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Research article

Oscillation of solutions for second-order neutral multi-delay differential equations

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Abstract: This paper investigates the oscillatory behavior of solutions for a class of second-order neutral differential equations with multiple delays. By employing rigorous analytical methods, we establish sufficient oscillation criteria that guarantee solutions exhibit oscillatory characteristics under prescribed conditions. The main results extend existing theory by incorporating multiple delay terms and nonlinear effects, thereby broadening the applicability of oscillation theory to more complex neutral delay differential equations. A detailed mathematical analysis is conducted to explore the influence of nonlinear terms and various delay functions on the solutions. It is shown that, despite these factors influencing the specific trajectory and amplitude of oscillations, the global oscillatory nature of the solutions remains intact provided that the proposed conditions are met. Numerical experiments illustrate interaction effects among multiple delays; in some settings, the amplitude envelope may show transitions in decay rate. This research thus provides both theoretical insights and practical implications, forming a solid foundation for future studies into more general nonlinear equations, higher-order neutral equations, and systems involving neutral differential equations with delays.

Keywords: neutral differential equations; oscillation theory; delay function; nonlinear terms
Mathematics Subject Classification: 34K11

1. Introduction

Second-order neutral delay differential equations arise in various scientific and engineering contexts. Understanding the oscillatory behavior of these equations is crucial, as oscillations often indicate important dynamics, such as persistent fluctuations in biological or physical models [1–3]. Consequently, the oscillation theory for functional differential equations has been extensively developed in recent decades [4–6]. Numerous researchers have established oscillation criteria for various classes of delay and neutral equations [7–9]. For example, special cases such as half-linear and Emden–Fowler neutral equations have been thoroughly studied in the literature [10, 11]. Significant advances have also been achieved for higher-order equations and other variants: oscillation results have been established for fourth-order and advanced-type delay equations using novel analytical techniques [12, 13], as well as for equations involving distributed or integral delay arguments via extended Riccati transformations and averaging methods [14, 15]. In addition to oscillatory behaviors, other qualitative properties of delayed equations, such as positive almost periodic solutions and stability criteria in neural network models with mixed delays, and stability analyses of neutral delay equations in population dynamics, have attracted considerable attention [16–18]. These developments underscore a broad and ongoing interest in the qualitative analysis of delay differential equations.

Despite this progress, there remains a significant research gap concerning second-order neutral functional differential equations (NFDEs) that incorporate multiple delays and mixed-sign coefficients [19, 20]. Most existing oscillation results address simplified scenarios, such as equations with a single discrete delay or with coefficients restricted to a fixed sign (nonnegative or nonpositive) [21]. When neutral equations allow coefficients (such as the neutral term or forcing terms) to vary in sign and involve multiple delays (possibly time-varying or differing in length), standard oscillation criteria no longer apply easily [22]. The presence of neutral terms with sign-changing coefficients fundamentally complicates the analysis, as it becomes difficult to determine the sign of solution derivatives or apply classical comparison arguments directly [23, 24]. Consequently, the general scenario involving mixed-sign neutral coefficients and multiple delays has been considerably less explored and lacks comprehensive results ensuring oscillation. To our knowledge, very few results exist in the literature that guarantee oscillation under such general conditions [25]. Thus, there is a clear need to develop oscillation criteria that accommodate sign-indefinite coefficients and multiple delays under relaxed assumptions.

In this paper, we bridge this gap by establishing new sufficient conditions for the oscillation of second-order neutral differential equations with multiple delays under more relaxed conditions. Unlike earlier results that require restrictive assumptions, such as positivity, monotonicity, or boundedness of coefficients and delays [26, 27], we significantly weaken these constraints. Specifically, we allow neutral terms with mixed-sign coefficients and multiple distinct delay arguments in both neutral and non-neutral terms, thereby substantially generalizing previous oscillation criteria (cf. Moaaz et al. [28], Bazighifan et al. [15]). Our criteria are both minimal and practically verifiable, enhancing their applicability. Illustrative examples and numerical simulations are provided to demonstrate the novelty and effectiveness of our results.

We consider the following general form of the second-order neutral differential equation:

$$\left(r(t)\left(x(t)+\sum_{i=1}^{n} p_i(t)x[\tau_i(t)]\right)'\right)' + \int_a^b q(t,s)f[x(\eta(t,s))]\,ds + x(t) = 0, \quad t \geq t_0. \tag{1.1}$$

We posit the following assumptions for Eq (1.1):

(H1) $r, p_i \in C^1([t_0,\infty))$, $r'(t) \geq 0$, $p_i \geq 0$, $0 \leq \sum_{i=1}^n p_i(t) < 1$, $\sup_{t\geq t_0} \sum_{i=1}^n p_i(t) \neq 1, i = 1, ..., n$, and for any starting point $t \geq t_0$, the function q always has non-zero values in the interval $[t,\infty)$.

(H2) Let

$$\int_{t_0}^{\infty} \frac{ds}{r(s)} < \infty \tag{1.2}$$

and $q(t,s) \geq 0,\ q \in C([t_0,\infty)\times(a,b),[0,\infty))$.

(H3) $f \in C(\mathbb{R},\mathbb{R})$, $\alpha > 1$ is the ratio of two odd positive integers, and there exists a constant $m > 0$ such that, for $x \neq 0$, $\frac{f(x)}{x^\alpha} \geq m$ holds.

(H4) $\tau_i \in C([t_0,\infty),(0,\infty)),\ \tau_i(t) \leq t,\ \lim_{t\to\infty}\tau_i(t) = \infty$.

(H5) $\eta \in C([t_0,\infty)\times(a,b),(0,\infty)),\ \eta \geq t,\ \lim_{t\to\infty}\eta(t,s) = \infty$, η has nonnegative partial derivatives.

The purpose of establishing the above assumptions is to build a fundamental oscillation theory framework, ensuring that the structure of the equation satisfies the fundamental conditions for oscillation analysis. Here,

(H1) Restricting the neutral coefficients ($0 \leq \sum p_i(t) < 1$) and the properties of the delay functions ensures that the solution can be normalized (e.g., it prevents the neutral term from dominating the solution behavior).

(H2)-(H3) Requiring $r(t)$ to be integrable, $q(t,s) \geq 0$, and the strong positivity of the nonlinear term $f(x)$ ($f(x)/x^\alpha \geq m$ for some $m > 0, \alpha > 0$) guarantees energy dissipation in the equation and provides a suitable premise for applying the Riccati transform.

(H4)-(H5) Standardizing the asymptotic behavior of the delay terms ($\tau_i(t) \leq t$) and the advance terms ($\eta(t,s) \geq t$) ensures that comparison lemmas (e.g., Lemmas 2.1 and 2.2) can establish the monotonicity patterns of solutions.

Let x satisfy (1.1). We introduce

(a) For sufficiently large t_1, if there always exists $t \geq t_1$ such that $x(t) = 0$, then $x(t)$ is termed an oscillatory solution.

(b) We call (1.1) an oscillatory equation if it has at least one oscillatory solution.

(c) Equation (1.1) has oscillatory solutions if $x(t)$ is neither eventually positive nor eventually negative.

Remark 1. *The selection of delay functions $\tau_i(t)$ and $\eta(t,s)$ must satisfy the dual requirements specified in Assumptions* (H4) *and* (H5)*: For neutral delay terms, functions $\tau_i(t)$ must maintain $\tau_i(t) \leq t$ with* $\lim_{t\to\infty}\tau_i(t) = \infty$ *to ensure causality and solution definability, while requiring $\tau_i \in C([t_0,\infty),(0,\infty))$ for functional regularity. For nonlinear terms involving distributed delays, functions $\eta(t,s)$ require $\eta(t,s) \geq t$ with* $\lim_{t\to\infty}\eta(t,s) = \infty$ *to preserve the existence of the solution forward, coupled with the derivative constraints $\frac{\partial\eta}{\partial t} \geq 0$ and $\frac{\partial\eta}{\partial s} \geq 0$ ensuring monotonicity in delay variations. These collectively guarantee that delay operators preserve the integrity of the solution when composed of nonlinearity $f(x)$ under integration, while preventing degeneracy in the oscillation criteria established in Theorems 3.1 and 3.2.*

2. Basic lemmas

To gain deeper insights into the oscillatory properties of solutions to Eq (1.1), it is essential to first establish several auxiliary results. We begin by introducing the following notation:

$$z(t) = x(t) + \sum_{i=1}^{n} p_i(t)x[\tau_i(t)],$$

$$\pi(t) = \int_t^\infty \frac{ds}{r(s)}.$$

Next, we impose the following assumptions:

$$\int_{t_2}^\infty \int_a^b q(t,s)dsdt = \infty, \quad t_2 \geq t_0. \tag{2.1}$$

$$\pi(t_0) = \int_{t_0}^\infty \frac{ds}{r(s)} < \infty. \tag{2.2}$$

Lemma 2.1. *Assume that $x(t) > 0$ is a solution to (1.1) defined in $[t_1, \infty)$, with $t_1 > t_0$ sufficiently large. Then*

$$z(t) > 0, z'(t) \text{ is oscillatory, or } z'(t) \leq 0, [r(t)z'(t)]' \leq 0, \quad t \geq t_1. \tag{2.3}$$

Moreover, when $z'(t) \leq 0$, we have

$$\left(\frac{z(t)}{\pi(t)}\right)' \geq 0, \quad t \geq t_1. \tag{2.4}$$

Proof. Assume that $x(t) > 0$ is a non-oscillatory solution of (1.1), when $x[\tau_i(t)] > 0$ and $x[\eta(t,s)] > 0$, from (H3), we obtain

$$\begin{aligned}[r(t)z'(t)]' &= -x(t) - \int_a^b q(t,s)f[x(\eta(t,s))]ds \\ &\leq -x(t) - m\int_a^b q(t,s)x^\alpha[\eta(t,s)]ds \\ &\leq -m\int_a^b q(t,s)x^\alpha[\eta(t,s)]ds \leq 0, \quad t \geq t_1 \geq t_0.\end{aligned} \tag{2.5}$$

Therefore, $z'(t)$ is not eventually positive. Otherwise, suppose that there exists a sufficiently large $t \geq t_2 \geq t_1$ such that $z'(t) > 0$, then

$$\begin{aligned}x(t) &= z(t) - \sum_{i=1}^n p_i(t)x[\tau_i(t)] \\ &\geq z(t) - \sum_{i=1}^n p_i(t)z[\tau_i(t)] \\ &\geq z(t)\left(1 - \sum_{i=1}^n p_i(t)\right),\end{aligned}$$

let $\eta(t,s) = t$, we get

$$x[\eta(t,s)] \geq z[\eta(t,s)]\left(1 - \sum_{i=1}^n p_i[\eta(t,s)]\right), \tag{2.6}$$

from (2.5) and (2.6), and with $\eta(t,s) \geq \eta(t,a)$, we obtain

$$\begin{aligned}[r(t)z'(t)]' &\leq -m\int_a^b qz^\alpha(\eta)\left(1 - \sum_{i=1}^n p_i[\eta(t,s)]\right)^\alpha ds \\ &\leq -mz^\alpha[\eta(t,a)]\int_a^b q\left(1 - \sum_{i=1}^n p_i[\eta(t,s)]\right)^\alpha ds.\end{aligned} \tag{2.7}$$

Due to $r(t)z'(t) > 0$ and $z^\alpha[\eta(t,a)] > 0$, we can let $\theta(t) = \frac{r(t)z'(t)}{z^\alpha[\eta(t,a)]} > 0$; therefore;

$$\begin{aligned} \theta'(t) &= \frac{[r(t)z'(t)]'}{z^\alpha[\eta(t,a)]} - \frac{\alpha r(t)z'(t)z'[\eta(t,a)]\eta'(t,a)}{z^{\alpha+1}[\eta(t,a)]} \\ &= \frac{[r(t)z'(t)]'}{z^\alpha[\eta(t,a)]} - \theta(t)\frac{\alpha z'[\eta(t,a)]\eta'(t,a)}{z[\eta(t,a)]}. \end{aligned} \tag{2.8}$$

Combining (2.7) with (2.8), we obtain

$$\theta'(t) \le -m\int_a^b q\left(1-\sum_{i=1}^n p_i[\eta(t,s)]\right)^\alpha ds. \tag{2.9}$$

Integrate Eq (2.9) from t_2 to t

$$\theta(t)-\theta(t_2) = -m\int_{t_2}^t\int_a^b q\left(1-\sum_{i=1}^n p_i[\eta(u,s)]\right)^\alpha dsdu$$

$$\theta(t) \le \theta(t_2) - m\inf_{t\ge t_2}\left(1-\sum_{i=1}^n p_i[\eta(t,b)]\right)^\alpha \int_{t_2}^t\int_a^b q(u,s)dsdu. \tag{2.10}$$

Due to $\int_{t_2}^\infty\int_a^b q(t,s)dsdt = \infty$, thus from (2.10), we know that $\theta(t) > 0$ does not hold universally, contradicting the assumption. Therefore, $z'(t)$ is not ultimately positive; that is, $z(t)$ satisfies (2.3). Moreover, when $z'(t) \le 0$, the monotonic decrease of $z(t)$ implies

$$z(t) \ge -\int_t^\infty z'(s)ds \ge -r(t)z'(t)\pi(t). \tag{2.11}$$

Now, from $\pi(t) = \int_t^\infty \frac{ds}{r(s)}$ and (2.11), we obtain

$$\begin{aligned} \left(\frac{z(t)}{\pi(t)}\right)' &= \frac{\pi(t)z'(t)-\pi'(t)z(t)}{\pi^2(t)} \\ &\ge \frac{-z(t)-r(t)\left(-\frac{1}{r(t)}\right)z(t)}{r(t)\pi^2(t)} \\ &= 0. \end{aligned}$$

□

Lemma 2.2. *Assume that $x(t) < 0$ is a solution of (1.1) in $[t_1,\infty)$, with $t_1 > t_0$ being sufficiently large. Then*

$$z(t) < 0, z'(t) \text{ is oscillatory, or } z'(t) \ge 0, [r(t)z'(t)]' \ge 0, \quad t \ge t_1. \tag{2.12}$$

Moreover, when $z'(t) \ge 0$, we have

$$\left(\frac{z(t)}{\pi(t)}\right)' \le 0, \quad t \ge t_1. \tag{2.13}$$

Proof. Assume that $x(t) < 0$ is a non-oscillatory solution of (1.1), when $x[\tau_i(t)] < 0$ and $x[\eta(t,s)] < 0$, from (H3), we obtain

$$\begin{aligned}[r(t)z'(t)]' &= -x(t) - \int_a^b q(t,s)f[x(\eta(t,s))]ds \\ &\geq -x(t) - m\int_a^b q(t,s)x^\alpha[\eta(t,s)]ds \\ &\geq -m\int_a^b q(t,s)x^\alpha[\eta(t,s)]ds \leq 0, \quad t \geq t_1 \geq t_0.\end{aligned} \tag{2.14}$$

Therefore, $z'(t)$ is not ultimately negative. Otherwise, suppose that there exists a sufficiently large $t \geq t_2 \geq t_1$ such that $z'(t) < 0$; then

$$x[\eta(t,s)] \leq z[\eta(t,s)]\left(1 - \sum_{i=1}^n p_i[\eta(t,s)]\right). \tag{2.15}$$

By combining Eqs (2.14) and (2.15), and with $\eta(t,b) \geq \eta(t,s) \geq t$, we have

$$[r(t)z'(t)]' \geq -mz^\alpha[\eta(t,b)]\int_a^b q\left(1 - \sum_{i=1}^n p_i[\eta(t,s)]\right)^\alpha ds. \tag{2.16}$$

Due to $r(t)z'(t) < 0, z^\alpha[\eta(t,b)] > 0$, let $P(t) = \frac{r(t)z'(t)}{z^\alpha[\eta(t,b)]} < 0$, therefore

$$\begin{aligned}P'(t) &= \frac{[r(t)z'(t)]'}{z^\alpha[\eta(t,b)]} - \frac{\alpha r(t)z'(t)z'[\eta(t,b)]\eta'(t,b)}{z^{\alpha+1}[\eta(t,b)]} \\ &\geq -m\int_a^b q(t,s)\left(1 - \sum_{i=1}^n p_i[\eta(t,s)]\right)^\alpha - P(t)\frac{\alpha z'[\eta(t,b)]\eta'(t,b)}{z[\eta(t,b)]} \\ &\geq -m\int_a^b q(t,s)\left(1 - \sum_{i=1}^n p_i[\eta(t,s)]\right)^\alpha.\end{aligned} \tag{2.17}$$

Integrate Eq (2.17) from t_2 to t

$$P(t) \geq P(t_2) - m\sup_{t\geq t_2}\left(1 - \sum_{i=1}^n p_i[\eta(t,a)]\right)^\alpha \int_{t_2}^t \int_a^b q(u,s)dsdu. \tag{2.18}$$

Due to $\int_{t_2}^\infty \int_a^b q(t,s)dsdt = \infty$, thus from (2.18), we know that $P(t) < 0$ does not hold universally, contradicting the assumption. Therefore, $z'(t)$ is not ultimately negative; that is, $z(t)$ satisfies (2.12). Moreover, when $z'(t) \geq 0$, it can be derived from the monotonically decreasing nature of $z(t)$ itself that

$$z(t) \leq -\int_t^\infty z'(s)ds \leq -r(t)z'(t)\pi(t). \tag{2.19}$$

Therefore

$$\begin{aligned}\left(\frac{z(t)}{\pi(t)}\right)' &= \frac{r(t)\pi(t)z'(t) - r(t)\pi'(t)z(t)}{r(t)\pi^2(t)} \\ &\leq \frac{-z(t) - r(t)\left(-\frac{1}{r(t)}\right)z(t)}{r(t)\pi^2(t)} \\ &= 0.\end{aligned}$$

□

Remark 2. *The existence of solutions for the neutral differential equation* (1.1) *is established through a contradiction argument against non-oscillatory solutions, leveraging the integral condition* $\int_{t_2}^{\infty}\int_a^b q(t,s)dsdt = \infty$ *from* (2.1) *and the convergence requirement* $\int_{t_0}^{\infty}\frac{ds}{r(s)} < \infty$ *from (H2). By assuming a non-oscillatory solution exists, whether eventually positive* ($x(t) > 0$ *for* $t \geq t_1$) *or eventually negative* ($x(t) < 0$ *for* $t \geq t_1$) *we construct the auxiliary function* $z(t) = x(t) + \sum_{i=1}^{n} p_i(t)x[\tau_i(t)]$ *and analyze its derivative behavior through Lemmas 2.1 and 2.2. For eventually positive solutions, Lemma 2.1 demonstrates that* $z'(t)$ *cannot remain ultimately positive without violating the divergence condition of the double integral of* $q(t,s)$*, while for eventually negative solutions, Lemma 2.2 symmetrically shows that* $z'(t)$ *cannot be ultimately negative. When* $z'(t) \leq 0$ *(positive case) or* $z'(t) \geq 0$ *(negative case), the monotonicity relationship* $(z(t)/\pi(t))' \geq 0$ *from* (2.4) *or* $(z(t)/\pi(t))' \leq 0$ *from* (2.13) *produces asymptotic bounds* $z(t) \geq c\pi(t)$ *or* $z(t) \leq \tilde{c}\pi(t)$*, respectively, which upon substitution into the integrated form of the differential equation produce contradictions. This categorical elimination of non-oscillatory solutions implicitly confirms existence through the compulsory oscillation property under the theorem's hypotheses, where solutions persist globally while alternating sign infinitely often as* $t \to \infty$.

In the following, we use these two lemmas to establish the main findings of our study.

3. Main results

Theorem 3.1. *Under the assumptions of* (1.2) *and* (2.1)*, if*

$$0 \leq \sum_{i=1}^{n} p_i(t)\frac{\pi[\tau_i(t)]}{\pi(t)} < 1, \quad \inf\left(1 - \sum_{i=1}^{n} p_i(t)\frac{\pi[\tau_i(t)]}{\pi(t)}\right) > 0, \quad t \geq t_1, \tag{3.1}$$

if the solution $x(t)$ *of* (1.1) *satisfies* $z'(t) = \left(x(t) + \sum_{i=1}^{n} p_i(t)x[\tau_i(t)]\right)'$*, then* $x(t)$ *is oscillatory.*

Proof. Let $x(t) > 0$ be a non-oscillatory solution of (1.1), with $x[\tau_i(t)] > 0$ and $x[\eta(t,s)] > 0$. Combining (1.2) and (2.1), we obtain

$$\int_{t_0}^{\infty}\frac{1}{r(\xi)}\int_{t_0}^{\xi}\int_a^b q(u,s)\pi^{\alpha}[\eta(u,s)]dsdud\xi = \infty, \quad t \geq t_1 \geq t_0.$$

According to Lemma 2.1, $z(t)$ satisfies (2.3). When $z'(t) \leq 0$, from (2.4) we can find that there exists a positive number $c > 0$ such that $z(t) \geq c\pi(t)$. Due to $x(t) \leq z(t)$ and $\tau(t) \leq t$, we have

$$\begin{aligned} z(t) &= x(t) + \sum_{i=1}^{n} p_i(t)x[\tau_i(t)] \\ &\leq x(t) + \sum_{i=1}^{n} p_i(t)z[\tau_i(t)] \\ &\leq x(t) + \frac{\pi[\tau(t)]}{\pi(t)}\sum_{i=1}^{n} p_i(t)z(t) \end{aligned}$$

i.e.,

$$x(t) \geq z(t)\left(1 - \frac{\pi[\tau(t)]}{\pi(t)}\sum_{i=1}^{n} p_i(t)\right). \tag{3.2}$$

Combining $z(t) \geq c\pi(t)$, (1.1), and (3.2), we obtain

$$\begin{aligned}
[r(t)z'(t)]' &= -x(t) - \int_a^b q(t,s) f[x(\eta(t,s))]ds \leq -m \int_a^b q(t,s)x^\alpha[\eta(t,s)]ds \\
&\leq -m \int_a^b q(t,s)z^\alpha[\eta(t,s)] \left(1 - \sum_{i=1}^n p_i[\eta(t,s)] \frac{\pi[\tau(\eta(t,s))]}{\pi[\eta(t,s)]}\right)^\alpha ds \\
&\leq -m \int_a^b q(t,s)c^\alpha \pi^\alpha[\eta(t,s)] \left(1 - \sum_{i=1}^n p_i[\eta(t,s)] \frac{\pi[\tau(\eta(t,s))]}{\pi[\eta(t,s)]}\right)^\alpha ds.
\end{aligned} \tag{3.3}$$

Integrate Eq (3.3) from t_1 to t

$$z'(t) \leq -\frac{mc^\alpha}{r(t)} \int_{t_1}^t \int_a^b q\pi^\alpha[\eta(u,s)] \left(1 - \sum_{i=1}^n p_i[\eta(u,s)] \frac{\pi[\tau(\eta(u,s))]}{\pi[\eta(u,s)]}\right)^\alpha dsdu + \frac{r(t_1)z'(t_1)}{r(t)}.$$

Then, performing integration of this equation from t_1 to t, we obtain

$$\begin{aligned}
z(t) &\leq z(t_1) - \frac{mc^\alpha}{r(\xi)} \int_{t_1}^\xi \int_a^b q(u,s)\pi^\alpha[\eta(u,s)] \left(1 - \sum_{i=1}^n p_i[\eta(u,s)] \frac{\pi[\tau(\eta(u,s))]}{\pi[\eta(u,s)]}\right)^\alpha dsdud\xi \\
&\leq z(t_1) - mc^\alpha \inf_{t \geq t_1} \left(1 - \sum_{i=1}^n p_i[\eta(t,s)] \frac{\pi[\tau(\eta(t,s))]}{\pi[\eta(t,s)]}\right)^\alpha \int_{t_1}^t \frac{1}{r(\xi)} \int_{t_1}^\xi \int_a^b q\pi^\alpha[\eta(u,s)]dsdud\xi.
\end{aligned}$$

From (3.1), it can be seen that

$$\left(1 - \sum_{i=1}^n p_i[\eta(t,s)] \frac{\pi[\tau(\eta(t,s))]}{\pi[\eta(t,s)]}\right)^\alpha > 0.$$

Moreover, due to $mc^\alpha > 0$ and

$$\int_{t_1}^\infty \frac{1}{r(\xi)} \int_{t_1}^\xi \int_a^b q(u,s)\pi^\alpha[\eta(u,s)]dsdud\xi = \infty.$$

Therefore, $\lim_{t\to\infty} z(t) = -\infty$ contradicts $z(t) > 0$ in (2.3), so $x(t) > 0$ is an oscillatory solution of Eq (1.1). In the following, we also prove that $x(t) < 0$ is an oscillatory solution of Eq (1.1).

Let $x(t) < 0$ be a non-oscillatory solution of (1.1), and $x[\tau_i(t)] < 0, x[\eta(t,s)] < 0$. According to Lemma 2.2, $z(t)$ satisfies (2.12). When $z'(t) \leq 0$, from (2.13) we can obtain that there exists a negative constant $\tilde{c} < 0$ such that $z(t) \leq \tilde{c}\pi(t)$, and

$$x(t) \leq z(t)\left(1 - \frac{\pi[\tau(t)]}{\pi(t)} \sum_{i=1}^n p_i(t)\right). \tag{3.4}$$

Combining $z(t) \leq \tilde{c}\pi(t)$, (1.1), and (3.4) yields

$$\begin{aligned}
[r(t)z'(t)]' &\geq -m \int_a^b q(t,s)x^\alpha[\eta(t,s)]ds \\
&\geq -m \int_a^b q(t,s)\tilde{c}^\alpha \pi^\alpha[\eta(t,s)] \left(1 - \sum_{i=1}^n p_i[\eta(t,s)] \frac{\pi[\tau(\eta(t,s))]}{\pi[\eta(t,s)]}\right)^\alpha ds.
\end{aligned} \tag{3.5}$$

Double integration of Eq (3.5) from t_1 to t yields

$$\begin{aligned} z(t) &\geq z(t_1) - \frac{m\tilde{c}^{\alpha}}{r(\xi)} \int_{t_1}^{\xi} \int_a^b q(u,s)\pi^{\alpha}[\eta(u,s)] \left(1 - \sum_{i=1}^n p_i[\eta(u,s)] \frac{\pi[\tau(\eta(u,s))]}{\pi[\eta(u,s)]}\right)^{\alpha} dsdud\xi \\ &\geq z(t_1) - m\tilde{c}^{\alpha} \inf_{t \geq t_1} \left(1 - \sum_{i=1}^n p_i[\eta(t,s)] \frac{\pi[\tau(\eta(t,s))]}{\pi[\eta(t,s)]}\right)^{\alpha} \int_{t_1}^{t} \frac{1}{r(\xi)} \int_{t_1}^{\xi} \int_a^b q\pi^{\alpha}[\eta(u,s)]dsdud\xi. \end{aligned}$$

Since $\tilde{c} < 0$ and α is an odd number, it follows that $m\tilde{c}^{\alpha} < 0$ and

$$\int_{t_1}^{\infty} \frac{1}{r(\xi)} \int_{t_1}^{\xi} \int_a^b q(u,s)\pi^{\alpha}[\eta(u,s)]dsdud\xi = \infty.$$

Therefore, $\lim_{t\to\infty} z(t) = +\infty$ contradicts $z(t) < 0$ in (2.12), so $x(t) < 0$ is an oscillatory solution of Eq (1.1). □

Theorem 3.2. *Consider Eq* (1.1) *under the same hypotheses (H1)–(H5) as in the main theorem, except that the monotonicity condition* $r'(t) \geq 0$ *for the coefficient* $r(t)$ *is not required. Assume these additional conditions hold:*

(G1) $\alpha > 0$ *is the ratio of two odd positive integers.*

(G2) $r(t) \in C([t_0,\infty),(0,\infty))$ *and define* $\pi(t) := \int_t^{\infty} r(s)^{-1/\alpha}\, ds$, *which is finite for* $t = t_0$.

(G3) $\tau(t), \sigma(t) \in C^1([t_0,\infty),\mathbb{R})$ *with* $\tau(t) \leq t$, $\sigma(t) \leq t$, $\sigma'(t) > 0$, *and* $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$.

(G4) Let $p(t), q(t)$ *be nonnegative continuous functions on* $[t_0,\infty)$ *with* $p(t) \in [0,1)$ *and* $q(t) \not\equiv 0$ *beyond* T. *Furthermore, for all* $t \geq T$,

$$\frac{1 - p(\sigma(t))\pi(\tau(\sigma(t)))}{\pi(\sigma(t))} > 0.$$

Let

$$Q(t) = q(t)\left[\frac{1 - p(\sigma(t))\pi(\tau(\sigma(t)))}{\pi(\sigma(t))}\right]^{\alpha}$$

with $Q(t) > 0$ *for* $t \geq T$. *If*

$$\int_T^{\infty} \left(\frac{1}{r(t)} \int_T^t Q(s)[\pi(\sigma(s))]^{\alpha} ds\right)^{1/\alpha} dt = \infty,$$

then all solutions of (1.1) *are oscillatory.*

Proof. Suppose, contrary to our claim, that (1.1) admits a non-oscillatory solution $x(t)$. Restricting to sufficiently large t, say $t \geq t_1 \geq T$, we have $x(t) > 0$. Consider the transformed variable

$$z(t) = p(t)x[\tau(t)] + x(t).$$

Then $z(t) > 0$. By substituting into (1.1), we get

$$(\, r(t)[z'(t)]^{\alpha}\,)' = -\,q(t)[x(\sigma(t))]^{\alpha}.$$

For sufficiently large t, $x(\sigma(t)) > 0$, and hence $(r(t)[z'(t)]^\alpha)' \le 0$. Thus, $r(t)[z'(t)]^\alpha$ does not increase, which means that $z'(t)$ eventually has a constant sign.

(1) $z'(t) \le 0$ for all $t \ge t_1$. In this case, $z(t)$ is decreasing and bounded below by 0, so $\lim_{t\to\infty} z(t)$ exists (finite). Due to $z'(t) < 0$ for $t \ge t_1$, there exists some $\gamma > 0$ such that

$$-z'(t) \ge \left(\frac{\gamma}{r(t)}\right)^{1/\alpha}. \tag{3.6}$$

Integrating Eq (3.6) from t to ∞ yields

$$\int_t^\infty -z'(s)\,ds = z(t) \ge \gamma^{1/\alpha} \int_t^\infty r(s)^{-1/\alpha} ds = \gamma^{1/\alpha}\,\pi(t).$$

Next, from the differential equation,

$$(r(t)[z'(t)]^\alpha)' \le -q(t)[x(\sigma(t))]^\alpha \le -q(t)\Big(1 - p(\sigma(t))\,\frac{\pi(\tau(\sigma(t)))}{\pi(\sigma(t))}\Big)^\alpha [z(\sigma(t))]^\alpha.$$

Due to $z(\sigma(t)) \ge \gamma^{1/\alpha}\pi(\sigma(t))$ by the above inequality, therefore

$$(r(t)[z'(t)]^\alpha)' \le -\gamma\, Q(t)\,[\pi(\sigma(t))]^{2\alpha}. \tag{3.7}$$

By applying double integration to Eq (3.7) from t_1 to t

$$z(t_1) - z(t) \ge \gamma^{1/\alpha} \int_{t_1}^t \left(\frac{1}{r(s)} \int_{t_1}^s Q(u)\,[\pi(\sigma(u))]^\alpha du\right)^{1/\alpha} ds.$$

Divergence of the right-hand side as $t \to \infty$ would force $z(t)$ to become negative for sufficiently large t, violating the $z(t) > 0$ condition. Therefore, $x(t)$ cannot eventually be positive.

(2) Alternatively, suppose that $x(t) < 0$ eventually and $z'(t) \ge 0$ for all sufficiently large t.

Assume, for contradiction, that Eq (1.1) admits a non-oscillatory solution $x(t)$ and that there exists $T \ge t_1$ such that $x(t) < 0$ and $z'(t) \ge 0$ for all $t \ge T$, where $z(t) = x(t) + p(t)x[\tau(t)]$ is the neutral combination used in the proof of Theorem 3.2. By Lemma 2.2, on $[T, \infty)$ we have

$$z(t) < 0, \qquad z'(t) \ge 0, \qquad (r(t)z'(t))' \ge 0, \qquad \text{and} \qquad \left(\frac{z(t)}{\pi(t)}\right)' \le 0,$$

with $\pi(t) := \int_t^\infty r(s)^{-1/\alpha}\,ds$ defined in (G2). Hence $z(t)/\pi(t)$ is nonincreasing on $[T, \infty)$, so there exists a constant $\tilde{c} < 0$ such that

$$z(t) \le \tilde{c}\,\pi(t) \qquad \text{for all } t \ge T. \tag{3.8}$$

Set $c_0 := -\tilde{c} > 0$. By the neutral relation and (G4),

$$x(\sigma(t)) \le \theta(t)\,z(\sigma(t)) \qquad \text{for } t \ge T, \qquad \theta(t) := 1 - \frac{p(\sigma(t))\,\pi(\tau(\sigma(t)))}{\pi(\sigma(t))} > 0.$$

Multiplying by -1 and using (3.8) gives

$$-x(\sigma(t)) \ge \theta(t)\,(-z(\sigma(t))) \ge c_0\,\theta(t)\,\pi(\sigma(t)) \qquad (t \ge T).$$

Since $\alpha > 0$ is the ratio of two odd positive integers (G1), the map $u \mapsto u^\alpha$ preserves order on $\mathbb{R}$. Therefore,

$$[-x(\sigma(t))]^\alpha \;\geq\; c_0^\alpha\, \theta(t)^\alpha\, [\pi(\sigma(t))]^\alpha \qquad (t \geq T). \tag{3.9}$$

From the differential identity used in Theorem 3.2,

$$\big(r(t)[z'(t)]^\alpha\big)' \;=\; -\,q(t)\,[x(\sigma(t))]^\alpha,$$

and (3.9) we obtain the pointwise lower bound

$$\big(r(t)[z'(t)]^\alpha\big)' \;\geq\; c_0^\alpha\, q(t)\, \theta(t)^\alpha\, [\pi(\sigma(t))]^\alpha \qquad (t \geq T). \tag{3.10}$$

Integrating (3.10) from T to t yields

$$r(t)[z'(t)]^\alpha \;\geq\; r(T)[z'(T)]^\alpha \;+\; c_0^\alpha \int_T^t q(s)\, \theta(s)^\alpha\, [\pi(\sigma(s))]^\alpha\, ds \qquad (t \geq T). \tag{3.11}$$

Taking the $1/\alpha$-power and dividing by $r(t)^{1/\alpha}$, we find

$$z'(t) \;\geq\; \left(\frac{1}{r(t)}\right)^{1/\alpha} \left[r(T)[z'(T)]^\alpha + c_0^\alpha \int_T^t q(s)\, \theta(s)^\alpha\, [\pi(\sigma(s))]^\alpha\, ds \right]^{1/\alpha}.$$

A further integration from T to t gives

$$z(t) - z(T) \;\geq\; c_0 \int_T^t \left(\frac{1}{r(u)} \int_T^u q(s)\, \theta(s)^\alpha\, [\pi(\sigma(s))]^\alpha\, ds \right)^{1/\alpha} du, \qquad t \geq T, \tag{3.12}$$

where we absorbed the nonnegative constant $r(T)[z'(T)]^\alpha$ into the integrand by monotonicity. By the hypothesis of Theorem 3.2,

$$\int_T^\infty \left(\frac{1}{r(u)} \int_T^u q(s)\, \theta(s)^\alpha\, [\pi(\sigma(s))]^\alpha\, ds \right)^{1/\alpha} du \;=\; \int_T^\infty \left(\frac{1}{r(u)} \int_T^u Q(s)\, [\pi(\sigma(s))]^\alpha\, ds \right)^{1/\alpha} du \;=\; \infty,$$

because $Q(s) = q(s)[\theta(s)/\pi(\sigma(s))]^\alpha$ by (G4). Consequently, the right-hand side of (3.12) diverges as $t \to \infty$, so $z(t) \to +\infty$. This contradicts $z(t) < 0$ for all $t \geq T$. Therefore, the alternative $z'(t) \geq 0$ cannot occur in the non-oscillatory scenario, which completes the refinement of this part of the proof of Theorem 3.2.

Thus, in both alternatives a contradiction is derived, and hence all solutions must be oscillatory. □

Equation (1.1) contains the time delay term and the nonlinear term. The nonlinear component may induce oscillatory behavior, periodic solutions, or chaotic dynamics, substantially increasing the complexity of the solution characteristics. The interplay between temporal delays and nonlinear effects can potentially generate bifurcation scenarios. To effectively demonstrate that when Eq (1.1) satisfies the conditions of Theorems 3.1 and 3.2, the nature of its global solutions exhibits oscillatory behavior, it is essential that we conduct numerical simulations to validate the key findings derived from our theorems.

4. Numerical simulations

This section presents three numerical experiments validating the principal results established in Theorems 3.1 and 3.2.

Case 1. Fix

$$\int_a^b q(t,s) f[x(\eta(t,s))] ds = \int_0^1 x(t-s) ds$$

and $n = 2$, and let $p_1 = 0.3,\ p_2 = 0.2,\ \tau_1 = 1,\ \tau_2 = 2$. Then consider the following equation:

$$(r_k(t)\,[x(t) + 0.3x(t-1) + 0.2x(t-2)]')' + \int_0^1 x(t-s) ds + x(t) = 0.$$

Where $r_1(t) = 1$, $r_2(t) = 2$, $r_3(t) = t$, $r_4(t) = t^2$, i.e., four distinct values of $r(t)$ can yield four equations. It can be readily verified that all these equations satisfy Theorem 3.1. The oscillatory curves of the solutions for each of the four equations will be plotted separately, as shown in Figure 1.

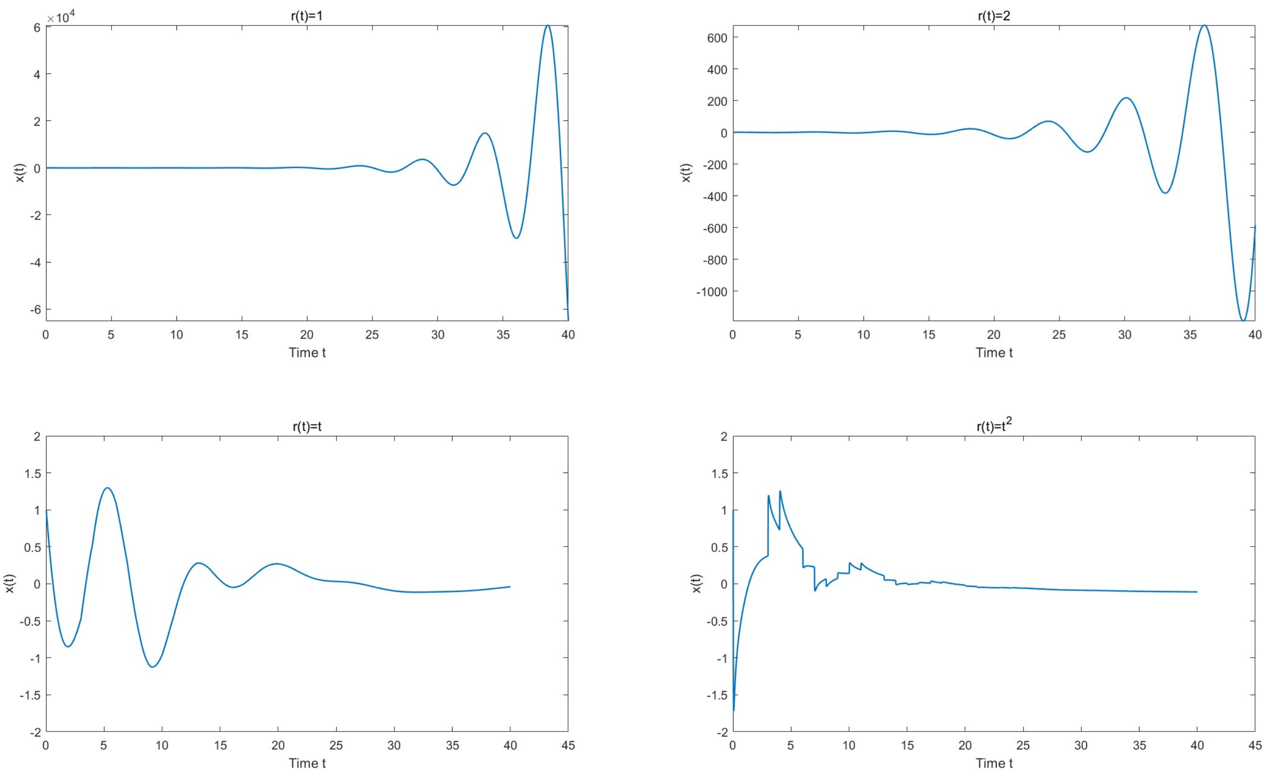

Figure 1. Solutions $x(t)$ for different $r(t)$ ($n = 2$).

Figure 1 illustrates that the solutions to the equation possess an oscillatory behavior, thus verifying Theorem 3.1. By comparing these four images, we observe that different coefficients exert a certain influence on the oscillatory trajectories of the solutions but do not affect their overall oscillatory nature.

Case 2. Fix

$$\int_a^b q(t,s)f[x(\eta(t,s))]\,ds = \int_0^1 x(t-s)\,ds,$$

choose $r(t) = (t+1)^\alpha$, $\alpha = 1.02, \tau_i(t) = t - i$ $(i = 1,\ldots,n)$, $p_i(t) \equiv \frac{0.8}{n}$ $(i = 1,\ldots,n)$, and consider $n \in \{1,2,3,4\}$. With these choices, we consider the following equation:

$$\left((t+1)^\alpha \left(x(t) + \frac{0.8}{n}\sum_{i=1}^{n} x(t-i)\right)'\right)' + \int_0^1 x(t-s)\,ds + x(t) = 0.$$

The numerical solutions $x(t)$ for $n = 1,2,3,4$ under this Case 2 setting are displayed in Figure 2.

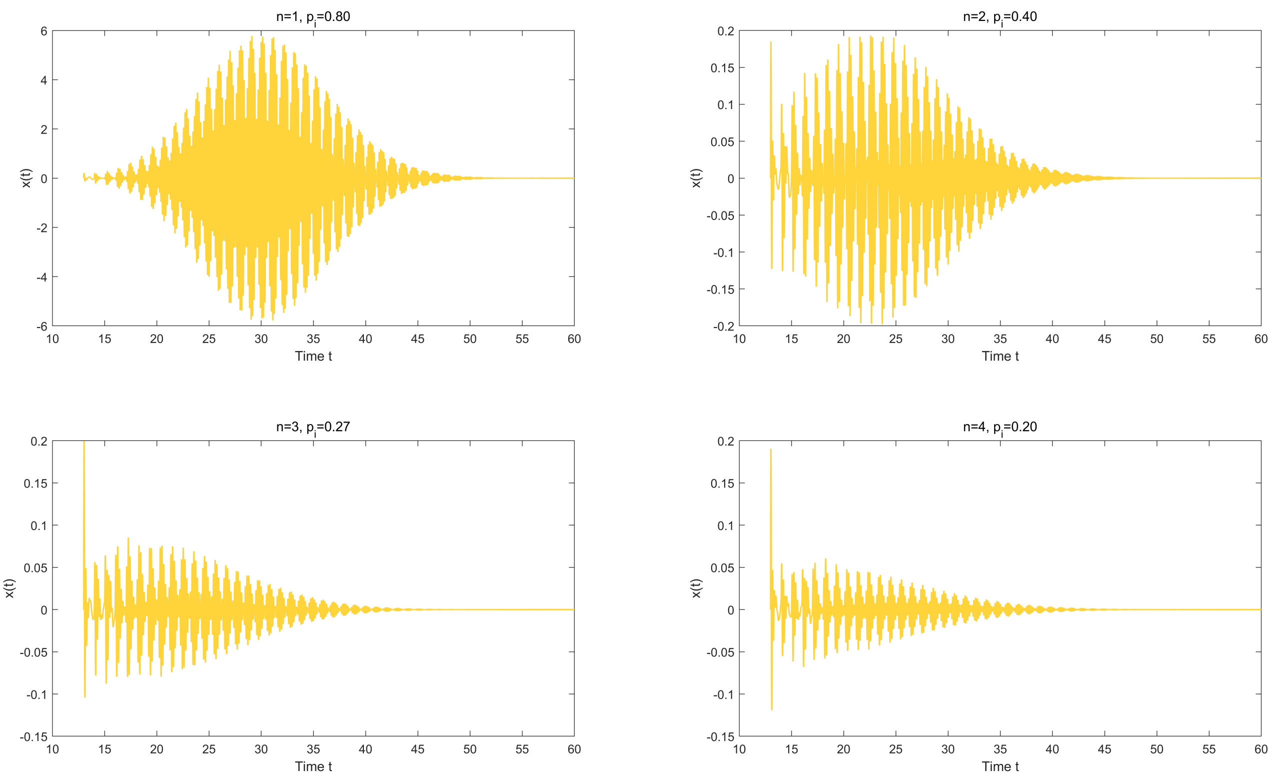

Figure 2. Solutions $x(t)$ for $n(n = 1,2,3,4)$ under the Case 2 setup.

As the number of delay channels increases from 1 to 4, the neutral feedback becomes dispersed and phase cancellation intensifies; under the time-growing damping $r(t)$, the forced response exhibits markedly reduced peak amplitude and faster decay, and all four panels share an envelope of "early build-up–mid-time peak–late rapid attenuation", consistent with the conclusion of Theorem 3.1.

Case 3. Now we continue to explore other examples of equations,

$$\left(t^2[x(t) + 0.2\,x(\tfrac{t}{2})]'\right)' + \int_0^1 \frac{1}{t+1}[x(t+s)]^3\,ds + x(t) = 0, \tag{4.1}$$

$$\left((t^2+1)\,[x(t)+0.3\,x(\tfrac{t}{3})]'\right)' \;+\; \int_0^1 \frac{1}{t+2}\,[x(t+s)]^3\,ds \;+\; x(t) \;=\; 0, \tag{4.2}$$

$$\left(t^2[x(t)+0.1\,x(\tfrac{t}{2})+0.1\,x(\tfrac{t}{3})]'\right)' \;+\; \int_0^1 \frac{1}{t+1}\,[x(t+s)]^3\,ds \;+\; x(t) \;=\; 0, \tag{4.3}$$

$$\left((t+1)^2\,[x(t)+0.25\,x(\tfrac{t}{2})]'\right)' \;+\; \int_0^1 \frac{1}{t+1}\,[x(t+s)]^3\,ds \;+\; x(t) \;=\; 0. \tag{4.4}$$

In these four equations, all neutral coefficients $r(t)$, delay coefficients $p_i(t)$, delay functions $\tau_i(t)$, integral kernels $q(t,s)$, nonlinearities $f(x)$, as well as delay mappings $\eta(t,s)$ simultaneously meet conditions (H1)–(H5) and satisfy Theorem 3.1, implying that all solutions to each equation are oscillatory. Now we plot the vibration images of the solutions for these four equations, as shown in Figure 3.

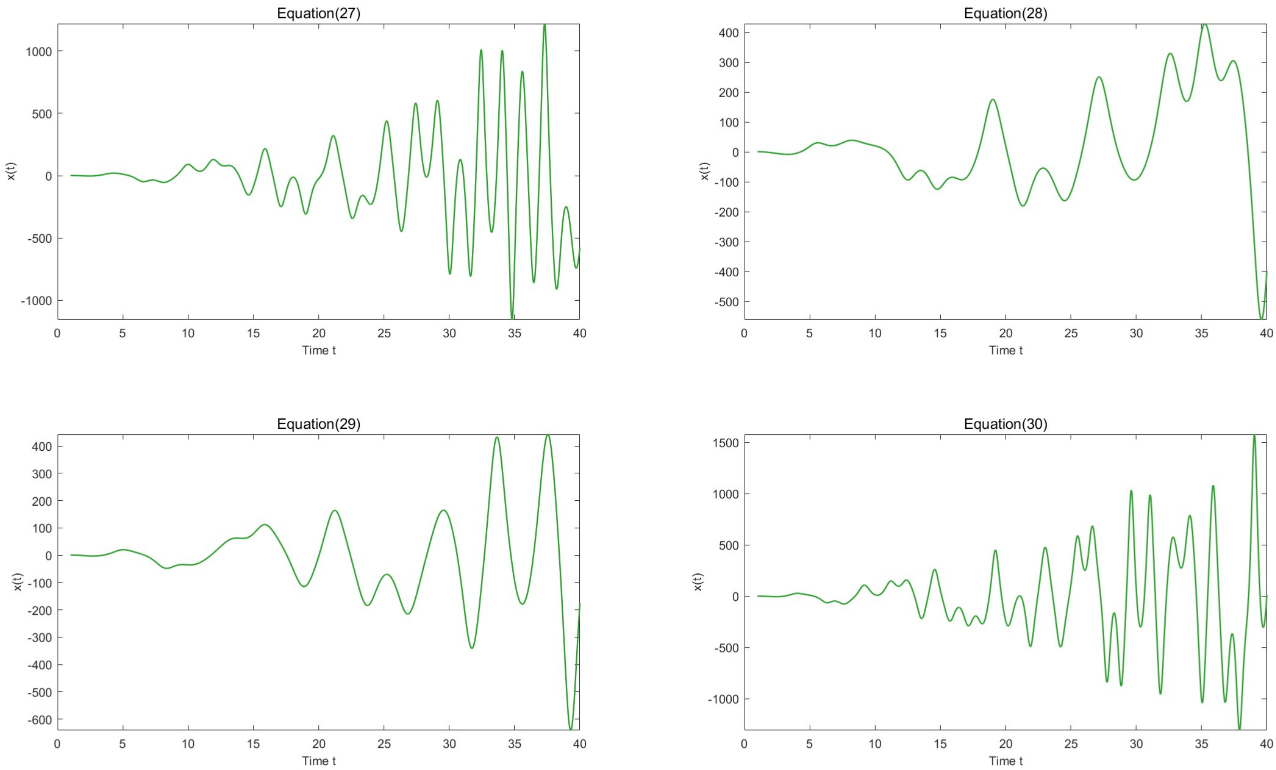

Figure 3. Oscillation diagrams of solutions $x(t)$ for Eqs (4.1)–(4.4).

5. Conclusions

This study presents a critical advancement in the oscillation theory for a class of second-order neutral differential equations characterized by multiple delays and nonlinear terms. Addressing the relative scarcity of oscillation results for such complex equations in the existing literature, we have successfully established a novel and more general criterion for oscillation, presented as Theorem 3.1.

The proof of this theorem is rigorously grounded in a series of lemmas, demonstrating via reductio ad absurdum that the existence of any non-oscillatory solution—whether eventually positive or negative—inevitably leads to a contradiction under the theorem's conditions.

Our numerical simulations jointly corroborate Theorem 3.1 and clarify what changes—and what does not—across the three cases. In Case 1 ($n = 2$), varying $r(t) \in \{1, 2, t, t^2\}$ modifies the transient trajectories but leaves the qualitative oscillation intact, as all four choices produce oscillatory solutions. In Case 2, with $r(t) = (t + 1)^{1.02}$ and a fixed total neutral weight $\sum_{i=1}^{n} p_i = 0.8$, increasing the number of delay channels $n = 1 \to 4$ disperses the neutral feedback, reduces peak amplitudes, and accelerates envelope decay; all panels display an "early build-up–mid-time peak–late rapid attenuation" profile consistent with Theorem 3.1. Finally, Case 3 assembles four representative neutral/distributed-delay models that satisfy (H1)–(H5) and thus fall under Theorem 3.1; their solutions are likewise oscillatory, underscoring the breadth and robustness of the criterion. Overall, the experiments indicate that the choice of $r(t)$ and the distribution of a fixed delay weight primarily shape transient features (amplitude and decay rate) without altering the oscillatory nature guaranteed by our theory.

In conclusion, this work provides both a more solid theoretical foundation and a deeper insight for the analysis of complex nonlinear dynamical systems. Our results also illuminate several clear and promising directions for future research. First, refining oscillation criteria by relaxing monotonicity constraints on $r(t)$, even allowing non-monotonicity or sign changes, will yield more robust and general criteria when combined with Riccati-type techniques. Second, investigating the interplay between fractional order derivatives and stochastic disturbances is crucial to characterizing the oscillation-stability boundary, with direct applications in fields like financial fluctuations and neural dynamics. Third, extending our conclusions to more general state-dependent and distributed time-delay structures will allow for better modeling of state-sensitive scenarios in biological systems and intelligent transportation.

Author contributions

Conceptualization, Lanshuo Hua, Wenjin Li and Jiaxuan Sun; methodology, Lanshuo Hua, Yanni Pang and Jiaxuan Sun; writing - original draft preparation, Lanshuo Hua, Jiaxuan Sun and Wenjin Li; writing - review and editing, Jiaxuan Sun and Yanni Pang. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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