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Research article

Adams-Bashforth methods on bigeometric calculus

Mehmet Çağrı Yilmazer[1], Sertac Goktas[2], Emrah Yilmaz[1,*] and Mikail Et[1]

[1] Department of Mathematics, Faculty of Science, Firat University, Elazığ, Türkiye
[2] Department of Mathematics, Faculty of Science, Mersin University, Mersin, Türkiye

* **Correspondence:** Email: emrah231983@gmail.com.

Abstract: In this study, bigeometric Adams-Bashforth methods were defined as a strong alternative to classical numerical methods. After methods were established theoretically, error estimation formulas for each method were developed. Finally, a classical differential equation was converted to a bigeometric multiplicative differential equation, and it was solved by using bigeometric Adams-Bashforth methods. Then, error estimations were given.

Keywords: multiplicative calculus; bigeometric Adams-Bashforth methods; error estimation
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1. Introduction

In the second half of the 17th century, Newton and Leibniz invented one of the most important mathematical theories; differential calculus. Afterwards calculus was reoriented by Euler, focusing on the idea of function, and therefore analysis was founded. Differentiation and integration are fundamental operations in calculus. The creation of multiplicative analysis originates from 1938, when Volterra and Hostinsky [1] put forward the Volterra analysis that was later specified as a particular case of multiplicative analysis. During 1967 and 1972, non-Newtonian calculus was presented by Grossman and Katz [2]. This theory has branches of bigeometric, geometric, biquadratic, and quadratic calculus, and so on. Moreover, this notion was extended to other fields in [3, 4] by Grossman.

Several mathematicians have comprehensively contributed to bigeometric calculus [5–8]. Georgiev et al. gave a comprehensive mathematical explanation of bigeometric calculus in [9–11]. Non-Newtonian, especially bigeometric, numerical estimation methods were put forward by many authors. For instance, Aniszewska [12] invented Runge-Kutta method in bigeometric format for implementations of dynamic systems. Also, Kadak et al. [13] presented a generalized form of Runge-Kutta techniques in a non-Newtonian sense and gave an example about second-order bigeometric Runge-Kutta as a specific case. Numan and Celik [14] introduced a novel technique called the bigeometric Newton method. In addition to numerical methods in bigeometric calculus, there are some crucial studies about them in multiplicative or geometric calculus. As an example, using multiplicative Taylor series, Riza and Aktore [15] developed geometric Runge-Kutta methods. Riza et al. [16] put forward multiplicative finite difference techniques to solve multiplicative and Volterra differential equations. The study of Adams Basforth-Moulton methods, which is also significant for this paper, was introduced by Misirli and Gurefe [17]. Apart from these studies, there are some important studies on numerical applications in multiplicative analysis in recent years [18–21].

This study is organized as follows. In section 2, basic concepts of bigeometric multiplicative analysis are expressed. In section 3, the multiplicative Adams-Bashforth method is given in detail. In the last section, a numerical example is given to support the theoretical part, and comparisons are made. In this study, the term multiplicative analysis refers to bigeometric multiplicative analysis.

2. Preliminaries

In this section, we present basic operations and concepts in bigeometric calculus, such as fundamental arithmetic operations, bigeometric derivative, and bigeometric integral formulas, to construct bigeometric Adams-Bashforth methods.

Definition 2.1. *[9] Multiplicative addition, subtraction, multiplication, division, and other basic operations in the set of* $\mathbb{R}_\star = (0, \infty)$ *are listed as below:*

$$\begin{array}{ll} a +_\star b = ab, & a^{k\star} = a^{(\log a)^k}, \\ a -_\star b = a/b, & {}_\star\sqrt{a} = e^{(\log a)^{\frac{1}{2}}}, \\ a \cdot_\star b = e^{\log a \log b}, & a^{-1\star} = e^{\frac{1}{\log a}}, \\ a/_\star b = e^{\frac{\log a}{\log b}}, & a \cdot_\star e = a, \\ a \cdot_\star a = a^{\log a}, & e^n \cdot_\star a = a^n, \end{array}$$

where $a, b \in \mathbb{R}_\star$*. Moreover,* $0_\star = 1$*,* $1_\star = e$*, and* $k_\star = e^k$*.*

Definition 2.2. *[9] Bigeometric factorial* $n!_\star$ *is described as follows:*

$$n!_\star = e^{n!},\ n \in \mathbb{N}.$$

Definition 2.3. *[9] Bigeometric binomial coefficient of* $n, k \in \mathbb{N}$ *under condition of* $n > k$ *is denoted as below:*

$$\binom{n}{k}_\star = e^{\binom{n}{k}}.$$

Definition 2.4. *[9] First-order multiplicative derivative in the bigeometric sense of* $f \in \mathcal{C}^1(A)$ *at* $x \in A \subseteq \mathbb{R}_\star$ *represented by the notation of* $f^\star(x)$ *is defined as below*

$$f^\star(x) = \lim_{h \to 0_\star} \left(f(x +_\star h) -_\star f(x) \right) /_\star h.$$

This formula also can be expressed as follows:

$$f^\star(x) = e^{\frac{x f'(x)}{f(x)}} \in \mathcal{C}^1_\star(A). \tag{2.1}$$

The space of all functions $f : A \to \mathbb{R}$ that are continuous on A and have continuous first multiplicative derivatives on A is denoted by $\mathcal{C}_\star^1(A)$.

Corollary 2.5. *[9] Let $f, g \in \mathcal{C}_\star^1(A)$ and $a \in \mathbb{R}_\star$. Some features of the first-order multiplicative derivative are listed below:*

- $(a \cdot_\star f)^\star(x) = a \cdot_\star f^\star(x)$,.
- $(af)^\star = f^\star(x)$,
- $(f +_\star g)^\star(x) = f^\star(x) +_\star g^\star(x)$,
- $(f -_\star g)^\star(x) = f^\star(x) -_\star g^\star(x)$,
- $(f \cdot_\star g)^\star(x) = f^\star(x) \cdot_\star g(x) +_\star f(x) \cdot_\star g^\star(x)$,
- $(f /_\star g)^\star(x) = (f^\star(x) \cdot_\star g(x) -_\star f(x) \cdot_\star g^\star(x)) /_\star (g(x))^{2\star}$,
- $(f \circ g)^\star(x) = (f^\star(g(x))) \cdot_\star g^\star(x)$,

where $x \in A$.

Definition 2.6. *[9] Let $A \subseteq \mathbb{R}_\star$, $f \in \mathcal{C}_\star^k(A)$, and $k \in \mathbb{N}$. Then, the $k-th$ order bigeometric multiplicative derivative formula is described as*

$$f^{\star(k)}(x) = \left(f^{\star(k-1)}\right)^\star(x), \ x \in A.$$

Definition 2.7. *[9] Let $A \subseteq \mathbb{R}_\star$, $f \in \mathcal{C}_\star^k(A)$, $k \in \mathbb{N}$, and $k \geq 2$. Then, the bigeometric multiplicative differential of x and $f(x)$ is defined as below, respectively:*

$$d_\star x = e^{d(\log x)} = e^{\frac{1}{x}dx},$$

and

$$d_\star f(x) = e^{d(\log f(x))} = e^{\frac{f'(x)}{f(x)}dx},$$

and

$$f^\star(x) = d_\star f(x) /_\star d_\star x,$$

where $x \in A$. By definition in the above, the following is obtained:

$$d_\star f(x) = f^\star(x) \cdot_\star d_\star x.$$

Definition 2.8. *[9] Assume that $a, b \in \mathbb{R}_\star$ and $f \in \mathcal{C}(\mathbb{R}_\star)$. Then, a bigeometric indefinite integral can be represented as follows:*

$$\int_\star f(x) \cdot_\star d_\star x = e^{\int \frac{1}{x} \log f(x) dx}, x \in \mathbb{R}_\star.$$

Bigeometric Cauchy integral is defined as below:

$$\int_{\star a}^{b} f(x) \cdot_\star d_\star x = e^{\int_a^b \frac{1}{x} \log f(x) dx}.$$

For $x \in \mathbb{R}_\star$, following holds:

$$d_\star \left(\int_{\star a}^{b} f(s) \cdot_\star d_\star s \right) /_\star d_\star x = f(x) \in \mathbb{R}_\star .$$

Furthermore, below equality is satisfied:

$$\int_{\star a}^{b} f^\star(s) \cdot_\star d_\star s = f(b) -_\star f(a) .$$

Corollary 2.9. *[9] Assume that $a, b, c \in \mathbb{R}_\star$ and $f, g \in \mathcal{C}(\mathbb{R}_\star)$. Then, the following features hold:*

- $\int_\star (a \cdot_\star f(x) +_\star b \cdot_\star g(x)) \cdot_\star d_\star x = a \cdot_\star \int_\star f(x) \cdot_\star d_\star x +_\star b \cdot_\star \int_\star g(x) \cdot_\star d_\star x$
- $\int_\star (a \cdot_\star f(x) -_\star b \cdot_\star g(x)) \cdot_\star d_\star x = a \cdot_\star \int_\star f(x) \cdot_\star d_\star x -_\star b \cdot_\star \int_\star g(x) \cdot_\star d_\star x$
- $\int_{\star a}^{a} f(x) \cdot_\star d_\star x = 0_\star$
- $\int_{\star a}^{b} f(x) \cdot_\star d_\star x = -_\star \int_{\star b}^{a} f(x) \cdot_\star d_\star x$
- $\int_{\star a}^{b} f(x) \cdot_\star d_\star x = \int_{\star a}^{c} f(x) \cdot_\star d_\star x +_\star \int_{\star c}^{b} f(x) \cdot_\star d_\star x$

All concepts and theorems related to the structure of bigeometric multiplicative analysis can be accessed from the studies numbered [2, 3, 9, 10]. After expressing basic concepts and rules in bigeometric calculus, the following numerical estimation method, which is also important for our novel bigeometric Adams-Bashforth techniques, is described in the below definition.

In [13], Runge-Kutta method for ordinary differential equations was developed in the frameworks of non-Newtonian calculus.

Definition 2.10. *Selecting $\alpha = e^x$ and $\beta = e^x$ in the generalized non-Newtonian Runge-Kutta method in [13] for the multiplicative differential equation defined as*

$$y^\star(x) = f(x, y), \ y(x_0) = y_0,$$

following second-order bigeometric Runge-Kutta formula is obtained:

$$\begin{aligned} y_0 &= \lambda, \\ y_{i+1} &= y_i f\left(x_i \sqrt{h}, y_i f(x_i, y_i)^{(\ln h)/2} \right)^{\ln h} . \end{aligned}$$

In this study, while bigeometric multiplicative Adams-Bashforth methods are being created, the second-order bigeometric Runge-Kutta method in definition 2.10 will also be used.

3. Main results

Before explaining bigeometric Adams-Bashforth methods, bigeometric Newton's backward interpolation formula will be given first. While doing this, the techniques of revealing the classical Adams-Bashforth methods were taken as a basis.

3.1. Bigeometric Newton's backward interpolation formula

The bigeometric backward interpolation formula is a mathematical tool underlying the bigeometric Adams-Bashforth method. With this formula, the bigeometric multiplicative derivative function is represented by known values at past points, and bigeometric multiplicative integral of this representation is used to calculate y_{m+1}. Therefore, the bigeometric Adams-Bashforth method is an explicit multistep method based on bigeometric backward interpolation.

Assume that $y_k = f(x_k)$ and $x_k = x_0 +_\star k_\star \cdot_\star h$, where $k = 0, 1, 2, \cdots, m$ and h is step size. Then, we define a polynomial of degree not exceeding m as follows:

$$\begin{aligned}\psi(x) &= c_0 +_\star c_1 \cdot_\star (x -_\star x_m) +_\star c_2 \cdot_\star (x -_\star x_m) \cdot_\star (x -_\star x_{m-1}) \\ &\quad +_\star c_3 \cdot_\star (x -_\star x_m) \cdot_\star (x -_\star x_{m-1}) \cdot_\star (x -_\star x_{m-2}) \\ &\quad +_\star \cdots +_\star c_n \cdot_\star (x -_\star x_n) \cdot_\star (x -_\star x_{m-1}) \cdot_\star \cdots \cdot_\star (x -_\star x_1),\end{aligned} \tag{3.1}$$

where c_k's are defined as constants, and we determine their values utilizing the following condition:

$$y_k = \psi(x_k), \ i = 0, 1, \cdots, m.$$

By inserting $x = x_m, x_{m-1}, \cdots x_1$ in 3.1, values of c_k's can be obtained. Substituting x_m into x, we have

$$\psi(x_m) = c_0, \ c_0 = y_m.$$

For $x = x_{m-1}$,

$$\psi(x_{m-1}) = c_0 +_\star c_1 \cdot_\star (x_{m-1} -_\star x_m), \text{ i.e. } y_{m-1} = y_m -_\star c_1 \cdot_\star h. \tag{3.2}$$

By 3.2, below equality is achieved:

$$c_1 = \frac{y_m -_\star y_{m-1}}{h}^\star = \frac{\nabla^\star y_m}{h}^\star.$$

By inserting x_{m-2} into x, we obtain

$$\begin{aligned}\psi(x_{m-2}) &= c_0 +_\star c_1 \cdot_\star (x_{m-2} -_\star x_m) +_\star c_2 \cdot_\star (x_{m-2} -_\star x_m) \cdot_\star (x_{m-2} -_\star x_{m-1}) \\ &= y_m -_\star \frac{y_m -_\star y_{m-1}}{h}^\star \cdot_\star 2_\star \cdot_\star h +_\star c_2 \cdot_\star (2_\star \cdot_\star h) \cdot_\star h.\end{aligned}$$

In other words, $y_{m-2} = 2_\star \cdot_\star y_{m-1} -_\star y_m +_\star c_2 \cdot_\star 2!_\star \cdot_\star h^{2\star}$. Therefore,

$$c_2 = \frac{y_m -_\star 2_\star \cdot_\star y_{m-1} +_\star y_{m-2}}{2_\star \cdot_\star h^{2\star}}^\star = \frac{\nabla^{2\star} y_m}{2!_\star \cdot_\star h^{2\star}}^\star.$$

In this way, all of c_k's can be achieved. Hence, we get generalize form as follows:

$$c_i = \frac{\nabla^{k\star} y_m}{i!_\star \cdot_\star h^{i\star}}^\star, \ i = 0, 1, \cdots, m.$$

By utilizing these values, $\psi(x)$ is obtained as below:

$$\psi(x) = y_m +_\star (x -_\star x_m) \cdot_\star \frac{\nabla^\star y_m}{h}^\star +_\star (x -_\star x_m) \cdot_\star (x -_\star x_{m-1}) \cdot_\star \frac{\nabla^{2\star} y_m}{2!_\star \cdot_\star h^{2\star}}$$

$$+_\star (x -_\star x_m) \cdot_\star (x -_\star x_{m-1}) \cdot_\star (x -_\star x_{m-2}) \cdot_\star \frac{\nabla^{3\star} y_m}{3!_\star \cdot_\star h^{3\star}}^\star +_\star \cdots$$
$$+_\star (x -_\star x_m) \cdot_\star (x -_\star x_{m-1}) \cdot_\star (x -_\star x_{m-2}) \cdot_\star \cdots \cdot_\star (x -_\star x_1) \cdot_\star \frac{\nabla^{m\star} y_m}{m!_\star \cdot_\star h^{m\star}}. \tag{3.3}$$

We can get a more simple version of formula 3.3 by setting $x = x_m +_\star P \cdot_\star h$ and $x_k = x_0 +_\star k_\star \cdot_\star h$. Therefore,

$$\begin{aligned} x -_\star x_{m-k} &= (x_m +_\star P \cdot_\star h) -_\star (x_0 +_\star [m_\star -_\star k_\star] \cdot_\star h) \\ &= (x_m -_\star x_0) +_\star (P -_\star [m_\star -_\star k_\star]) \cdot_\star h \\ &= (P +_\star k_\star) \cdot_\star h \end{aligned}$$

where $k = 0, 1, 2, \cdots m$. By implementing these outcomes, 3.3 can be reduced to the following formula:

$$\begin{aligned} \psi_m (P) = y_m &+_\star P \cdot_\star \nabla^\star y_m +_\star \frac{P \cdot_\star (P +_\star 1_\star)}{2^\star!}^\star \cdot_\star \nabla^{2\star} y_m \\ &+_\star \frac{P \cdot_\star (P +_\star 1_\star) \cdot_\star (P +_\star 2_\star)}{3^\star!}^\star \cdot_\star \nabla^{3\star} y_m +_\star \cdots \\ &+_\star \frac{P \cdot_\star (P +_\star 1_\star) \cdot_\star (P +_\star 2_\star) \cdot_\star \cdots \cdot_\star (P +_\star m_\star -_\star 1_\star)}{m^\star!}^\star \cdot_\star \nabla^{m\star} y_m \end{aligned} \tag{3.4}$$

We call the equation in 3.4 as bigeometric Newton's backward interpolation formula. Using the bigeometric backward interpolation formula, we can express the general version of the error approach that we will use in the following section as follows.

$$\begin{aligned} E(x) &= (x -_\star x_m) \cdot_\star (x -_\star x_{m-1}) \cdot_\star \cdots \cdot_\star (x -_\star x_1) \cdot_\star (x -_\star x_0) \cdot_\star \frac{f^{\star(m+1)}(\gamma)}{(m+1)!\star}_\star \\ &= P \cdot_\star (P +_\star 1_\star) \cdot_\star (P +_\star 2_\star) \cdot_\star \cdots \cdot_\star (P +_\star m_\star) \cdot_\star h^{n+1\star} \cdot_\star \frac{f^{\star(m+1)}(\gamma)}{(m+1)!\star}_\star \\ &= \binom{P+m}{m+1}_\star \cdot_\star \nabla^{(m+1)\star} y_m = C_\star (P+m, m+1) \cdot_\star \nabla^{(m+1)\star} y_m \end{aligned} \tag{3.5}$$

where $C_\star(\cdot,\cdot)$ is multiplicative combination [9], $P = \frac{x -_\star x_m}{h}^\star$ and γ is placed between x_{min} and x_{max}. Here, x_{min} and x_{max} are defined as below, respectively:

$$x_{min} = \min\{x_0, x_1, \cdots, x_m, x\},$$
$$x_{max} = \max\{x_0, x_1, \cdots, x_m, x\}.$$

3.2. Bigeometric Adams-Bashforth methods

Bigeometric Runge-Kutta methods are named one-step techniques since they utilize only the data from the prior point to calculate consecutive point; in other words, only initial point (x_0, y_0) is utilized to calculate (x_1, y_1) and generally, y_m is required to calculate y_{m+1}. Then various points have been obtained; it is applicable to implement several previous points in computation. As an illustration, we construct the bigeometric Adams-Bashforth four-step method, which needs y_{m-3}, y_{m-2}, y_{m-1}, and y_m in the computation of y_{m+1}.

3.2.1. Bigeometric Adams-Bashforth algorithms

Based on [22, 23], we can establish bigeometric Adams-Bashforth techniques for bigeometric differential equations defined as follows:

$$y^{\star}(x) = f(x, y(x)), \ y(x_0) = y_0. \tag{3.6}$$

It is named as a bigeometric initial value problem. Both sides of formula 3.6 are integrated as below:

$$\int_{\star x_m}^{x_{m+1}} y^{\star}(x) \cdot_{\star} d_{\star}x = \int_{\star x_m}^{x_{m+1}} f(x, y) \cdot_{\star} d_{\star}x.$$

By substituting $\psi_m(P)$ into $f(x, y)$, we achieve the following formula:

$$y_{m+1} = y_m +_{\star} h \cdot_{\star} \int_{\star 0}^{1} \psi_m(P) \cdot_{\star} d_{\star}P, \tag{3.7}$$

or

$$y_{m+1} = y_m \left(\int_{\star 0}^{1} \psi_m(P) \cdot_{\star} d_{\star}P \right)^{\ln h}, \tag{3.8}$$

where $P = 1_{\star}$ when $x = x_{m+1}$, $P = 0_{\star}$ when $x = x_m$, and $d_{\star}x = h \cdot_{\star} d_{\star}P$. By utilizing bigeometric approximation $\psi_m(P)$ for $f(x, y)$, which depends on points of $f_m = f(x_m, y_m)$, $f_{m-1} = f(x_{m-1}, y_{m-1})$, $f_{m-2} = f(x_{m-2}, y_{m-2})$, $f_{m-3} = f(x_{m-3}, y_{m-3})$, and bigeometric integration of 3.7, bigeometric Adams-Bashforth algorithms also called as bigeometric predictor formulas, are obtained as below, respectively:

(1) For $m = 0$,

$$y_{m+1} = y_m (f_m)^{\ln h}. \tag{3.9}$$

(2) For $m = 1$,

$$y_{m+1} = y_m \left(f_m^{\frac{3}{2}} f_{m-1}^{-\frac{1}{2}} \right)^{\ln h}. \tag{3.10}$$

(3) For $m = 2$,

$$y_{m+1} = y_m \left(f_m^{\frac{23}{12}} f_{m-1}^{-\frac{16}{12}} f_{m-2}^{\frac{5}{12}} \right)^{\ln h}. \tag{3.11}$$

(4) For $m = 3$,

$$y_{m+1} = y_m \left(f_m^{\frac{55}{24}} f_{m-1}^{-\frac{59}{24}} f_{m-2}^{\frac{37}{24}} f_{m-3}^{-\frac{9}{24}} \right)^{\ln h}. \tag{3.12}$$

3.2.2. Error analysis of bigeometric Adams-Bashforth algorithms

Here, $\psi_m(P)$ and $\psi_{m+1}(P)$ are utilized in formula 3.4 to evaluate the errors of bigeometric multi-step techniques. As bigeometric subtraction $\psi_{m+1}(P)$ from $\psi_m(P)$ in 3.5 yields the truncation errors of these numerical techniques for $\psi_m(P)$, we consider

$$y_{m+1} = y_m \left(\int_{\star 0}^{1} \psi_{m+1}(P) \cdot_\star d_\star P \right)^{\ln h}, \tag{3.13}$$

and

$$y_{m+1} = y_m \left(\int_{\star 0}^{1} \psi_m(P) \cdot_\star d_\star P \right)^{\ln h}. \tag{3.14}$$

Then, a formula can be achieved by subtracting 3.13 from 3.14 as below:

$$\int_{\star 0}^{1} C_\star (P+m, m+1) \cdot_\star \nabla^{(m+1)\star} y_m \cdot_\star d_\star P \cong 1.$$

Hence,

$$Error(\psi_m(P)) = \left(f^{\star(m+1)}(\gamma)\right)^{(\ln h)^{m+1} \int_0^1 \frac{C(P+m,m+1)}{P} dP}, \tag{3.15}$$

where $\nabla^{(m+1)\star} y_m = \left(f^{\star(m+1)}(\gamma)\right)^{(\ln h)^{n+1}}$ for $x_{i-m} \le \gamma_i \le x_{i+1}$. We can calculate errors by using the formula of 3.15 as below, respectively:

$$Error(\psi_0(P)) = \left(f^\star(\gamma)\right)^{\ln h}, \tag{3.16}$$

$$Error(\psi_1(P)) = \left(f^{\star(2)}(\gamma)\right)^{\frac{3(\ln h)^2}{4}}, \tag{3.17}$$

$$Error(\psi_2(P)) = \left(f^{\star(3)}(\gamma)\right)^{\frac{23(\ln h)^3}{36}}, \tag{3.18}$$

$$Error(\psi_3(P)) = \left(f^{\star(4)}(\gamma)\right)^{\frac{55(\ln h)^4}{96}}. \tag{3.19}$$

3.2.3. Numerical example

For easy and direct implementation of bigeometric Adams-Bashforth methods, the solution of the following initial value problem will be taken into account as follows:

$$y'(x) = 1 - \frac{1}{x},\ y(1) = 1. \tag{3.20}$$

Clearly, the analytic solution of 3.20 is $y(x) = x - \ln x$. We will demonstrate the dynamics of Adams-Bashforth methods in a bigeometric sense and will analyze outcomes for feasible step increments. Ahead of that, by considering the definition of the bigeometric derivative in 2.1, the equivalent bigeometric initial value problem of 3.20 is described as below:

$$y^\star(x) = e^{\frac{x-1}{y}},\ y(1) = 1. \tag{3.21}$$

Applying bigeometric numerical techniques of 3.9, 3.10, 3.11, and 3.12 to the differential equation with respect to bigeometric analysis in 3.21, the following results and graphs are achieved. In Figures 1–4, the solutions of problem 3.20 with first, second, third, and fourth-order bigeometric Adams-Bashforth methods, respectively, are compared with the analytical solution. These solutions are examined together with error analysis in Tables 1–4.

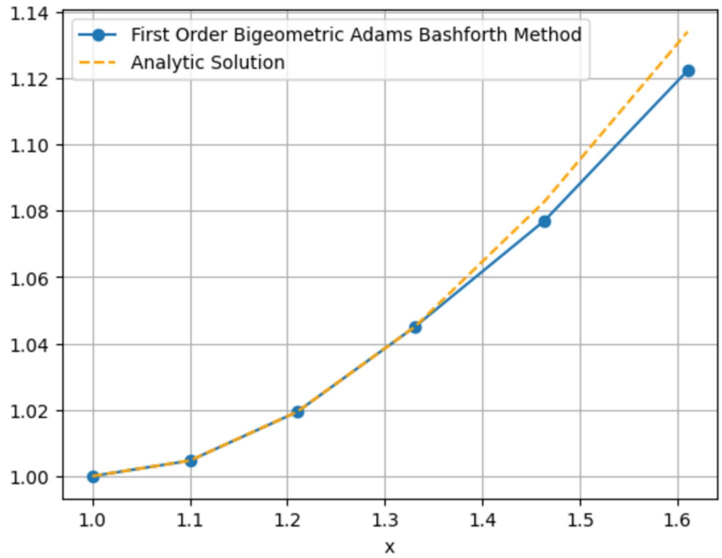

Figure 1. Comparison between the graphs of first-order bigeometric solution and exact solution is drawn using Python version 3.12.5.

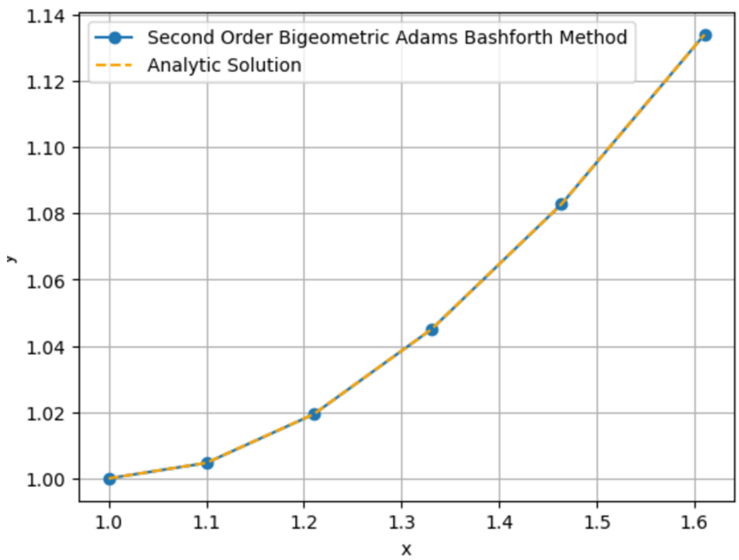

Figure 2. Comparison between the graphs of second-order bigeometric solution and exact solution is drawn using Python version 3.12.5.

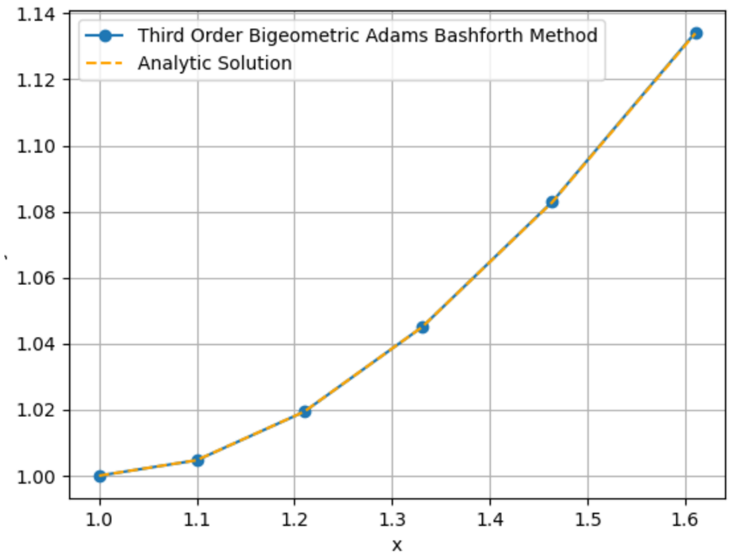

Figure 3. Comparison between the graphs of third-order bigeometric solution and exact solution is drawn using Python version 3.12.5.

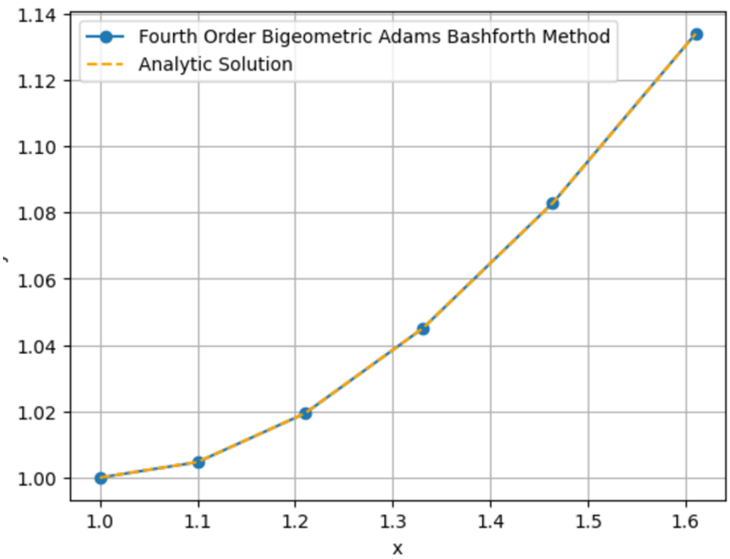

Figure 4. Comparison between the graphs of fourth-order bigeometric solution and exact solution is drawn using Python version 3.12.5.

We illustrate results of analytic solution, (BABM1) first order bigeometric Adams-Bashforth method and errors between them in the following:

Table 1. Comparison of outcomes of BABM1 method with exact values and its relative errors computed by Python version 3.12.5.

x	$y_{analytic}$	y_{BABM1}	*Relative Error*
1.1	≈ 1.0046898201	≈ 1.0046628173	$\approx 2.70029 \times 10^{-5}$
$(1.1)^{10}$	≈ 1.6406406619	≈ 1.6004174865	≈ 0.0402231754
$(1.1)^{20}$	≈ 4.8212963532	≈ 4.7593753719	≈ 0.0619209813
$(1.1)^{30}$	≈ 14.590096874	≈ 14.560179101	≈ 0.0299177731
$(1.1)^{40}$	≈ 41.446848376	≈ 41.491868356	≈ 0.0450199809
$(1.1)^{50}$	≈ 112.62534388	≈ 112.78351191	≈ 0.1581680284
$(1.1)^{60}$	≈ 298.76302875	≈ 299.07112772	≈ 0.3080989686
$(1.1)^{70}$	≈ 783.07524421	≈ 783.56930915	≈ 0.4940649411

We demonstrate outcomes of analytic solution, (BABM2) second order bigeometric Adams-Bashforth method and errors between them in the below:

Table 2. Comparison of outcomes of BABM2 method with exact values and its relative errors computed by Python version 3.12.5.

x	$y_{analytic}$	y_{BABM2}	*Relative Error*
1.1	≈ 1.0046898201	≈ 1.0046628173	$\approx 2.70029 \times 10^{-5}$
$(1.1)^{10}$	≈ 1.6406406619	≈ 1.6430279814	≈ 0.0023873193
$(1.1)^{20}$	≈ 4.8212963532	≈ 4.8212963532	≈ 0.0091472449
$(1.1)^{30}$	≈ 14.590096874	≈ 14.601033192	≈ 0.0109363182
$(1.1)^{40}$	≈ 41.446848376	≈ 41.455209474	≈ 0.0083610985
$(1.1)^{50}$	≈ 112.62534388	≈ 112.62771647	≈ 0.0023725842
$(1.1)^{60}$	≈ 298.76302875	≈ 298.76302875	≈ 0.0068324182
$(1.1)^{70}$	≈ 783.07524421	≈ 783.07524421	≈ 0.0193327382

We show values of analytic solution, (BABM3) third order bigeometric Adams- Bashforth method and errors between them in the following:

Table 3. Comparison of outcomes of BABM3 method with exact values and its relative errors computed by Python version 3.12.5.

x	$y_{analytic}$	y_{BABM3}	*Relative Error*
1.1	≈ 1.0046898201	≈ 1.0046628173	$\approx 2.70029 \times 10^{-5}$
$(1.1)^{10}$	≈ 1.6406406619	≈ 1.6412899167	≈ 0.0006492546
$(1.1)^{20}$	≈ 4.8212963532	≈ 4.8210619971	≈ 0.0002343561
$(1.1)^{30}$	≈ 14.590096874	≈ 14.589139891	≈ 0.0009569845
$(1.1)^{40}$	≈ 41.446848376	≈ 41.445775378	≈ 0.0010729971
$(1.1)^{50}$	≈ 112.62534388	≈ 112.62452232	≈ 0.0008215604
$(1.1)^{60}$	≈ 298.76302875	≈ 298.76274241	≈ 0.0002863529
$(1.1)^{70}$	≈ 783.07524421	≈ 783.07576859	≈ 0.0005243844

We demonstrate values of analytic solution, (BABM4) fouth order bigeometric Adams-Bashforth method and errors between them in the following:

Table 4. Comparison of outcomes of BABM4 method with exact values and its relative errors computed by Python version 3.12.5.

x	$y_{analytic}$	y_{BABM4}	*Relative Error*
1.1	≈ 1.0046898201	≈ 1.0046628173	$\approx 2.70029 \times 10^{-5}$
$(1.1)^{10}$	≈ 1.6406406619	≈ 1.6405228642	≈ 0.0001177977
$(1.1)^{20}$	≈ 4.8212963532	≈ 4.8211330645	≈ 0.0001632886
$(1.1)^{30}$	≈ 14.590096874	≈ 14.590087375	≈ 0.0000094989
$(1.1)^{40}$	≈ 41.446848376	≈ 41.446899357	≈ 0.0000509818
$(1.1)^{50}$	≈ 112.62534388	≈ 112.62540205	≈ 0.0000581688
$(1.1)^{60}$	≈ 298.76302875	≈ 298.76306401	≈ 0.0000352609
$(1.1)^{70}$	≈ 783.07524421	≈ 783.07523197	≈ 0.0000122363

It can be seen from these tables and figures that the solutions obtained by the fourth-order bigeometric Adams-Bashforth method are closer to the classical solution than other bigeometric Adams-Bashforth methods.

4. Conclusions

In this study, we introduced Adams-Bashforth methods in a bigeometric sense to solve first-order bigeometric differential equations. These results show that the bigeometric Adams-Bashforth methods we obtained can be more effective and useful than other bigeometric numerical methods used in the literature.

Author contributions

Mehmet Çağrı Yilmazer: Conceptualization, writing-original draft; Sertac Goktas: Editing, writing-original draft; Emrah Yilmaz: Funding acquisition, editing, writing-original draft; Mikail Et: Editing, writing-original draft. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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