(1,5,7,10),(2,8,12,19)\}, \tilde{\tilde{x}}_4 = \{(0,0,0,0),(0,0,0,0),(0,0,0,0)\}, \tilde{\tilde{x}}_5 = \{(0,0,0,0),(0,0,0,0),(0,0,0,0)\}, \tilde{\tilde{x}}_6 = \{(12,15,16,17),(15,15,20,20),(9,18,28,35)\}$ and $\tilde{\tilde{x}}_7 = \{(0,0,0,0),(0,0,0,0),(0,0,0,0)\}$.

According to Step 3 of the PrAlApp, (i), (ii), and (iii) represents optimal solutions of the HFLPP (P_5).

Step 4: According to Step 4 of the PrAlApp, the unique optimal solution of the HFLPP (P_5) can be obtained as follows.

Step 4(a): From the obtained optimal solution $x_1 = 0, x_2 = 100, x_3 = 85, x_4 = 0, x_5 = 0, x_6 = \frac{55}{3}, x_7 = 0$, x_2, x_3, and x_6 are optimal basic variables. Therefore, the optimal basis matrix $B = \begin{bmatrix} 1 & 1 & 0 \\ 5 & 3 & 1 \\ 4 & 10 & 0 \end{bmatrix}$.

Step 4(b): It can be verified that the multiplicative inverse of the optimal basis matrix B is $B^{-1} = \begin{bmatrix} \frac{5}{3} & 0 & -\frac{1}{6} \\ -\frac{19}{3} & 1 & \frac{1}{3} \\ -\frac{2}{3} & 0 & \frac{1}{6} \end{bmatrix}$.

Step 4(c): The optimal solution matrix $\begin{bmatrix} \tilde{\tilde{x}}_2 \\ \tilde{\tilde{x}}_6 \\ \tilde{\tilde{x}}_3 \end{bmatrix} = B^{-1} \times \tilde{\tilde{b}}'_i = \begin{bmatrix} \frac{5}{3} & 0 & -\frac{1}{6} \\ -\frac{19}{3} & 1 & \frac{1}{3} \\ -\frac{2}{3} & 0 & \frac{1}{6} \end{bmatrix} \times$

$$\begin{bmatrix} \{(10,14,16,20),(12,15,15,20),(8,13,18,24)\} \\ \{(35,70,90,120),(40,80,90,120),(40,80,90,120)\} \\ \{(70,100,100,130),(80,95,110,140),(80,95,110,140)\} \end{bmatrix} =$$

$$\begin{bmatrix} \left\{\left(-5,\frac{20}{3},10,\frac{65}{3}\right),\left(-\frac{10}{3},\frac{20}{3},\frac{55}{6},20\right),\left(-10,\frac{10}{3},\frac{85}{6},\frac{80}{3}\right)\right\} \\ \left\{\left(-\frac{205}{3},2,\frac{104}{3},100\right),\left(-60,\frac{50}{3},\frac{95}{3},\frac{272}{3}\right),\left(-\frac{256}{3},-\frac{7}{3},\frac{133}{3},116\right)\right\} \\ \left\{\left(-\frac{5}{3},6,\frac{22}{3},15\right),\left(0,\frac{35}{6},\frac{25}{3},\frac{46}{3}\right),\left(-\frac{8}{3},\frac{23}{6},\frac{29}{3},18\right)\right\} \end{bmatrix}$$

Hence, $\tilde{\tilde{x}}_2 = \left\{\left(-\frac{10}{3},\frac{20}{3},\frac{55}{6},20\right),\left(-5,\frac{20}{3},10,\frac{65}{3}\right),\left(-10,\frac{10}{3},\frac{85}{6},\frac{80}{3}\right)\right\}, \tilde{\tilde{x}}_6 = \left\{\left(-\frac{205}{3},2,\frac{104}{3},100\right)\left(-\frac{256}{3},-\frac{7}{3},\frac{133}{3},116\right),\left(-60,\frac{50}{3},\frac{95}{3},\frac{272}{3}\right)\right\}$ and $\tilde{\tilde{x}}_3 = \left\{\left(-\frac{5}{3},6,\frac{22}{3},15\right),\left(-\frac{8}{3},\frac{23}{6},\frac{29}{3},18\right),\left(0,\frac{35}{6},\frac{25}{3},\frac{46}{3}\right)\right\}$.

9. Advantages of the PrAlApp over Saghi et al.'s approach

It is better to solve the HFLPP (P_4) by the PrAlApp compared to Saghi et al.'s [9] approach due to the following reasons.

(i) Less computational efforts are required to solve the HFLPP (P_4) by the PrAlApp compared to Saghi et al.'s [9] approach.

(ii) In the PrAlApp, an OS of the HFLPP (P_4) is obtained with the help of an OS of the CLPP (P_8). Since existing tools like MATLAB, TORA, MAPLE, and LINGO can be used to find an OS of the CLPP (P_8), the HFLPP (P_4) corresponding to large-scaled real-life problems can be easily solved by the PrAlApp.

(iii) If there exist an OS of the HFLPP (P_4), then, there will exist an infinite number of optimal solutions of the HFLPP (P_4). All these infinite number of optimal solutions can be obtained by the PrAlApp.

If a unique OS exists, then the HFLPP (P_4) is obtained by Saghi et al.'s [9] approach.

10. Conclusions

An alternative approach is proposed to solve the HFLPP (P_4). Also, it is pointed out that it is better to use the PrAlApp compared to Saghi et al.'s [9] approach. Furthermore, to illustrate the PrAlApp, an existing HFLPP is solved by the PrAlApp. It is pertinent to mention that neither Saghi et al.'s [9] approach nor the PrAlApp can be used to fully solve HFLPPs. To propose an approach to fully solve HFLPPs may be considered a challenging, open research problem.

Author contributions

Raina Ahuja: Conceptualization, methodology, writing- original draft; Meraj Ali Khan: Funding acquisition, visualization, writing – review & editing; Parul Tomar: Software, writing-review & editing; Amit Kumar: Visualization, supervision; S. S. Appadoo: Visualization, supervision; Ibrahim Al-Dayel: Funding acquisition, visualization, writing – review & editing.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interest.

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