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Research article

On the singular integral representation of the fractional powers of Jacobi differential operators

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Abstract: In this paper, we introduce the fractional Jacobi operator and present its formulation in terms of a pseudo-differential operator via the Fourier–Jacobi transform. Furthermore, by employing the generalized shift operator related to the Jacobi operator, we establish a singular integral representation of the fractional Jacobi operator.

Keywords: fractional Jacobi differential operators; Fourier–Jacobi integral transform; operational calculus
Mathematics Subject Classification: 26A33, 42B10

1. Introduction

Fractional calculus, which extends classical differentiation and integration to non-integer orders, has emerged as a powerful framework for modeling a wide variety of complex phenomena in physics, engineering, and applied mathematics [3, 20, 28]. Its origins can be traced back to the pioneering ideas of Leibniz, Euler, Laplace, Lacroix, and Fourier, who speculated about the meaning of derivatives of arbitrary order. The first rigorous application was by N. H. Abel (1823), who employed fractional integration to solve an integral equation arising from the tautochrone problem—a landmark in the history of fractional calculus. This was followed by foundational contributions from Liouville (1832), Grünwald (1867), Riemann (posthumously published in 1892), and Letnikov (1868–1872), which established the theoretical underpinnings of the subject [29, 32].

Fractional derivatives have proved particularly effective in describing systems with memory effects, hereditary properties, and anomalous diffusion, often outperforming integer-order models. For example, Riewe [30, 31] extended the calculus of variations to fractional derivatives in order to better capture nonconservative mechanical systems. In another domain, fractional Laplace–Beltrami operators naturally arise in mathematical physics, notably in relativistic quantum mechanics and geometric analysis [2, 4, 13, 14, 19, 26, 27]. These operators are closely linked to pure-jump stochastic processes, serving as infinitesimal generators of stable Lévy processes—a connection elegantly developed in Bertoin's monograph [6]. Moreover, Feynman's path integral formulation has been extended to this probabilistic framework via Lévy processes [25].

In this work, we study the fractional powers of the one-dimensional Jacobi–Laplacian, a second-order differential operator defined on the interval $(0, \infty)$ as

$$\mathscr{L}_{\alpha,\beta} := \frac{1}{A_{\alpha,\beta}(x)} \frac{d}{dx}\Big(A_{\alpha,\beta}(x)\, \frac{d}{dx}\Big), \tag{1.1}$$

where the weight function $A_{\alpha,\beta}(x)$ is given by

$$A_{\alpha,\beta}(x) = 2^{2\rho}(\sinh x)^{2\alpha+1}(\cosh x)^{2\beta+1}, \qquad \rho := \alpha + \beta + 1.$$

Equivalently,

$$\mathscr{L}_{\alpha,\beta} = \frac{d^2}{dx^2} + \big[(2\alpha + 1)\coth(x) + (2\beta + 1)\tanh(x)\big]\frac{d}{dx}. \tag{1.2}$$

For the parameters $\alpha \geq \beta \geq -\frac{1}{2}$, the operator $-\mathscr{L}_{\alpha,\beta}$ is formally self-adjoint in the Hilbert space $L^2(A_{\alpha,\beta}(x)\, dx)$ of square-integrable functions with respect to the measure $A_{\alpha,\beta}(x)\, dx$ on $(0, \infty)$. We define

$$D_{\alpha,\beta} := \Big\{ u \in L^2\!\big(A_{\alpha,\beta}(x)\, dx\big) \;\Big|\; u,\ u' \text{ are absolutely continuous on every compact subset of } (0,\infty),$$
$$\mathscr{L}_{\alpha,\beta}(u) \in L^2\!\big(A_{\alpha,\beta}(x)\, dx\big), \quad \lim_{x\to 0^+} x^{2\alpha+1}\, u'(x) = 0 \Big\}.$$

The operator $\mathscr{L}_{\alpha,\beta}$, endowed with the domain $D_{\alpha,\beta}$, is self-adjoint and has a simple, continuous spectrum covering $[\rho^2, \infty)$, as established in the Jacobi setting in [17, §2] and further detailed in [12] within the framework of weighted Sturm–Liouville theory. This spectral structure underlies the Fourier–Jacobi transform, which emerges from the spectral decomposition of $-\mathscr{L}_{\alpha,\beta}$; see Koornwinder [23]. The harmonic analysis associated with $\mathscr{L}_{\alpha,\beta}$ has been extensively developed since the 1970s, largely owing to the pioneering work of M. F. Jensen and T. H. Koornwinder [18, 22, 23]. Their contributions laid the foundation for a robust theoretical framework that integrates convolution structures, heat and Poisson semigroups, and a wide range of integral transforms. Building on this foundation, B. Salem and Dachraoui [34] conducted a detailed study of pseudo-differential operators associated with the Jacobi differential operator, introducing novel symbol classes adapted to this setting and proving continuity results on certain subspaces of the Schwartz space. They also derived integral representations and obtained sharp L^1-norm inequalities by exploiting the dual convolution structure of Jacobi functions. Extending this work, Ben Salem and Samaali [33] explored the spectral theory of the Fourier–Jacobi transform, defining fractional powers $(-\mathscr{L}_{\alpha,\beta})^s$ via spectral calculus (see Definition 3.12). For a broader perspective on the spectral methods for fractional operators; see [35]. More recently, analogous results have been established for a wider class of operators, including second-order differential and partial differential operators of the Bessel type [7], Bessel–Laplacian operators [8, 9], and fractional Dunkl–Laplacians [10].

In the present work, we introduce a novel singular–integral representation for the fractional one-dimensional Jacobi–Laplacian $\mathscr{L}_{\alpha,\beta}$, formulated within the framework of the generalized translation operator and the Jacobi heat kernel. For the special parameters $\alpha = \frac{1}{2}(p-1)$ and $\beta = \frac{1}{2}(q-1)$, with $p \geq q > 0$, the operator $\mathscr{L}_{\alpha,\beta}$ coincides with the radial part of the Laplace–Beltrami operator on Riemannian symmetric spaces of the noncompact type and rank one [22]. Fractional-order operators on Riemannian manifolds have been widely studied (see, e.g., [2, 4, 5] and the references therein), with particular emphasis on the fractional powers of the Laplacian on hyperbolic spaces. While the functional calculus provides a natural definition in this context, Banica et al. [5] proposed an alternative characterization as a Dirichlet-to-Neumann map for a degenerate elliptic extension problem, in the spirit of the celebrated Caffarelli–Silvestre construction [11].

The principal result of this paper is stated in Theorem 2.3, which asserts that for every $f \in C_0(\mathbb{R})$ and $0 < s < 1$, the fractional operator $(-\mathscr{L}_{\alpha,\beta})^s$ admits the singular–integral representation

$$(-\mathscr{L}_{\alpha,\beta})^s f(x) = \frac{s}{\Gamma(1-s)} \int_0^\infty [f(x) - T_x^{(\alpha,\beta)} f(y)]\, w_s^{(\alpha,\beta)}(y)\, dy, \tag{1.3}$$

where $T_x^{(\alpha,\beta)}$ is the generalized translation operator associated with $\mathscr{L}_{\alpha,\beta}$ and

$$w_s^{(\alpha,\beta)}(y) := A_{\alpha,\beta}(y) \int_0^\infty \frac{E_t^{(\alpha,\beta)}(y)}{t^{1+s}}\, dt,$$

with $E_t^{(\alpha,\beta)}(y)$ denoting the Jacobi heat kernel.

Theorem 2.3 exhibits several genuinely new features: (i) The singular–integral representation (1.3) is, to the best of our knowledge, entirely new and has not appeared previously in the literature. (ii) Both the generalized translation operator $T_x^{(\alpha,\beta)}$ and the kernel $w_s^{(\alpha,\beta)}$ are intrinsically linked to the operator $\mathscr{L}_{\alpha,\beta}$, in contrast to Bessel or Bessel–Laplacian fractional analog, where the generalized translation is specific to those operators. Moreover, the formula (1.3) holds for the full parameter range $\alpha \geq \beta \geq -\frac{1}{2}$, thereby extending beyond the restrictions commonly found in the Jacobi fractional Laplacian literature (see, e.g., [2,5]). (iii) The proof employs a two–regime integrability analysis (Lemma 3.3), a technical ingredient absent from earlier studies of the Laplacian or Laplace–Bessel settings.

As in the case of the classical fractional Laplacian (see, e.g., [24]), one can also obtain for $(-\mathscr{L}_{\alpha,\beta})^s$ alternative representations, such as a Bochner-type formula derived directly from the Lévy–Khintchine identity in Lemma 3.3, or an extension problem in the spirit of Banica et al. [5]. However, since these follow readily from our framework, we restrict our attention here to the singular–integral formulation, leaving such equivalent descriptions to the reader.

2. Notation and main result

For $\lambda \in \mathbb{C}$, the Jacobi function $x \mapsto \varphi_\lambda^{(\alpha,\beta)}(x)$ is defined as the unique solution to the following boundary value problem:

$$\mathscr{L}_{\alpha,\beta} u(x) = -(\lambda^2 + \rho^2)\, u(x), \tag{2.1}$$

subject to the initial conditions:

$$u(0) = 1, \quad u'(0) = 0.$$

By substituting $z = -\sinh^2 x$ into the first equation of problem (2.1), we obtain the standard hypergeometric differential equation (cf. [15, 2.1(1)]), which reads as follow:

$$z(1-z)\frac{d^2u}{dz^2} + (c - (a+b+1)z)\frac{du}{dz} - ab\, u = 0,$$

with the parameter:

$$a = \frac{1}{2}(\rho - i\lambda), \quad b = \frac{1}{2}(\rho + i\lambda), \quad c = \alpha + 1.$$

Hence, if $\alpha \notin \{-1, -2, -3, \dots\}$, the function $\varphi_\lambda^{(\alpha,\beta)}(x)$ can be expressed in the form

$$\varphi_\lambda^{(\alpha,\beta)}(x) = F\left(\tfrac{1}{2}(\rho - i\lambda),\ \tfrac{1}{2}(\rho + i\lambda);\ \alpha + 1;\ -\sinh^2(x)\right), \quad x \geq 0. \tag{2.2}$$

Here, $F(a, b; c; z)$ denotes the unique analytic continuation (for $z \notin [1, \infty)$) of the Gauss hypergeometric series

$$\sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{(c)_k\,k!}\,z^k, \quad |z| < 1,$$

where $a, b, c \in \mathbb{C}$, $c \notin -\mathbb{N}$, and $(a)_k$ is the Pochhammer symbol, defined as

$$(a)_0 := 1, \quad (a)_k := a(a+1)\cdots(a+k-1), \quad \text{for } k \geq 1.$$

For $\lambda \notin \{-i, -2i, -3i, \dots\}$, a second linearly independent solution of Eq (2.1) (see [15, 2.9(9)]) is given by:

$$\phi_\lambda^{(\alpha,\beta)}(x) = (2\sinh(x))^{i\lambda-\rho}\ F\left(\tfrac{1}{2}(-\alpha+\beta+1-i\lambda),\ \tfrac{1}{2}(\alpha+\beta+1-i\lambda);\ 1-i\lambda;\ -(\sinh x)^{-2}\right).$$

This solution is characterized by the asymptotic behavior

$$\phi_\lambda^{(\alpha,\beta)}(x) = \mathrm{e}^{(i\lambda-\rho)t}(1 + o(1)) \quad \text{as } x \to \infty.$$

One can show that

$$\varphi_\lambda^{(\alpha,\beta)}(x) = c^{(\alpha,\beta)}(\lambda)\,\phi_\lambda^{(\alpha,\beta)}(x) + c^{(\alpha,\beta)}(-\lambda)\,\phi_{-\lambda}^{(\alpha,\beta)}(x), \tag{2.3}$$

where

$$c^{(\alpha,\beta)}(\lambda) = \frac{2^{\rho-i\lambda}\Gamma(i\lambda)\Gamma(\alpha+1)}{\Gamma(\frac{1}{2}(\rho+i\lambda))\Gamma(\frac{1}{2}(\alpha-\beta+i\lambda))}. \tag{2.4}$$

For special values of α and β, the Jacobi functions $\varphi_\lambda^{(\alpha,\beta)}(x)$ are interpreted as spherical functions on non-compact Riemannian symmetric spaces of rank one. For further details, the reader is referred to the insightful survey by T. H. Koornwinder [22].

Let $C_0(\mathbb{R})$ be the class of all even and infinitely differentiable functions on $\mathbb{R}$ with compact support. The Fourier–Jacobi transform (also called the Jacobi transform) is defined for a function $f \in C_0(\mathbb{R})$ by [18]

$$\mathcal{F}_{\alpha,\beta}(f)(\lambda) = \int_0^\infty \varphi_\lambda^{(\alpha,\beta)}(x)\,f(x)\,A_{\alpha,\beta}(x)\,dx.$$

In particular $\mathcal{F}_{-1/2,-1/2}f$ is the Fourier cosine function transform

$$\mathcal{F}_{-1/2,-1/2}(f)(\lambda) = \int_0^\infty \cos(\lambda x)\,f(x)\,dx.$$

We recall the inversion formula

$$\mathcal{F}_{\alpha,\beta}^{-1}(g)(x) = \frac{1}{2\pi}\int_0^\infty g(\lambda)\,\varphi_\lambda^{(\alpha,\beta)}(x)\,\frac{d\lambda}{|c_{\alpha,\beta}(\lambda)|^2},$$

and the Plancherel formula

$$\int_0^\infty |f(x)|^2 A_{\alpha,\beta}(x)\,dx = \frac{1}{2\pi}\int_0^\infty |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 \frac{d\lambda}{|c_{\alpha,\beta}(\lambda)|^2}.$$

It is worth noting that the factor $|c_{\alpha,\beta}(\lambda)|^2$ appearing in the Plancherel formula is computed using the identity

$$\Gamma(i\lambda)\,\Gamma(-i\lambda) = \frac{\pi}{\lambda\,\sinh(\pi\lambda)}.$$

Let $\mathscr{A}$ be the class of even entire, and rapidly decreasing functions of the exponential type, that is, $g \in \mathscr{A}$ if and only if g is even and entire, and, for all $n \in \mathbb{N}$, $B, K_n > 0$ exists such that for all $\lambda \in \mathbb{C}$

$$|g(\lambda)| \le K_n(1 + |\lambda|)^{-n} e^{B|\operatorname{Im}\lambda|}. \tag{2.5}$$

Theorem 2.1. *[23] For all $\alpha,\beta \in \mathbb{C}$, the mapping $f \mapsto \mathcal{F}_{\alpha,\beta}(f)$ is a bijection from $C_0(\mathbb{R})$ onto $\mathscr{A}$.*

The generalized translation operator was introduced by Flensted-Jensen and Koornwinder [18, Formula (5.1)], and is defined as

$$(T_x^{\alpha,\beta} f)(y) = \int_0^\infty f(z)\,K(x,y,z)\,A_{\alpha,\beta}(z)\,dz, \tag{2.6}$$

where the kernel $K(x,y,z)$ is explicitly given by

$$\begin{aligned} K(x,y,z) = 2^{-2\rho}\, &\frac{\Gamma(\alpha+1)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\alpha+\frac{1}{2}\right)}(\cosh x\,\cosh y\,\cosh z)^{\alpha-\beta-1} \\ &\times(\sinh x\,\sinh y\,\sinh z)^{2\alpha}(1-B^2)^{\frac{\alpha-1}{2}}\,{}_2F_1\left(\alpha+\beta,\,\alpha-\beta;\,\alpha+\tfrac{1}{2};\,\tfrac{1}{2}(1-B)\right), \end{aligned} \tag{2.7}$$

with

$$B = \frac{\cosh^2 x + \cosh^2 y + \cosh^2 z - 1}{2\,\cosh x\,\cosh y\,\cosh z}. \tag{2.8}$$

The kernel $K(x,y,z)$ has compact support given by

$$\operatorname{supp}(K) = \{\, z \in \mathbb{R}_+ \;:\; |x-y| \le z \le x+y \,\}.$$

Furthermore, the Jacobi function satisfies the following product formula [22]:

$$\varphi_\lambda(x)\,\varphi_\lambda(y) = \int_0^\infty \varphi_\lambda(z)\,K(x,y,z)\,A_{\alpha,\beta}(z)\,dz, \tag{2.9}$$

which plays a fundamental role in the harmonic analysis associated with the Jacobi–Laplacian.

Finally, as shown in [22], the generalized translation operator satisfies the following Fourier–Jacobi transform relation:

$$\mathcal{F}_{\alpha,\beta}(T_x^{\alpha,\beta} f)(\lambda) = \varphi_\lambda(x)\,\mathcal{F}_{\alpha,\beta} f(\lambda). \tag{2.10}$$

In [22, Sections 7.1 and 7.2], several properties of the generalized Jacobi translation were established

- $T_0^{(\alpha,\beta)} f = f$ for every $f \in L^p\left(\mathbb{R}_+,\ A_{\alpha,\beta}(x)dx\right)$, with $1 \leq p \leq \infty$.
- If $f \in C_0(\mathbb{R})$, then $T_y^{\alpha,\beta} f \in C_0(\mathbb{R})$.
- For all $f \in L^p\left(\mathbb{R}_+,\ A_{\alpha,\beta}(x)dx\right)$, with $1 \leq p \leq \infty$, the following inequality holds:

$$\|T_y^{(\alpha,\beta)} f\|_{L^p\left(\mathbb{R}_+,\ A_{\alpha,\beta}(x)dx\right)} \leq \|f\|_{L^p\left(\mathbb{R}_+,\ A_{\alpha,\beta}(x)dx\right)}. \tag{2.11}$$

Next, we recall the Gaussian kernel associated with the operator $\mathscr{L}_{\alpha,\beta}$ and summarize some of its essential properties needed in what follows.

For $t > 0$, define the Gaussian kernel as

$$E_t^{(\alpha,\beta)}(x) := E^{(\alpha,\beta)}(t,x) = \int_0^\infty e^{-t(\lambda^2+\rho^2)}\,\varphi_\lambda(x)\,\frac{d\lambda}{2\pi|c(\lambda)|^2}. \tag{2.12}$$

The associated kernel $E^{(\alpha,\beta)}(t,x,y)$ is defined via the generalized translation operator $T_x^{\alpha,\beta}$ as

$$E^{(\alpha,\beta)}(t,x,y) := T_x^{(\alpha,\beta)} E_t^{(\alpha,\beta)}(y).$$

For each $t > 0$, the Jacobi-type heat kernel $E_t^{(\alpha,\beta)}(x)$ is an even, strictly positive, and C^∞-smooth function on $\mathbb{R}$. Moreover, it satisfies the following fundamental identities (see [1, Theorem II.2 and Corollary III.1] and [16, 36]):

$$\int_0^\infty E_t^{(\alpha,\beta)}(x)\,A_{\alpha,\beta}(x)\,dx = 1, \qquad \mathscr{F}_{\alpha,\beta}\Big(E_t^{(\alpha,\beta)}\Big)(\lambda) = e^{-t(\lambda^2+\rho^2)}. \tag{2.13}$$

Here, the first identity expresses the conservation of mass in the Jacobi setting, while the second characterizes the kernel as the Fourier–Jacobi transform of the heat semigroup $e^{-t(\mathscr{L}_{\alpha,\beta})}$. The heat semigroup $(H_t)_{t\geq 0}$ associated with $\mathscr{L}_{\alpha,\beta}$ is defined for $f \in \mathscr{D}_0(\mathbb{R})$ by

$$H_0 f(x) = f(x), \qquad H_t f(x) = \int_0^\infty E^{(\alpha,\beta)}(t,x,y)\,f(y)\,A_{\alpha,\beta}(y)\,dy, \quad t > 0. \tag{2.14}$$

This operator extends to a bounded operator on $L^p(\mathbb{R}_+ A_{\alpha,\beta}(x)\,dx)$ for all $1 \leq p \leq \infty$. Additionally, $H_t f \geq 0$ whenever $f \geq 0$.

In [21], the following pointwise estimate of the Gaussian kernel is established.

Theorem 2.2. *Let $\alpha \geq \beta \geq -\frac{1}{2}$, $\alpha \neq -1/2$, and $x > 0$. Then the heat kernel $E_t^{(\alpha,\beta)}(x)$ satisfies the following estimate:*

$$E_t^{(\alpha,\beta)}(x) \asymp t^{-\alpha-1} e^{-\rho^2 t - \rho x - \frac{x^2}{4t}} (1+t+x)^{\alpha-1/2}(1+x),$$

where $f \asymp g$ means that constants $C_1, C_2 > 0$ exist such that

$$C_1\, g \leq f \leq C_2\, g.$$

We now recall the definition and some useful properties of the modified Bessel function of the second kind. For $\alpha \notin \mathbb{Z}$, the function $K_\alpha(x)$ is defined by [37]

$$K_\alpha(x) = \frac{\pi}{2\sin(\alpha\pi)}\left(I_{-\alpha}(x) - I_\alpha(x)\right), \tag{2.15}$$

where $I_\alpha(x)$ denotes the modified Bessel function of the first kind, given by the series expansion:

$$I_\alpha(x) = \sum_{n=0}^{\infty} \frac{1}{n!\,\Gamma(\alpha+n+1)} \left(\frac{x}{2}\right)^{2n+\alpha}, \quad x > 0. \tag{2.16}$$

An integral representation of $K_\alpha(x)$, valid for $\Re(x) > 0$, is

$$K_\alpha(x) = 2^{-1-\alpha} x^\alpha \int_0^\infty e^{-t} e^{-\frac{x^2}{4t}} t^{-1-\alpha}\, dt. \tag{2.17}$$

The asymptotic behavior of $K_\alpha(x)$ is characterized as follows:

- As $x \to \infty$, the function exhibits exponential decay

$$K_\alpha(x) \sim \sqrt{\frac{\pi}{2x}}\, e^{-x}. \tag{2.18}$$

- As $x \to 0^+$, the behavior depends on the value of α

$$K_\alpha(x) \sim \begin{cases} 2^{\alpha-1}\Gamma(\alpha) x^{-\alpha}, & \text{if } \alpha > 0, \\ -\log\left(\frac{x}{2}\right) - \gamma, & \text{if } \alpha = 0, \end{cases} \tag{2.19}$$

  where γ is an Euler–Mascheroni-type constant defined by

$$\gamma = \lim_{n\to\infty} \left(-\log n + \sum_{k=1}^{n} \frac{1}{k} \right) = \int_1^\infty \left(-\frac{1}{x} + \frac{1}{\lfloor x \rfloor} \right) dx. \tag{2.20}$$

A natural approach to defining the fractional powers of the operator $-\mathscr{L}_{\alpha,\beta}$ is via the generalized Fourier transform. For a parameter $0 < s < 1$, we define the fractional power operator $(-\mathscr{L}_{\alpha,\beta})^s$ as a pseudo-differential operator within the framework of generalized Fourier–Jacobi analysis. To ensure the well-posedness of this operator, we first introduce a suitable fractional Sobolev-type space

$$\mathcal{H}_s := \left\{ f \in L^2(\mathbb{R}_+, A_{\alpha,\beta}(x)\, dx) \;\middle|\; (\lambda^2 + \rho^2)^{s/2}\, \mathscr{F} f(\lambda) \in L^2\left(\mathbb{R}_+, \frac{d\lambda}{2\pi |c(\lambda)|^2} \right) \right\}. \tag{2.21}$$

For any $f \in \mathcal{H}_s$, the fractional relativistic Bessel operator is then defined by

$$(-\mathscr{L}_{\alpha,\beta})^{s/2} f = \mathcal{F}_{\alpha,\beta}^{-1} \left((\lambda^2 + \rho^2)^{s/2}\, \mathcal{F}_{\alpha,\beta} f \right). \tag{2.22}$$

It is well known that the space $C_0(\mathbb{R})$ of continuous functions with compact support is dense in the Hilbert space $\mathcal{H}_s$.

The following theorem constitutes a central result of this work, providing a singular integral representation for the fractional powers of the Jacobi–Laplacian.

Theorem 2.3. *Let $\alpha \geq \beta \geq -\frac{1}{2}$, and let $0 < s < 1$. Then, for every $f \in C_0(\mathbb{R})$, the fractional operator $(-\mathscr{L}_{\alpha,\beta})^s$ admits the singular–integral representation*

$$(-\mathscr{L}_{\alpha,\beta})^s f(x) = \frac{s}{\Gamma(1-s)} \int_0^\infty [f(x) - T_x^{(\alpha,\beta)} f(y)]\, w_s^{(\alpha,\beta)}(y)\, dy, \tag{2.23}$$

where the weight kernel $w_s^{(\alpha,\beta)}(y)$ is given explicitly by

$$w_s^{(\alpha,\beta)}(y) := A_{\alpha,\beta}(y) \int_0^\infty \frac{E_t^{(\alpha,\beta)}(y)}{t^{1+s}}\, dt, \tag{2.24}$$

and $E_t^{(\alpha,\beta)}(y)$ denotes the Jacobi heat kernel.

3. Proof of the main result

Before presenting the proof of the main theorem, we first develop several auxiliary lemmas that will be essential for the subsequent analysis. The proof itself will be given in this section, where we restrict our attention to the case $\alpha \geq \beta \geq -\frac{1}{2}$ with $\alpha \neq -\frac{1}{2}$. The remaining case will be treated separately in Section 4.

Lemma 3.1. *Let $f \in C^2(\mathbb{R})$ be an even function. Then for all $x \geq 0$, we have*

$$f(x) = f(0) + \int_0^x \vartheta^{(\alpha,\beta)}(x,\xi)\, \mathscr{L}_{\alpha,\beta} f(\xi)\, A_{\alpha,\beta}(\xi)\, d\xi,$$

where

$$\vartheta^{(\alpha,\beta)}(x,\xi) = \int_\xi^x \frac{d\eta}{A_{\alpha,\beta}(\eta)}, \quad 0 < \xi \leq x.$$

Proof. Since $f \in C^2(\mathbb{R})$, we have

$$f(x) - f(0) = \int_0^x f'(\xi)\, d\xi.$$

Noting that

$$f'(\xi) = -\frac{d}{d\xi}\left(\int_\xi^x \frac{d\eta}{A_{\alpha,\beta}(\eta)} \right) \cdot f'(\xi) \cdot A_{\alpha,\beta}(\xi),$$

we integrate by parts to obtain

$$f(x) - f(0) = \int_0^x \vartheta^{(\alpha,\beta)}(x,\xi)\, \mathscr{L}_{\alpha,\beta} f(\xi)\, A_{\alpha,\beta}(\xi)\, d\xi.$$

□

Lemma 3.2. *The Jacobi function $\varphi_\lambda^{(\alpha,\beta)}(x)$ possesses the following properties:*

(i) $\left|\varphi_\lambda^{(\alpha,\beta)}(x)\right| \leq 1$,
(ii) $1 - \varphi_\lambda^{(\alpha,\beta)}(x) \leq \frac{1}{2}(\lambda^2 + \rho^2)x^2$.

Proof. The first assertion (i) is a special case of a known result (see [18, p. 248]); its proof is given in [17, p. 153, Lemma 11]. Mentioning this fact does not affect the novelty of the paper.

(ii) Since $\varphi_\lambda^{(\alpha,\beta)}(x) \in C^2(\mathbb{R})$ and is even, by Lemma 3.1 and Eq (2.1), we have

$$1 - \varphi_\lambda^{(\alpha,\beta)}(x) = (\lambda^2 + \rho^2) \int_0^x \int_\xi^x \frac{d\eta}{A_{\alpha,\beta}(\eta)} \varphi_\lambda^{(\alpha,\beta)}(\xi) A_{\alpha,\beta}(\xi)\, d\xi.$$

Since $A'_{\alpha,\beta}(x) \geq 0$ for all $x \geq 0$, the weight function $A_{\alpha,\beta}(x)$ is non-decreasing. Hence, for any $0 \leq \xi \leq \eta$, we have

$$A_{\alpha,\beta}(\xi) \leq A_{\alpha,\beta}(\eta).$$

Using this monotonicity property, we obtain the following estimate:

$$1 - \varphi_\lambda^{(\alpha,\beta)}(x) \leq (\lambda^2 + \rho^2) \int_0^x \int_\xi^x |\varphi_\lambda^{(\alpha,\beta)}(\xi)|\, d\eta\, d\xi.$$

Applying property (i), $|\varphi_\lambda^{(\alpha,\beta)}(\xi)| \leq 1$, we deduce

$$1 - \varphi_\lambda^{(\alpha,\beta)}(x) \leq (\lambda^2 + \rho^2) \int_0^x \int_\xi^x d\eta\, d\xi = \frac{1}{2}(\lambda^2 + \rho^2) x^2.$$

This completes the proof. □

Lemma 3.3. *Let $0 < s < 1$, and let $\lambda \in \mathbb{R}$ with $\lambda^2 + \rho^2 \neq 0$. Then the following integral representation holds:*

$$(\lambda^2 + \rho^2)^s = \frac{s}{\Gamma(1-s)} \int_0^\infty \left(1 - \varphi_\lambda^{(\alpha,\beta)}(x)\right) w_s^{\alpha,\beta}(x)\, dx, \tag{3.1}$$

where the weight function $w_s^{\alpha,\beta}(x)$ is given by

$$w_s^{\alpha,\beta}(x) = A_{\alpha,\beta}(x) \int_0^\infty \frac{E_t^{(\alpha,\beta)}(x)}{t^{1+s}}\, dt. \tag{3.2}$$

Proof. We begin by recalling a classical identity involving the Gamma function, obtained via integration by parts:

$$\int_0^\infty t^{-(1+s)} \left(1 - e^{-t(\lambda^2+\rho^2)}\right) dt = \frac{\Gamma(1-s)}{s} (\lambda^2 + \rho^2)^s, \quad 0 < s < 1. \tag{3.3}$$

On the other hand, by the spectral representation of the heat semigroup (see Eq (2.13)), we have the following for every $t > 0$:

$$1 - e^{-t(\lambda^2+\rho^2)} = \int_0^\infty E_t^{(\alpha,\beta)}(x) \left(1 - \varphi_\lambda^{(\alpha,\beta)}(x)\right) A_{\alpha,\beta}(x)\, dx.$$

Multiplying both sides by $t^{-(1+s)}$ and integrating over $t \in (0, \infty)$, we obtain

$$\int_0^\infty t^{-(1+s)} \left(1 - e^{-t(\lambda^2+\rho^2)}\right) dt = \int_0^\infty \int_0^\infty \frac{E_t^{(\alpha,\beta)}(x)}{t^{1+s}} \left(1 - \varphi_\lambda^{(\alpha,\beta)}(x)\right) A_{\alpha,\beta}(x)\, dx\, dt.$$

Multiplying both sides by $t^{-(1+s)}$ and integrating over $t \in (0, \infty)$, we obtain

$$\int_0^\infty t^{-(1+s)} \left(1 - e^{-t(\lambda^2+\rho^2)}\right) dt = \int_0^\infty \int_0^\infty \frac{E_t^{(\alpha,\beta)}(x)}{t^{1+s}} \left(1 - \varphi_\lambda^{(\alpha,\beta)}(x)\right) A_{\alpha,\beta}(x)\, dx\, dt.$$

The interchange of the order of integration is justified by the Fubini–Tonelli theorem, since the integrand is non-negative for all $t > 0$ and $x \geq 0$. Specifically, we use the facts that

$$E_t^{(\alpha,\beta)}(x) \geq 0 \quad \text{and} \quad 1 - \varphi_\lambda^{(\alpha,\beta)}(x) \geq 0,$$

as established in [1, Theorem II.2] and part (i) of Lemma 3.2. So, the integrand in the above double integral

$$\frac{E_t^{(\alpha,\beta)}(x)}{t^{1+s}} \left(1 - \varphi_\lambda^{(\alpha,\beta)}(x)\right) A_{\alpha,\beta}(x)$$

is positive for all $t > 0$ and $x \geq 0$, and hence the Fubini–Tonelli theorem ensures the validity of the interchange in the order of integration.

Comparing this result with (3.3), we deduce the desired representation:

$$(\lambda^2 + \rho^2)^s = \frac{s}{\Gamma(1-s)} \int_0^\infty \left(1 - \varphi_\lambda^{(\alpha,\beta)}(x)\right) w_s^{\alpha,\beta}(x)\, dx,$$

where the weight function $w_s^{\alpha,\beta}(x)$ is given by

$$w_s^{\alpha,\beta}(x) = A_{\alpha,\beta}(x) \int_0^\infty \frac{E_t^{(\alpha,\beta)}(x)}{t^{1+s}}\, dt.$$

This concludes the proof. □

From Theorem 2.2, an elementary computation gives the following result.

Lemma 3.4. *Let $\alpha \geq \beta \geq -\frac{1}{2}$. Then the function $E_t^{(\alpha,\beta)}(x)$ satisfies the following upper bounds:*

- *If $\alpha \geq \frac{1}{2}$, then*

$$E_t^{(\alpha,\beta)}(x) \lesssim e^{-\rho^2 t - \rho x - \frac{x^2}{4t}} \cdot \begin{cases} (x+1)^{\alpha+\frac{1}{2}}\, t^{-\alpha-1}, & \text{if } t \leq x+1, \\ t^{-\frac{1}{2}}, & \text{if } t \geq x+1. \end{cases}$$

- *If $-\frac{1}{2} \leq \alpha \leq \frac{1}{2}$, then*

$$E_t^{(\alpha,\beta)}(x) \lesssim (x+1)^{\alpha+\frac{1}{2}}\, t^{-\alpha-1}\, e^{-\rho^2 t - \rho x - \frac{x^2}{4t}}.$$

Lemma 3.5. *Let $\alpha \geq \beta \geq -\frac{1}{2}$ with $\alpha \neq -\frac{1}{2}$, and let $0 < s < 1$. Then*

$$\int_0^\infty \frac{x^2}{x^2+1}\, w_s^{(\alpha,\beta)}(x)\, dx < \infty,$$

where $w_s^{(\alpha,\beta)}(x)$ denotes the fractional weight associated with the Jacobi-type heat kernel, as defined in Eq (3.2).

Proof. We treat the two parameter ranges $\alpha \geq \frac{1}{2}$ and $-\frac{1}{2} < \alpha \leq \frac{1}{2}$.
Case 1. ($\alpha \geq \frac{1}{2}$) From Lemma 3.2, we have

$$\int_0^\infty \frac{E_t^{(\alpha,\beta)}(x)}{t^{1+s}}\,dt = \int_0^{x+1} \frac{E_t^{(\alpha,\beta)}(x)}{t^{1+s}}\,dt + \int_{x+1}^\infty \frac{E_t^{(\alpha,\beta)}(x)}{t^{1+s}}\,dt$$
$$\lesssim (x+1)^{\alpha+1/2} e^{-\rho x} \int_0^{x+1} \frac{e^{-\rho^2 t - \frac{x^2}{4t}}}{t^{2+s+\alpha}}\,dt$$
$$+ e^{-\rho x} \int_{x+1}^\infty \frac{e^{-\rho^2 t - \frac{x^2}{4t}}}{t^{1+s+1/2}}\,dt.$$

Extending both integrals above to $(0,\infty)$ for an upper bound gives

$$\int_0^\infty \frac{E_t^{(\alpha,\beta)}(x)}{t^{1+s}}\,dt \lesssim (x+1)^{\alpha+1/2} e^{-\rho x} \int_0^\infty \frac{e^{-\rho^2 t - \frac{x^2}{4t}}}{t^{2+s+\alpha}}\,dt + e^{-\rho x} \int_0^\infty \frac{e^{-\rho^2 t - \frac{x^2}{4t}}}{t^{1+s+1/2}}\,dt.$$

By the integral representation of the modified Bessel function of the second kind K_ν (see (2.17)), we deduce

$$\int_0^\infty \frac{E_t^{(\alpha,\beta)}(x)}{t^{1+s}}\,dt \lesssim (x+1)^{\alpha+1/2} e^{-\rho x} \frac{K_{1+s+\alpha}(\rho x)}{x^{1+s+\alpha}} + e^{-\rho x} \frac{K_{s+1/2}(\rho x)}{x^{s+1/2}}.$$

Hence,

$$w_s^{(\alpha,\beta)}(x) \lesssim A_{\alpha,\beta}(x) e^{-\rho x} \left[(x+1)^{\alpha+1/2} \frac{K_{1+s+\alpha}(\rho x)}{x^{1+s+\alpha}} + \frac{K_{s+1/2}(\rho x)}{x^{s+1/2}} \right]. \tag{3.4}$$

Case 2. ($-\frac{1}{2} < \alpha \leq \frac{1}{2}$) In this range, the behavior of $w_s^{(\alpha,\beta)}(x)$ is dominated by the second term in (3.4). From Lemma 3.4 and similar estimates, we have

$$w_s^{(\alpha,\beta)}(x) \lesssim A_{\alpha,\beta}(x) e^{-\rho x} \frac{K_{s+1/2}(\rho x)}{x^{s+1/2}}. \tag{3.5}$$

Using the asymptotic behavior of the modified Bessel function $K_\alpha(x)$ both near infinity Eq (2.18) and near zero Eq (2.19), together with the asymptotics of $A_{\alpha,\beta}(x)$

$$A_{\alpha,\beta}(x) \sim x^{2\alpha+1} \quad \text{as } x \to 0^+, \qquad A_{\alpha,\beta}(x) \sim e^{-2\rho x} \quad \text{as } x \to \infty,$$

the bounds in (3.4) and (3.5) imply that

$$w_s^{(\alpha,\beta)}(x) \lesssim \frac{1}{x^{2s+1}} \quad \text{as } x \to 0^+, \qquad w_s^{(\alpha,\beta)}(x) \lesssim \frac{1}{x^{s+1}} \quad \text{as } x \to \infty.$$

Under the given conditions on s, it follows that

$$\int_0^\infty \frac{x^2}{x^2+1} w_s^{(\alpha,\beta)}(x)\,dx < \infty.$$

□

Now to the proof of Theorem 2.3.

Proof of Theorem 2.3. Let $f \in C_0(\mathbb{R})$. We first demonstrate that the integral on the right-hand side of (2.23) is well-defined. To this end, we split the integral into two parts as follow

$$\int_0^\infty \left|f(x) - T_x^{(\alpha,\beta)} f(y)\right| w_s^{(\alpha,\beta)}(y)\, dy = J_1 + J_2,$$

where we define

$$J_1 := \int_0^1 \left|f(x) - T_x^{(\alpha,\beta)} f(y)\right| w_s^{(\alpha,\beta)}(y)\, dy, \quad J_2 := \int_1^\infty \left|f(x) - T_x^{(\alpha,\beta)} f(y)\right| w_s^{(\alpha,\beta)}(y)\, dy.$$

Using the representation of the generalized translation operator $T_x^{(\alpha,\beta)}$ given in (2.11) and applying the inversion formula for the Jacobi transform, we have

$$f(x) - T_x^{(\alpha,\beta)} f(y) = \int_0^\infty \varphi_\lambda^{(\alpha,\beta)}(x)\left(1 - \varphi_\lambda^{(\alpha,\beta)}(y)\right) \mathcal{F}_{\alpha,\beta} f(\lambda)\, \frac{d\lambda}{2\pi |c(\lambda)|^2}.$$

By Lemma 3.2, we obtain the following estimate:

$$|f(x) - T_x^{(\alpha,\beta)} f(y)| \leq \frac{1}{2} y^2 \int_0^\infty (\lambda^2 + \rho^2)\, |\mathcal{F}_{\alpha,\beta} f(\lambda)|\, \frac{d\lambda}{2\pi |c(\lambda)|^2}. \tag{3.6}$$

Using the asymptotic estimate from [18], we have

$$|c_{\alpha,\beta}(\lambda)|^{-2} \asymp (1 + |\lambda|)^{2\alpha+1} \quad \text{as } \lambda \to +\infty,$$

we deduce that an integer $n_0 \in \mathbb{N}$ exists such that

$$C_0 := \int_0^\infty \frac{1}{(1 + \lambda^2 + \rho^2)^{n_0}}\, \frac{d\lambda}{2\pi |c(\lambda)|^2} < \infty.$$

Since $f \in C_0(\mathbb{R})$, the Paley–Wiener theorem for the Jacobi transform (see [23]) implies that $\mathcal{F}_{\alpha,\beta} f \in \mathscr{A}$, the space of rapidly decreasing functions. Therefore, a constant $A > 0$ exists such that

$$|\mathcal{F}_{\alpha,\beta} f(\lambda)| \leq \frac{A}{(1 + \lambda^2 + \rho^2)^{n_0+1}}.$$

Combining these estimates, we obtain

$$\int_0^\infty (\lambda^2 + \rho^2)\, |\mathcal{F}_{\alpha,\beta} f(\lambda)|\, \frac{d\lambda}{2\pi |c(\lambda)|^2} \leq AC_0 < \infty.$$

As a result, the estimate

$$|f(x) - T_x^{(\alpha,\beta)} f(y)| \leq \frac{1}{2} A C_0 y^2$$

holds for all $x, y \geq 0$.

Therefore

$$J_1 := \int_0^1 \left|f(x) - T_x^{(\alpha,\beta)} f(y)\right| w_s^{(\alpha,\beta)}(y)\, dy \leq \frac{1}{2} A C_0 \int_0^1 y^2 w_s^{(\alpha,\beta)}(y)\, dy.$$

Finally, from Lemma 3.5, we conclude that $J_1 < \infty$.

For J_2, we use the fact that $f \in C_0(\mathbb{R})$, along with the bound $|T_x^{(\alpha,\beta)} f(y)| \le \|f\|_\infty$. It follows that

$$|f(x) - T_x^{(\alpha,\beta)} f(y)| \le 2\|f\|_\infty. \tag{3.7}$$

Since the weight function $w_s^{(\alpha,\beta)}(y)$ is integrable over the interval $(1, \infty)$, we conclude that $J_2 < \infty$.

Thus, the integral on the right-hand side of (2.23) is finite, and the formula is well-defined.

Applying the generalized Fourier transform to both sides of (2.23), and using Lemma 3.3 together with the symmetry relation $T_x^{(\alpha,\beta)} f(y) = T_y^{(\alpha,\beta)} f(x)$, we deduce, by the identity

$$\mathscr{F}_{\alpha,\beta}\left(T_y^{(\alpha,\beta)} f\right)(\lambda) = \varphi^{(\alpha,\beta)}(y)\,\mathscr{F}_{\alpha,\beta} f(\lambda),$$

that

$$\begin{aligned}
&\mathscr{F}_{\alpha,\beta}\left[\frac{1}{|\Gamma(-s)|}\int_0^\infty \left\{f(x) - T_x^{(\alpha,\beta)} f(y)\right\} w_s^{(\alpha,\beta)}(y)\,dy\right](\lambda)\\
&= \left(\frac{1}{|\Gamma(-s)|}\int_0^\infty \left(1 - \varphi^{(\alpha,\beta)}(y)\right) w_s^{(\alpha,\beta)}(y)\,dy\right)\mathscr{F}_{\alpha,\beta} f(\lambda)\\
&= (\lambda^2 + \rho^2)^s\,\mathscr{F}_{\alpha,\beta} f(\lambda).
\end{aligned}$$

To complete the proof, it remains to justify the interchange of the Jacobi transform and the integral in the equation above. Specifically, we need to show that

$$x \mapsto \int_0^\infty \left|f(x) - T_x^{(\alpha,\beta)} f(y)\right| w_s^{(\alpha,\beta)}(y)\,dy \in L^1(\mathbb{R}_+, A_{\alpha,\beta}(x)\,dx). \tag{3.8}$$

Applying Lemma 3.1 to the function $y \mapsto T_x^{(\alpha,\beta)} f(y)$, and noting that $T_x^{(\alpha,\beta)} f(0) = f(x)$, we obtain

$$T_x^{(\alpha,\beta)} f(y) = f(x) + \int_0^y \vartheta^{(\alpha,\beta)}(y,\xi)\,T_x^{(\alpha,\beta)}\mathscr{L}_{\alpha,\beta} f(\xi)\,A_{\alpha,\beta}(\xi)\,d\xi. \tag{3.9}$$

Since the operator $T_x^{(\alpha,\beta)}$ commutes with $\mathscr{L}_{\alpha,\beta}$ and $T_x^{(\alpha,\beta)} f(y)$ is symmetric in its variables x and y, it follows from (2.11) that

$$\|\mathscr{L}_{\alpha,\beta} T_x^{(\alpha,\beta)} f\|_{L^1(\mathbb{R}_+, A_{\alpha,\beta} dx)} = \|T_x^{(\alpha,\beta)}\mathscr{L}_{\alpha,\beta} f\|_{L^1(\mathbb{R}_+, A_{\alpha,\beta} dx)} \le \|\mathscr{L}_{\alpha,\beta} f\|_{L^1(\mathbb{R}_+, A_{\alpha,\beta} dx)}.$$

Combining this estimate with (3.9) and using the monotonicity property of $A_{\alpha,\beta}$, namely

$$A_{\alpha,\beta}(\xi) \le A_{\alpha,\beta}(\eta) \quad \text{for} \quad 0 \le \xi \le \eta,$$

we deduce

$$\begin{aligned}
\|f - T_x^{(\alpha,\beta)} f(y)\|_{L^1(\mathbb{R}_+, A_{\alpha,\beta}\,dx)} &\le \|\mathscr{L}_{\alpha,\beta} f\|_{L^1(\mathbb{R}_+, A_{\alpha,\beta}\,dx)} \int_0^y \vartheta^{(\alpha,\beta)}(y,\xi)\,A_{\alpha,\beta}(\xi)\,d\xi\\
&\le \frac{1}{2}\,\|\mathscr{L}_{\alpha,\beta} f\|_{L^1(\mathbb{R}_+, A_{\alpha,\beta}\,dx)}\,y^2.
\end{aligned}$$

In view of (3.7), we therefore conclude that

$$\begin{aligned}\int_0^\infty &\|f - T_x^{(\alpha,\beta)} f(y)\|_{L^1(\mathbb{R}_+, A_{\alpha,\beta}\,dx)}\, w_s^{(\alpha,\beta)}(y)\,dy \\ &\le \|\mathscr{L}_{\alpha,\beta} f\|_{L^1(\mathbb{R}_+, A_{\alpha,\beta}\,dx)} \int_0^1 y^2\, w_s^{(\alpha,\beta)}(y)\,dy \\ &\quad + \ 2\|f\|_{L^1(\mathbb{R}_+, A_{\alpha,\beta}\,dx)} \int_1^\infty w_s^{(\alpha,\beta)}(y)\,dy \\ &< \infty,\end{aligned}$$

which justifies the interchange of the Jacobi transform with the integral. □

4. Examples

4.1. The case $\alpha = \beta = -\frac{1}{2}$

An important special case arises when $\alpha = \beta = -\frac{1}{2}$. In this setting, the Jacobi–Laplacian simplifies drastically and coincides with the standard second-order Laplacian on the real line:

$$\mathscr{L}_{-1/2,-1/2} = \frac{d^2}{dx^2}.$$

Consequently, its fractional powers reproduce the well-known (even) fractional Laplacian on $\mathbb{R}$. More precisely, for $0 < s < 1$, the operator

$$\big(-\mathscr{L}_{-1/2,-1/2}\big)^{s/2} = \big(-\tfrac{d^2}{dx^2}\big)^{s/2}$$

is defined as a pseudo-differential operator with the symbol λ^s. In the case of an even function $f : \mathbb{R} \to \mathbb{R}$, one has

$$\mathcal{F}_c\Big[\big(-\tfrac{d^2}{dx^2}\big)^{s/2} f\Big](\lambda) = \lambda^s\, \mathcal{F}_c f(\lambda), \qquad \lambda \ge 0,$$

where $\mathcal{F}_c$ denotes the cosine Fourier transform:

$$\mathcal{F}_c f(\lambda) := \int_0^\infty f(x)\, \cos(\lambda x)\,dx.$$

Furthermore, a singular integral representation derived from its Fourier symbol is given by [7]

$$\big(-\tfrac{d^2}{dx^2}\big)^{s/2} f(x) = \frac{\Gamma(1+s)}{2\pi}\, \sin\big(\tfrac{\pi s}{2}\big) \int_0^\infty \frac{f(x) - T_x f(y)}{y^{1+s}}\,dy,$$

where T_x denotes the even translation operator, defined as

$$T_x f(y) := \frac{f(x+y) + f(x-y)}{2}.$$

4.2. The case of $\alpha = \frac{1}{2}$ and $\beta = -\frac{1}{2}$

In the case of $\alpha = \frac{1}{2}$ and $\beta = -\frac{1}{2}$, the unique solution to the boundary value problem

$$\begin{cases} \dfrac{d^2u}{dx^2} + \tanh(x)\,\dfrac{du}{dx} = -(\lambda^2+1)\,u, \\ u(0) = 1, \quad u'(0) = 0 \end{cases}$$

is explicitly given by

$$\varphi_\lambda^{(1/2,-1/2)}(x) = \begin{cases} \dfrac{\sin(\lambda x)}{\lambda \sinh x}, & \text{if } \lambda \neq 0, \\ \dfrac{x}{\sinh x}, & \text{if } \lambda = 0. \end{cases}$$

The associated Fourier–Jacobi transform pair is

$$\begin{cases} \mathcal{F}_{(1/2,-1/2)} f(\lambda) = 4\displaystyle\int_0^\infty f(x)\,\varphi_\lambda^{(1/2,-1/2)}(x)\,\sinh^2(x)\,dx, \\ \qquad\qquad f(x) = \dfrac{1}{2\pi}\displaystyle\int_0^\infty \mathcal{F}_{(1/2,-1/2)} f(\lambda)\,\varphi_\lambda^{(1/2,-1/2)}(x)\,\lambda^2\,d\lambda. \end{cases} \tag{4.1}$$

The generalized translation operator $T_x^{(1/2,-1/2)} f(y)$ is given by

$$T_x^{(1/2,-1/2)} f(y) = \frac{1}{2\sinh x\,\sinh y}\int_{|x-y|}^{x+y} f(u)\,\sinh u\,du.$$

The heat kernel $E_t^{(1/2,-1/2)}(x)$ corresponding to this setting is

$$E_t^{(1/2,-1/2)}(x) = \frac{1}{2\sqrt{\pi}\,t^{3/2}} \cdot \frac{x}{\sinh x} \cdot e^{-t-\frac{x^2}{4t}}.$$

Indeed, from (2.13) and the inversion formula for the Jacobi function transform (4.1), we have

$$\begin{aligned} E_t^{(1/2,-1/2)}(x) &= \frac{1}{2\pi}\int_0^\infty e^{-t(\lambda^2+1)}\,\varphi_\lambda^{(1/2,-1/2)}(x)\,\lambda^2\,d\lambda \\ &= \frac{e^{-t}}{2\pi \sinh x}\int_0^\infty e^{-\lambda^2 t}\,\sin(\lambda x)\,\lambda\,d\lambda. \end{aligned}$$

Using the classical identity (see Watson [37, Chapters 12 and 13])

$$\int_0^\infty J_\nu(at)\,t^{\nu+1} e^{-p^2t^2}\,dt = \frac{a^\nu}{(2p^2)^{\nu+1}} e^{-\frac{a^2}{4p^2}}, \quad \Re(\nu) > -1,$$

and the known formula

$$J_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin x,$$

where $J_\nu(x)$ is the Bessel function of the first kind, we obtain

$$E_t^{(1/2,-1/2)}(x) = \frac{1}{2\sqrt{\pi}\,t^{3/2}} \cdot \frac{x}{\sinh x} \cdot e^{-t-\frac{x^2}{4t}}.$$

The associated density function $w_s^{(1/2,-1/2)}(x)$, defined by (2.24), is given by

$$\begin{aligned} w_s^{(1/2,-1/2)}(x) &= 4\sinh^2(x)\int_0^\infty \frac{E_t^{(1/2,-1/2)}(x)}{t^{1+s}}\,dt \\ &= \frac{1}{2\sqrt{\pi}}\,x\sinh x\int_0^\infty \frac{e^{-t-\frac{x^2}{4t}}}{t^{5/2+s}}\,dt \\ &= \frac{2^{s+\frac{3}{2}}}{\sqrt{\pi}}\,x^{-s-\frac{1}{2}}\sinh x\,K_{s+\frac{3}{2}}(x), \end{aligned}$$

where $K_\nu(x)$ is the modified Bessel function of the second kind.

Corollary 4.1. *Let $0 < s < 1$, and suppose that $f \in \mathcal{D}(\mathbb{R})$, the space of smooth functions with compact support. Then*

$$\left(-\frac{d^2}{dx^2} - 2\tanh(x)\,\frac{d}{dx}\right)^s f(x) = \gamma(s)\int_0^\infty \left\{f(x) - T_x^{(1/2,-1/2)}f(y)\right\}\,w_s^{(1/2,-1/2)}(y)\,dy,$$

where:

- *$T_x^{(1/2,-1/2)}f(y)$ is the generalized translation operator*

$$T_x^{(1/2,-1/2)}f(y) = \frac{1}{2\sinh x\,\sinh y}\int_{|x-y|}^{x+y} f(u)\,\sinh u\,du.$$

- *The weight function $w_s^{(1/2,-1/2)}(y)$, involving the modified Bessel function $K_{s+\frac{3}{2}}$, is given by*

$$w_s^{(1/2,-1/2)}(y) = y^{-s-\frac{1}{2}}\sinh y\,K_{s+\frac{3}{2}}(y).$$

- *The normalization constant $\gamma(s)$ is*

$$\gamma(s) = \frac{2^{s+\frac{3}{2}}}{\sqrt{\pi}}\cdot\frac{s}{\Gamma(1-s)}.$$

5. Conclusions

In this work, we have investigated fractional powers of the one-dimensional Jacobi–Laplacian $\mathscr{L}_{\alpha,\beta}$ and developed a novel singular integral representation based on harmonic analysis techniques adapted to the Jacobi setting. Building on the Fourier–Jacobi transform and the generalized translation operator, we have established an explicit formula for $(-\mathscr{L}_{\alpha,\beta})^{s/2}$, thereby extending the classical framework of fractional operators to this family of second-order differential operators with nontrivial weights.

Our approach complements existing definitions via spectral calculus and pseudo-differential operator theory, offering a more transparent, kernel-based interpretation of fractional Jacobi operators. The proposed representation paves the way for further analysis of mapping properties, boundedness results, and potential applications to evolution equations with memory effects on noncompact Riemannian symmetric spaces and related geometries.

Future research directions include the following:

- Extending the current results to higher-dimensional Jacobi settings and more general rank-one symmetric spaces.
- Investigating numerical methods for solving equations driven by fractional Jacobi operators.
- Exploring connections between the singular integral formulation, the extension problem à la Caffarelli–Silvestre, and probabilistic representations via Lévy processes.

Use of Generative-AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that there are no conflicts of interest regarding the publication of this work.

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