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Research article

Ergodic property and density function of a stochastic Kawasaki disease model

Ying He[1,*] **and Bo Bi**[2]

[1] School of Mathematics and Statistics, Northeast Petroleum University, Daqing163318, China

[2] School of Public Health, Hainan Medical University, Haikou 571199, China

* **Correspondence:** Email: heying65332015@163.com.

Abstract: In this paper, a stochastic Kawasaki disease model was investigated. First, it was theoretically proved that the solution of the stochastic model is positive and global, as well as the existence of an ergodic stationary distribution. Subsequently, we derived the exact expression of the probability density function around a quasi-equilibrium point through the four-dimensional Fokker-Planck equation. Finally, some numerical simulations were introduced to validate the theoretical findings.

Keywords: Kawasaki disease; ergodicity; probability density; Fokker–Planck equation
Mathematics Subject Classification: 37H05, 37H30, 60H10

1. Introduction

Kawasaki disease (KD) is an acute self-limited febrile systemic vasculitis that occurs predominantly in infants and young children. KD is reported to be the leading cause of acquired heart disease in children in developed countries and has now been researched worldwide [1,2]. It is unfortunate that, at present, there is no clear pathogenesis for KD, and no specific or sensitive biomarkers exist for KD, making diagnosis and treatment challenging [3]. Some researchers suggest that the cross reaction between external infection and organ tissue composition is the cause of KD, thereby causing an immune imbalance and the generation of multiple cytokines [4]. Qiang et al. [5] recently presented the following mathematical model for analyzing the interaction between important cytokines and endothelial cells in the lesion area of patients with KD:

$$
\begin{cases}
\dfrac{\mathrm{d}E}{\mathrm{d}t} = r + \dfrac{k_6 V(t)E(t)}{1+V(t)} - k_1 E(t)P(t) - d_1 E(t), \\
\dfrac{\mathrm{d}V}{\mathrm{d}t} = k_2 E(t)P(t) - d_2 V(t), \\
\dfrac{\mathrm{d}C}{\mathrm{d}t} = k_3 E(t)P(t) + k_4 V(t) - d_3 C(t), \\
\dfrac{\mathrm{d}P}{\mathrm{d}t} = k_5 C(t) - d_4 P(t).
\end{cases}
\tag{1.1}
$$

Here, $E(t), V(t), C(t)$ and $P(t)$ represent concentrations of normal endothelial cells, vascular endothelial growth factors, activated adhesion factors/chemokines and inflammatory factors in the lesion area at time t in the acute stage of patients with Kawasaki disease, respectively. $k_j\,(j = 1, 2, ..., 6)$, $d_i\,(i = 1, 2, 3, 4)$ and r are positive parameters of model (1.1), and their descriptions are given in Table 1. The basic reproduction number of the deterministic model (1.1) is given by $R_0 = \frac{rk_5(k_2k_4+d_2k_3)}{d_1d_2d_3d_4}$.

Table 1. Biological meanings and units of the parameters.

Parameter	Biological meanings	Unit
r	Proliferation rate of normal endothelial cells	pg/ml/day
d_1	Apoptosis rate of normal endothelial cells	/day
d_2	Hydrolytic rate of endothelial growth factors	/day
d_3	Hydrolytic rate of adhesion factors and chemokines	/day
d_4	Hydrolytic rate of inflammatory factors	/day
k_1	The rate of injury of endothelial cells caused by inflammatory factors	pg/ml/day
k_2	Production rate of endothelial growth factors caused by inflammatory factors	pg/ml/day
k_3	Production rate of activated adhesion factors and chemokines caused by inflammatory factors	pg/ml/day
k_4	Production rate of activated adhesion factors and chemokines caused by endothelial growth factors	pg/ml/day
k_5	Production rate of inflammatory factors by increasing abnormally activated immune cells	/day
k_6	Proliferation rate of endothelial cells promoted by endothelial growth factors	/day

On the other hand, environmental noises are commonly thought to be damaging, resulting in dynamical disorder [6,7]. It is worth noting that noises also make a positive contribution to the dynamics of complex nonlinear systems, particularly in biomathematics and interdisciplinary physical models, such as loss-induced phenomena, information processing in biochemical signaling systems and so on [8–13]. Environmental perturbations, like climate changes, habitats, health habits, medical quality, etc., are obvious factors that influence the evolution of KD over time. In order to accommodate some random factors, Chen et al. [14] incorporated white noise and color noise into model (1.1), as well as took into account the existence of the ergodic stationary distribution of a stochastic KD model. In comparison with [14], we give the exact expression of the probability density function of model (1.2) with white noise, around the quasi-equilibrium point by solving the four-dimensional Fokker-Planck equation. This is the key aim of the paper. Thus, we introduce multiplicative white noise into the deterministic KD model to establish a stochastic KD model and study its dynamic behavior. Corresponding to the deterministic model (1.1), the stochastic system takes the following form:

$$\begin{cases} \mathrm{d}E(t) = \Big(r + \dfrac{k_6 V(t) E(t)}{1+V(t)} - k_1 E(t)P(t) - d_1 E(t)\Big)\mathrm{d}t + \sigma_1 E(t)\mathrm{d}B_1(t), \\ \mathrm{d}V(t) = \Big(k_2 E(t)P(t) - d_2 V(t)\Big)\mathrm{d}t + \sigma_2 V(t)\mathrm{d}B_2(t), \\ \mathrm{d}C(t) = \Big(k_3 E(t)P(t) + k_4 V(t) - d_3 C(t)\Big)\mathrm{d}t + \sigma_3 C(t)\mathrm{d}B_3(t), \\ \mathrm{d}P(t) = \Big(k_5 C(t) - d_4 P(t)\Big)\mathrm{d}t + \sigma_4 P(t)\mathrm{d}B_4(t), \end{cases} \tag{1.2}$$

where $B_1(t)$, $B_2(t)$, $B_3(t)$ and $B_4(t)$ are mutually independent standard Browntain motions and $\sigma_i^2 > 0$ represent the intensity of white noise $B_i(t)$ $(i = 1, 2, 3, 4)$, respectively.

The rest of this paper is organized as follows. In Section 2, we demonstrate that the stochastic system (1.2) has a unique global positive solution for any positive initial value. In Section 3, we investigate the existence of a unique stationary distribution to the stochastic model (1.2). In Section 4, we give the precise expression of the probability density function. In Section 5, numerical simulations are given to confirm our analytical findings.

2. Existence and uniqueness of the global positive solution

Throughout the paper, based on the condition (H) in Ref. [5], we also assume that $k_6 < d_1$.

Theorem 2.1. *For any initial value* $(E(0), V(0), C(0), P(0)) \in \mathbb{R}^4_+$, *there exists a unique solution* $(E(t), V(t), C(t), P(t))$ *to system (1.2) on* $t \geq 0$ *and the solution will remain in* $\mathbb{R}^4_+$ *with probability one, namely,* $(E(t), V(t), C(t), P(t)) \in \mathbb{R}^4_+$ *for all* $t \geq 0$ *almost surely (a.s.).*

Proof. Since the coefficients of system (1.2) are locally Lipschitz continuous, thus there is a unique local solution $(E(t), V(t), C(t), P(t))$ on $t \in [0, \tau_0)$ for any initial value $(E(0), V(0), C(0), P(0)) \in \mathbb{R}^4_+$, where τ_0 is an explosion time. We verify that this solution is global, i.e., $\tau_0 = \infty$. Let $n_0 \geq 1$ be sufficiently large such that $E(0), V(0), C(0)$ and $P(0)$ all lie within the interval $[\frac{1}{n_0}, n_0]$. For $n > n_0$, we define a stopping time by:

$$\tau_n = \inf\Big\{t \in [0, \tau_0) : \ \min\{E(t), V(t), C(t), P(t)\} \leq \frac{1}{n} \ \text{ or } \ \max\{E(t), V(t), C(t), P(t)\}\} \geq n\Big\},$$

where throughout this paper we set $\inf\emptyset = \infty$. Evidently, τ_n is increasing as $n \to \infty$. Set $\tau_\infty = \lim_{n\to\infty} \tau_n$, thus $\tau_\infty \leq \tau_0$ a.s. If we show that $\tau_\infty = \infty$ a.s., then $\tau_0 = \infty$ a.s., which means that $(E(t), V(t), C(t), P(t)) \in \mathbb{R}^4_+$ a.s., for all $t \geq 0$. If $\tau_\infty < \infty$, then there exists a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that

$$\mathbb{P}\{\tau_\infty \leq T\} > \varepsilon.$$

Accordingly, there exists an integer $n_1 > n_0$ such that

$$\mathbb{P}\{\tau_n \leq T\} > \varepsilon, \quad \forall n \geq n_1.$$

Based on the fact that $u - 1 - \ln u \geq 0$, for any $u > 0$, we have

$$c_0 u - 1 - \ln c_0 u = c_0 u - (1 + \ln c_0) - \ln u \geq 0, \text{ for any } c_0 > 0.$$

Now a non-negative C^2-function $U(E, V, C, P)$ is constructed as

$$U(E,V,C,P) = E - e_1 - e_1 \ln \frac{E}{e_1} + e_2 V - (1 + \ln e_2) - \ln V + e_3 C - (1 + \ln e_3) - \ln C + e_4 P - (1 + \ln e_4) - \ln P,$$

where e_1, e_2, e_3, e_4 are positive constants to be determined later. Applying Itô's formula to U, we have

$$\mathrm{d}U = LU\mathrm{d}t + \sigma_1(E - e_1)\mathrm{d}B_1 + \sigma_2(e_2 V - 1)dB_2 + \sigma_3(e_3 C - 1)\mathrm{d}B_3 + \sigma_4(e_4 P - 1)\mathrm{d}B_4,$$

where

$$\begin{aligned} LU(E,V,C,P) = {} & r + \frac{k_6 VE}{1+V} - k_1 EP - d_1 E + e_1\Big(- \frac{r}{E} - \frac{k_6 V}{1+V} + k_1 p + d_1 + \frac{1}{2}\sigma_1^2\Big) + e_2(k_2 EP - d_2 V) \\ & - \frac{k_2 EP}{V} + d_2 + \frac{1}{2}\sigma_2^2 + e_3(k_3 EP + k_4 V - d_3 C) - \frac{k_3 EP}{C} - k_4 \frac{V}{C} + d_3 + \frac{1}{2}\sigma_3^2 \\ & + e_4(k_5 C - d_4 P) - \frac{k_5 C}{P} + d_4 + \frac{1}{2}\sigma_4^2 \\ \leq {} & (e_1 k_1 - e_4 d_4)P + (e_4 k_5 - e_3 d_3)C + (e_3 k_4 - e_2 d_2)V + (e_2 k_2 + e_3 k_3 - k_1)EP \\ & + (k_6 - d_1)E + r + e_1 d_1 + \frac{1}{2}e_1\sigma_1^2 + d_2 + \frac{1}{2}\sigma_2^2 + d_3 + \frac{1}{2}\sigma_3^2 + d_4 + \frac{1}{2}\sigma_4^2. \end{aligned}$$

Choose the parameters e_1, e_2, e_3 and e_4 such that

$$\begin{cases} e_1 k_1 - e_4 d_4 = 0, \\ e_4 k_5 - e_3 d_3 = 0, \\ e_3 k_4 - e_2 d_2 = 0, \\ e_2 k_2 + e_3 k_3 - k_1 = 0. \end{cases}$$

More precisely, we can obtain that $e_1 = \frac{d_2 d_3 d_4}{k_5(k_2 k_4 + k_3 d_2)}$, $e_2 = \frac{k_1 k_4}{k_2 k_4 + k_3 d_2}$, $e_3 = \frac{k_1 d_2}{k_2 k_4 + k_3 d_2}$ and $e_4 = \frac{k_1 d_2 d_3}{k_5(k_2 k_4 + k_3 d_2)}$. By the assumption $k_6 < d_1$, then we derive that

$$LU(E,V,C,P) \leq r + e_1 d_1 + \frac{1}{2}e_1\sigma_1^2 + d_2 + \frac{1}{2}\sigma_2^2 + d_3 + \frac{1}{2}\sigma_3^2 + d_4 + \frac{1}{2}\sigma_4^2 := K,$$

where K is a positive constant that is independence of E, V, C and P. The rest of this proof is mostly similar to Theorem 2.1 of Liu and Jiang [14], so we therefore omit it here. □

3. Stationary distribution

Let $X(t)$ be a regular time-homogeneous Markov process in $\mathbb{R}^n$ described by the following stochastic differential equation:

$$\mathrm{d}X(t) = f(x(t))\mathrm{d}t + \sum_{r=1}^{m} g_r(x)\mathrm{d}B_r(t).$$

The diffusion matrix is defined as follows: $A(x) = (a_{ij}(x)),\ \ a_{ij}(x) = \sum_{r=1}^{m} g_r^i(x)g_r^j(x)$, where $X(t)$ is nonsingular.

Lemma 3.1. *([15]) The Markov process $X(t)$ has a unique ergodic stationary distribution $\pi(\cdot)$, if there exists a bounded domain $\widetilde{D} \subset \mathbb{R}^n$ with a regular boundary Γ and*

(T_1) there is a positive number ρ_0 such that $\sum\limits_{i,j=1}^{n} a_{ij}(x)\delta_i\delta_j \geq \rho_0 \mid \delta^2 \mid,\ \ x \in \widetilde{D},\ \delta \in \mathbb{R}^n$;

(T_2) there is a non-negative C^2- function V such that LV is negative for any $x \in \mathbb{R}^n \setminus \widetilde{D} = \widetilde{D}^C$.

Then for all $x \in \mathbb{R}^n$, it follows that

$$\mathbb{P}\Big\{\lim_{T\to\infty}\frac{1}{T}\int_0^T \nu(X(t))\mathrm{d}t = \int_{\mathbb{R}^n}\nu(x)\pi(\mathrm{d}x)\Big\},$$

where $\nu(\cdot)$ is an integral function with respect to the measure $\pi(\cdot)$.

Theorem 3.1. *Suppose that $R_0^s > 1$, then for any initial value $(E(0), V(0), C(0), P(0)) \in \mathbb{R}_+^4$, the system* (1.2) *has a unique stationary distribution $\pi(\cdot)$ and it has the ergodic property, where*

$$R_0^s = \frac{rk_3k_5}{(d_1+\frac{1}{2}\sigma_1^2)(d_3+\frac{1}{2}\sigma_3^2)(d_4+\frac{1}{2}\sigma_4^2)} + \frac{rk_2k_4k_5}{(d_1+\frac{1}{2}\sigma_1^2)(d_2+\frac{1}{2}\sigma_2^2)(d_3+\frac{1}{2}\sigma_3^2)(d_4+\frac{1}{2}\sigma_4^2)}.$$

Proof. In order to prove Theorem 3.1, we only need to verify the conditions (T_1) and (T_2) in Lemma 3.1. First, we verify that condition (T_2) holds. We define

$$V_1(E, V, C, P) = -(a_1 + a_2)\ln E - c_2 \ln v - \ln C - (b_1 + b_2)\ln P,$$

where a_1, a_2, b_1, b_2, c_2 are positive constants to be determined later. Making use of Itô's formula, we get

$$\begin{aligned}
LV_1 = \ & -a_1\frac{r}{E} - k_3\frac{EP}{C} - b_1k_5\frac{C}{P} + a_1(d_1+\frac{1}{2}\sigma_1^2) + b_1(d_4+\frac{1}{2}\sigma_4^2)\\
& - a_2\frac{r}{E} - b_2k_5\frac{C}{P} - c_2\frac{k_2EP}{V} - k_4\frac{V}{C} + a_2(d_1+\frac{1}{2}\sigma_1^2) + b_2(d_4+\frac{1}{2}\sigma_4^2) + c_2(d_2+\frac{1}{2}\sigma_2^2)\\
& - (a_1+a_2)\frac{k_6V}{1+V} + (a_1+a_2)k_1P + d_3 + \frac{1}{2}\sigma_3^2\\
\leq \ & -3\sqrt[3]{rk_3k_5a_1b_1} + a_1(d_1+\frac{1}{2}\sigma_1^2) + b_1(d_4+\frac{1}{2}\sigma_4^2) - 4\sqrt[4]{rk_2k_4k_5a_2b_2c_2} + a_2(d_1+\sigma_1^2) + b_2(d_4+\sigma_4^2)\\
& + c_2(d_2+\frac{1}{2}\sigma_2^2) + d_3 + \frac{1}{2}\sigma_3^2 + (a_1+a_2)k_1P.
\end{aligned}$$

Letting

$$a_1 = \frac{rk_3k_5}{(d_1+\frac{1}{2}\sigma_1^2)^2(d_4+\frac{1}{2}\sigma_4^2)}, \qquad b_1 = \frac{rk_3k_5}{(d_1+\frac{1}{2}\sigma_1^2)(d_4+\frac{1}{2}\sigma_4^2)^2},$$
$$a_2 = \frac{rk_2k_4k_5}{(d_1+\frac{1}{2}\sigma_1^2)^2(d_2+\frac{1}{2}\sigma_2^2)(d_4+\frac{1}{2}\sigma_4^2)}, \quad b_2 = \frac{rk_2k_4k_5}{(d_1+\frac{1}{2}\sigma_1^2)(d_2+\frac{1}{2}\sigma_2^2)(d_4+\frac{1}{2}\sigma_4^2)^2},$$
$$c_2 = \frac{rk_2k_4k_5}{(d_1+\frac{1}{2}\sigma_1^2)(d_2+\frac{1}{2}\sigma_2^2)^2(d_4+\frac{1}{2}\sigma_4^2)}.$$

We have that

$$\begin{aligned}
LV_1 &\leq -\frac{rk_3k_5}{(d_1+\frac{1}{2}\sigma_1^2)(d_4+\frac{1}{2}\sigma_4^2)} - \frac{rk_2k_4k_5}{(d_1+\frac{1}{2}\sigma_1^2)(d_2+\frac{1}{2}\sigma_2^2)(d_4+\frac{1}{2}\sigma_4^2)} + d_3 + \frac{1}{2}\sigma_3^2 + (a_1+a_2)k_1P \\
&= -(d_3+\frac{1}{2}\sigma_3^2)\Big(\frac{rk_3k_5}{(d_1+\frac{1}{2}\sigma_1^2)(d_3+\frac{1}{2}\sigma_3^2)(d_4+\frac{1}{2}\sigma_4^2)} + \frac{rk_2k_4k_5}{(d_1+\sigma_1^2)(d_2+\frac{1}{2}\sigma_2^2)(d_3+\frac{1}{2}\sigma_3^2)(d_4+\frac{1}{2}\sigma_4^2)} - 1\Big) \\
&\quad + (a_1+a_2)k_1P \\
&= -(d_3+\frac{1}{2}\sigma_3^2)(R_0^s-1) + (a_1+a_2)k_1P \\
&= -\lambda + (a_1+a_2)k_1P,
\end{aligned}$$

where $\lambda = (d_3+\frac{1}{2}\sigma_3^2)(R_0^s-1)$. We define the following C^2-function $W(E,V,C,P)$:

$$W = MV_1 - \ln E - \ln V - \ln C + \frac{1}{\theta+1}\Big(\frac{d_2k_3+2k_2k_4}{2k_1k_4}E + V + \frac{d_2}{2k_4}C + \frac{d_2d_3}{4k_4k_5}P\Big)^{\theta+1},$$

where $\theta > 0$ is a constant satisfying

$$\Big[(d_1-k_6)\wedge\frac{d_2}{2}\wedge\frac{d_3}{2}\wedge d_4\Big] > \frac{\theta}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2),$$

and $M > 0$ is a sufficiently large number satisfying the following condition

$$-M\lambda + d \leq -2,$$

where

$$\lambda = (d_3+\frac{1}{2}\sigma_3^2)(R_0^s-1) > 0,$$

$$\begin{aligned}
B = \sup_{(E,V,C,P)\in\mathbb{R}_+^4} &\Big\{\frac{d_2k_3+2k_2k_4}{2k_1k_4}r(\frac{d_2k_3+2k_2k_4}{2k_1k_4}E + V + \frac{d_2}{2k_4}C + \frac{d_2d_3}{4k_4k_5}P)^{\theta} \\
&- \frac{e}{2}(\frac{d_2k_3+2k_2k_4}{2k_1k_4}E + V + \frac{d_2}{2k_4}C + \frac{d_2d_3}{4k_4k_5}P)^{\theta+1}\Big\},
\end{aligned}$$

$$e = \Big[(d_1-k_6)\wedge\frac{d_2}{2}\wedge\frac{d_3}{2}\wedge d_4\Big] - \frac{\theta}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2),$$

and

$$d = B + d_1 + d_2 + d_3 + \frac{1}{2}(\sigma_1^2+\sigma_2^2+\sigma_3^2).$$

We easily notice that

$$\liminf_{n\to\infty,\,(E,V,C,P)\in\mathbb{R}_+^4\setminus U_n} W(E,V,C,P) = +\infty,$$

where $U_n = (\frac{1}{n}, n) \times (\frac{1}{n}, n) \times (\frac{1}{n}, n) \times (\frac{1}{n}, n)$. Let $W(E_0, V_0, C_0, P_0)$ be the minimal value point of $W(E, V, C, P)$. Then we can define a non-negative C^2-function V as follows:

$$\begin{aligned}
V(E,V,C,P) &= W(E,V,C,P) - W(E_0,V_0,C_0,P_0)\\
&= MV_1 - \ln E - \ln V - \ln C + \frac{1}{\theta+1}\Big(\frac{d_2k_3+2k_2k_4}{2k_1k_4}E + V + \frac{d_2}{2k_4}C + \frac{d_2d_3}{4k_4k_5}P\Big)^{\theta+1}\\
&\quad - W(E_0,V_0,C_0,P_0)\\
&:= MV_1 + V_2 + V_3 - W(E_0,V_0,C_0,P_0),
\end{aligned}$$

where

$$V_2 = -\ln E - \ln V - \ln C,\ V_3 = \frac{1}{\theta+1}\Big(\frac{d_2k_3+2k_2k_4}{2k_1k_4}E + V + \frac{d_2}{2k_4}C + \frac{d_2d_3}{4k_4k_5}P\Big)^{\theta+1}.$$

By a direct calculation, we get

$$LV_2 \le -\frac{r}{E} - \frac{k_2EP}{V} - \frac{k_3EP}{C} + k_1P + d_1 + d_2 + d_3 + \frac{1}{2}(\sigma_1^2+\sigma_2^2+\sigma_3^2).$$

$$\begin{aligned}
LV_3 = &\Big(\frac{d_2k_3+2k_2k_4}{2k_1k_4}E + V + \frac{d_2}{2k_4}C + \frac{d_2d_3}{4k_4k_5}P\Big)^{\theta}\cdot\Big\{\frac{d_2k_3+2k_2k_4}{2k_1k_4}(r + \frac{k_6VE}{1+V} - k_1EP - d_1E) + k_2EP\\
&- d_2V + \frac{d_2}{2k_4}(k_3EP + k_4V - d_3C) + \frac{d_2d_3}{4k_4k_5}(k_5C - d_4P)\Big\}\\
&+ \frac{1}{2}\theta(\frac{d_2k_3+2k_2k_4}{2k_1k_4}E + V + \frac{d_2}{2k_4}C + \frac{d_2d_3}{4k_4k_5}P)^{\theta-1}\\
&\cdot[(\frac{d_2k_3+2k_2k_4}{2k_1k_4})^2\sigma_1^2E^2 + \sigma_2^2V^2 + (\frac{d_2}{2k_4})^2\sigma_3^2C^2 + (\frac{d_2d_3}{4k_4k_5})^2\sigma_4^2P^2]\\
\le &\Big(\frac{d_2k_3+2k_2k_4}{2k_1k_4}E + V + \frac{d_2}{2k_4}C + \frac{d_2d_3}{4k_4k_5}P\Big)^{\theta}\Big\{\frac{d_2k_3+2k_2k_4}{2k_1k_4}r - [(d_1-k_6)\wedge\frac{d_2}{2}\wedge\frac{d_3}{2}\wedge d_4]\\
&\cdot\Big(\frac{d_2k_3+2k_2k_4}{2k_1k_4}E + V + \frac{d_2}{2k_4}C + \frac{d_2d_3}{4k_4k_5}P\Big)\Big\} + \frac{\theta}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)\\
&\cdot\Big(\frac{d_2k_3+2k_2k_4}{2k_1k_4}E + V + \frac{d_2}{2k_4}C + \frac{d_2d_3}{4k_4k_5}P\Big)^{\theta+1}\\
= &\frac{d_2k_3+2k_2k_4}{2k_1k_4}r\Big(\frac{d_2k_3+2k_2k_4}{2k_1k_4}E + V + \frac{d_2}{2k_4}C + \frac{d_2d_3}{4k_4k_5}P\Big)^{\theta} - \Big[\Big((d_1-k_6)\wedge\frac{d_2}{2}\wedge\frac{d_3}{2}\wedge d_4\Big)\\
&- \frac{\theta}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)\Big]\cdot\Big(\frac{d_2k_3+2k_2k_4}{2k_1k_4}E + V + \frac{d_2}{2k_4}C + \frac{d_2d_3}{4k_4k_5}P\Big)^{\theta+1}\\
\le &B - \frac{e}{2}\Big(\frac{d_2k_3+2k_2k_4}{2k_1k_4}E + V + \frac{d_2}{2k_4}C + \frac{d_2d_3}{4k_4k_5}P\Big)^{\theta+1}\\
\le &B - \frac{e}{2}\Big[(\frac{d_2k_3+2k_2k_4}{2k_1k_4})^{\theta+1}E^{\theta+1} + V^{\theta+1} + (\frac{d_2}{2k_4})^{\theta+1}C^{\theta+1} + (\frac{d_2d_3}{4k_4k_5})^{\theta+1}P^{\theta+1}\Big],
\end{aligned}$$

where

$$B=\sup_{(E,V,C,P)\in\mathbb{R}_+^4}\Big\{\frac{d_2k_3+2k_2k_4}{2k_1k_4}r(\frac{d_2k_3+2k_2k_4}{2k_1k_4}E+V+\frac{d_2}{2k_4}C+\frac{d_2d_3}{4k_4k_5}P)^{\theta}$$
$$-\frac{e}{2}(\frac{d_2k_3+2k_2k_4}{2k_1k_4}E+V+\frac{d_2}{2k_4}C+\frac{d_2d_3}{4k_4k_5}P)^{\theta+1}\Big\},$$

$$e=\Big[(d_1-k_6)\wedge\frac{d_2}{2}\wedge\frac{d_3}{2}\wedge d_4)\Big]-\frac{\theta}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)>0,$$

and θ is a positive constant and satisfies the condition $0<\theta<\min\Big\{1,\frac{2\Big[\big(d_1-k_6)\wedge\frac{d_2}{2}\wedge\frac{d_3}{2}\wedge d_4\big)\Big]}{\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2}\Big\}$.

From above analysis, we have

$$\begin{aligned}LV\le&-M\lambda+[M(a_1+a_2)+1]k_1P-\frac{r}{E}-\frac{k_2EP}{V}-\frac{k_3EP}{C}+B+d_1+d_2+d_3+\frac{1}{2}(\sigma_1^2+\sigma_2^2+\sigma_3^2)\\&-\frac{e}{2}\Big[(\frac{d_2k_3+2k_2k_4}{2k_1k_4})^{\theta+1}E^{\theta+1}+V^{\theta+1}+(\frac{d_2}{2k_4})^{\theta+1}C^{\theta+1}+(\frac{d_2d_3}{4k_4k_5})^{\theta+1}P^{\theta+1}\Big]\\\le&-M\lambda+[M(a_1+a_2)+1]k_1P-\frac{r}{E}-\frac{k_2EP}{V}-\frac{k_3EP}{C}-\frac{e}{4}\Big[(\frac{d_2k_3+2k_2k_4}{2k_1k_4})^{\theta+1}E^{\theta+1}+V^{\theta+1}\\&+(\frac{d_2}{2k_4})^{\theta+1}C^{\theta+1}+(\frac{d_2d_3}{4k_4k_5})^{\theta+1}P^{\theta+1}\Big]\\&-\frac{e}{4}\Big[(\frac{d_2k_3+2k_2k_4}{2k_1k_4})^{\theta+1}E^{\theta+1}+V^{\theta+1}+(\frac{d_2}{2k_4})^{\theta+1}C^{\theta+1}+(\frac{d_2d_3}{4k_4k_5})^{\theta+1}P^{\theta+1}\Big]+d,\end{aligned}$$

where

$$d=B+d_1+d_2+d_3+\frac{1}{2}(\sigma_1^2+\sigma_2^2+\sigma_3^2).$$

Define a bounded closed set as follows:

$$D=\{(E,V,C,P)\in\mathbb{R}_+^4:\ \epsilon<E<\frac{1}{\epsilon},\ \epsilon<P<\frac{1}{\epsilon},\ \epsilon^3<V<\frac{1}{\epsilon^3},\ \epsilon^3<C<\frac{1}{\epsilon^3}\},$$

where $\varepsilon>0$ is a sufficiently small parameter such that

$$-\frac{r}{\varepsilon}+F\le-1,\tag{3.1}$$

$$\varepsilon<\frac{1}{[M(a_1+a_2)+1]k_1},\tag{3.2}$$

$$-\frac{k_2}{\varepsilon}+F\le-1,\tag{3.3}$$

$$-\frac{k_3}{\varepsilon}+F\le-1,\tag{3.4}$$

$$-\frac{e}{4}(\frac{d_2k_3+2k_2k_4}{2k_1k_4\varepsilon})^{\theta+1}+F\le-1,\tag{3.5}$$

$$-\frac{e}{4}(\frac{d_2d_3}{4k_4k_5\varepsilon})^{\theta+1}+F\le-1,\tag{3.6}$$

$$
-\frac{e}{4}\frac{1}{\varepsilon^{3(\theta+1)}}+F\leq -1, \tag{3.7}
$$

$$
-\frac{e}{4}(\frac{d_2}{2k_4\varepsilon^3})^{\theta+1}+F\leq -1, \tag{3.8}
$$

where F is a positive constant which will be given explicitly in expression (3.9). For convenience, we can divide $(E,V,C,P)\in\mathbb{R}^4_+\setminus D$ into eight domains:

$$
\begin{aligned}
&D_1=\{(E,V,C,P)\in\mathbb{R}^4_+ : 0<E<\varepsilon\},\ D_2=\{(E,V,C,P)\in\mathbb{R}^4_+ : 0<P<\varepsilon\},\\
&D_3=\{(E,V,C,P)\in\mathbb{R}^4_+ : 0<V<\varepsilon^3,\ E>\varepsilon,\ P>\varepsilon\},\\
&D_4=\{(E,V,C,P)\in\mathbb{R}^4_+ : 0<C<\varepsilon^3,\ E>\varepsilon,\ P>\varepsilon\},\\
&D_5=\{(E,V,C,P)\in\mathbb{R}^4_+ : E>\frac{1}{\varepsilon}\},\ D_6=\{(E,V,C,P)\in\mathbb{R}^4_+ : P>\frac{1}{\varepsilon}\},\\
&D_7=\{(E,V,C,P)\in\mathbb{R}^4_+ : V>\frac{1}{\varepsilon^3}\},\\
&D_8=\{(E,V,C,P)\in\mathbb{R}^4_+ : C>\frac{1}{\varepsilon^3}\}.
\end{aligned}
$$

Then, we will show that

$$
LV(E,V,C,P)\leq -1,\quad \text{for } (E,V,C,P)\in\mathbb{R}^4_+\setminus D=\bigcup_{i=1}^{8}D_i.
$$

Case 1. If $(E,V,C,P)\in D_1$, we can obtain that

$$
\begin{aligned}
LV\leq& -\frac{r}{E}+[M(a_1+a_2)+1]k_1P\\
&-\frac{e}{4}\Big[(\frac{d_2k_3+2k_2k_4}{2k_1k_4})^{\theta+1}E^{\theta+1}+V^{\theta+1}+(\frac{d_2}{2k_4})^{\theta+1}C^{\theta+1}+(\frac{d_2d_3}{4k_4k_5})^{\theta+1}P^{\theta+1}\Big]+d\\
\leq& -\frac{r}{E}+F\\
\leq& -\frac{r}{\varepsilon}+F\\
\leq& -1,\ \text{ by (3.1)},
\end{aligned}
$$

where

$$
\begin{aligned}
F=\sup_{(E,V,C,P)\in\mathbb{R}^4_+}&\Big\{[M(a_1+a_2)+1]k_1P-\frac{e}{4}\Big[(\frac{d_2k_3+2k_2k_4}{2k_1k_4})^{\theta+1}E^{\theta+1}+V^{\theta+1}+(\frac{d_2}{2k_4})^{\theta+1}C^{\theta+1}\\
&+(\frac{d_2d_3}{4k_4k_5})^{\theta+1}P^{\theta+1}\Big]+d\Big\}.
\end{aligned} \tag{3.9}
$$

Case 2. If $(E,V,C,P)\in D_2$, we have

$$
\begin{aligned}
LV\leq& -M\lambda+[M(a_1+a_2)+1]k_1P+d\\
\leq& -2+[M(a_1+a_2)+1]k_1\varepsilon\\
\leq& -1,\ \text{ by (3.2)}.
\end{aligned}
$$

Case 3. If $(E, V, C, P) \in D_3$, one can derive that

$$\begin{aligned}
LV \le & -\frac{k_2 EP}{V} + [M(a_1+a_2)+1]k_1 P \\
& - \frac{e}{4}\Big[(\frac{d_2k_3+2k_2k_4}{2k_1k_4})^{\theta+1}E^{\theta+1} + V^{\theta+1} + (\frac{d_2}{2k_4})^{\theta+1}C^{\theta+1} + (\frac{d_2d_3}{4k_4k_5})^{\theta+1}P^{\theta+1}\Big] + d \\
\le & -\frac{k_2 EP}{V} + F \\
\le & -\frac{k_2}{\varepsilon} + F \\
\le & -1, \quad \text{by (3.3)}.
\end{aligned}$$

Case 4. If $(E, V, C, P) \in D_4$, one can see that

$$\begin{aligned}
LV \le & -\frac{k_3 EP}{C} - \frac{e}{4}\Big[(\frac{d_2k_3+2k_2k_4}{2k_1k_4})^{\theta+1}E^{\theta+1} + V^{\theta+1} + (\frac{d_2}{2k_4})^{\theta+1}C^{\theta+1} + (\frac{d_2d_3}{4k_4k_5})^{\theta+1}P^{\theta+1}\Big] \\
& + [M(a_1+a_2)+1]k_1 P + d \\
\le & -\frac{k_3 EP}{C} + F \\
\le & -\frac{k_3}{\varepsilon} + F \\
\le & -1, \quad \text{by (3.4)}.
\end{aligned}$$

Case 5. If $(E, V, C, P) \in D_5$, it follows that

$$\begin{aligned}
LV \le & -\frac{e}{4}(\frac{d_2k_3+2k_2k_4}{2k_1k_4})^{\theta+1}E^{\theta+1} - \frac{e}{4}\Big[(\frac{d_2k_3+2k_2k_4}{2k_1k_4})^{\theta+1}E^{\theta+1} + V^{\theta+1} + (\frac{d_2}{2k_4})^{\theta+1}C^{\theta+1} + (\frac{d_2d_3}{4k_4k_5})^{\theta+1}P^{\theta+1}\Big] \\
& + [M(a_1+a_2)+1]k_1 P + d \\
\le & -\frac{e}{4}(\frac{d_2k_3+2k_2k_4}{2k_1k_4\varepsilon})^{\theta+1} + F \\
\le & -1, \quad \text{by (3.5)}.
\end{aligned}$$

Case 6. If $(E, V, C, P) \in D_6$, we have

$$\begin{aligned}
LV \le & -\frac{e}{4}(\frac{d_2d_3}{4k_4k_5})^{\theta+1}P^{\theta+1} - \frac{e}{4}\Big[(\frac{d_2k_3+2k_2k_4}{2k_1k_4})^{\theta+1}E^{\theta+1} + V^{\theta+1} + (\frac{d_2}{2k_4})^{\theta+1}C^{\theta+1} + (\frac{d_2d_3}{4k_4k_5})^{\theta+1}P^{\theta+1}\Big] \\
& + [M(a_1+a_2)+1]k_1 P + d \\
\le & -\frac{e}{4}(\frac{d_2d_3}{4k_4k_5\varepsilon})^{\theta+1} + F \\
\le & -1, \quad \text{by (3.6)}.
\end{aligned}$$

Case 7. If $(E, V, C, P) \in D_7$, we derive

$$\begin{aligned}
LV \le & -\frac{e}{4}V^{\theta+1} - \frac{e}{4}\Big[(\frac{d_2k_3+2k_2k_4}{2k_1k_4})^{\theta+1}E^{\theta+1} + V^{\theta+1} + (\frac{d_2}{2k_4})^{\theta+1}C^{\theta+1} + (\frac{d_2d_3}{4k_4k_5})^{\theta+1}P^{\theta+1}\Big] \\
& + [M(a_1+a_2)+1]k_1 P + d \\
\le & -\frac{e}{4}\frac{1}{\varepsilon^{3(\theta+1)}} + F \\
\le & -1, \quad \text{by (3.7)}.
\end{aligned}$$

Case 8. If $(E, V, C, P) \in D_8$, one can see that

$$
\begin{aligned}
LV \leq & -\frac{e}{4}(\frac{d_2}{2k_4})^{\theta+1}C^{\theta+1} - \frac{e}{4}\Big[(\frac{d_2k_3 + 2k_2k_4}{2k_1k_4})^{\theta+1}E^{\theta+1} + V^{\theta+1} + (\frac{d_2}{2k_4})^{\theta+1}C^{\theta+1} + (\frac{d_2d_3}{4k_4k_5})^{\theta+1}P^{\theta+1}\Big] \\
& + [M(a_1 + a_2) + 1]k_1P + d \\
\leq & -\frac{e}{4}(\frac{d_2}{2k_4\varepsilon^3})^{\theta+1} + F \\
\leq & -1, \text{ by (3.8).}
\end{aligned}
$$

Consequently, there exists a sufficiently small ε satisfying

$$LV(E, V, C, P) \leq -1, \quad \text{for all } (E, V, C, P) \in \mathbb{R}^4_+ \setminus D.$$

Thus the condition (T_2) in Lemma 3.1 holds. In addition, the corresponding diffusion matrix of system (1.2) is described as

$$A = diag(\sigma_1^2E^2, \sigma_2^2V^2, \sigma_3^2C^2, \sigma_4^2P^2).$$

Then there exists a positive constant $M_0 = \min_{(E,V,C,P)\in D\subset\mathbb{R}^4_+}\{\sigma_1^2E^2, \sigma_2^2V^2, \sigma_3^2C^2, \sigma_4^2P^2\}$ such that

$$\sum_{i,j=1}^{4} a_{ij}(E, V, C, P)\delta_i\delta_j = \sigma_1^2E^2\delta_1^2 + \delta_2^2V^2\delta_2^2 + \sigma_3^2C^2\delta_3^2 + \sigma_3^2P^2\delta_4^2 \geq M_0 \mid \delta \mid^2,$$

for any $(E, V, C, P) \in D$ and $\delta = (\delta_1, \delta_2, \delta_3, \delta_4) \in \mathbb{R}^4_+$. This means that the condition (T_1) in Lemma 3.1 also holds. According to Lemma 3.1, we can get that system (1.2) has a unique stationary distribution which is ergodic. The proof is complete. □

4. Probability density function

From the results in Theorem 3.1, we obtain that the stochastic solution follows a unique stationary distribution $\pi(\cdot)$ which has the ergodicity property. In this section, we try our best to derive the explicit expression of the density function with respect to $\pi(\cdot)$

First, let $u_1 = \ln E$, $u_2 = \ln V$, $u_3 = \ln C$, $u_4 = \ln P$, then by Itô's formula, system (1.2) is transformed into the following form:

$$
\begin{cases}
\mathrm{d}u_1 = \Big(re^{-u_1} + \dfrac{k_6e^{u_2}}{1 + e^{u_2}} - k_1e^{u_4} - m_1\Big)\mathrm{d}t + \sigma_1\mathrm{d}B_1(t), \\
\mathrm{d}u_2 = \Big(k_2e^{u_1+u_4-u_2} - m_2\Big)\mathrm{d}t + \sigma_2\mathrm{d}B_2(t), \\
\mathrm{d}u_3 = \Big(k_3e^{u_1+u_4-u_3} + k_4e^{u_2-u_3} - m_3\Big)\mathrm{d}t + \sigma_3\mathrm{d}B_3(t), \\
\mathrm{d}u_4 = \Big(k_5e^{u_3-u_4} - m_4\Big)\mathrm{d}t + \sigma_4\mathrm{d}B_4(t),
\end{cases}
\tag{4.1}
$$

where $m_i = d_i + \frac{\sigma_i^2}{2}$ $(i = 1, 2, 3, 4)$.

Let $R_0^p = \frac{rk_5(k_2k_4+m_2k_3)}{m_1m_2m_3m_4} = R_0^s > 1$, which is the same R_0 in (1.1), when $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0$. Then there is a unique quasi-endemic equilibrium $P^* = (E^*, V^*, C^*, P^*) = (e^{u_1^*}, e^{u_2^*}, e^{u_3^*}, e^{u_4^*})$ which is determined by the following equation:

$$\begin{cases} re^{-u_1^*} + \dfrac{k_6 e^{u_2^*}}{1+e^{u_2^*}} - k_1 e^{u_4^*} - m_1 = 0, \\ k_2 e^{u_1^*+u_4^*-u_2^*} - m_2 = 0, \\ k_3 e^{u_1^*+u_4^*-u_3^*} + k_4 e^{u_2^*-u_3^*} - m_3 = 0, \\ k_5 e^{u_3^*-u_4^*} - m_4 = 0. \end{cases} \tag{4.2}$$

Therefore, in view of (4.2), we can obtain that

$$V^* = \frac{rk_2k_6 - m_1m_2k_1R_0^s + rm_1k_2(R_0^s - 1) + \sqrt{\Delta_1}}{2m_1m_2k_1R_0^s},$$
$$\Delta_1 = [rk_2k_6 - m_1m_2k_1R_0^s + rm_1k_2(R_0^s - 1)]^2 + 4rm_1^2m_2k_1k_2R_0^s(R_0^s - 1),$$
$$E^* = \frac{m_2m_3m_4}{k_5(k_2k_4 + m_2k_3)}, C^* = \frac{k_2k_4 + m_2k_3}{m_3k_2}V^*, P^* = \frac{k_5}{m_4}C^* = \frac{k_5(k_2k_4 + m_2k_3)}{k_2m_3m_4}V^*.$$

Let $x_k = u_k - u_k^*$ $(k = 1, 2, 3, 4)$, we therefore derive the following linearized equation of system (4.1):

$$\begin{cases} \mathrm{d}x_1 = (-a_{11}x_1 + a_{12}x_2 - a_{14}x_4)\mathrm{d}t + \sigma_1 \mathrm{d}B_1(t), \\ \mathrm{d}x_2 = (a_{21}x_1 - a_{21}x_2 + a_{21}x_4)\mathrm{d}t + \sigma_2 \mathrm{d}B_2(t), \\ \mathrm{d}x_3 = (a_{31}x_1 + a_{32}x_2 - a_{33}x_3 + a_{31}x_4)\mathrm{d}t + \sigma_3 \mathrm{d}B_3(t), \\ \mathrm{d}x_4 = (a_{43}x_3 - a_{43}x_4)\mathrm{d}t + +\sigma_4 \mathrm{d}B_4(t), \end{cases} \tag{4.3}$$

where

$$a_{11} = re^{-u_1^*} > 0,\ a_{12} = \frac{k_6 e^{u_2^*}}{(1+e^{u_2^*})^2},\ a_{14} = k_1 e^{u_4^*} > 0, a_{21} = k_2 e^{u_1^*+u_4^*-u_2^*} > 0,$$
$$a_{31} = k_3 e^{u_1^*+u_4^*-u_3^*} > 0,\ a_{32} = k_4 e^{u_2^*-u_3^*} > 0, a_{33} = a_{31} + a_{32} > 0,\ a_{43} = k_5 e^{u_3^*-u_4^*} > 0.$$

Next, we give three lemmas to determine the positive definiteness or semi-positive definiteness of a four-dimensional matrix.

Lemma 4.1. *([16]) For the algebraic equation $\Lambda_0^2 + A_0\Sigma_0 + \Sigma_0 A_0^T = 0$, where $\Lambda_0 = diag(1, 0, 0, 0)$ and Σ_0 is a real symmetric matrix, the standard matrix is*

$$A_0 = \begin{pmatrix} -\tau_1 & -\tau_2 & -\tau_3 & -\tau_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

If $\tau_1 > 0,\ \tau_3 > 0,\ \tau_4 > 0$ and $\tau_1\tau_2\tau_3 - \tau_3^2 - \tau_1^2\tau_4 > 0$, then Σ_0 is a positive definite matrix, where

$$\Sigma_0 = \begin{pmatrix} \frac{\tau_2\tau_3-\tau_1\tau_4}{2(\tau_1\tau_2\tau_3-\tau_3^2-\tau_1^2\tau_4)} & 0 & -\frac{\tau_3}{2(\tau_1\tau_2\tau_3-\tau_3^2-\tau_1^2\tau_4)} & 0 \\ 0 & \frac{\tau_3}{2(\tau_1\tau_2\tau_3-\tau_3^2-\tau_1^2\tau_4)} & 0 & -\frac{\tau_1}{2(\tau_1\tau_2\tau_3-\tau_3^2-\tau_1^2\tau_4)} \\ -\frac{\tau_3}{2(\tau_1\tau_2\tau_3-\tau_3^2-\tau_1^2\tau_4)} & 0 & \frac{\tau_1}{2(\tau_1\tau_2\tau_3-\tau_3^2-\tau_1^2\tau_4)} & 0 \\ 0 & -\frac{\tau_1}{2(\tau_1\tau_2\tau_3-\tau_3^2-\tau_1^2\tau_4)} & 0 & \frac{\tau_1\tau_2-\tau_3}{2\tau_4(\tau_1\tau_2\tau_3-\tau_3^2-\tau_1^2\tau_4)} \end{pmatrix}.$$

Here A_0 in this form is called the standard R_1 matrix.

Lemma 4.2. *([16]) For the algebraic equation* $\Lambda_0^2 + B_0\Omega_0 + \Omega_0 B_0^T = 0$*, where* $\Lambda_0 = diag(1,0,0,0)$ *and* Ω_0 *is a real symmetric matrix, the standard matrix is*

$$B_0 = \begin{pmatrix} -\eta_1 & -\eta_2 & -\eta_3 & -\eta_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \eta_5 \end{pmatrix}.$$

If $\eta_1 > 0,\ \eta_3 > 0,\ \eta_1\eta_2 - \eta_3 > 0$*, then the matrix* Ω_0 *is a semi-positive definite matrix, which follows*

$$\Omega_0 = \begin{pmatrix} \frac{\eta_2}{2(\eta_1\eta_2-\eta_3)} & 0 & -\frac{1}{2(\eta_1\eta_2-\eta_3)} & 0 \\ 0 & \frac{1}{2(\eta_1\eta_2-\eta_3)} & 0 & 0 \\ -\frac{1}{2(\eta_1\eta_2-\eta_3)} & 0 & \frac{\eta_1}{2\eta_3(\eta_1\eta_2-\eta_3)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Lemma 4.3. *([16]) For the algebraic equation* $\Lambda_0^2 + C_0\Theta_0 + \Theta_0 C_0^T = 0$*, where* $\Lambda_0 = diag(1,0,0,0)$ *and* Θ_0 *is a real symmetric matrix, the standard matrix is*

$$C_0 = \begin{pmatrix} -\xi_1 & -\xi_2 & -\xi_3 & -\xi_4 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \xi_5 & \xi_6 \\ 0 & 0 & \xi_7 & \xi_8 \end{pmatrix}.$$

If $\xi_1 > 0,\ \xi_2 > 0$*, then the matrix* Θ_0 *is a semi-positive definite matrix, which follows*

$$\Theta_0 = \begin{pmatrix} \frac{1}{2\xi_1} & 0 & 0 & 0 \\ 0 & \frac{1}{2\xi_1\xi_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Theorem 4.1. *If* $R_0^s > 1$*, then the stationary solution* $(E(t), V(t), C(t), P(t))$ *of system (1.2) around* P^* *follows a unique log-normal probability density* $\Phi(E, V, C, P)$*, which takes the form:*

$$\Phi(E,V,C,P) = (2\pi)^{-2}|\Sigma|^{-\frac{1}{2}}(E\cdot V\cdot C\cdot P)^{-1}e^{-\frac{1}{2}(\ln\frac{E}{E^*},\ln\frac{V}{V^*},\ln\frac{C}{C^*},\ln\frac{P}{P^*})\Sigma^{-1}(\ln\frac{E}{E^*},\ln\frac{V}{V^*},\ln\frac{C}{C^*},\ln\frac{P}{P^*})^T},$$

where the covariance matrix $\Sigma = \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4$ *is a positive definite matrix,* $\Sigma_1, \Sigma_2, \Sigma_3,$ *and* Σ_4 *are defined as follows:*

$$\Sigma_1 = \begin{cases} (a_{21}w_1w_2\sigma_1)^2(J_3J_2J_1)^{-1}\Sigma_0[(J_3J_2J_1)^{-1}]^T, & \text{if } w_1 \neq 0,\ w_2 \neq 0, \\ (a_{21}w_1\sigma_1)^2(J_4J_2J_1)^{-1}\Omega_1[(J_4J_2J_1)^{-1}]^T, & \text{if } w_1 \neq 0,\ w_2 = 0, \\ (a_{31}a_{43}\sigma_1)^2(J_6J_5J_1)^{-1}\Omega_2[(J_6J_5J_1)^{-1}]^T, & \text{if } w_1 = 0, \end{cases}$$

$$\Sigma_2 = \begin{cases} [-a_{43}(a_{12}a_{31} + a_{14}a_{32})\sigma_2]^2(J_9J_8J_7)^{-1}\Sigma_0[(J_9J_8J_7)^{-1}]^T, & \text{if } w_3 = 0, \\ (a_{43}a_{32}\sigma_2)^2(J_{11}J_{10}J_8J_7)^{-1}\Omega_3[(J_{11}J_{10}J_8J_7)^{-1}]^T, & \text{if } w_3 \neq 0,\ w_4 = 0, \\ (a_{43}a_{32}w_4\sigma_2)^2(J_{12}J_{10}J_8J_7)^{-1}\Sigma_0[(J_{12}J_{10}J_8J_7)^{-1}]^T, & \text{if } w_3 \neq 0,\ w_4 \neq 0, \end{cases}$$

$$\Sigma_3 = \begin{cases} (-a_{14}a_{43}\sigma_3)^2(J_{15}J_{14}J_{13})^{-1}\Omega_4[(J_{15}J_{14}J_{13})^{-1}]^T, & \textit{if } w_5 = 0, \\ (a_{14}a_{43}w_5\sigma_3)^2(J_{16}J_{14}J_{13})^{-1}\Sigma_0[(J_{16}J_{14}J_{13})^{-1}]^T, & \textit{if } w_5 \neq 0, \end{cases}$$

$$\Sigma_4 = \begin{cases} (a_{31}\sigma_4)^2(J_{19}J_{18}J_{17})^{-1}\Theta_1[(J_{19}J_{18}J_{17})^{-1}]^T, & \textit{if } w_6 = w_7 = 0, \\ (a_{31}w_7\sigma_4)^2(J_{21}J_{20}J_{18}J_{17})^{-1}\Omega_5[(J_{21}J_{20}J_{18}J_{17})^{-1}]^T, & \textit{if } w_6 = 0, w_7 \neq 0, \\ [(a_{12}a_{31} + a_{14}a_{32})w_6\sigma_4]^2(J_{22}J_{18}J_{17})^{-1}\Sigma_0[(J_{22}J_{18}J_{17})^{-1}]^T, & \textit{if } w_6 \neq 0, w_7 = 0, \\ (a_{31}w_6\sigma_4)^2(J_{24}J_{23}J_{18}J_{17})^{-1}\Omega_6[(J_{24}J_{23}J_{18}J_{17})^{-1}]^T, & \textit{if } w_6 \neq 0, w_7 \neq 0, w_8 = 0, \\ (a_{31}w_6w_8\sigma_4)^2(J_{25}J_{23}J_{18}J_{17})^{-1}\Sigma_0[(J_{25}J_{23}J_{18}J_{17})^{-1}]^T, & \textit{if } w_6 \neq 0, w_7 \neq 0, w_8 \neq 0, \end{cases}$$

where

$$w_1 = a_{33} - \frac{a_{33}a_{31}}{a_{21}},\ w_2 = a_{43} + \frac{a_{31}a_{43}(a_{33} - a_{43})}{a_{21}w_1},\ w_3 = -\frac{a_{12}[a_{12}a_{31} + a_{32}(a_{11} - a_{33})]}{a_{32}^2},$$

$$w_4 = w_3 - (a_{14} + \frac{a_{12}a_{31}}{a_{32}}) - (a_{11} + \frac{a_{12}a_{31}}{a_{32}})\frac{w_2}{a_{43}}, w_5 = a_{21} + \frac{a_{21}(a_{21} - a_{11})}{a_{14}} - \frac{a_{12}a_{21}^2}{a_{14}^2},\ w_6 = \frac{a_{21}a_{33}(a_{31} - a_{21})}{a_{31}^2},$$

$$w_7 = \frac{a_{11}a_{14}a_{31} - a_{14}^2a_{31} + a_{12}a_{21}a_{31} + a_{14}a_{21}a_{32} - a_{14}a_{31}a_{33}}{a_{31}^2}, w_8 = a_{12} + \frac{a_{14}a_{32}}{a_{31}} + \frac{a_{21}a_{33}w_7}{a_{31}w_6} + (a_{14} - a_{11})\frac{w_7}{w_6},$$

and the matrices $J_1, \ldots J_{24}$, $\Omega_1, \ldots, \Omega_6$, Θ_1 *are defined in the following proof.*

Proof. Let $X = (x_1, x_2, x_3, x_4)^T$, $B(t) = (B_1(t), B_2(t), B_3(t), B_4(t))^T$, $\Lambda = diag(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ and

$$A = \begin{pmatrix} -a_{11} & a_{12} & 0 & -a_{14} \\ a_{21} & -a_{21} & 0 & a_{21} \\ a_{31} & a_{32} & -a_{33} & a_{31} \\ 0 & 0 & a_{43} & -a_{43} \end{pmatrix},$$

then, we can rewrite model (4.3) as:

$$\mathrm{d}X(t) = AX(t)\mathrm{d}t + \Lambda \mathrm{d}B(t).$$

In view of the theory of Gardiner et al. [17], the corresponding density function $\Phi(x_1, x_2, x_3, x_4)$ of the quasi-stationary distribution of model (4.3) can be described by the four-dimensional Fokker-Plank equation:

$$\begin{aligned} &-\sum_{i=1}^{4}\frac{\sigma_i^2}{2}\frac{\partial^2\Phi(z(t), t)}{\partial x_i^2} + \frac{\partial}{\partial x_1}[(-a_{11}x_1 - a_{12}x_2 - a_{14}x_4)\Phi] + \frac{\partial}{\partial x_2}[(a_{21}x_1 - a_{21}x_2 + a_{21}x_4)\Phi] \\ &+ \frac{\partial}{\partial x_3}[(a_{31}x_1 + a_{32}x_2 - a_{33}x_3 + a_{31}x_4)\Phi] + \frac{\partial}{\partial x_4}[(a_{43}x_3 - a_{43}x_4)\Phi] = 0. \end{aligned}$$

By the relevant results of Roozen [18], $\Phi(x_1, x_2, x_3, x_4)$ can be derived as

$$\Phi(x_1, x_2, x_3, x_4) = q_0 e^{-\frac{1}{2}(x_1, x_2, x_3, x_4)Q(x_1, x_2, x_3, x_4)^T},$$

where q_0 is a constant which is determined by $\int_{\mathbb{R}^4} \Phi(x_1, x_2, x_3, x_4)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4 = 1$. The real symmetry matrix Q satisfies the following algebraic equation:

$$Q\Lambda^2 Q + A^T Q + QA = 0.$$

If Q is positive definite, then it is invertible. We denote $Q^{-1} = \Sigma$, then we equivalently obtain

$$\Lambda^2 + A\Sigma + \Sigma A^T = 0. \tag{4.4}$$

In addition, we can calculate that the corresponding constants

$$q_0 = (2\pi)^{-2}|\Sigma|^{-\frac{1}{2}}.$$

By the finite independent superposition principle [17], Eq (4.4) is equivalent to the sum of the following four algebraic sub-equations:

$$\Lambda_i^2 + A\Sigma_i + \Sigma_i A^T = 0, \;\; i = 1, 2, 3, 4,$$

where

$$\begin{aligned} &\Lambda_1 = diag(\sigma_1, 0, 0, 0), \;\; \Lambda_2 = diag(0, \sigma_2, 0, 0), \\ &\Lambda_3 = diag(0, 0, \sigma_3, 0), \;\; \Lambda_4 = diag(0, 0, 0, \sigma_4), \\ &\Sigma = \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4, \;\; \Lambda^2 = \Lambda_1^2 + \Lambda_2^2 + \Lambda_3^2 + \Lambda_4^2. \end{aligned}$$

In order to obtain the positive definiteness of Σ, we define the characteristic equation of matrix A as

$$\varphi_A(\lambda) = \lambda^4 + \tau_1\lambda^3 + \tau_2\lambda^2 + \tau_3\lambda + \tau_4 = 0, \tag{4.5}$$

where

$$\begin{aligned} &\tau_1 = a_{11} + a_{21} + a_{33} + a_{43}, \tau_2 = a_{21}(a_{11} - a_{12}) + a_{33}(a_{11} + a_{21}) + a_{43}(a_{11} + a_{21} + a_{32}), \\ &\tau_3 = a_{21}a_{33}(a_{11} - a_{12}) + a_{43}[a_{21}(a_{11} - a_{12}) + a_{11}a_{32} + a_{14}a_{31}], \tau_4 = (a_{14} - a_{12})a_{21}a_{33}a_{43}. \end{aligned}$$

According to the expressions of E^*, V^*, C^*, P^* and $R_0^s > 1$, we can prove that $\tau_1 > 0, \tau_3 > 0, \tau_4 > 0, \tau_1\tau_2 > \tau_3, \tau_1(\tau_2\tau_3 - \tau_1\tau_4) - \tau_3^2 > 0$, which implies that all the roots of characteristic equation (4.5) have negative real-parts and so the matrix A is a Hurwitz matrix. In view of the related theory of matrix's similar transformation, we known that the corresponding characteristic equations are still similarity invariant after a series of elementary transformations on A, and there exists a unique standard R_1 matrix of A.

Now, we prove the positive definiteness of Σ by solving Eq (4.4).

Step 1. We first consider the algebraic equation

$$\Lambda_1^2 + A\Sigma_1 + \Sigma_1 A^T = 0. \tag{4.6}$$

Let $A_1 = J_1 A J_1^{-1}$, where the elimination matrix J_1 is given by

$$J_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{a_{31}}{a_{21}} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Direct calculation leads to that

$$A_1 = \begin{pmatrix} -a_{11} & a_{12} & 0 & -a_{14} \\ a_{21} & -a_{21} & 0 & a_{21} \\ 0 & w_1 & -a_{33} & 0 \\ 0 & \frac{a_{31}a_{43}}{a_{21}} & a_{43} & -a_{43} \end{pmatrix},$$

where $w_1 = a_{33} - \frac{a_{33}a_{31}}{a_{21}}$.

Subcase (1.1) If $w_1 \neq 0$. Let $A_2 = J_2 A_1 J_2^{-1}$, where the elimination J_2 is given by

$$J_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{a_{31}a_{43}}{w_1 a_{21}} & 1 \end{pmatrix}.$$

Then

$$A_2 = \begin{pmatrix} -a_{11} & a_{12} & -\frac{a_{14}a_{31}a_{43}}{a_{21}w_1} & -a_{14} \\ a_{21} & -a_{21} & \frac{a_{31}a_{43}}{w_1} & a_{21} \\ 0 & w_1 & -a_{33} & 0 \\ 0 & 0 & w_2 & -a_{43} \end{pmatrix},$$

where $w_2 = a_{43} + \frac{a_{31}a_{43}(a_{33}-a_{43})}{a_{21}w_1}$.

Subcase (1.1.1) If $w_1 \neq 0$, $w_2 \neq 0$. Let $P_1 = J_3 A_2 J_3^{-1}$, where the standardized transformation matrix J_3 takes the form

$$J_3 = \begin{pmatrix} a_{21}w_1w_2 & -w_1w_2(a_{21}+a_{33}+a_{43}) & w_2[a_{33}^2 + a_{33}a_{43} + a_{43}(a_{31}+a_{43})] & -a_{43}^3 + a_{21}w_1w_2 \\ 0 & w_1w_2 & -w_2(a_{33}+a_{43}) & a_{43}^2 \\ 0 & 0 & w_2 & -a_{33} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By direct computation we have

$$P_1 = \begin{pmatrix} -\tau_1 & -\tau_2 & -\tau_3 & -\tau_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

where τ_1, τ_2, τ_3 and τ_4 are the same as above. Then we can transform Eq (4.6) into

$$(J_3J_2J_1)\Lambda_1^2(J_3J_2J_1)^T + P_1[(J_3J_2J_1)\Sigma_1(J_3J_2J_1)^T] + [(J_3J_2J_1)\Sigma_1(J_3J_2J_1)^T]P_1^T = 0,$$

$$\text{i.e.,}\ \ \Lambda_0^2 + P_1\Sigma_0 + \Sigma_0 P_1^T = 0,$$

where $\Sigma_0 = \varrho_1^{-2}(J_3J_2J_1)\Sigma_1(J_3J_2J_1)^T$, $\varrho_1 = a_{21}w_1w_2\sigma_1$. According to Lemma 4.1, we can obtain that the matrix Σ_0 is positive definite whose explicite form is

$$\Sigma_0 = \begin{pmatrix} \frac{\tau_2\tau_3-\tau_1\tau_4}{2(\tau_1\tau_2\tau_3-\tau_3^2-\tau_1^2\tau_4)} & 0 & -\frac{\tau_3}{2(\tau_1\tau_2\tau_3-\tau_3^2-\tau_1^2\tau_4)} & 0 \\ 0 & \frac{\tau_3}{2(\tau_1\tau_2\tau_3-\tau_3^2-\tau_1^2\tau_4)} & 0 & -\frac{\tau_1}{2(\tau_1\tau_2\tau_3-\tau_3^2-\tau_1^2\tau_4)} \\ -\frac{\tau_3}{2(\tau_1\tau_2\tau_3-\tau_3^2-\tau_1^2\tau_4)} & 0 & \frac{\tau_1}{2(\tau_1\tau_2\tau_3-\tau_3^2-\tau_1^2\tau_4)} & 0 \\ 0 & -\frac{\tau_1}{2(\tau_1\tau_2\tau_3-\tau_3^2-\tau_1^2\tau_4)} & 0 & \frac{\tau_1\tau_2-\tau_3}{2\tau_4(\tau_1\tau_2\tau_3-\tau_3^2-\tau_1^2\tau_4)} \end{pmatrix}.$$

Therefore the matrix $\Sigma_1 = \varrho^2(J_3J_2J_1)^{-1}\Sigma_0[(J_3J_2J_1)^{-1}]^T$ is also positive definite.

Subcase (1.1.2) If $w_1 \neq 0$, $w_2 = 0$. Let $P_2 = J_4A_2J_4^{-1}$, where the standardized transformation matrix J_4 takes the form

$$J_4 = \begin{pmatrix} a_{21}w_1 & -w_1(a_{21}+a_{33}) & a_{33}^2 + a_{31}a_{43} & a_{21}w_1 \\ 0 & w_1 & -a_{33} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then we obtain

$$P_2 = \begin{pmatrix} -\eta_{11} & -\eta_{12} & -\eta_{13} & -\eta_{14} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -a_{43} \end{pmatrix},$$

where $\eta_{11} = a_{33} - a_{21}(-1+w_1)$, $\eta_{12} = a_{21}a_{33} - a_{31}a_{43}$, $\eta_{13} = 0$, $\eta_{14} = a_{21}a_{43}w_1$. Therefore, we have

$$(J_4J_2J_1)\Lambda_1^2(J_4J_2J_1)^T + P_2[(J_4J_2J_1)\Sigma_1(J_4J_2J_1)^T] + [(J_4J_2J_1)\Sigma_1(J_4J_2J_1)^T]P_2^T = 0,$$

$$\text{i.e.,}\quad \Lambda_0^2 + P_2\Omega_1 + \Omega_1P_2^T = 0,$$

where $\Omega_1 = \varrho_2^{-2}(J_4J_2J_1)\Sigma_1(J_4J_2J_1)^T$, $\varrho_2 = a_{21}w_1\sigma_1$. By Lemma 4.2, we obtain the semi-positive definite matrix Ω_1 with explicit form as

$$\Omega_1 = \begin{pmatrix} \frac{\eta_{12}}{2(\eta_{11}\eta_{12}-\eta_{13})} & 0 & -\frac{1}{2(\eta_{11}\eta_{12}-\eta_{13})} & 0 \\ 0 & \frac{1}{2(\eta_{11}\eta_{12}-\eta_{13})} & 0 & 0 \\ -\frac{1}{2(\eta_{11}\eta_{12}-\eta_{13})} & 0 & \frac{\eta_{11}}{2\eta_{13}(\eta_{11}\eta_{12}-\eta_{13})} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, the matrix $\Sigma_1 = \varrho_2^2(J_4J_2J_1)^{-1}\Omega_1[(J_4J_2J_1)^{-1}]^T$ is also semi-positive definite. And a constant ν_1 exists satisfying

$$\Sigma_1 \geq \nu_1 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Subcase (1.2) If $w_1 = 0$. Let $A_3 = J_5A_1J_5^{-1}$, where

$$J_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -a_{11} & a_{12} & -a_{14} & 0 \\ a_{21} & -a_{21} & a_{21} & 0 \\ 0 & \frac{a_{31}a_{43}}{a_{21}} & -a_{43} & a_{43} \\ 0 & 0 & 0 & -a_{33} \end{pmatrix}.$$

Let $P_3 = J_6A_3J_6^{-1}$, where the standardized transformation matrix J_6 is given by

$$J_6 = \begin{pmatrix} a_{31}a_{43} & -a_{31}a_{43} - \frac{a_{31}a_{43}^2}{a_{21}} & a_{31}a_{43} + a_{43}^2 & -a_{33}a_{43} - a_{43}^2 \\ 0 & \frac{a_{31}a_{43}}{a_{21}} & -a_{43} & a_{43} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By simple computation, we obtain

$$P_3=\begin{pmatrix}-\eta_{21} & -\eta_{22} & -\eta_{23} & -\eta_{24}\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & -a_{33}\end{pmatrix},$$

where

$$\eta_{21}=a_{11}+a_{21}+a_{43},\eta_{22}=-a_{12}a_{21}-(-a_{21}+a_{31})a_{43}+a_{11}(a_{21}+a_{43}),$$
$$\eta_{23}=-a_{43}[a_{12}a_{21}-a_{14}a_{31}+a_{11}(-a_{21}+a_{31})],\eta_{24}=-a_{43}[(a_{11}-a_{12})a_{21}-(a_{11}+a_{21})a_{33}+a_{33}^2].$$

So Eq (4.6) can be equivalently transformed into the following form:

$$(J_6J_5J_1)\Lambda_1^2(J_6J_5J_1)^T+P_3[(J_6J_5J_1)\Sigma_1(J_6J_5J_1)^T]+[(J_6J_5J_1)\Sigma_1(J_6J_5J_1)^T]P_3^T=0,$$
$$\text{i.e.,}\quad \Lambda_0^2+P_3\Omega_2+\Omega_2P_3^T=0,$$

where $\Omega_2=\varrho_3^{-2}(J_6J_5J_1)\Sigma_1(J_6J_5J_1)^T$, $\varrho_3=a_{31}a_{43}\sigma_1$. By Lemma 4.2, we obtain that the matrix Ω_2 is semi-positive definite whose explicit form is

$$\Omega_2=\begin{pmatrix}\frac{\eta_{22}}{2(\eta_{21}\eta_{22}-\eta_{23})} & 0 & -\frac{1}{2(\eta_{21}\eta_{22}-\eta_{23})} & 0\\ 0 & \frac{1}{2(\eta_{21}\eta_{22}-\eta_{23})} & 0 & 0\\ -\frac{1}{2(\eta_{21}\eta_{22}-\eta_{23})} & 0 & \frac{\eta_{21}}{2\eta_{23}(\eta_{21}\eta_{22}-\eta_{23})} & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

Hence, the matrix $\Sigma_1=\varrho_3^2(J_6J_5J_1)^{-1}\Omega_2[(J_6J_5J_1)^{-1}]^T$ is also semi-positive definite. Also, a constant ν_2 exists satisfying

$$\Sigma_1\geq\nu_2\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}.$$

Step 2. Consider the algebraic equation

$$\Lambda_2^2+A\Sigma_2+\Sigma_2A^T=0. \tag{4.7}$$

Let $A_4=J_7AJ_7^{-1}$, where

$$J_7=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{pmatrix},\quad A_4=\begin{pmatrix}-a_{21}&0&a_{21}&a_{21}\\a_{32}&-a_{33}&a_{31}&a_{31}\\0&a_{43}&-a_{43}&0\\a_{12}&0&-a_{14}&-a_{11}\end{pmatrix}.$$

Let $A_5=J_8A_4J_8^{-1}$, where the elimination J_8 takes the form

$$J_8=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&-\frac{a_{12}}{a_{32}}&0&1\end{pmatrix}.$$

Direct calculation leads to that

$$A_5 = \begin{pmatrix} -a_{21} & \frac{a_{12}a_{21}}{a_{32}} & a_{21} & a_{21} \\ a_{32} & \frac{a_{12}a_{31}}{a_{32}} - a_{33} & a_{31} & a_{31} \\ 0 & a_{43} & -a_{43} & 0 \\ 0 & w_3 & -a_{14} - \frac{a_{12}a_{31}}{a_{32}} & -a_{11} - \frac{a_{12}a_{31}}{a_{32}} \end{pmatrix},$$

where $w_3 = -\frac{a_{12}[a_{12}a_{31}+a_{32}(a_{11}-a_{33})]}{a_{32}^2}$.

Subcase (2.1) If $w_3 = 0$. Let $P_4 = J_9 A_5 J_9^{-1}$, where the standardized transformation matrix J_9 is given by

$$J_9 = \begin{pmatrix} -a_{43}(a_{12}a_{31} + a_{14}a_{32}) & j_9(1) & j_9(2) & j_9(3) \\ 0 & -\frac{a_{43}(a_{12}a_{31}+a_{14}a_{32})}{a_{32}} & \frac{(a_{12}a_{31}+a_{14}a_{32})[a_{12}a_{31}+a_{32}(a_{11}+a_{43})]}{a_{32}^2} & \frac{(a_{12}a_{31}+a_{11}a_{32})^2}{a_{32}^2} \\ 0 & 0 & -a_{14} - \frac{a_{12}a_{31}}{a_{32}} & -a_{11} - \frac{a_{12}a_{31}}{a_{32}} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$j_9(1) = \frac{(a_{12}a_{31} + a_{14}a_{32})a_{43}(a_{11} + a_{33} + a_{43})}{a_{32}},$$

$$j_9(2) = -\frac{(a_{12}a_{31} + a_{14}a_{32})\{(a_{12}a_{31} + a_{11}a_{32})^2 + a_{32}a_{43}[a_{12}a_{31} + (a_{11} + a_{31})a_{32}] + a_{32}^2 a_{43}^2\}}{a_{32}^3},$$

$$j_9(3) = -\frac{(a_{12}a_{31} + a_{11}a_{32})^3 + a_{31}a_{32}^2(a_{12}a_{31} + a_{14}a_{32})a_{43}}{a_{32}^3}.$$

By direct computation we have

$$P_4 = \begin{pmatrix} -\tau_1 & -\tau_2 & -\tau_3 & -\tau_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then, Eq (4.7) can be equivalently transformed into the following form:

$$(J_9J_8J_7)\Lambda_2^2(J_9J_8J_7)^T + P_4[(J_9J_8J_7)\Sigma_2(J_9J_8J_7)^T] + [(J_9J_8J_7)\Sigma_2(J_9J_8J_7)^T]P_4^T = 0,$$

$$\text{that is,} \quad \Lambda_2^2 + P_4\Sigma_0 + \Sigma_0 P_4^T = 0.$$

According to Lemma 4.1, this means that the matrix Σ_0 is positive definite which takes the form

$$\Sigma_0 = \begin{pmatrix} \frac{\tau_2\tau_3-\tau_1\tau_4}{2(\tau_1\tau_2\tau_3-\tau_3^2-\tau_1^2\tau_4)} & 0 & -\frac{\tau_3}{2(\tau_1\tau_2\tau_3-\tau_3^2-\tau_1^2\tau_4)} & 0 \\ 0 & \frac{\tau_3}{2(\tau_1\tau_2\tau_3-\tau_3^2-\tau_1^2\tau_4)} & 0 & -\frac{\tau_1}{2(\tau_1\tau_2\tau_3-\tau_3^2-\tau_1^2\tau_4)} \\ -\frac{\tau_3}{2(\tau_1\tau_2\tau_3-\tau_3^2-\tau_1^2\tau_4)} & 0 & \frac{\tau_1}{2(\tau_1\tau_2\tau_3-\tau_3^2-\tau_1^2\tau_4)} & 0 \\ 0 & -\frac{\tau_1}{2(\tau_1\tau_2\tau_3-\tau_3^2-\tau_1^2\tau_4)} & 0 & \frac{\tau_1\tau_2-\tau_3}{2\tau_4(\tau_1\tau_2\tau_3-\tau_3^2-\tau_1^2\tau_4)} \end{pmatrix},$$

where

$$\Sigma_0 = \varrho_4^{-2}(J_9J_8J_7)\Sigma_2(J_9J_8J_7)^T,\ \varrho_4 = -a_{43}(a_{12}a_{31} + a_{14}a_{32})\sigma_2.$$

Thus, the matrix

$$\Sigma_2 = \varrho_4^2 (J_9 J_8 J_7)^{-1} \Sigma_0 [(J_9 J_8 J_7)^{-1}]^T$$

is also positive definite.

Subcase (2.2) If $w_3 \neq 0$. Let $A_6 = J_{10} A_5 J_{10}^{-1}$, where

$$J_{10} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{w_3}{a_{43}} & 1 \end{pmatrix}, A_6 = \begin{pmatrix} -a_{21} & \frac{a_{12}a_{21}}{a_{32}} & a_{21} + \frac{a_{21}w_3}{a_{43}} & a_{21} \\ a_{32} & \frac{a_{12}a_{31}}{a_{32}} - a_{33} & a_{31} + \frac{a_{31}w_3}{a_{43}} & a_{31} \\ 0 & a_{43} & -a_{43} & 0 \\ 0 & 0 & w_4 & -a_{11} - \frac{a_{12}a_{31}}{a_{32}} \end{pmatrix},$$

and $w_4 = w_3 - (a_{14} + \frac{a_{12}a_{31}}{a_{32}}) - (a_{11} + \frac{a_{12}a_{31}}{a_{32}})\frac{w_3}{a_{43}}$.

Subcase (2.2.1) If $w_3 \neq 0, w_4 = 0$. Let $P_5 = J_{11} A_6 J_{11}^{-1}$, where the standardized transformation matrix J_{11} is given by

$$J_{11} = \begin{pmatrix} a_{43}a_{32} & a_{43}(\frac{a_{12}a_{31}}{a_{32}} - a_{33}) - a_{43}^2 & a_{43}^2 + a_{31}(a_{43} + w_3) & a_{31}a_{43} \\ 0 & a_{43} & -a_{43} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By simple computation, we obtain

$$P_5 = \begin{pmatrix} -\eta_{31} & -\eta_{32} & -\eta_{33} & -\eta_{34} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -a_{11} - \frac{a_{12}a_{31}}{a_{32}} \end{pmatrix},$$

where

$$\begin{aligned} \eta_{31} &= -\frac{a_{12}a_{31} - a_{32}(a_{21} + a_{33} + a_{43})}{a_{32}}, \\ \eta_{32} &= -a_{31}a_{43} + a_{33}a_{43} + a_{21}(a_{33} + a_{43}) - \frac{a_{12}[a_{21}(a_{31} + a_{32}) + a_{31}a_{43}]}{a_{32}} - a_{31}w_3, \\ \eta_{33} &= -a_{21}[\frac{a_{12}a_{43}(a_{31} + a_{32})}{a_{32}} + (a_{31} + a_{32} - a_{33})a_{43} + (a_{31} + a_{32})w_3], \\ \eta_{34} &= -a_{43}[-a_{11}a_{31} - \frac{a_{12}a_{31}^2}{a_{32}} + a_{21}(a_{31} + a_{32})]. \end{aligned}$$

So Eq (4.7) can be equivalently transformed into the following form:

$$(J_{11}J_{10}J_8J_7)\Lambda_2^2(J_{11}J_{10}J_8J_7)^T + P_5[(J_{11}J_{10}J_8J_7)\Sigma_2(J_{11}J_{10}J_8J_7)^T] + [(J_{11}J_{10}J_8J_7)\Sigma_2(J_{11}J_{10}J_8J_7)^T]P_5^T = 0,$$

$$\text{i.e.,}\quad \Lambda_0^2 + P_5\Omega_3 + \Omega_3 P_5^T = 0,$$

where $\Omega_3 = \varrho_5^{-2}(J_{11}J_{10}J_8J_7)\Sigma_2(J_{11}J_{10}J_8J_7)^T, \varrho_5 = a_{43}a_{32}\sigma_2$. By Lemma 4.2, we obtain that the matrix Ω_3 is semi-positive definite whose explicit form is

$$\Omega_3 = \begin{pmatrix} \frac{\eta_{32}}{2(\eta_{31}\eta_{32} - \eta_{33})} & 0 & -\frac{1}{2(\eta_{31}\eta_{32} - \eta_{33})} & 0 \\ 0 & \frac{1}{2(\eta_{31}\eta_{32} - \eta_{33})} & 0 & 0 \\ -\frac{1}{2(\eta_{31}\eta_{32} - \eta_{33})} & 0 & \frac{\eta_{31}}{2\eta_{33}(\eta_{31}\eta_{32} - \eta_{33})} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, the matrix $\Sigma_2 = \varrho_5^2(J_{11}J_{10}J_8J_7)^{-1}\Omega_3[(J_{11}J_{10}J_8J_7)^{-1}]^T$ is also semi-positive definite. Also, a constant v_3 exists satisfying

$$\Sigma_2 \geq v_3 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Subcase (2.2.2) If $w_3 \neq 0, w_4 \neq 0$. Let $P_6 = J_{12}A_6J_{12}^{-1}$, where elimination matrix J_{12} is given by

$$J_{12} = \begin{pmatrix} a_{32}a_{43}w_4 & -a_{43}w_4(a_{11}+a_{33}+a_{43}) & j_{12}(1) & j_{12}(2) \\ 0 & a_{43}w_4 & -\frac{w_4[a_{12}a_{31}+a_{32}(a_{11}+a_{43})]}{a_{32}} & \frac{(a_{12}a_{31}+a_{11}a_{32})^2}{a_{32}^2} \\ 0 & 0 & w_4 & -a_{11}-\frac{a_{12}a_{31}}{a_{32}} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where

$$j_{12}(1) = \frac{w_4\{a_{12}^2a_{31}^2 + a_{12}a_{31}a_{32}(2a_{11}+a_{43}) + a_{32}^2[a_{11}^2 + a_{11}a_{43} + a_{43}(a_{31}+a_{43}) + a_{31}w_3]\}}{a_{32}^2},$$

$$j_{12}(2) = -\frac{(a_{12}a_{31}+a_{11}a_{32})^3}{a_{32}^3} + a_{31}a_{43}w_4.$$

By direct computation we have

$$P_6 = \begin{pmatrix} -\tau_1 & -\tau_2 & -\tau_3 & -\tau_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then, Eq (4.7) can be equivalently transformed into the following form:

$$\Lambda_0^2 + P_6\Sigma_0 + \Sigma_0 P_6^T = 0,$$

where $\Sigma_2 = \varrho_6^2(J_{12}J_{10}J_8J_7)^{-1}\Sigma_0[(J_{12}J_{10}J_8J_7)^{-1}]^T$, $\varrho_6 = a_{32}a_{43}w_4\sigma_2$, and Σ_2 is positive definite by Lemma 4.1.

Step 3. Consider the algebraic equation

$$\Lambda_3^2 + A\Sigma_3 + \Sigma_3A^T = 0. \tag{4.8}$$

Let $A_7 = J_{13}AJ_{13}^{-1}$, where

$$J_{13} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, A_7 = \begin{pmatrix} -a_{33} & a_{31} & a_{31} & a_{32} \\ a_{43} & -a_{43} & 0 & 0 \\ 0 & -a_{14} & -a_{11} & a_{12} \\ 0 & a_{21} & a_{21} & -a_{21} \end{pmatrix}.$$

Let $A_8 = J_{14}A_7J_{14}^{-1}$, where the elimination J_{14} is given by

$$J_{14} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{a_{21}}{a_{14}} & 1 \end{pmatrix}.$$

Direct calculation leads to that

$$A_8=\begin{pmatrix}-a_{33} & a_{31} & a_{31}-\frac{a_{32}a_{21}}{a_{14}} & a_{32}\\ a_{43} & -a_{43} & 0 & 0\\ 0 & -a_{14} & -a_{11}-\frac{a_{12}a_{21}}{a_{14}} & a_{12}\\ 0 & 0 & w_5 & -a_{21}+\frac{a_{12}a_{21}}{a_{14}}\end{pmatrix},$$

where $w_5=a_{21}+\frac{a_{21}(a_{21}-a_{11})}{a_{14}}-\frac{a_{12}a_{21}^2}{a_{14}^2}$.

Subcase (3.1) If $w_5=0$. Let $P_7=J_{15}A_8J_{15}^{-1}$, where

$$J_{15}=\begin{pmatrix}-a_{14}a_{43} & a_{12}a_{21}+a_{14}(a_{11}+a_{43}) & (\frac{a_{11}a_{14}+a_{12}a_{21}}{a_{14}})^2 & -a_{12}(a_{11}+a_{21})\\ 0 & -a_{14} & -a_{11}-\frac{a_{12}a_{21}}{a_{14}} & a_{12}\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix},$$

$$P_7=\begin{pmatrix}-\eta_{41} & -\eta_{42} & -\eta_{43} & -\eta_{44}\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & -a_{21}+\frac{a_{12}a_{21}}{a_{14}}\end{pmatrix},$$

with

$$\begin{aligned}\eta_{41}&=\frac{a_{12}a_{21}+a_{14}(a_{11}+a_{33}+a_{43})}{a_{14}},\\ \eta_{42}&=-(a_{31}-a_{33})a_{43}+a_{11}(a_{33}+a_{43})+\frac{a_{12}a_{21}(a_{33}+a_{43})}{a_{14}},\\ \eta_{43}&=-a_{43}[-a_{14}a_{31}+a_{21}a_{32}+a_{11}(a_{31}-a_{33})+\frac{a_{12}a_{21}(a_{31}-a_{33})}{a_{14}}],\\ \eta_{44}&=-\frac{a_{12}^3a_{21}^2}{a_{14}^2}-a_{12}a_{21}(a_{21}-a_{33})+a_{14}a_{32}a_{43}+a_{12}(a_{21}+a_{31}-a_{33})a_{43}-\frac{a_{12}^2a_{21}(-2a_{21}+a_{33}+a_{43})}{a_{14}}.\end{aligned}$$

In the meanwhile, Eq (4.8) can be equivalent transformed into

$$\Lambda_0^2+P_7\Omega_4+\Omega_4P_7^T=0,$$

where $\Omega_4=\varrho_7^{-2}(J_{15}J_{14}J_{13})\Sigma_3(J_{15}J_{14}J_{13})^T$, $\varrho_7=-a_{14}a_{43}\sigma_3$. From Lemma 4.2 we have that Ω_4 is semi-positive definite and

$$\Omega_4=\begin{pmatrix}\frac{\eta_{42}}{2(\eta_{41}\eta_{42}-\eta_{43})} & 0 & -\frac{1}{2(\eta_{41}\eta_{42}-\eta_{43})} & 0\\ 0 & \frac{1}{2(\eta_{41}\eta_{42}-\eta_{43})} & 0 & 0\\ -\frac{1}{2(\eta_{41}\eta_{42}-\eta_{43})} & 0 & \frac{\eta_{41}}{2\eta_{43}(\eta_{41}\eta_{42}-\eta_{43})} & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

Thus, the matrix $\Sigma_3=\varrho_7^2(J_{15}J_{14}J_{13})^{-1}\Omega_4[(J_{15}J_{14}J_{13})^{-1}]^T$ is also semi-positive definite. Also, a constant v_4 exists satisfying

$$\Sigma_3\geq v_4\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}.$$

Subcase (3.2) If $w_5 \neq 0$. Let $P_8 = J_{16}A_8J_{16}^{-1}$, where J_{16} is given by

$$J_{16} = \begin{pmatrix} -a_{14}a_{43}w_5 & J_{16}(1) & J_{16}(2) & J_{16}(3) \\ 0 & -a_{14}w_5 & -(a_{11}+a_{21})w_5 & (-1+\frac{a_{12}}{a_{14}})^2a_{21}^2 + a_{12}w_5 \\ 0 & 0 & w_5 & -a_{21} + \frac{a_{12}a_{21}}{a_{14}} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$P_8 = \begin{pmatrix} -\tau_1 & -\tau_2 & -\tau_3 & -\tau_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

with

$$J_{16}(1) = w_5[a_{12}a_{21} - a_{21}a_{12} + a_{14}(a_{11} + a_{21} + a_{43})],$$
$$J_{16}(2) = \frac{1}{a_{14}^2}w_5\{(a_{11}a_{14} + a_{12}a_{21})[a_{11}a_{14} + (a_{12} + a_{14})a_{21} - a_{21}a_{12}] + (-a_{14}a_{21} + a_{21}a_{12})^2 + a_{12}a_{14}^2w_5\},$$
$$J_{16}(3) = -\frac{1}{a_{14}^3}[(a_{14}a_{21} - a_{21}a_{12})^3 + a_{12}a_{14}^2w_5(a_{11}a_{14} + a_{12}a_{21} + 2a_{14}a_{21} - 2a_{21}a_{12})].$$

Similarity, we transformed Eq (4.8) into the following equation:

$$\Lambda_0^2 + P_8\Sigma_0 + \Sigma_0P_8^T = 0,$$

where $\Sigma_0 = \varrho_8^{-2}(J_{16}J_{14}J_{13})\Sigma_3[(J_{16}J_{14}J_{13})]^T$ and $\varrho_8 = a_{14}a_{43}w_5\sigma_3$. From Lemma 4.1 we can obtain that Σ_0 is definite and so the matrix $\Sigma_3 = \varrho_8^2(J_{16}J_{14}J_{13})^{-1}\Sigma_0[(J_{16}J_{14}J_{13})^{-1}]^T$ is also positive definite. Also, a positive constant v_5 exists satisfying

$$\Sigma_3 \geq v_5\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Step 4. Consider the algebraic equation

$$\Lambda_4^2 + A\Sigma_4 + \Sigma_4A^T = 0. \tag{4.9}$$

Let $A_9 = J_{17}AJ_{17}^{-1}$, where

$$J_{17} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad A_9 = \begin{pmatrix} -a_{43} & a_{43} & 0 & 0 \\ a_{31} & -a_{33} & a_{32} & a_{31} \\ a_{21} & 0 & -a_{21} & a_{21} \\ -a_{14} & 0 & a_{12} & -a_{11} \end{pmatrix}.$$

Let $A_{10} = J_{18}A_9J_{18}^{-1}$, where the elimination J_{18} is given by

$$J_{18} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{a_{21}}{a_{31}} & 1 & 0 \\ 0 & \frac{a_{14}}{a_{31}} & 0 & 1 \end{pmatrix}.$$

Direct calculation leads to that

$$A_{10} = \begin{pmatrix} -a_{43} & a_{43} & 0 & 0 \\ a_{31} & -a_{33} + \frac{a_{21}a_{32}}{a_{31}} - a_{14} & a_{32} & a_{31} \\ 0 & w_6 & -\frac{a_{21}a_{33}}{a_{31}} & 0 \\ 0 & w_7 & a_{12} + \frac{a_{14}a_{32}}{a_{31}} & -a_{11} + a_{14} \end{pmatrix},$$

where

$$w_6 = \frac{a_{21}a_{33}(a_{31} - a_{21})}{a_{31}^2},\ w_7 = \frac{a_{11}a_{14}a_{31} - a_{14}^2 a_{31} + a_{12}a_{21}a_{31} + a_{14}a_{21}a_{32} - a_{14}a_{31}a_{33}}{a_{31}^2}.$$

Subcase (4.1) If $w_6 = 0, w_7 = 0$. Let $P_9 = J_{19}A_{10}J_{19}^{-1}$, where elimination matrix J_{19} is given by

$$J_{19} = \begin{pmatrix} a_{31} & -a_{33} + \frac{a_{21}a_{32}}{a_{31}} - a_{14} & a_{32} & a_{31} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$P_9 = \begin{pmatrix} -\xi_{11} & -\xi_{12} & -\xi_{13} & -\xi_{14} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{a_{21}a_{33}}{a_{31}} & 0 \\ 0 & 0 & a_{12} + \frac{a_{14}a_{32}}{a_{31}} & -a_{11} + a_{44} \end{pmatrix},$$

with

$$\begin{aligned} \xi_{11} &= -\frac{a_{21}a_{32} - a_{31}(a_{14} + a_{33} + a_{43})}{a_{31}}, \\ \xi_{12} &= -a_{43}[-a_{14} + a_{31} + \frac{a_{21}a_{32}}{a_{31}} - a_{33}], \\ \xi_{13} &= -a_{12}a_{31} - a_{32}(a_{14} - \frac{a_{21}a_{33}}{a_{31}} + a_{43}), \\ \xi_{14} &= -a_{31}(-a_{11} + a_{14} + a_{43}). \end{aligned}$$

Then, Eq (4.9) can be equivalently transformed into the following form:

$$\Lambda_0^2 + P_9\Theta_1 + \Theta_1 P_9^T = 0,$$

where

$$\Theta_1 = \varrho_9^{-2}(J_{19}J_{18}J_{17})\Sigma_4(J_{19}J_{18}J_{17})^T, \varrho_9 = a_{31}\sigma_4.$$

In view of Lemma 4.3 we get that Θ_1 is semi-positive definite whose explicit form is as follows:

$$\Theta_1 = \begin{pmatrix} \frac{1}{2\xi_{11}} & 0 & 0 & 0 \\ 0 & \frac{1}{2\xi_{11}\xi_{12}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, the matrix $\Sigma_4 = \varrho_9^2(J_{19}J_{18}J_{17})^{-1}\Theta_1(J_{19}J_{18}J_{17})^{-1}]^T$ is also semi-positive definite. Also, a positive constant ν_6 exists satisfying

$$\Sigma_4 \geq \nu_6 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Subcase (4.2) If $w_6 = 0, w_7 \neq 0$. Let $A_{11} = J_{20}A_{10}J_{20}^{-1}$, where

$$J_{20} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},\ A_{11} = \begin{pmatrix} -a_{43} & a_{43} & 0 & 0 \\ a_{31} & -a_{33} + \frac{a_{21}a_{32}}{a_{31}} - a_{14} & a_{31} & a_{32} \\ 0 & w_7 & -a_{11} + a_{14} & a_{12} + \frac{a_{14}a_{32}}{a_{31}} \\ 0 & 0 & 0 & -\frac{a_{21}a_{33}}{a_{31}} \end{pmatrix}.$$

Let $P_{10} = J_{21}A_{11}J_{21}^{-1}$, where the standardized transformed matrix J_{21} is given by

$$J_{21} = \begin{pmatrix} a_{31}w_7 & w_7(-a_{11} + \frac{a_{21}a_{32}}{a_{31}} - a_{33}) & (a_{11} - a_{14})^2 + a_{31}w_7 & a_{32}w_7 + \frac{(a_{12}a_{31}+a_{14}a_{32})(-a_{11}a_{31}+a_{14}a_{31}-a_{21}a_{33})}{a_{31}^2} \\ 0 & w_7 & -a_{11} + a_{14} & a_{12} + \frac{a_{14}a_{32}}{a_{31}} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$P_{10} = \begin{pmatrix} -\eta_{51} & -\eta_{52} & -\eta_{53} & -\eta_{54} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{a_{21}a_{33}}{a_{31}} \end{pmatrix},$$

with $\eta_{51} = -\frac{a_{21}a_{32}-a_{31}(a_{11}+a_{33}+a_{43})}{a_{31}}$, $\eta_{52} = -\frac{1}{a_{31}}[a_{14}^2a_{31} + a_{11}a_{21}a_{32} + a_{14}(-a_{21}a_{32} + a_{31}a_{33}) + (a_{31}^2 + a_{21}a_{32} - a_{31}a_{33})a_{43} - a_{11}a_{31}(a_{14} + a_{33} + a_{43}) + a_{31}^2w_7]$, $\eta_{53} = \frac{a_{43}(a_{11}-a_{14})[-a_{21}a_{32}+a_{31}(a_{14}-a_{31}+a_{33})]}{a_{31}} - a_{31}a_{43}w_7$, $\eta_{54} = -\frac{(a_{12}a_{31}+a_{14}a_{32})\{a_{21}a_{33}[-a_{31}(a_{14}+a_{33})+a_{21}(a_{32}+a_{33})]-a_{31}a_{43}[a_{31}(-a_{14}+a_{31}-a_{33})+a_{21}(a_{32}+a_{33})]\}+a_{31}^2a_{32}(-a_{21}a_{33}+a_{31}a_{43})w_7}{a_{31}^3}$.

In the meanwhile, Eq (4.9) can be transformed into the following equivalent form:

$$\Lambda_0^2 + P_{10}\Omega_5 + \Omega_5P_{10}^T = 0,$$

where

$$\Omega_5 = \varrho_9^{-2}(J_{21}J_{20}J_{18}J_{17})\Sigma_4(J_{21}J_{20}J_{18}J_{17})^T, \varrho_9 = a_{31}w_7\sigma_4.$$

In view of Lemma 4.2, we obtain that the matrix Ω_5 is semi-positive definite whose explicit form is as follows:

$$\Omega_5 = \begin{pmatrix} \frac{\eta_{52}}{2(\eta_{51}\eta_{52}-\eta_{53})} & 0 & -\frac{1}{2(\eta_{51}\eta_{52}-\eta_{53})} & 0 \\ 0 & \frac{1}{2(\eta_{51}\eta_{52}-\eta_{53})} & 0 & 0 \\ -\frac{1}{2(\eta_{51}\eta_{52}-\eta_{53})} & 0 & \frac{\eta_{51}}{2\eta_{53}(\eta_{51}\eta_{52}-\eta_{53})} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, the matrix $\Sigma_4 = \varrho_9^2(J_{21}J_{20}J_{18}J_{17})^{-1}\Omega_5[(J_{21}J_{20}J_{18}J_{17})^{-1}]^T$ is also semi-positive definite. Also, a constant ν_7 exists satisfying

$$\Sigma_4 \geq \nu_7\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}.$$

Subcase (4.3) If $w_6 \neq 0, w_7 = 0$. Let $P_{11} = J_{22}A_{10}J_{22}^{-1}$, where

$$J_{22} = \begin{pmatrix}(a_{12}a_{31}+a_{14}a_{32})w_6 & j_{22}(1) & j_{22}(2) & j_{22}(3)\\ 0 & (a_{12}+\frac{a_{14}a_{32}}{a_{31}})w_6 & \frac{(a_{12}a_{31}+a_{14}a_{32})(-a_{11}a_{31}+a_{14}a_{31}-a_{21}a_{33})}{a_{31}^2} & (a_{11}-a_{14})^2\\ 0 & 0 & a_{12}+\frac{a_{14}a_{32}}{a_{31}} & -a_{11}+a_{14}\\ 0&0&0&1\end{pmatrix},$$

$$P_{11} = \begin{pmatrix}-\tau_1 & -\tau_2 & -\tau_3 & -\tau_4\\ 1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix},$$

and

$$\begin{aligned}
j_{22}(1) &= -\frac{w_6(a_{12}a_{31}+a_{14}a_{32})[(a_{11}a_{31}-a_{21}a_{32}+(a_{21}+a_{31})a_{33}]}{a_{31}^2},\\
j_{22}(2) &= \frac{1}{a_{31}^3}(a_{12}a_{31}+a_{14}a_{32})[a_{11}^2a_{31}^2+a_{14}^2a_{31}^2-a_{14}a_{21}a_{31}a_{33}+a_{21}^2a_{33}^2\\
&\quad +a_{11}a_{31}(-2a_{14}a_{31}+a_{21}a_{33})+a_{31}^2a_{32}w_6],\\
j_{22}(3) &= (-a_{11}+a_{14})^3+(a_{12}a_{31}+a_{14}a_{32})w_6.
\end{aligned}$$

Then, Eq (4.9) is transformed into

$$\Lambda_0^2 + P_{11}\Sigma_0 + \Sigma_0 P_{11}^T = 0.$$

Therefore $\Sigma_4 = \varrho_{10}^2(J_{22}J_{18}J_{17})^{-1}\Sigma_0[(J_{22}J_{18}J_{17})^{-1}]^T$, where $\varrho_{10} = (a_{12}a_{31}+a_{14}a_{32})w_6\sigma_4$. Finally we know that Σ_4 is positive definite.

Subcase (4.4) If $w_6 \neq 0, w_7 \neq 0$. Let $A_{12} = J_{23}A_{10}J_{23}^{-1}$, where

$$J_{23} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&-\frac{w_7}{w_6}&0\end{pmatrix},\quad A_{12} = \begin{pmatrix}-a_{43} & a_{43} & 0 & 0\\ a_{31} & -a_{33}+\frac{a_{21}a_{32}}{a_{31}}-a_{14} & a_{32}+\frac{a_{31}w_7}{w_6} & a_{31}\\ 0 & w_6 & -\frac{a_{21}a_{33}}{a_{31}} & 0\\ 0&0&w_8&-a_{11}+a_{14}\end{pmatrix},$$

with $w_8 = a_{12}+\frac{a_{14}a_{32}}{a_{31}}+\frac{a_{21}a_{33}w_7}{a_{31}w_6}+(a_{14}-a_{11})\frac{w_7}{w_6}$.

Subcase (4.4.1) If $w_6 \neq 0, w_7 \neq 0, w_8 = 0$. Let $P_{12} = J_{24}A_{12}J_{24}^{-1}$, where the standardized transformed matrix J_{24} is given by

$$J_{24} = \begin{pmatrix}a_{31}w_6 & -\frac{[a_{14}a_{31}-a_{21}a_{32}+(a_{21}+a_{31})a_{33}]w_6}{a_{31}} & \frac{a_{21}^2a_{33}^2}{a_{31}^2}+a_{32}w_6+a_{31}w_7 & a_{31}w_6\\ 0 & w_6 & -\frac{a_{21}a_{33}}{a_{31}} & 0\\ 0&0&1&0\\0&0&0&1\end{pmatrix}.$$

Direct calculation leads to that

$$P_{12} = \begin{pmatrix} -\eta_{61} & -\eta_{62} & -\eta_{63} & -\eta_{64} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -a_{11} + a_{14} \end{pmatrix},$$

with

$$\begin{aligned} \eta_{61} &= -\frac{a_{21}(a_{32} - a_{33}) - a_{31}(a_{14} + a_{33} + a_{43})}{a_{31}}, \\ \eta_{62} &= -\frac{1}{a_{31}^2}\Big\{a_{21}^2 a_{32} a_{33} - a_{14} a_{31}(a_{21} a_{33} + a_{31} a_{43}) + a_{21} a_{31}[a_{32} a_{43} - a_{33}(a_{33} + a_{43})] \\ &\quad + a_{31}^2[-a_{33} a_{43} + a_{32} w_6 + a_{31}(a_{43} + w_7)]\Big\}, \\ \eta_{63} &= -\frac{1}{a_{31}^2} a_{43}[-a_{14} a_{21} a_{31} a_{33} + a_{21}^2 a_{32} a_{33} + a_{21} a_{31}(a_{31} - a_{33}) a_{33} + a_{31}^2(a_{32} w_6 + a_{31} w_7)], \\ \eta_{64} &= -a_{31} w_6(-a_{11} + a_{14} + a_{43}). \end{aligned}$$

Then we can transform Eq (4.9) into the following equivalent form:

$$\Lambda_0^2 + P_{12}\Omega_6 + \Omega_6 P_{12}^T = 0,$$

where

$$\Omega_6 = \begin{pmatrix} \frac{\eta_{62}}{2(\eta_{61}\eta_{62}-\eta_{63})} & 0 & -\frac{1}{2(\eta_{61}\eta_{62}-\eta_{63})} & 0 \\ 0 & \frac{1}{2(\eta_{61}\eta_{62}-\eta_{63})} & 0 & 0 \\ -\frac{1}{2(\eta_{61}\eta_{62}-\eta_{63})} & 0 & \frac{\eta_{61}}{2\eta_{63}(\eta_{61}\eta_{62}-\eta_{63})} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Further, we can obtain the matrix

$$\Sigma_4 = \varrho_{11}^2 (J_{24} J_{23} J_{18} J_{17})^{-1} \Omega_6 [(J_{24} J_{23} J_{18} J_{17})^{-1}]^T,$$

where $\varrho_{11} = a_{31} w_6 \sigma_4$ and the semi-positive definite of Σ_4 is obtained by Lemma 4.2. Also, there exists a constant v_8 such that

$$\Sigma_4 \geq v_8 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Subcase (4.4.2) If $w_6 \neq 0, w_7 \neq 0, w_8 \neq 0$. Let $P_{13} = J_{25} A_{12} J_{25}^{-1}$, where

$$J_{25} = \begin{pmatrix} a_{31} w_6 w_8 & j_{25}(1) & j_{25}(2) & j_{25}(3) \\ 0 & w_6 w_8 & w_8(-a_{11} + a_{14} - \frac{a_{21} a_{33}}{a_{31}}) & (a_{11} - a_{14})^2 \\ 0 & 0 & w_8 & -a_{11} + a_{14} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with

$$j_{25}(1) = -\frac{w_6 w_8[a_{11}a_{31} - a_{21}a_{32} + (a_{21} + a_{31})a_{33}]}{a_{31}},$$
$$j_{25}(2) = w_8[(a_{11} - a_{14})^2 + \frac{a_{21}a_{33}(a_{11}a_{31} - a_{14}a_{31} + a_{21}a_{33})}{a_{31}^2} + a_{32}w_6 + a_{31}w_7],$$
$$j_{25}(3) = (-a_{11} + a_{14})^3 + a_{31}w_6 w_8.$$

Next, the algebraic Eq (4.9) is transformed into

$$\Lambda_0^2 + P_{13}\Sigma_0 + \Sigma_0 P_{13}^T = 0.$$

The specific form of Σ_4 is

$$\Sigma_4 = \varrho_{12}^2 (J_{25}J_{23}J_{18}J_{17})^{-1}\Sigma_0[(J_{25}J_{23}J_{18}J_{17})^{-1}]^T,$$

where $\varrho_{12} = a_{31}w_6 w_8 \sigma_4$. Then by Lemma 4.1, the positive definiteness of Σ_4 is verified. Now we are in the position to prove the matrix $\Sigma = \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4$ is a positive definite matrix. If $w_1 = 0$, $w_3 \neq 0$, $w_4 = 0$, $w_5 = 0$, $w_6 = 0$ and $w_7 \neq 0$, the covariance matrix

$$\begin{aligned}\Sigma &= \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 \\ &\geq v_2\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} + v_3\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix} + v_4\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix} + v_5\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}.\end{aligned}$$

Then the probability density function $\Phi(x_1, x_2, x_3, x_4)$ can be expressed in the following form:

$$\Phi(x_1, x_2, x_3, x_4) = (2\pi)^{-2}|\Sigma|^{-\frac{1}{2}} e^{-\frac{1}{2}(x_1,x_2,x_3,x_4)\Sigma^{-1}(x_1,x_2,x_3,x_4)^T}.$$

Considering the transformation $(x_1, x_2, x_3, x_4) = (\ln\frac{E}{E^*}, \ln\frac{V}{V^*}, \ln\frac{C}{C^*}, \ln\frac{P}{P^*})$, we have the following from density function of model (1.2):

$$\Phi(E, V, C, P) = (2\pi)^{-2}|\Sigma|^{-\frac{1}{2}}(E\cdot V\cdot C\cdot P)^{-1} e^{-\frac{1}{2}(\ln\frac{E}{E^*}, \ln\frac{V}{V^*}, \ln\frac{C}{C^*}, \ln\frac{P}{P^*})\Sigma^{-1}(\ln\frac{E}{E^*}, \ln\frac{V}{V^*}, \ln\frac{C}{C^*}, \ln\frac{P}{P^*})^T}.$$

□

5. Extinction

Define

$$R_0^E = \left(\frac{2k_4k_2}{d_2} + k_3\right)\frac{r}{d_1 - k_6} - \frac{1}{3}\min\left\{\frac{d_2}{2} + \frac{\sigma_2^2}{2}, \frac{d_3}{2} + \frac{\sigma_3^2}{2}, d_4 + \frac{\sigma_4^2}{2}\right\},$$

and

$$H(t) = \frac{2k_4}{d_2}V(t) + C(t) + \frac{d_3}{2k_5}P(t).$$

Theorem 5.1. *Assume that $R_0^E < 0$, then for any initial value $(E(0), V(0), C(0), P(0)) \in \mathbb{R}_+^4$ the solution $(E(t), V(t), C(t), P(t))$ of system (1.2) will follow:*

$$\lim_{t\to\infty}\sup \frac{\ln H(t)}{t} \le R_0^E < 0, \qquad a.s.,$$

which means that the KD disease of system (1.2) will die out with probability 1.

Proof. Employing Itô's formula to $\ln H(t)$, one has

$$\begin{aligned}
\mathrm{d}\ln H(t) =\ & \frac{1}{H}\Big\{\Big(\frac{2k_4k_2}{d_2}+k_3\Big)EP - k_4V - \frac{d_3}{2}C - \frac{d_3d_4}{2k_5}P\Big\}\mathrm{d}t - \frac{1}{2H^2}\Big\{(\frac{2k_4}{d_2})^2\sigma_2^2V^2 + \sigma_3^2C^2 + (\frac{d_3}{2k_5})^2\sigma_4^2P^2\Big\}\mathrm{d}t \\
& + \frac{\frac{2k_4}{d_2}\sigma_2V}{H}\mathrm{d}B_2(t) + \frac{\sigma_3C}{H}\mathrm{d}B_3(t) + \frac{\frac{d_3}{2k_5}\sigma_4P}{H}\mathrm{d}B_4(t) \\
\le\ & \Big(\frac{2k_4k_2}{d_2}+k_3\Big)E\mathrm{d}t - \frac{\Big(k_4V + \frac{d_3}{2}C + \frac{d_3d_4}{2k_5}P\Big)\Big(\frac{2k_4}{d_2}V + C + \frac{d_3}{2k_5}P\Big)}{\Big(\frac{2k_4}{d_2}V + C + \frac{d_3}{2k_5}P\Big)^2}\mathrm{d}t \\
& - \frac{1}{2H^2}\Big\{(\frac{2k_4}{d_2})^2\sigma_2^2V^2 + \sigma_3^2C^2 + (\frac{d_3}{2k_5})^2\sigma_4^2P^2\Big\}\mathrm{d}t \\
& + \frac{\frac{2k_4}{d_2}\sigma_2V}{H}\mathrm{d}B_2(t) + \frac{\sigma_3C}{H}\mathrm{d}B_3(t) + \frac{\frac{d_3}{2k_5}\sigma_4P}{H}\mathrm{d}B_4(t) \\
\le\ & \Big(\frac{2k_4k_2}{d_2}+k_3\Big)E\mathrm{d}t - \frac{\frac{2k_4^2}{d_2}V^2 + \frac{d_3}{2}C^2 + \frac{d_3^2d_4}{2k_5}P^2}{\Big(\frac{2k_4}{d_2}V + C + \frac{d_3}{2k_5}P\Big)^2}\mathrm{d}t - \frac{1}{2H^2}\Big\{(\frac{2k_4}{d_2})^2\sigma_2^2V^2 + \sigma_3^2C^2 + (\frac{d_3}{2k_5})^2\sigma_4^2P^2\Big\}\mathrm{d}t \\
& + \frac{\frac{2k_4}{d_2}\sigma_2V}{H}\mathrm{d}B_2(t) + \frac{\sigma_3C}{H}\mathrm{d}B_3(t) + \frac{\frac{d_3}{2k_5}\sigma_4P}{H}\mathrm{d}B_4(t) \\
=\ & \Big(\frac{2k_4k_2}{d_2}+k_3\Big)E\mathrm{d}t - \frac{1}{\Big(\frac{2k_4}{d_2}V + C + \frac{d_3}{2k_5}P\Big)^2}\Big[\Big(\frac{2k_4}{d_2}\Big)^2\Big(\frac{d_2}{2}+\frac{\sigma_2^2}{2}\Big)V^2 + \Big(\frac{d_3}{2}+\frac{\sigma_3^2}{2}\Big)C^2 \\
& + \Big(\frac{d_3}{2k_5}\Big)^2\Big(d_4 + \frac{\sigma_4^2}{2})P^2\Big]\mathrm{d}t + \frac{\frac{2k_4}{d_2}\sigma_2V}{H}\mathrm{d}B_2(t) + \frac{\sigma_3C}{H}\mathrm{d}B_3(t) + \frac{\frac{d_3}{2k_5}\sigma_4P}{H}\mathrm{d}B_4(t).
\end{aligned}$$

From $(a+b+c)^2 \le 3(a^2+b^2+c^2)$, $(a>0, b>0, c>0)$, we have $-\frac{\left(\frac{2k_4}{d_2}\right)^2V^2+C^2+\left(\frac{d_3}{2k_5}\right)^2P^2}{\left(\frac{2k_4}{d_2}V+C+\frac{d_3}{2k_5}P\right)^2} \le -\frac{1}{3}$. Then

$$\begin{aligned}
\mathrm{d}\ln H(t) \le & \Big(\frac{2k_4k_2}{d_2}+k_3\Big)E\mathrm{d}t - \frac{1}{3}\min\Big\{\frac{d_2}{2}+\frac{\sigma_2^2}{2}, \frac{d_3}{2}+\frac{\sigma_3^2}{2}, d_4+\frac{\sigma_4^2}{2}\Big\}\mathrm{d}t + \frac{\frac{2k_4}{d_2}\sigma_2V}{H}\mathrm{d}B_2(t) \\
& + \frac{\sigma_3C}{H}\mathrm{d}B_3(t) + \frac{\frac{d_3}{2k_5}\sigma_4P}{H}\mathrm{d}B_4(t).
\end{aligned} \tag{5.1}$$

From the first equation of system (1.2), we have

$$\mathrm{d}E \le \Big[r - \Big(d_1 - k_6\Big)E\Big]\mathrm{d}t + \sigma_1E\mathrm{d}B_1.$$

For the stochastic differential equation with the initial value $E(0) = \widetilde{E}(0)$,

$$\mathrm{d}\widetilde{E} = \left[r - \left(d_1 - k_6\right)\widetilde{E}\right] + \sigma_1 \widetilde{E}\mathrm{d}B_1,$$

and we have $\lim_{t\to\infty} \frac{1}{t}\int_0^t \widetilde{E}(s)\mathrm{d}s = \frac{r}{d_1-k_6}$. According to the comparison principle, we have $E(t) \leq \widetilde{E}(t), a.s.$ So

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t E(s)\mathrm{d}s \leq \lim_{t\to\infty} \frac{1}{t}\int_0^t \widetilde{E}(s)\mathrm{d}s = \frac{r}{d_1 - k_6}.$$

Integrating (5.1) from 0 to t, and dividing by t on both sides, it can be seen that

$$\begin{aligned}\frac{\ln H(t)}{t} \leq &\frac{\ln H(0)}{t} + \left(\frac{2k_4k_2}{d_2} + k_3\right)\frac{1}{t}\int_0^t E(s)\mathrm{d}s - \frac{1}{3}\min\left\{\frac{d_2}{2} + \frac{\sigma_2^2}{2}, \frac{d_3}{2} + \frac{\sigma_3^2}{2}, d_4 + \frac{\sigma_4^2}{2}\right\}\mathrm{d}t \\ &+ \frac{1}{t}\int_0^t \frac{\frac{2k_4}{d_2}\sigma_2 V}{H}\mathrm{d}B_2(t) + \frac{1}{t}\int_0^t \frac{\sigma_3 C}{H}\mathrm{d}B_3(t) + \frac{1}{t}\int_0^t \frac{\frac{d_3}{2k_5}\sigma_4 P}{H}\mathrm{d}B_4(t).\end{aligned}$$

Next, by the strong of large numbers, one derives

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{\frac{2k_4}{d_2}\sigma_2 V}{H}\mathrm{d}B_2(t) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{\sigma_3 C}{H}\mathrm{d}B_3(t) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{\frac{d_3}{2k_5}\sigma_4 P}{H}\mathrm{d}B_4(t) = 0. \tag{5.2}$$

From (5.1) and (5.2), we can deduce that

$$\begin{aligned}\lim_{t\to\infty}\sup \frac{\ln H(t)}{t} \leq &\left(\frac{2k_4k_2}{d_2} + k_3\right)\cdot \frac{r}{d_1 - k_6} - \frac{1}{3}\min\left\{\frac{d_2}{2} + \frac{\sigma_2^2}{2}, \frac{d_3}{2} + \frac{\sigma_3^2}{2}, d_4 + \frac{\sigma_4^2}{2}\right\} \\ := &R_0^E.\end{aligned}$$

If $R_0^E < 0$, then

$$\lim_{t\to\infty} P(t) = 0, \quad a.s.$$

□

6. Numerical simulation

Using the well-known high-order numerical method of Milstein [19], we get the discretization equation of system (1.2):

$$\begin{cases} E_{k+1} = E_k + \left[r + \dfrac{k_6 V_k E_k}{1 + V_k} - k_1 E_k P_k - d_1 E_k\right]\triangle t + \sigma_1 E_k \sqrt{\triangle t}\zeta_{1,k} + \dfrac{\sigma_1^2}{2} E_k(\zeta_{1,k}^2 - 1)\triangle t, \\ V_{k+1} = V_k + \left[k_2 E_k P_k - d_2 V_k\right]\triangle t + \sigma_2 V_k \sqrt{\triangle t}\zeta_{2,k} + \dfrac{\sigma_2^2}{2} V_k(\zeta_{2,k}^2 - 1)\triangle t, \\ C_{k+1} = C_k + \left[k_3 E_k P_k + k_4 V_k - d_3 C_k\right]\triangle t + \sigma_3 C_k \sqrt{\triangle t}\zeta_{3,k} + \dfrac{\sigma_3^2}{2} C_k(\zeta_{3,k}^2 - 1)\triangle t, \\ P_{k+1} = P_k + \left[k_5 C_k - d_4 P_k\right]\triangle t + \sigma_4 P_k \sqrt{\triangle t}\zeta_{4,k} + \dfrac{\sigma_4^2}{2} P_k(\zeta_{4,k}^2 - 1)\triangle t, \end{cases}$$

where $\zeta_{i,k}, (i = 1, 2, 3, 4;\ k = 1, 2 \ldots, n)$ are independent Gaussian random variables $N(0, 1)$.

Example 6.1. *Let* $(E(0), V(0), C(0), P(0)) = (0.5, 2.5, 0.5, 2.5)$. *We choose* $r = 2, k_1 = 2, k_3 = 0.5, k_4 = 0.5, k_6 = 0.4, k_2 = 3, k_5 = 0.3, \sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0.15, d_1 = d_2 = 1, d_3 = 0.6, d_4 = 0.8$. *Direct calculation leads to*

$$R_0^s = \frac{rk_3k_5}{(d_1 + \frac{1}{2}\sigma_1^2)(d_3 + \frac{1}{2}\sigma_3^2)(d_4 + \frac{1}{2}\sigma_4^2)} + \frac{rk_2k_4k_5}{(d_1 + \frac{1}{2}\sigma_1^2)(d_2 + \frac{1}{2}\sigma_2^2)(d_3 + \frac{1}{2}\sigma_3^2)(d_4 + \frac{1}{2}\sigma_4^2)} = 2.4855.$$

Then from Theorem 4.1 we can conclude that system (1.2) admits a global positive stationary solution on $\mathbb{R}_+^4$, *see Figure 1.*

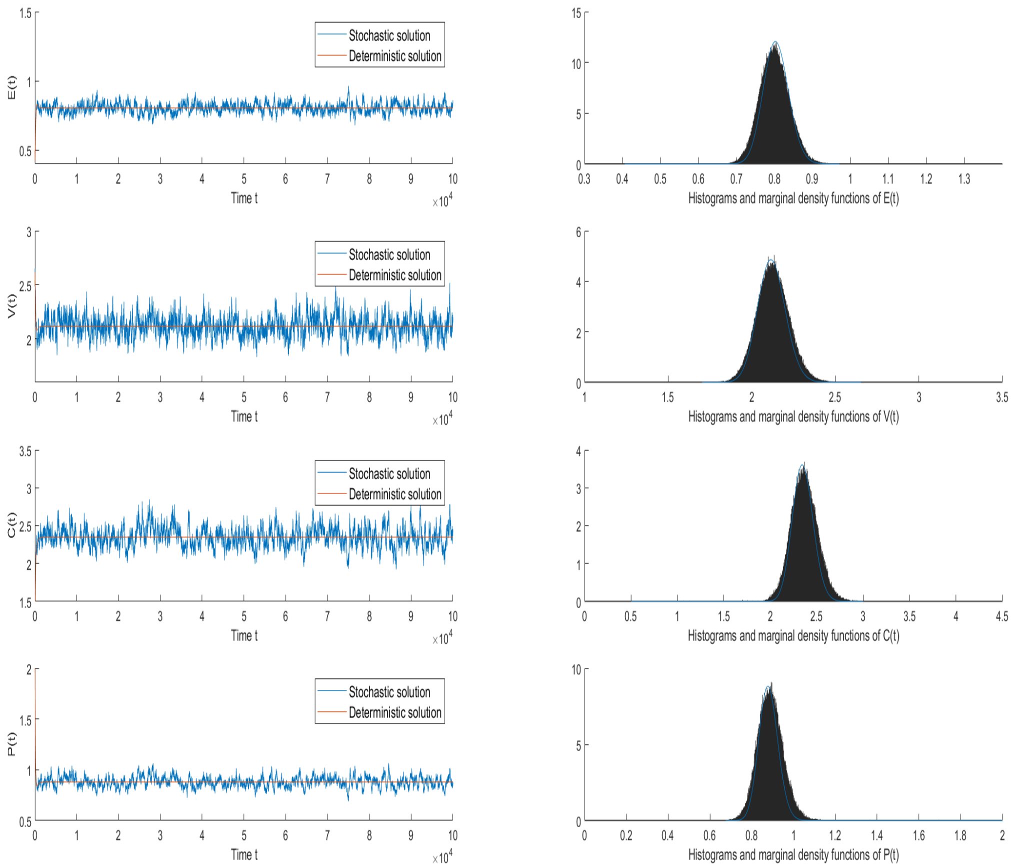

Figure 1. Graphs on the left show the trajectory of the stochastic system and the deterministic system under perturbation $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0.05$, and graphs on the right show histograms and marginal density functions of the solution.

Example 6.2. *We obtain quasi-stable equilibrium*

$$Q^* = (E^*, V^*, C^*, P^*) = (0.8037, 2.1179, 2.3490, 0.8795), R_0^s = 2.4855.$$

Furthermore, we have $w_1 = 0.5109, w_2 = 0.7541, w_3 = -0.3705, w_4 = -0.9945, w_5 = 0.1264, w_6 = -22.6266, w_7 = 37.1546, w_8 = -0.0147$. *Thus from Theorem 4.1, we calculate the specific expression* *of the covariance matrix:*

$$\begin{aligned}\Sigma =\ & (a_{21}w_1w_2\sigma_1)^2(J_3J_2J_1)^{-1}\Sigma_0[(J_3J_2J_1)^{-1}]^T + (a_{43}a_{32}w_4\sigma_2)^2(J_{12}J_{10}J_8J_7)^{-1}\Sigma_0[(J_{12}J_{10}J_8J_7)^{-1}]^T \\ & + (a_{14}a_{43}w_5\sigma_3)^2(J_{16}J_{14}J_{13})^{-1}\Sigma_0[(J_{16}J_{14}J_{13})^{-1}]^T \\ & + [(a_{12}a_{31} + a_{14}a_{32})w_6\sigma_4]^2(J_{22}J_{18}J_{17})^{-1}\Sigma_0[(J_{22}J_{18}J_{17})^{-1}]^T \\ =\ & \begin{pmatrix} 0.0017 & -3.1610e^{-4} & -7.7961e^{-4} & -0.0016 \\ -3.1610e^{-4} & 0.0015 & 6.7900e^{-4} & 7.7383e^{-4} \\ -7.7961e^{-4} & 6.7900e^{-4} & 0.0022 & 0.0014 \\ -0.0016 & 7.7383e^{-4} & 0.0014 & 0.0027 \end{pmatrix}.\end{aligned}$$

Then the corresponding marginal density functions of E, V, C, P are separately given as follows:

$$\begin{aligned}\Phi_E &= \frac{1}{0.1034E}e^{-\frac{(\ln E+0.2186)^2}{0.0034}},\ \Phi_V = \frac{1}{0.0974V}e^{-\frac{(\ln V-0.7504)^2}{0.003}}, \\ \Phi_C &= \frac{1}{0.1182C}e^{-\frac{(\ln C-0.854)^2}{0.0044}},\ \Phi_P = \frac{1}{0.1291P}e^{-\frac{(\ln P+0.1284)^2}{0.0053}}.\end{aligned}$$

See the right-hand figures of Figure 1. We also plot the marginal density functions and the frequency histogram fitting curves of E, V, P and C. See Figure 2.

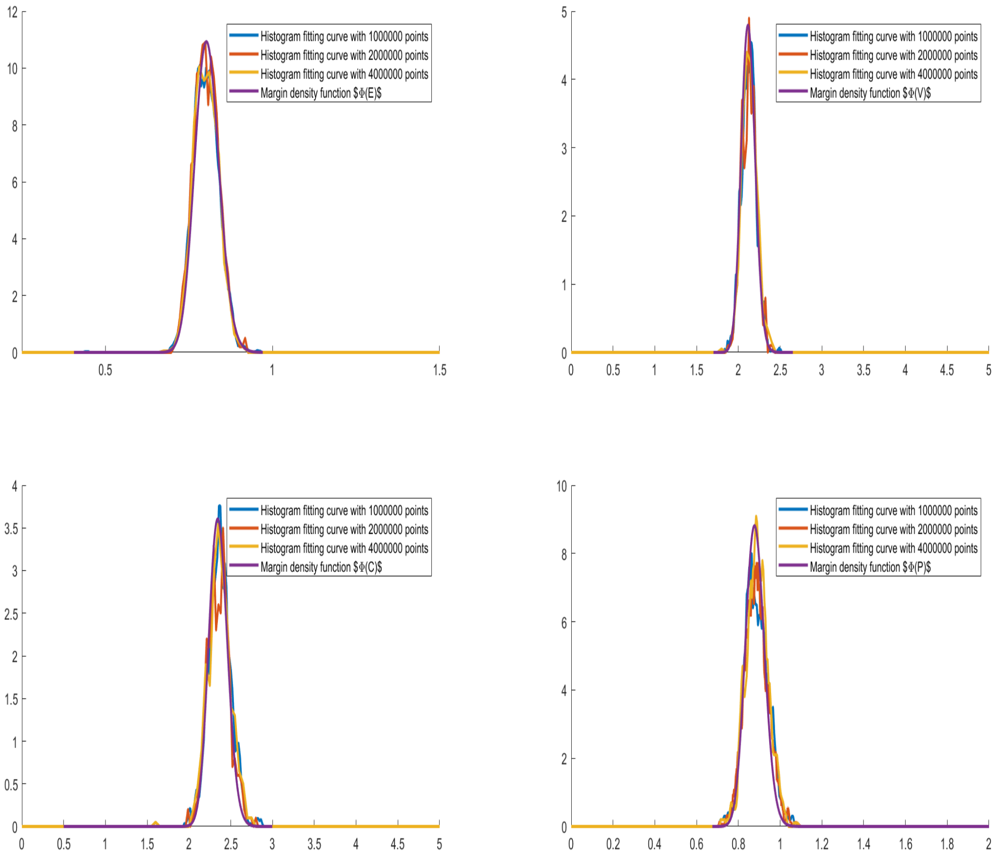

Figure 2. Blue, red, yellow lines represent the frequency histogram fitting curves of E, V, P and C at the iteration times equal to $1000000, 2000000$ and 4000000, respectively. Purple lines represent the theoretical marginal density functions $E(t), V(t), P(t)$ and $C(t)$.

Example 6.3. *We choose* $r = 1, k_2 = 0.3, k_3 = 0.1, k_4 = 0.2, k_6 = 0.2, d_2 = 2, d_4 = 0.8, \sigma_4 = 0.5$, *then from Theorem 5.1 we have*

$$R_0^E = \left(\frac{2k_4k_2}{d_2} + k_3\right)\frac{r}{d_1 - k_6} - \frac{1}{3}\min\left\{\frac{d_2}{2} + \frac{\sigma_2^2}{2}, \frac{d_3}{2} + \frac{\sigma_3^2}{2}, d_4 + \frac{\sigma_4^2}{2}\right\} = -0.1083 < 0.$$

We can conclude that KD disease of system (1.2) will disappear, see Figure 3.

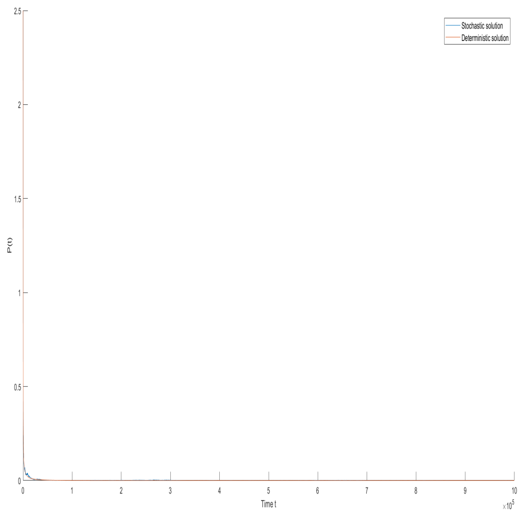

Figure 3. Blue and red lines represent the deterministic solution and stochastic solution of P, where $r = 1, k_2 = 0.3, k_3 = 0.1, k_4 = 0.2, k_6 = 0.2, d_2 = 2, d_4 = 0.8, \sigma_4 = 0.5$.

7. Conclusions

This paper is concerned with the stochastic Kawasaki disease. First, we proved that the global positive solution of model (1.2) is unique. Next, we obtained sufficient conditions for the ergodic stationary distribution. More importantly, we gave the local normal probability density function around the quasi-endemic equilibrium of system (1.2), by solving the corresponding Fokker-Planck equation. Finally, several numerical simulations were provided to verify our analytical results.

Author contributions

Ying He: Conceptualization, investigation, formal analysis, writing – review, editing; Bo Bi: Numerical simulation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no competing interests.

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