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Research article

On the dynamics of the Fermat-like Diophantine equation involving Pell numbers

Ahmet Emin*

Department of Mathematics, Karabuk University, Karabuk 78050, Turkey

* **Correspondence:** Email: ahmetemin@karabuk.edu.tr; Tel: +903704189212.

Abstract: In this study, we investigated a Fermat-like Diophantine equation involving Pell numbers, specifically examining powers of two that can be represented as the sum of the x-th powers of any two Pell numbers. To solve the equation $P_m^x + P_n^x = 2^a$ for $x \geq 3$, where m, n, x, and a are non-negative integers, we employed Baker's theory of linear forms in logarithms, a modified version of the Baker-Davenport reduction method, and properties of continued fractions. Our results extend previous findings for $x = 1$ and $x = 2$, shedding light on the interplay between Pell numbers and powers of two in this exponential context.

Keywords: Pell number; Diophantine equation; linear forms in logarithms; reduction method
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1. Introduction and motivation

Let $(P_n)_{n\geq 0}$ be the sequence of Pell numbers defined by the recurrence relation $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$, with the initial terms given as $(P_0, P_1) = (0, 1)$. In addition, Pell numbers can be generated using Binet's formulas as follows:

$$P_n = \frac{\zeta^n - \xi^n}{2\sqrt{2}}, \tag{1.1}$$

for $n \geq 0$, where $\zeta = 1 + \sqrt{2}$ and $\xi = 1 - \sqrt{2}$, which are the roots of the characteristic equation $x^2 - 2x - 1 = 0$. Clearly, we have $\zeta + \xi = 2$ and $\zeta\xi = -1$. Pell numbers have been extensively studied in various disciplines, including number theory, algebra, and even applications in geometry and physics. For a comprehensive exploration of notable examples and applications of Pell numbers in various scientific fields, one can consult [1].

Recently, there has been growing interest in studying various Diophantine equations involving integer sequences such as Fibonacci, Lucas, or Pell numbers. In particular, considerable attention has been given to cases where the sum or difference of two or more Fibonacci, Lucas, or Pell numbers results in another Fibonacci, Lucas, or Pell number, or a perfect power. As an example, in [2], Bravo and Luca solved the Diophantine equation representing powers of two as the sum of any two Fibonacci numbers, i.e., $F_m + F_n = 2^a$. Later, in [3], Bravo and Bravo investigated a related problem and determined all solutions to the Diophantine equation representing powers of two as the sum of any three Fibonacci numbers, i.e., $F_m + F_n + F_l = 2^a$. Likewise, Bravo and Luca explored an analogous problem involving Lucas numbers. In [4], they studied the Diophantine equation representing powers of two as the sum of any two Lucas numbers, i.e., $L_m + L_n = 2^a$, and identified all its solutions. In [5], Edjeou et al. explored an extended version of the previous work and determined all the solutions to the Diophantine equation representing powers of two as the sum of three Lucas numbers, i.e., $L_k + L_l + L_t = 2^d$. In [6], Bravo et al. studied the same problem for Pell numbers: expressing powers of two as the sum of any three Pell numbers, that is, $P_l + P_m + P_n = 2^a$, and found all its solutions. Finally, In [7], Emin investigated the sum of the squares of any two Lucas numbers equals a power of two, i.e., $L_m^2 + L_n^2 = 2^a$.

From the findings of the literature review, it appears that no prior research has explored whether powers of two can be expressed as the sum of the x-th powers of two Pell numbers for $x \geq 2$. This gap in the literature led us to investigate exponential Pell numbers. Initially, we focused on cases where the sum of the squares of any two Pell numbers equals a power of two, i.e., $P_m^2 + P_n^2 = 2^a$, as investigated in [8]. In this study, our goal is to determine all solutions of the Diophantine equation

$$P_m^x + P_n^x = 2^a, \tag{1.2}$$

where m, n, x, and a are integers, with $a \geq 0$, $0 \leq m \leq n$, and $x \geq 3$.

This equation is structurally similar to Fermat's famous equation $x^n + y^n = z^n$. In particular, by interpreting Pell numbers P_m and P_n as counterparts to x and y, and powers of two as analogues of z, Eq (1.2) can be viewed as a "Fermat-like" Diophantine equation. Unlike Fermat's equation, which concerns sums of n-th powers of integers, this study explores powers of terms from the Pell sequence—a sequence with rich arithmetic properties.

In the spirit of Fermat's Last Theorem, where the number of solutions to the equation $x^n + y^n = z^n$ decreases drastically as the exponent n increases—being infinite for $n = 1$, still infinite (though fewer) for $n = 2$, and nonexistent for $n > 2$—Eq (1.2) appears to exhibit a similar pattern. Specifically, while Eq (1.2) admits a few solutions for small values of x (namely, four solutions when $x = 1$ and four when $x = 2$), no further solutions have been found for $x > 2$. This pattern supports the hypothesis that Eq (1.2) has only finitely many solutions for large exponents, reinforcing its classification as a Fermat-like Diophantine equation.

The solutions to this equation have been thoroughly studied for $x = 1$ and $x = 2$. When $x = 1$, all solutions of Eq (1.2) were examined in [6]. These solutions are:

$$\begin{aligned}
P_0^1 + P_1^1 &= P_0 + P_1 = 0 + 1 = 1 = 2^0,\\
P_0^1 + P_2^1 &= P_0 + P_2 = 0 + 2 = 2 = 2^1,\\
P_1^1 + P_1^1 &= P_1 + P_1 = 1 + 1 = 2 = 2^1,\\
P_2^1 + P_2^1 &= P_2 + P_2 = 2 + 2 = 4 = 2^2.
\end{aligned}$$

Similarly, when $x = 2$, all solutions of Eq (1.2) were examined in [8]. These solutions are:

$$P_0^2 + P_1^2 = 0^2 + 1^2 = 0 + 1 = 1 = 2^0,$$
$$P_0^2 + P_2^2 = 0^2 + 2^2 = 0 + 4 = 4 = 2^2,$$
$$P_1^2 + P_1^2 = 1^2 + 1^2 = 1 + 1 = 2 = 2^1,$$
$$P_2^2 + P_2^2 = 2^2 + 2^2 = 4 + 4 = 8 = 2^3.$$

Building on these results, this paper investigates the case where $x \geq 3$. By exploring Eq (1.2) in this broader context, we aim to determine whether additional solutions exist and to further understand the interplay between Pell numbers and powers of two. This investigation not only extends existing results but also highlights the structural resemblance between Eq (1.2) and Fermat's equation.

Previous studies have mainly focused on equations such as $F_{n+1}^x - F_{n-1}^x = F_m$ in [9], $P_n^x + P_{n+1}^x = P_m$ in [10], $F_m^x \pm F_n^x = F_r$ in [11], and $F_n^x + F_k^x = F_m^y$ in [12], and have investigated whether sums of powers of Fibonacci or Pell numbers yield terms (or powers of terms) of the same sequence. However, the question of whether the sum of powers of terms from one sequence can equal a term (or power of a term) of a different integer sequence has not been studied. For example, $F_m^x + F_n^x = P_s$, $F_m^x + F_n^x = P_s^y$, and $P_m^x + P_n^x = F_s$ remain open problems in the literature.

This paper takes a first step in this direction by investigating whether the sum of powers of two Pell numbers can equal a power of 2, a value outside the Pell sequence. Determining the solutions to this equation extends known results for higher powers. It lays the groundwork for future studies of mixed-sequence exponential Diophantine equations such as $P_m^x + P_n^x = F_s$ or $P_m^x + P_n^x = L_s$. Although the classical methods used—such as Matveev's theorem and the Baker–Davenport reduction—are well-established, their application requires careful adaptation due to the specific properties of the Pell sequence. Thus, this work contributes by solving a nontrivial problem and providing a framework for investigating a broader class of exponential Diophantine equations involving generalized linear recurrence sequences.

2. Auxiliary results

This section of the paper outlines various lemmas, theorems, corollaries, and notations from algebraic number theory.

First, we present the Lemma that provides the bounds within which any term of the Pell number sequence lies, as given in [6]:

Lemma 1. *The inequality*

$$\zeta^{n-2} \leq P_n \leq \zeta^{n-1} \tag{2.1}$$

satisfies for $n \geq 1$.

An essential property of Pell numbers, which we will use in the proof of our main theorem, is the following. We observe that the inequality

$$\frac{P_{n-1}}{P_n} \leq \frac{5}{12}$$

satisfies for all $n \geq 3$. As a consequence, we conclude that for any $m < n$ and $n \geq 3$, the inequality $\frac{P_m}{P_n} \leq \frac{5}{12}$ also holds.

Let δ be an algebraic number of degree α and

$$\sum_{i=0}^{\alpha} \beta_i x^{\alpha-i}$$

be its minimal polynomial in $\mathbb{Z}[x]$, where the β_i s are relatively prime integers with $\beta_0 > 0$. The logarithmic height of δ is represented by $h(\delta)$ and defined by

$$h(\delta) = \alpha^{-1}\left(\log\beta_0 + \sum_{i=1}^{\alpha} \log\left(\max\left\{\left|\delta^{(i)}\right|, 1\right\}\right)\right), \tag{2.2}$$

where the $\delta^{(i)}$ s are the conjugates of δ.

The following features related to the logarithmic height can be found in various works listed in the references:

$$h(\delta_1 + \delta_2) \leq h(\delta_1) + h(\delta_2) + \log 2, \tag{2.3}$$

$$h\left(\delta_1\delta_2^{\pm 1}\right) \leq h(\delta_1) + h(\delta_2), \tag{2.4}$$

$$h(\delta^v) = |v|\, h(\delta), v \in \mathbb{Z}. \tag{2.5}$$

Let $\delta_1, \delta_2, \ldots, \delta_{\ell-1}, \delta_\ell$ be non-zero real algebraic numbers in a number field $\mathbb{L}$ of degree $d_{\mathbb{L}}$, and let $s_1, s_2, \ldots, s_{\ell-1}, s_\ell$ be non-zero rational integers. Also

$$\Lambda = {\delta_1}^{s_1}{\delta_2}^{s_2}\ldots{\delta_{\ell-1}}^{s_{\ell-1}}{\delta_\ell}^{s_\ell} - 1 \text{ and } B \geq \max\{|s_1|, |s_2|, \ldots, |s_{\ell-1}|, |s_\ell|\}.$$

Let $A_1, A_2, \ldots, A_{\ell-1}, A_\ell$ be the positive real numbers such that

$$A_i \geq \max\left\{d_{\mathbb{L}} h(\delta_i), \left|\log\delta_i\right|, 0.16\right\} \text{ for all } i = 1, 2, \ldots, \ell. \tag{2.6}$$

Using the previously introduced notations, Theorem 9.4 in [13] by Bugeaud et al., which follows from Corollary 2.3 in [14] by Matveev, is stated as follows:

Theorem 1. *[13] If $\Lambda \neq 0$ and $\mathbb{L}$ is a real number field, then,*

$$\log(|\Lambda|) > -1.4 \times 30^{\ell+3} \times \ell^{4.5} \times d_{\mathbb{L}}^2 \times (1 + \log d_{\mathbb{L}}) \times (1 + \log B) \times A_1 \times A_2 \times \ldots \times A_\ell.$$

There are many applications of Matveev's Theorem, such as Theorem 2.1 in [14]. For instance, one may refer to Lemma 4 in [15].

In [[16], Lemma 5 (a)], Dujella and Pethő introduced a variation of the reduction method based on the Baker-Davenport Lemma in [17]. Later, in [18], Bravo et al. formulated the following lemma, which directly modifies the result presented by Dujella and Pethő in [16].

Lemma 2. *[18] Let A be a positive integer, p/q be a convergent of the continued fraction of the irrational ψ such that $q > 6A$, and let φ, γ, and ϑ be real numbers where φ is positive and γ is greater than one. Let $\epsilon := \|\vartheta q\| - A\,\|\psi q\|$, where $\|\cdot\|$ is the distance from the nearest integer. If $\epsilon > 0$, then there is no integer solution (a, b, c) of inequality*

$$0 < |a\psi - b + \vartheta| < \frac{\varphi}{\gamma^c}$$

with

$$a \leq A \text{ and } c \geq \frac{\log(\varphi q/\epsilon)}{\log\gamma}.$$

Legendre's theorem, which is presented below, will be applied in several sections of our study.

Theorem 2 (Legendre, [19]). *Let ψ be a real number, p, q be integers, and let $\psi = [a_0, a_1, ...]$. If*

$$\left|\frac{p}{q} - \psi\right| < \frac{1}{2q^2},$$

then $\frac{p}{q}$ is a convergent of the continued fraction of ψ. In addition, let A and B be non-negative integers with $q_B > A > q_{B-1}$. Put $\ell := max\{a_i\}$ for $i = 0, 1, 2, ..., B$, and then

$$\frac{1}{(\ell + 2)q^2} < \left|\frac{p}{q} - \psi\right|.$$

The two lemmas presented below will be applied multiple times throughout this paper, and additional details regarding these lemmas can be found in [20].

Lemma 3. *Let $u > e$ and $R \geq 3$ be real numbers. If $\frac{u}{\log u} < R$, then $u < 2R \log R$.*

Lemma 4. *Let $u > e^2$ and $R > 100$ be real numbers. If $\frac{u}{(\log u)^2} < R$, then $u < 4R (\log R)^2$.*

Theorem 3 (A. Pethő, [21]). *The only perfect powers of an exponent larger than 1 in the Pell numbers are*

$$P_1 = 1 \text{ and } P_7 = 13^2 = 169.$$

Based on the definition of Pell sequences and Theorem 3, the following corollary can be derived:

Corollary 1. *The only powers of two in the Pell numbers are*

$$P_1 = 2^0 = 1 \text{ and } P_2 = 2^1 = 2.$$

3. Main theorem

The main result of the paper is presented below:

Theorem 4. *Let m, n, a, and x be integers such that $0 \leq m \leq n$, $a \geq 0$, and $x \geq 3$. Then, all solutions of the Diophantine equation $P_m^x + P_n^x = 2^a$ are:*

$$\begin{aligned} P_0^x + P_1^x &= 0^x + 1^x = 1 = 2^0, \\ P_0^x + P_2^x &= 0^x + 2^x = 2^x = 2^x, \\ P_1^x + P_1^x &= 1^x + 1^x = 2 = 2^1, \\ P_2^x + P_2^x &= 2^x + 2^x = 2 \cdot 2^x = 2^{x+1}. \end{aligned} \tag{3.1}$$

Proof. Assume that Eq (1.2) holds.

If $n = m$, then Eq (1.2) takes the form $P_n = 2^{\frac{a-1}{x}}$. Considering Corollary 1, the solutions of Eq (1.2) can be identified as $(m, n, a, x) \in \{(1, 1, 1, x), (2, 2, x + 1, x)\}$. Hence, from now on, we will suppose that $m < n$ for the remainder of the paper.

If $n = 1$, then $m = 0$. Thus, we have $P_0^x + P_1^x = 2^a$, which implies $1 = 2^a$. Therefore, the equation is satisfied for all $x \geq 3$ when $a = 0$. Consequently, $(m, n, a, x) \in \{(0, 1, 0, x)\}$.

If $n = 2$, then m can be either 1 or 0.

Case 1: When we consider $n = 2$ and $m = 0$, we have $P_0^x + P_2^x = 2^a$, which gives $2^x = 2^a$. Thus, the equation is satisfied for all $a = x \geq 3$. Consequently, $(m, n, a, x) \in \{(0, 2, x, x)\}$.

Case 2: When we consider $n = 2$ and $m = 1$, we have $P_1^x + P_2^x = 2^a$, leading to $1 + 2^x = 2^a$. This equation is satisfied only when $x = 0$ and $a = 1$. When $x \geq 3$, the equation is never satisfied, since the right-hand side of the equation is even, while the left-hand side is odd. Given that our assumption is $x \geq 3$, there are no quadruples (m, n, a, x).

Thus, suppose now that $n \geq 3$, $m \geq 3$, and $x \geq 3$. Based on Wiles' proof of Fermat's Last Theorem in [22], the equation $p_m^x + p_n^x = 2^x$ has no solution; therefore, there is no quadruple (m, n, x, x) that satisfies this equation.

Now, by applying Lemma 1 and considering Eq (1.2), we can establish a relationship between n, a, and x as follows:

$$2^a = P_m^x + P_n^x \leq \zeta^{(m-1)x} + \zeta^{(n-1)x} < 2^{2(n-1)x+1} < 2^{2nx} \tag{3.2}$$

and

$$\zeta^{(m-2)x} < \zeta^{(m-2)x} + \zeta^{(n-2)x} \leq P_m^x + P_n^x = 2^a < \zeta^a. \tag{3.3}$$

Therefore, based on Eqs (3.2) and (3.3), we conclude that

$$x \leq (m-2)x < a < 2nx. \tag{3.4}$$

This relation will be frequently used throughout various sections of the paper.

We rewrite Eq (1.2) as

$$P_m^x + P_n^x = 2^a \Rightarrow 2^a - P_n^x = P_m^x. \tag{3.5}$$

By dividing both sides of the last equation by P_n^x, taking the absolute values, and using the fact that $\frac{P_m}{P_n} \leq \frac{5}{12}$ for $m < n$ and $n \geq 3$, we have that

$$\left|2^a P_n^{-x} - 1\right| = \left(\frac{P_m}{P_n}\right)^x = \left(\frac{1}{\frac{P_n}{P_m}}\right)^x \leq \frac{1}{2.4^x}.$$

Consequently, we have

$$|\Lambda_1| \leq \frac{1}{2.4^x}, \text{ where } \Lambda_1 := 2^a P_n^{-x} - 1. \tag{3.6}$$

To achieve this, applying Theorem 1, we can take $\ell := 2$, $\delta_1 := 2$, $\delta_2 := P_n$, $s_1 := a$, and $s_2 := -x$. It can be observed that $\delta_1, \delta_2 \in \mathbb{Q}$, so we can take $\mathbb{L} := \mathbb{Q}$ with degree $d_{\mathbb{L}} := 1$. It can be checked that $\Lambda_1 \neq 0$. Indeed, if $\Lambda_1 = 0$, we get $2^a = P_n^x$. However, based on Corollary 1, this equation has no solution for $n \geq 3$ and $x \geq 3$. Hence, $\Lambda_1 \neq 0$. Using Eq (2.2), Inequality (2.6), and Lemma 1, we get that $h(\delta_1) = \log 2$ and $h(\delta_2) = \log P_n \leq (n-1)\log\zeta < n\log\zeta$. Then, we can take $A_1 := \log 2$ and $A_2 := n\log\zeta$, since $A_i \geq max\left\{d_{\mathbb{L}} h(\delta_i), \left|\log\delta_i\right|, 0.16\right\}$, $i = 1, 2$. From the Inequality (3.4) and $B \geq \max\{a, |-x|\}$, we obtain $B := a$. As a consequence, based on Theorem 1, we get

$$\log(|\Lambda_1|) > -1.4 \times 30^5 \times 2^{4.5} \times 1^2 \times (1 + \log 1) \times (1 + \log a) \times \log 2 \times (n \log\zeta).$$

By making certain mathematical simplifications to the above inequality, we get

$$\log(|\Lambda_1|) > -9.5 \times 10^8 \times n \log a > -9.5 \times 10^8 \times n \log 2nx, \tag{3.7}$$

where we used the fact that $1+\log a < 2\log a$, for $a \geq 3$, and $a < 2nx$. In addition, from Inequality (3.6), we have

$$\log(|\Lambda_1|) < -x\log 2.4. \tag{3.8}$$

Considering Inequality (3.7) together with Inequality (3.8), we get

$$x < 1.1\times 10^9 n\log 2nx. \tag{3.9}$$

3.1. Bounding x in terms of n

We now establish a lower bound for x in terms of n. Inequality (3.9) can be rewritten as

$$\frac{2nx}{\log 2nx} < 2.2\times 10^9 n^2. \tag{3.10}$$

By applying Lemma 3, we obtain that

$$\begin{aligned} 2nx &< 2\cdot\left(2.2\times 10^9 n^2\right)\log\left(2.2\times 10^9 n^2\right)\\ &< 4.4\times 10^9 n^2\cdot 22\log n\\ &= 9.68\times 10^{10} n^2\log n, \end{aligned}$$

where we used the fact that $\log\left(2.2\times 10^9 n^2\right) < 22\log n$ for $n\geq 3$. So, we infer that

$$x < 5\times 10^{10} n\log n. \tag{3.11}$$

3.2. The case when $n\in[3,500]$

From Inequality (3.11), it is evident that

$$x < 1.6\times 10^{14}$$

holds. We set

$$t_1 = -x\log P_n + a\log 2. \tag{3.12}$$

Hence, Inequality (3.6) can be rewritten as

$$|\Lambda_1| = \left|e^{t_1}-1\right| \leq \frac{1}{2.4^x}. \tag{3.13}$$

Since $\Lambda_1 \neq 0$, it follows that $t_1 \neq 0$. Therefore, we now consider the following cases separately. Suppose that $t_1 > 0$. Then, $e^{t_1}-1 > 0$. Applying this to Inequality (3.13), and using the fact that $p \leq e^p - 1$ for all $p\in\mathbb{R}$, we obtain:

$$0 < t_1 < e^{t_1}-1 \leq \frac{1}{2.4^x}.$$

Suppose now that $t_1 < 0$. It is straightforward to observe that $\frac{1}{2.4^x} < \frac{1}{2}$ for all $x\geq 3$. Therefore, from (3.13), we have $\left|e^{t_1}-1\right| < \frac{1}{2}$. Hence, $\left|e^{t_1}\right| < 2$. Since $t_1 < 0$, we can conclude that

$$0 < |t_1| < e^{|t_1|}-1 = e^{|t_1|}\left|e^{t_1}-1\right| \leq \frac{2}{2.4^x}.$$

In either case, the inequality

$$0 < |t_1| < \frac{2}{2.4^x}$$

holds for all $x \geq 3$. By replacing t_1 in the preceding inequality with its formula (3.12) and dividing both sides by $a \log P_n$, we obtain

$$0 < \left|\frac{\log 2}{\log P_n} - \frac{x}{a}\right| < \frac{2}{2.4^x \cdot a \cdot \log P_n} < \frac{1.25}{a \cdot 2.4^x} \tag{3.14}$$

for $n \geq 3$. Now, we use Theorem 2, known as Legendre's Theorem, to determine a better bound for x. In fact $x < 60$. On the contrary, suppose that $x \geq 60$. Then

$$2.4^x \geq 2.4^{60} > 10^{22} > 2.5 \cdot 2 \cdot 500 \cdot \left(1.6 \times 10^{14}\right) > 2.5 \cdot 2nx > 2.5 \cdot a,$$

where $a < 2nx < 1.6 \times 10^{17}$. From Inequality (3.14), we get

$$\left|\frac{\log 2}{\log P_n} - \frac{x}{a}\right| < \frac{1.25}{a2.4^x} < \frac{1.25}{2.5a^2} = \frac{1}{2a^2}$$

which means that $\frac{x}{a}$ is a convergent of continued fractions of $\frac{\log 2}{\log P_n}$.

Let $\psi := \frac{\log 2}{\log P_n}, a < 2nx < A = 1.6 \times 10^{17}$ by Legendre's Theorem, and $[a_0, a_1, a_2, a_3, ...] = [0, 50, 1, 1, 1, 2, ...]$ be the continued fraction expansion of ψ. Also, let p_k/q_k denote its $k-$th convergent, where p_k and q_k are relatively prime integers. Following a quick analysis using Mathematica©, it is apparent that

$$101682324803030480 = q_{30} < A = 1.6 \times 10^{17} < q_{31} = 280191282839616811.$$

Moreover, $\ell := \max\{a_i\} = a_{27} = 4124$ for $i = 1, 2, 3, ..., 31$. Thus, from Theorem 2, we conclude that

$$\frac{1}{(4124+2)a^2} < \left|\frac{\log 2}{\log P_n} - \frac{x}{a}\right| < \frac{1.25}{2.4^x a}. \tag{3.15}$$

Consequently, the last inequality yields

$$a > \frac{2.4^x}{1.25 \cdot 4126} \geq \frac{2.4^{60}}{1.25 \cdot 4126} > 1.25 \times 10^{19},$$

which is a contradiction because $a < 1.6 \times 10^{17}$. Therefore, it follows that $x < 60$.

Considering Inequality (3.4), a short calculation via Mathematica© indicates to us that the equation $P_m^x + P_n^x = 2^a$ does not have any solutions for $n \in [3, 500]$ and $x \in [3, 59]$. Thus, the investigation is finished for the case $n \in [3, 500]$.

3.3. The case when $n > 500$

From now on, we suppose that $n > 500$. At this stage, we establish an absolute upper bound for x. Set $\Omega_1 := x/\zeta^{2n}$. When we consider (3.11) and $n > 500$, we get that

$$\Omega_1 < \frac{5 \times 10^{10} n \cdot \log n}{\zeta^{2n}} < \frac{1}{\zeta^n}. \tag{3.16}$$

In particular, $\Omega_1 < \zeta^{-500} < 10^{-191}$. Incidentally, it is clear that

$$P_n^x = \frac{\zeta^{nx}}{8^{x/2}}\left(1 - \frac{(-1)^n}{\zeta^{2n}}\right)^x.$$

Moreover, by using the fact that $1+s < exp\,(s) < 1+2s$ and $-2s < \log(1-s) < -s$ for $s \in (0, 0.79]$, we deduce that if n is even, then

$$1 > \left(1 - \frac{1}{\zeta^{2n}}\right)^x = exp\left(x \log\left(1 - \frac{1}{\zeta^{2n}}\right)\right) > exp\left(-2\frac{x}{\zeta^{2n}}\right) = exp\,(-2\Omega_1) > 1 - 2\Omega_1, \tag{3.17}$$

whereas if n is odd, then

$$1 < \left(1 + \frac{1}{\zeta^{2n}}\right)^x = exp\left(x \log\left(1 + \frac{1}{\zeta^{2n}}\right)\right) < exp\left(\frac{x}{\zeta^{2n}}\right) = exp\,(\Omega_1) < 1 + 2\Omega_1. \tag{3.18}$$

Thus, taking into account the last two Inequalities (3.17) and (3.18), and the fact that $\Omega_1 < 10^{-191}$ is extremely small, we conclude that

$$\left|P_n^x - \frac{\zeta^{nx}}{8^{x/2}}\right| < 2\Omega_1 \frac{\zeta^{nx}}{8^{x/2}}. \tag{3.19}$$

When we rearrange Eq (1.2), it leads us to

$$P_m^x + P_n^x = 2^a \Rightarrow P_n^x - \frac{\zeta^{nx}}{8^{x/2}} + P_m^x = 2^a - \frac{\zeta^{nx}}{8^{x/2}}. \tag{3.20}$$

If we take the absolute values after a small adjustment to Eq (3.20), we get that

$$\left|2^a - \frac{\zeta^{nx}}{8^{x/2}}\right| = \left|P_m^x + P_n^x - \frac{\zeta^{nx}}{8^{x/2}}\right| \le \left|P_m^x\right| + \left|P_n^x - \frac{\zeta^{nx}}{8^{x/2}}\right|.$$

Dividing both sides of the last inequality by $\frac{\zeta^{nx}}{8^{x/2}}$, we obtain that

$$\left|2^a 8^{x/2} \zeta^{-nx} - 1\right| \le \frac{P_m^x 8^{x/2}}{\zeta^{nx}} + 2\Omega_1 < \frac{\left(\frac{2\sqrt{2}}{\zeta}\right)^x}{\zeta^{(n-m)x}} + \frac{2}{\zeta^n} < \frac{1.7}{\zeta^{(n-m)x}} + \frac{2}{\zeta^n} \tag{3.21}$$

for $x \ge 3, n > 500$, and $n > m$. Owing to $x \ge 3$ and $n > 500$, we have that

$$\left|2^{a+1.5x} \zeta^{-nx} - 1\right| < \frac{1.7}{\zeta^{(n-m)x}} + \frac{2}{\zeta^n} < \frac{1}{8}. \tag{3.22}$$

So, $2^{a+1.5x}\zeta^{-nx} \in \left(1, \frac{9}{8}\right)$. Specifically, after carrying out the necessary calculations, it is observed that

$$0.1a + 0.15x < nx < 0.8a + 1.2x < a + 1.5x < a + 1.5a < 3a, \tag{3.23}$$

where we used the fact that $x < a$ from Inequality (3.4). Let $\lambda = min\,\{n, (n-m)\,x\}$. It follows that (3.21) can be rewritten as

$$\left|2^{a+1.5x} \zeta^{-nx} - 1\right| \le \frac{3.7}{\zeta^{\lambda}}. \tag{3.24}$$

We now proceed to apply Theorem 1. Set

$$|\Lambda_2| < \frac{3.7}{\zeta^\lambda}, \text{ where } \Lambda_2 := 2^{a+1.5x} \cdot \zeta^{-nx} - 1. \tag{3.25}$$

To achieve this, we can take $\ell = 2$, $\delta_1 = 2$, $\delta_2 = \zeta$, $s_1 = a + 1.5x$, and $s_2 = -nx$. It can be observed that $\delta_1, \delta_2 \in \mathbb{Q}\left(\sqrt{2}\right)$, so we can take $\mathbb{L} = \mathbb{Q}\left(\sqrt{2}\right)$ with degree $d_{\mathbb{L}} = 2$. We can verify that $\Lambda_2 \neq 0$. Indeed, if $\Lambda_2 = 0$, then $2^{a+1.5x} = \zeta^{nx}$. For some positive integer a and x, the expression $a + 1.5x$ may not always be an integer. However, if this expression is multiplied by 2, it becomes $2a + 3x$, which is an integer for all positive integers a and x. Consequently, 2^{2a+3x} is also an integer. Squaring both sides of the equation $2^{a+1.5x} = \zeta^{nx}$ gives $2^{2a+3x} = \zeta^{2nx}$. This is impossible because 2^{2a+3x} is an integer, while ζ^{2nx} is not. Thus, $\Lambda_2 \neq 0$. Considering Inequality (3.23) and $B \geq \max\{a + 1.5x, |-nx|\}$, we have $B = 3a$. Under these conditions, we proceed to evaluate the following:

$$h(\delta_1) = \log 2, \ h(\delta_2) = \frac{1}{2}\log\zeta, A_1 := 2\log 2, \text{ and } A_2 := \log\zeta.$$

According to Theorem 1, we can express this as

$$\log(|\Lambda_2|) > -1.4 \times 30^5 \times 2^{4.5} \times 2^2 \times (1 + \log 2) \times (1 + \log 3a) \times 2\log 2 \times \log\zeta.$$

By applying mathematical simplifications to the inequality above, we deduce

$$\log(|\Lambda_2|) > -2.6 \times 10^{10} \times \log nx, \tag{3.26}$$

where we used the fact that $a < 2nx$ and $1 + \log 3a < 1 + \log 6nx < 4\log nx$, for $n > 500$, $x \geq 3$. Considering the right-hand side of Inequality (3.25), we derive

$$\log(|\Lambda_2|) < \log 3.7 - \lambda \log\zeta. \tag{3.27}$$

Comparing the resulting inequality with (3.26), we deduce that

$$\lambda < 3 \times 10^{10} \log nx \tag{3.28}$$

for $n > 500$ and $x \geq 3$.

Case 1: Let $\lambda = n$. Then, Inequality (3.28) becomes $n < 3 \times 10^{10} \log nx$. Hence, we can rewrite it as

$$\frac{nx}{\log nx} < 3 \times 10^{10} x \tag{3.29}$$

and, by using Lemma 3 once more, we obtain

$$nx < 2 \cdot \left(3 \times 10^{10} x\right) \log\left(3 \times 10^{10} x\right) < 6 \times 10^{10} x \cdot 23 \log x < 1.4 \times 10^{12} x \log x$$

where we used the fact that $\log\left(3 \times 10^{10} x\right) < 23 \log x$ for $x \geq 3$. Therefore, we conclude that

$$n < 1.4 \times 10^{12} \log x. \tag{3.30}$$

Substituting this upper bound for n into Inequality (3.11) leads to

$$x < 1.4 \times 10^{26} \text{ and } n < 8.5 \times 10^{13}. \tag{3.31}$$

Case 2: Let $\lambda = (n-m)x$. Then, $3(n-m) \leq (n-m)x < n$. This implies that $n < 1.5m < 2m$, i.e. we can take $n < 2m$. Therefore, $m > 250$ and $n > 500$. Now, set $\Omega_2 := x/\zeta^{2m}$. Considering Inequality (3.11) along with $n > 500$, $m > 250$, and $n < 2m$, we obtain

$$\Omega_2 < \frac{5 \times 10^{10} \cdot 2m \cdot \log 2m}{\zeta^{2m}} < \frac{1}{\zeta^m}. \tag{3.32}$$

Specifically, $\Omega_2 < \zeta^{-250} < 10^{-95}$. Furthermore, it is evident that

$$P_n^x = \frac{\zeta^{nx}}{8^{x/2}}\left(1 - \frac{(-1)^n}{\zeta^{2n}}\right)^x.$$

Following the reasoning outlined after Inequality (3.19), it becomes clear that

$$max\left\{\left|P_n^x - \frac{\zeta^{nx}}{8^{x/2}}\right|, \left|P_m^x - \frac{\zeta^{mx}}{8^{x/2}}\right|\right\} < 2\Omega_2 \frac{\zeta^{nx}}{8^{x/2}}. \tag{3.33}$$

If we rearrange Eq (1.2), we obtain

$$P_m^x + P_n^x = 2^a \Rightarrow P_n^x - \frac{\zeta^{nx}}{8^{x/2}} + P_m^x - \frac{\zeta^{mx}}{8^{x/2}} = 2^a - \frac{\zeta^{nx}}{8^{x/2}} - \frac{\zeta^{mx}}{8^{x/2}}. \tag{3.34}$$

If we take the absolute values after a small adjustment to (3.34) and use Inequality (3.33), we get

$$\left|2^a - \frac{\zeta^{nx}}{8^{x/2}} - \frac{\zeta^{mx}}{8^{x/2}}\right| = \left|P_m^x + P_n^x - \frac{\zeta^{nx}}{8^{x/2}} - \frac{\zeta^{mx}}{8^{x/2}}\right| \leq \left|P_m^x - \frac{\zeta^{mx}}{8^{x/2}}\right| + \left|P_n^x - \frac{\zeta^{nx}}{8^{x/2}}\right| < 4\Omega_2 \frac{\zeta^{nx}}{8^{x/2}}$$

which leads to

$$\left|2^a - \frac{\zeta^{nx}}{8^{x/2}}\left(1 + \zeta^{(m-n)x}\right)\right| < 4\Omega_2 \frac{\zeta^{nx}}{8^{x/2}}.$$

Now, dividing both sides of the last inequality by $\frac{\zeta^{nx}}{8^{x/2}}\left(1 + \zeta^{(m-n)x}\right)$, we obtain that

$$\begin{aligned} \left|2^{a+1.5x}\zeta^{-nx}\left(1 + \zeta^{(m-n)x}\right)^{-1} - 1\right| &< \frac{4\Omega_2}{1 + \zeta^{(m-n)x}} \\ &< \frac{4}{\zeta^m\left(1 + \zeta^{(m-n)x}\right)} \\ &< \frac{4}{\zeta^m\left(1 - \zeta^{-2}\right)} \\ &< \frac{5}{\zeta^m}, \end{aligned} \tag{3.35}$$

where we used the fact that $1 - \zeta^{-2} \leq 1 - \zeta^{2(m-n)} < 1 + \zeta^{(m-n)x}$, and thus we obtain $\frac{1}{1+\zeta^{(m-n)x}} < \frac{1}{1-\zeta^{-2}}$ for $x \geq 3$, $m > 250$, and $n > 500$.

We now proceed to apply Theorem 1 once again. Set

$$|\Lambda_3| < \frac{5}{\zeta^m}, \text{ where } \Lambda_3 := 2^{a+1.5x}\zeta^{-nx}\left(1 + \zeta^{(m-n)x}\right)^{-1} - 1. \tag{3.36}$$

To achieve this, we can take $\ell = 3$, $\delta_1 = 2$, $\delta_2 = \zeta$, $\delta_3 = \left(1 + \zeta^{(m-n)x}\right)$, $s_1 = a + 1.5x$, $s_2 = -nx$, and $s_3 = -1$. Observe that $\delta_1, \delta_2, \delta_3 \in \mathbb{Q}\left(\sqrt{2}\right)$, so we can take $\mathbb{L} = \mathbb{Q}\left(\sqrt{2}\right)$ with degree $d_{\mathbb{L}} = 2$. We verify that $\Lambda_3 \neq 0$. Indeed, if $\Lambda_3 = 0$, then $2^{a+1.5x} = \zeta^{nx} + \zeta^{mx}$ which leads to a contradiction. Since $2^{a+1.5x} = \zeta^{nx} + \zeta^{mx}$, and by conjugating this relation in $\mathbb{L}$, we have that $2^{a+1.5x} = \xi^{nx} + \xi^{mx}$. Combining these two relations, we get

$$\zeta^{nx} < \zeta^{nx} + \zeta^{mx} = |\xi^{nx} + \xi^{mx}| \leq |\xi|^{nx} + |\xi|^{mx} < 1$$

which is impossible for $x \geq 3$, $m > 250$, and $n > 500$. Therefore, $\Lambda_3 \neq 0$. From Inequality (3.4) and $B \geq \max\{a + 1.5x, |-nx|, |-1|\}$, we obtain $B = 3nx$ for $x \geq 3$ and $n > 500$. Under these conditions, we proceed to evaluate the following:

$$h(\delta_1) = \log 2,\ \ h(\delta_2) = \frac{1}{2}\log\zeta, A_1 := 2\log 2,\ \text{ and } A_2 := \log\zeta.$$

Also, using (2.3)–(2.5) and (3.28), we get that

$$h(\delta_3) := h\left(1 + \zeta^{(m-n)x}\right) < \log 2 + \frac{1}{2}(n-m)x\log\zeta,\ \text{ and } A_3 := 2.8 \times 10^{10}\log nx.$$

According to Theorem 1, we can write

$$\log(|\Lambda_3|) > C \times (1 + \log 3nx) \times 2\log 2 \times \log\zeta \times 2.8 \times 10^{10} \times \log nx$$

where $C = -1.4 \times 30^6 \times 3^{4.5} \times 2^2 \times (1 + \log 2)$, and $1 + \log 3nx < 2\log nx$ holds for $n \geq 3$ and $x \geq 3$. Also, with specific mathematical simplifications of the inequality presented above, we obtain

$$\log(|\Lambda_3|) > -6.7 \times 10^{22} \times (\log nx)^2. \tag{3.37}$$

From the right-hand side of Inequality (3.36), we get

$$\log(|\Lambda_3|) < \log 5 - m\log\zeta. \tag{3.38}$$

Comparing the resulting inequality with (3.37), we deduce that

$$m < 7.8 \times 10^{22}(\log nx)^2 < 8 \times 10^{22}(\log 2mx)^2 \tag{3.39}$$

where we used the fact that $n < 2m$. Thus, we can rewrite it as

$$\frac{2mx}{(\log 2mx)^2} < 16 \times 10^{22}x.$$

Now, by using Lemma 4, we find an upper bound for m:

$$\begin{aligned} 2mx &< 4 \times 16 \times 10^{22}x\left(\log\left(16 \times 10^{22}x\right)\right)^2 \\ &= 64 \times 10^{22}x \cdot \left(\log\left(16 \times 10^{22}\right) + \log x\right)^2 \\ &< 64 \times 10^{22}x \cdot (54 + \log x)^2 \end{aligned}$$

$$< 64 \times 10^{22} x \cdot 54^2 \log x,$$

where we used the fact that $(54 + \log x)^2 < 54^2 \log x$ for $x \geq 3$. Hence, using Inequality (3.11) together with $n < 2m$, we deduce that

$$m < 9 \times 10^{28}. \tag{3.40}$$

Furthermore, from (3.4) and (3.11), along with the conditions $n < 2m$ and $a < 2nx$, we obtain that

$$n < 1.8 \times 10^{29}, x < 6.1 \times 10^{41}, \text{ and } a < 2.2 \times 10^{71}.$$

Consequently, by considering Cases 1 and 2 together with the above results, we summarize these findings in the following lemma.

Lemma 5. *All the possible solutions of Eq* (1.2) *are over the ranges* $m < n$, $n > 500$, $m < 9 \times 10^{28}$, $n < 1.8 \times 10^{29}$, $x < 6.1 \times 10^{41}$, *and* $a < 2.2 \times 10^{71}$.

3.4. Reducing the bounds on n

Now, first, we set

$$t_2 := (a + 1.5x) \log 2 - nx \log \zeta. \tag{3.41}$$

Then, Eq (3.25) can be rewritten as:

$$|\Lambda_2| = \left| e^{t_2} - 1 \right| < \frac{1.7}{\zeta^{(n-m)x}} + \frac{2}{\zeta^n}. \tag{3.42}$$

Since $\Lambda_2 \neq 0$, it follows that $t_2 \neq 0$. Therefore, we consider the following cases separately. Suppose that $t_2 > 0$. Then, $e^{t_2} - 1 > 0$. Applying this to Inequality (3.42), and using the fact that $p \leq e^p - 1$ for all $p \in \mathbb{R}$, we obtain:

$$0 < t_2 < e^{t_2} - 1 < \frac{1.7}{\zeta^{(n-m)x}} + \frac{2}{\zeta^n}. \tag{3.43}$$

Suppose now that $t_2 < 0$. It is straightforward to observe that from (3.22), $\frac{1.7}{\zeta^{(n-m)x}} + \frac{2}{\zeta^n} < \frac{1}{8}$. Therefore, from (3.42), we have $\left| e^{t_2} - 1 \right| < \frac{1}{8}$. Hence, $\left| e^{t_2} \right| < 2$. Since $t_2 < 0$, we can conclude that

$$0 < |t_2| < e^{|t_2|} - 1 = e^{|t_2|} \left| e^{t_2} - 1 \right| < \frac{3.4}{\zeta^{(n-m)x}} + \frac{4}{\zeta^n}. \tag{3.44}$$

In either case, the inequality

$$0 < |t_2| < \frac{3.4}{\zeta^{(n-m)x}} + \frac{4}{\zeta^n} \tag{3.45}$$

holds for all $x \geq 3$ and $n \geq 3$. By replacing t_2 in the preceding inequality with its formula (3.41) and dividing both sides by $(a + 1.5x) \log \zeta$, we obtain

$$\begin{aligned} 0 < \left| \frac{\log 2}{\log \zeta} - \frac{nx}{a + 1.5x} \right| &< \frac{3.4}{\zeta^{(n-m)x} (a + 1.5x) \log \zeta} + \frac{4}{\zeta^n (a + 1.5x) \log \zeta} \\ &< \frac{4}{(a + 1.5x)\, \zeta^{(n-m)x}} + \frac{5}{(a + 1.5x)\, \zeta^n}. \end{aligned} \tag{3.46}$$

Since $n > 500$, we obtain the following inequality:

$$\begin{aligned}\zeta^n > \zeta^{500} > 10^{191} > 7.5 \times 10^{76} &> 3 \times 10^4 \cdot 3 \cdot \left(1.8 \times 10^{29}\right) \cdot \left(6.1 \times 10^{41}\right) \\ &> 3 \times 10^4 \cdot 3nx \\ &> 3 \times 10^4 \cdot (a + 1.5x).\end{aligned}$$

Next, we derive a better bound for $(n-m)x$. Indeed, $(n-m)x \leq 201$. On the contrary, suppose that $(n-m)x > 201$. Thus, we observe the following inequality:

$$\begin{aligned}\zeta^{(n-m)x} > \zeta^{201} > 7 \times 10^{75} &> 2 \times 10^4 \cdot 3 \cdot \left(1.8 \times 10^{29}\right) \cdot \left(6.1 \times 10^{41}\right) \\ &> 2 \times 10^4 \cdot 3nx \\ &> 2 \times 10^4 \cdot (a + 1.5x).\end{aligned}$$

From the last two inequalities, we can rewrite (3.46) as follows:

$$0 < \left|\frac{\log 2}{\log \zeta} - \frac{nx}{a + 1.5x}\right| < \frac{4}{(a + 1.5x)\,\zeta^{(n-m)x}} + \frac{5}{(a + 1.5x)\,\zeta^{n}} < \frac{1}{2100\,(a + 1.5x)^2}.$$

Thus, $\left|\frac{\log 2}{\log \zeta} - \frac{nx}{a+1.5x}\right| < \frac{1}{2(a+1.5x)^2}$, implying that $\frac{nx}{a+1.5x}$ is a convergent of the continued fractions of $\frac{\log 2}{\log \zeta}$ by Theorem 2. Let $\psi := \frac{\log 2}{\log \zeta}, a + 1.5x < 3nx < 3.3 \times 10^{71} = A$ by Lemma 5, and $[a_0, a_1, a_2, a_3, ...] = [0, 1, 3, 1, 2, 6, ...]$ be the continued fraction expansion of ψ. Also, let p_l/q_l denote its l-th convergent, where p_l and q_l are relatively prime integers. Let $\frac{p_l}{q_l} = \frac{nx}{a+1.5x}$ for some integer l. After a brief examination with Mathematica©, it becomes apparent that $q_{138} < A = 3.3 \times 10^{71} < q_{139}$. So, $l \in \{1, 2, 3, \ldots, 139\}$. Also, $\ell := \max\{a_i\} = a_{94} = 2030$ for $i = 1, 2, 3, ..., 139$. Consequently, from Theorem 2, we obtain that

$$\frac{1}{(2030 + 2)\,(a + 1.5x)^2} < \left|\frac{\log 2}{\log \zeta} - \frac{nx}{a + 1.5x}\right| < \frac{1}{2100\,(a + 1.5x)^2}, \tag{3.47}$$

which leads to a contradiction. Hence, $(n-m)x \leq 201$. Therefore, it follows that $n - m \leq 67$, and so $m > 433$ and $n > 500$. Thus, Inequality (3.35) is satisfied.

Now, second, we set

$$t_3 := (a + 1.5x)\log 2 - nx \log \zeta + \log\left(\frac{1}{1 + \zeta^{(m-n)x}}\right) \tag{3.48}$$

and

$$|\Lambda_3| = \left|e^{t_3} - 1\right| < \frac{5}{\zeta^m}. \tag{3.49}$$

Since $\Lambda_3 \neq 0$, it follows that $t_3 \neq 0$. As previously demonstrated, for any value of t_3, where either $t_3 > 0$ or $t_3 < 0$, we have the following inequality:

$$0 < |t_3| < \frac{10}{\zeta^m} \tag{3.50}$$

which holds for all $m > 433$. By replacing t_3 in the previous inequality with its formula (3.48) and dividing both sides by $\log\zeta$, we obtain

$$0 < \left|(a+1.5x)\frac{\log 2}{\log\zeta} - nx + \frac{\log\left(1/\left(1+\zeta^{(m-n)x}\right)\right)}{\log\zeta}\right| < \frac{10}{\zeta^m\cdot\log\zeta} < \frac{12}{\zeta^m}.$$

By Lemma 5, we take $A := 3.3\times 10^{71} > 3nx > a+1.5x$. We also take

$$\psi := \frac{\log 2}{\log\zeta}, \vartheta := \frac{\log\left(1/\left(1+\zeta^{(m-n)x}\right)\right)}{\log\zeta}, \varphi := 12, \gamma := \zeta, \text{ and } c := m.$$

It is apparent that ψ is an irrational number, and that q_{141} is the denominator of the 141-st convergent of the continued fraction expansion of ψ, where $6A < q_{141}$. We therefore apply Lemma 2 for all $n-m$ values in the range $[1,67]$. Consequently, we obtain that

$$m \le \frac{\log\left(\varphi q_{141}/\epsilon\right)}{\log\gamma} < 203.68$$

where $6A < q_{141}$ and $\epsilon := \|\vartheta q_{141}\| - A\|\psi q_{141}\| > 0$, $\epsilon = 0.0002298796261$. If (m,n,a) is a possible solution of Eq (1.2), then $m \le 203$, which implies that $n < 270$. However, this contradicts our assumption that $n > 500$.

Remark 1. *The expression* $\vartheta := \frac{\log\left(1/\left(1+\zeta^{-(n-m)x}\right)\right)}{\log\zeta}$ *has been studied for all values of* $(n-m)x$ *within the interval* $[1,201]$ *using Mathematica©. It was observed, however, that* ϑ *does not take integer values for any* $(n-m)x$ *in this range. Should* ϑ *equal an integer at any point in this interval, such instances will be considered separately.*

□

Remark 2. *We note that all four solutions listed in Eq* (3.1) *correspond to the following identities:*

$$\begin{aligned}
P_0^x + P_1^x &= 0^x + 1^x = 1 = 2^0,\\
P_0^x + P_2^x &= 0^x + 2^x = 2^x,\\
P_1^x + P_1^x &= 1^x + 1^x = 2 = 2^1,\\
P_2^x + P_2^x &= 2^x + 2^x = 2\cdot 2^x = 2^{x+1}.
\end{aligned}$$

This shows that Eq (1.2) *has no further solutions for* $x \ge 3$ *beyond those given in Eq* (3.1)*. In particular, the last case highlights that*

$$P_2^x + P_2^x = 2^{x+1}$$

is a direct consequence of the identity $2^x + 2^x = 2\cdot 2^x = 2^{x+1}$*. Thus, these solutions essentially arise from basic exponential identities involving small Pell numbers.*

4. Conclusions

In this study, we investigated the representation of powers of two as the sum of the x-th powers of two Pell numbers, focusing on the Diophantine equation $P_m^x + P_n^x = 2^a$ for $x \geq 3$. This work builds on earlier studies for $x = 1$ and $x = 2$, where all solutions of the equation were completely characterized in [6] and [8], respectively.

We extended these results to the case $x \geq 3$, offering a generalization of previous findings. Using Baker's theory of linear forms in logarithms, we derived explicit bounds for the parameters involved in the equation. These bounds were further refined through a modified version of the Baker-Davenport reduction method, enhanced by the properties of continued fractions. This allowed us to effectively compute all possible solutions.

This study not only generalizes previous results but also provides a methodological framework that can be applied to other recurrence sequences. Future research could explore extensions of these techniques to broader recurrence relations or higher-order powers, revealing additional connections between recurrence sequences and exponential Diophantine equations.

Similar to previous studies investigating the sums, cardinality, approximate values, and differential equations of certain integer sequences, as presented in [23], some analogous properties of this Diophantine equation could also be examined in future research.

Use of AI tools declaration

The author declares that Artificial Intelligence (AI) tools were not used in the creation of this article.

Conflict of interest

The author declares no conflict of interest in this paper.

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