AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(8): 18494–18523.
DOI: 10.3934/math.2025826
Received: 02 July 2025
Revised: 04 August 2025
Accepted: 11 August 2025
Published: 15 August 2025

Research article

Pareto-optimal reinsurance design in a duopoly market with asymmetric information

Haonan Ma and Ying Fang*

School of Mathematics and Statistics, Shandong Normal University, Jinan 250358, China

* **Correspondence:** Email: fangying319@163.com.

Abstract: This work studied the optimal reinsurance design in a duopolistic market comprising two types of insurers and two reinsurers under asymmetric information, where reinsurers cannot directly observe insurers' risk types. We modeled reinsurers as risk-neutral agents maximizing expected net profit, subject to individual rationality, incentive compatibility, and convex preference constraints. We introduced the principle of Pareto optimality to formulate the objective function in a multi-agent setting. Applying the Lagrange dual approach, we derived optimal reinsurance menus for all cases. Under the Value-at-Risk (VaR) risk measure, we identified the globally optimal reinsurance menu by comparative analysis and provided its closed-form solution. Furthermore, we compared exponential and Pareto distributions with identical expected losses to study tail risk effects.

Keywords: Pareto-optimal reinsurance; asymmetric information; duopolistic market; distortion risk measures; individual rationality; incentive compatibility; convex preference
Mathematics Subject Classification: 91G05, 91G30

Abbreviations: VaR: Value-at-Risk; TVaR: Tail Value-at-Risk; GlueVaR: Glued Value-at-Risk; RVaR: Range Value-at-Risk; MIF: the Marginal Indemnity Function

1. Introduction

The optimal (re)insurance design problem has been extensively studied, with diverse optimization criteria examined for various purposes. Pareto optimality serves as a key criterion for determining optimal (re)insurance strategies when coordinating interests among market participants and optimizing risk allocation. As a fundamental concept in economics, Pareto optimality was first introduced into the field of insurance actuarial science by Borch [1] through cooperative game theory, studying efficient risk transfer between insurers and reinsurers under expected utility maximization. Arrow [2] explicitly characterized optimal contract forms, demonstrating that stop-loss treaties constitute the optimal solution under exponential utility assumptions. Bühlmann [3] generalized the framework to n-party risk pools with arbitrary convex risk measures. Modern extensions have enhanced and broadened the core framework. The analytical perspective shifts from unilateral viewpoints (either insurer or reinsurer) to a bilateral perspective that integrates both contractual parties [4]. To enhance characterization and comparison of decision-makers' risk aversion, distortion risk measures [5, 6] and belief heterogeneity [7, 8] have been incorporated into the Pareto-optimal (re)insurance framework.

The assumption of symmetric information between insurers and reinsurers is conventionally adopted in most research. However, the practical market rarely achieves this idealized state, as information asymmetry inherently exists between contracting parties in (re)insurance dealings. Arnott and Stiglitz [9, 10] pioneered the principal-agent framework with one principal and two agents under asymmetric information, laying the foundation for subsequent asymmetric information theories. Recent extensions by Cheung et al. [11, 12] introduced this framework to insurance markets, deriving optimal reinsurance contracts that maximize insurers' expected profits under VaR or concave distortion risk measures by policyholders, subject to incentive compatibility and individual rationality constraints. Boonen et al. [13] further characterized optimal reinsurance features through marginal indemnification functions based on heterogeneous risk preferences between two insurer types. Liang et al. [14] then incorporated tail risk overestimation and heterogeneous risk distributions under adverse selection.

Distortion risk measures are a fundamental class of risk quantification tools in financial mathematics, actuarial science, and risk management. The theoretical foundation of these measures traces back to Yaari's [15] Dual Theory of Choice and Wang's [16] formal definition. In practical applications, distortion risk measures are employed in insurance pricing (e.g., Wang's premium principle) and regulatory capital frameworks (e.g., Solvency II and Basel Accords). VaR and TVaR, as special cases of distortion measures, are now industry standards in risk management. Recent literature has expanded the theoretical and practical applications. Belles-Sampera et al. [17] developed the GlueVaR risk measure and studied its relationship with VaR and TVaR, while investigating its tail subadditivity. Santolino et al. [18] showed that extreme loss contributions to distortion risk measures exhibit subadditivity under tail-concave distortion functions. Pesenti et al. [19] quantified the robustness of distortion risk measures under distributional uncertainty through the Wasserstein distance, yielding quasi-explicit results for VaR and RVaR.

The existing literature primarily focuses on unilateral monopoly structures in (re)insurance markets, characterized by either a single seller or a single buyer (termed a *bilateral monopoly* when both sides consist of single entities). Cai et al. [4] investigated the optimal reinsurance in a bilateral monopoly market by minimizing a convex combination of VaR measures for both the insurer's and reinsurer's losses. Asimit et al. [20] developed a bilateral monopoly model with multiple risk environments and showed that layer-type indemnities are Pareto-optimal within each environment. Boonen et al. [13] and Liang et al. [14] employed the marginal indemnification function approach to study the optimal (re)insurance menus in markets comprising one insurer (reinsurer) and two classes of policyholders (insurers). Boonen et al. [21] investigated competitive pricing of reinsurance contracts in a market with one insurer and multiple reinsurers. Departing from prior studies, we developed a duopolistic reinsurance market model comprising two reinsurers and two types of insurers, which better reflects real-world markets.

This paper proceeds as follows. Section 2 presents some preliminaries and formulates the main problem in this paper. Section 3 shows the optimal reinsurance menus in each case through classified analysis. Section 4 derives the globally optimal reinsurance menu under the VaR risk measure, establishing its closed-form solution. Numerical examples are provided when the risk exposure follows either exponential or Pareto distributions under VaR. Section 5 presents the main conclusions, outlines study limitations, and proposes future research directions.

2. Preliminaries and problem formulation

2.1. Distortion risk measures

We assume that all insurers employ distortion risk measures to evaluate their respective risk exposure. Distortion risk measures are defined as follows.

Definition 2.1. *For a non-negative random variable Z with the distribution function $F_Z(\cdot)$ and the survival function $S_Z(\cdot) = 1 - F_Z(\cdot)$, the distortion risk measure is defined as*

$$\rho_g(Z) = \int_0^\infty g\left(S_Z(t)\right) \mathrm{d}t, \tag{2.1}$$

where $g \in \mathcal{G} = \{g : [0,1] \mapsto [0,1] \mid g(0) = 0,\ g(1) = 1,\ g \text{ is non-decreasing and left-continuous}\}$ is called the distortion function.

Distortion risk measures satisfy comonotonic additivity, translation invariance, and positive homogeneity:

(1) Comonotonic additivity: Any pair of comonotonic random variables (Z_1, Z_2)—meaning that Z_1 and Z_2 are increasing functions of a common random variable—satisfies the relation $\rho_g(Z_1 + Z_2) = \rho_g(Z_1) + \rho_g(Z_2)$.

(2) Translation invariance: Any random variable Z and any constant c satisfy $\rho_g(Z + c) = \rho_g(Z) + c$.

(3) Positive homogeneity: Any random variable Z and any constant $c \geq 0$ satisfy $\rho_g(cZ) = c\rho_g(Z)$.

VaR is a prominent example of distortion risk measures, which is widely used in global financial regulations. We define VaR as follows.

Definition 2.2. *The* VaR *of a random variable Z at confidence level $\alpha \in (0,1)$ is given by*

$$\mathrm{VaR}_\alpha(Z) = \inf\{z \in \mathbb{R}_+ : F_Z(z) \geq \alpha\}. \tag{2.2}$$

The distortion function of VaR *is the indicator function $g(t) = \mathbf{1}_{\{1-\alpha<t\leq 1\}}$, which takes value 1 if $1-\alpha < t \leq 1$ and 0 otherwise.*

2.2. Problem formulation

We model a duopoly reinsurance market comprising two types of insurers and two reinsurers with asymmetric information. Let the non-negative random variable X with the distribution function $F_X(x)$ and the survival function $S_X(x)$ denote the initial risk exposure assumed by an insurer during a given time period. The insurer pays a premium $0 \leq \pi \leq \omega$ to the reinsurer in exchange for transferring the risk $f(X)$. Since premiums cannot be negative or infinite in practice, we impose both lower and upper bounds on premium values. To address potential ex post moral hazard problems [22], we assume that $f \in \mathcal{F}$ is non-decreasing and 1-Lipschitz, with

$$\mathcal{F} = \{f : [0, M] \mapsto [0, M] \mid f(0) = 0, 0 \le f(x_1) - f(x_2) \le x_1 - x_2 \, , 0 \le x_2 \le x_1 \le M\},$$

where M denotes the supremum of X. According to Rademacher theorem [23], f is almost everywhere differentiable on $[0, M]$, which can be expressed in integral form as

$$f(x) = \int_0^x h(z)\,\mathrm{d}z, \; x \in [0, M], \tag{2.3}$$

where $h \in \mathcal{H} = \{h : [0, M] \mapsto [0, 1]\}$ is called the marginal indemnity function (MIF) of f [24].

$F_X(x)$ is known by both insurers and reinsurers. Reinsurers know the market contains two insurer types — type 1 insurers with risk preference $\rho_{g_1}(\cdot)$ and type 2 insurers with risk preference $\rho_{g_2}(\cdot)$ — in proportions p and $1 - p$, respectively, where $g_1, g_2 \in \mathcal{G}$. We assume that type 2 insurers are more risk-averse than type 1 insurers as follows, implying type 1's willingness-to-pay is weakly lower for all risks.

Assumption 2.1. $\rho_{g_1}(X) \le \rho_{g_2}(X)$.

Due to information asymmetry, reinsurers cannot immediately identify an individual insurer's risk type. The two reinsurers, labeled Reinsurer 1 and Reinsurer 2, respectively, offer the reinsurance menu containing two types of reinsurance contracts $\{(\pi_{11}, f_{11}); (\pi_{12}, f_{12})\}$ and $\{(\pi_{21}, f_{21}); (\pi_{22}, f_{22})\}$ to insurers, where premiums $0 \le \pi_{11}, \pi_{12}, \pi_{21}, \pi_{22} \le \omega$ and indemnity functions $f_{11}, f_{12}, f_{21}, f_{22} \in \mathcal{F}$. The reinsurance contract $\left(\pi_{ij}, f_{ij}\right)$ is designed by the Reinsurer i for type j insurers.

Assumption 2.2. *(1)* $\pi_{11} = \pi_{21} := \pi_{\cdot 1}$ *and* $\pi_{12} = \pi_{22} := \pi_{\cdot 2}$.

(2) $\rho_{g_1}(f_{11}(X)) = \rho_{g_1}(f_{21}(X)) := \rho_{g_1}(f_{\cdot 1}(X))$ *and* $\rho_{g_2}(f_{12}(X)) = \rho_{g_2}(f_{22}(X)) := \rho_{g_2}(f_{\cdot 2}(X))$.

The price convergence of similar products, driven by cost convergence, has become a common phenomenon in some industries, which is the result of long-term strategic interactions among market participants. In the type j insurer submarket, Reinsurer i assumes a risk exposure of $f_{ij}(X)$ through the reinsurance contract (π_{ij}, f_{ij}), resulting in a distorted marginal cost of $\rho_{g_i}(f_{ij}(X))$. In a specific sub-market, targeting the insurer group shares identical risk distributions and risk preferences, we reasonably assume that all policies sold have the same distorted marginal cost (i.e., Assumption 2.2 (2)). In the reinsurance market we model, two monopolistic reinsurers exploit scale economies to reduce policy costs to marginal levels [25]. Furthermore, marginal cost convergence induces premium convergence (i.e., Assumption 2.2 (1)). For example, in the type 1 insurer submarket, all buyers share identical risk distributions and risk preferences of type 1. We assume that all policies in this submarket, (π_{11}, f_{11}) and (π_{21}, f_{21}), have equal distorted marginal costs (i.e., $\rho_{g_1}(f_{11}(X)) = \rho_{g_1}(f_{21}(X)) = \rho_{g_1}(f_{\cdot 1}(X))$), which leads to equal corresponding premiums (i.e., $\pi_{11} = \pi_{21} = \pi_{\cdot 1}$). Moreover, the oligopolistic market structure helps sustain price convergence, which aligns with cartel theory's prediction of an inverse relationship between collusion stability and the number of participants.

This reinsurance market comprises two submarkets catering to type 1 and type 2 insurers, respectively. An insurer of type 1 selects a reinsurance contract from Reinsurer 1 with probability m and from Reinsurer 2 with probability $(1 - m)$. An insurer of type 2 selects a reinsurance contract from Reinsurer 1 with probability n and from Reinsurer 2 with probability $(1 - n)$.

Remark 2.1. *The parameters m and n denote the market shares of Reinsurer 1 in two submarkets. Different assignments of (m, n) characterize distinct market structures. For instance, when $m > \frac{1}{2}$ and $n < \frac{1}{2}$, Reinsurer 1 dominates Reinsurer 2 in the type-1-oriented submarket, while Reinsurer 2 dominates Reinsurer 1 in the type-2-oriented submarket.*

Consider two risk-neutral reinsurers who maximize their expected net profits. Their objective functions are defined as

$$\begin{aligned} P_1 &= \mathbb{E}\left[(\pi_{11} - f_{11}(X))\mathbf{1}_{\{i=1,j=1\}} + (\pi_{12} - f_{12}(X))\mathbf{1}_{\{i=1,j=2\}}\right] \\ &= pm(\pi_{11} - \mathbb{E}[f_{11}(X)]) + (1-p)n(\pi_{12} - \mathbb{E}[f_{12}(X)]), \end{aligned} \tag{2.4}$$

$$\begin{aligned} P_2 &= \mathbb{E}\left[(\pi_{21} - f_{21}(X))\mathbf{1}_{\{i=2,j=1\}} + (\pi_{22} - f_{22}(X))\mathbf{1}_{\{i=2,j=2\}}\right] \\ &= p(1-m)(\pi_{21} - \mathbb{E}[f_{21}(X)]) + (1-p)(1-n)(\pi_{22} - \mathbb{E}[f_{22}(X)]), \end{aligned} \tag{2.5}$$

where P_i denotes the expected net profit of Reinsurer i.

Introducing Pareto optimality is natural for multi-agent systems, where no one position can be improved without worsening the others. Define the admissible reinsurance menu set as $\mathcal{M} = \{\{(\pi_{11}, f_{11}), (\pi_{12}, f_{12}), (\pi_{21}, f_{21}), (\pi_{22}, f_{22})\} \mid f_{11}, f_{12}, f_{21}, f_{22} \in \mathcal{F} \text{ and } 0 \le \pi_{11}, \pi_{12}, \pi_{21}, \pi_{22} \le \omega\}$. In this paper, $\left\{\left(\pi_{11}^*, f_{11}^*\right), \left(\pi_{12}^*, f_{12}^*\right), \left(\pi_{21}^*, f_{21}^*\right), \left(\pi_{22}^*, f_{22}^*\right)\right\} \in \mathcal{M}$ is Pareto-optimal if there is no other $\{(\pi_{11}, f_{11}), (\pi_{12}, f_{12}), (\pi_{21}, f_{21}), (\pi_{22}, f_{22})\} \in \mathcal{M}$ such that $P_1 \ge P_1^*$ and $P_2 \ge P_2^*$ hold with at least one strict inequality.

The following proposition provides a method for obtaining a Pareto-optimal reinsurance menu.

Proposition 2.1. *An optimal reinsurance menu $\left\{\left(\pi_{11}^*, f_{11}^*\right), \left(\pi_{12}^*, f_{12}^*\right), \left(\pi_{21}^*, f_{21}^*\right), \left(\pi_{22}^*, f_{22}^*\right)\right\}$ can be derived by solving the convex combination maximization problem*

$$\max_{\{(\pi_{11}, f_{11}), (\pi_{12}, f_{12}), (\pi_{21}, f_{21}), (\pi_{22}, f_{22})\} \in \mathcal{M}} \lambda P_1 + (1-\lambda) P_2, \tag{2.6}$$

where $\lambda \in (0, 1)$.

Proof. As in [26], a method for determining a Pareto-optimal reinsurance menu involves any two positive constants, k_1 and k_2, and solving for

$$\max_{\{(\pi_{11}, f_{11}), (\pi_{12}, f_{12}), (\pi_{21}, f_{21}), (\pi_{22}, f_{22})\} \in \mathcal{M}} k_1 P_1 + k_2 P_2.$$

Without loss of generality, we set $k_1 = \lambda$ and $k_2 = (1-\lambda)$. Let $\tilde{f}_{ij}(X) = r\hat{f}_{ij}(X) + (1-r)\check{f}_{ij}(X)$ and $\tilde{\pi}_{ij} = r\hat{\pi}_{ij} + (1-r)\check{\pi}_{ij}$ for $\tilde{f}_{ij}(X)$, $\hat{f}_{ij}(X)$, $\check{f}_{ij}(X) \in \mathcal{F}$, $\tilde{\pi}_{ij}$, $\hat{\pi}_{ij}$, $\check{\pi}_{ij} \ge 0$, and $0 \le r \le 1$. Following condition C on page 90 of [26], the Pareto-optimal reinsurance menu can be derived when the following two conditions hold.

$$P_1\left(\left(\tilde{\pi}_{11}, \tilde{f}_{11}\right), \left(\tilde{\pi}_{12}, \tilde{f}_{12}\right)\right) \ge rP_1\left(\left(\hat{\pi}_{11}, \hat{f}_{11}\right), \left(\hat{\pi}_{12}, \hat{f}_{12}\right)\right) + (1-r)P_1\left(\left(\check{\pi}_{11}, \check{f}_{11}\right), \left(\check{\pi}_{12}, \check{f}_{12}\right)\right), \tag{2.7}$$

$$P_2\left(\left(\tilde{\pi}_{21}, \tilde{f}_{21}\right), \left(\tilde{\pi}_{22}, \tilde{f}_{22}\right)\right) \ge rP_2\left(\left(\hat{\pi}_{21}, \hat{f}_{21}\right), \left(\hat{\pi}_{22}, \hat{f}_{22}\right)\right) + (1-r)P_2\left(\left(\check{\pi}_{21}, \check{f}_{21}\right), \left(\check{\pi}_{22}, \check{f}_{22}\right)\right). \tag{2.8}$$

By applying the linearity property of expectation, we obtain

$$P_1\left(\left(\tilde{\pi}_{11}, \tilde{f}_{11}\right), \left(\tilde{\pi}_{12}, \tilde{f}_{12}\right)\right) = pm\left[\tilde{\pi}_{11} - \mathbb{E}\left[\tilde{f}_{11}(X)\right]\right] + (1-p)n\left[\tilde{\pi}_{12} - \mathbb{E}\left[\tilde{f}_{12}(X)\right]\right]$$

$$= pm\left[r\hat{\pi}_{11} + (1-r)\check{\pi}_{11} - \mathbb{E}\left[r\hat{f}_{11}(X) + (1-r)\check{f}_{11}(X)\right]\right] + (1-p)n\,[r\hat{\pi}_{12}$$
$$+ (1-r)\check{\pi}_{12} - \mathbb{E}\left[r\hat{f}_{12}(X) + (1-r)\check{f}_{12}\right]]$$
$$= r\left[pm\left(\hat{\pi}_{11} - \mathbb{E}\left[\hat{f}_{11}(X)\right]\right) + (1-p)n\left(\hat{\pi}_{12} - \mathbb{E}\left[\hat{f}_{12}(X)\right]\right)\right] + (1-r)\,[pm$$
$$\left(\check{\pi}_{11} - \mathbb{E}\left[\check{f}_{11}(X)\right]\right) + (1-p)n\left(\check{\pi}_{12} - \mathbb{E}\left[\check{f}_{12}(X)\right]\right)]$$
$$= rP_1\left(\left(\hat{\pi}_{11}, \hat{f}_{11}\right), \left(\hat{\pi}_{12}, \hat{f}_{12}\right)\right) + (1-r)\,P_1\left(\left(\check{\pi}_{11}, \check{f}_{11}\right), \left(\check{\pi}_{12}, \check{f}_{12}\right)\right),$$

which means that (2.7) holds. Similarly, (2.8) also holds. Therefore, the Pareto-optimal reinsurance menu can be obtained from Problem 2.6. □

Remark 2.2. *λ represents the bargaining power allocation: Reinsurer* 1 *dominated as $\lambda \in \left(\frac{1}{2}, 1\right)$, balanced at $\lambda = \frac{1}{2}$, and Reinsurer* 2 *dominated as $\lambda \in \left(0, \frac{1}{2}\right)$.*

We impose three classes of conditions to ensure the validity of the reinsurance menu. First, to incentivize insurers to purchase reinsurance, we impose individual rationality constraints (IR) into the design of the reinsurance contracts, which ensures that purchasing the reinsurance contract designed for the type of the insurer does not worsen the insurer's situation. Second, to encourage insurers to truthfully reveal their risk preferences, we impose incentive compatibility constraints (IC), which ensure that type 1 insurers are motivated to choose contract (π_{11}, f_{11}) or (π_{21}, f_{21}) over (π_{12}, f_{12}) and (π_{22}, f_{22}) and type 2 insurers to choose contract (π_{12}, f_{12}) or (π_{22}, f_{22}) over (π_{11}, f_{11}) and (π_{21}, f_{21}). Consequently, reinsurers can infer insurers' risk preferences based on their contract selection behavior. Finally, assume that insurers have convex preferences. In economics, the convexity of indifference curves implies a diminishing marginal rate of substitution. Economic theory models insurers' risk preferences as convex, satisfying the axiom: $\forall \alpha \in (0,1)$, *if $A \le C$ and $B \le C$, then $\alpha A + (1-\alpha) B \le C$*. This property guarantees that when neither Decision A nor Decision B is strictly preferred to Decision C, then no convex combination of A and B can be strictly preferred to C. Consequently, we impose convex preference constraints (CP) based on the preferences derived from individual rationality constraints and incentive compatibility constraints.

Above all, the main problem of this paper can be formulated as

$$\max_{\{(\pi_{11},f_{11}),(\pi_{12},f_{12}),(\pi_{21},f_{21}),(\pi_{22},f_{22})\}\in\mathcal{M}} \lambda P_1 + (1-\lambda)P_2, \tag{2.9}$$

$$\text{IR11}: \rho_{g_1}(X - f_{11}(X) + \pi_{11}) \le \rho_{g_1}(X),$$
$$\text{IR21}: \rho_{g_1}(X - f_{21}(X) + \pi_{21}) \le \rho_{g_1}(X),$$
$$\text{IR12}: \rho_{g_2}(X - f_{12}(X) + \pi_{12}) \le \rho_{g_2}(X),$$
$$\text{IR22}: \rho_{g_2}(X - f_{22}(X) + \pi_{22}) \le \rho_{g_2}(X),$$
$$\text{IC11}: \begin{cases} \rho_{g_2}(X - f_{12}(X) + \pi_{12}) \le \rho_{g_2}(X - f_{11}(X) + \pi_{11}), \\ \rho_{g_2}(X - f_{22}(X) + \pi_{22}) \le \rho_{g_2}(X - f_{11}(X) + \pi_{11}), \end{cases}$$
$$\text{IC12}: \begin{cases} \rho_{g_1}(X - f_{11}(X) + \pi_{11}) \le \rho_{g_1}(X - f_{12}(X) + \pi_{12}), \\ \rho_{g_1}(X - f_{21}(X) + \pi_{21}) \le \rho_{g_1}(X - f_{12}(X) + \pi_{12}), \end{cases}$$
$$\text{IC21}: \begin{cases} \rho_{g_2}(X - f_{12}(X) + \pi_{12}) \le \rho_{g_2}(X - f_{21}(X) + \pi_{21}), \\ \rho_{g_2}(X - f_{22}(X) + \pi_{22}) \le \rho_{g_2}(X - f_{21}(X) + \pi_{21}), \end{cases}$$

$$\text{IC22}: \begin{cases} \rho_{g_1}(X - f_{11}(X) + \pi_{11}) \le \rho_{g_1}(X - f_{22}(X) + \pi_{22}), \\ \rho_{g_1}(X - f_{21}(X) + \pi_{21}) \le \rho_{g_1}(X - f_{22}(X) + \pi_{22}), \end{cases}$$
$$\text{CP1}: \alpha_1\rho_{g_1}(X - f_{11}(X) + \pi_{11}) + (1-\alpha_1)\rho_{g_1}(X - f_{21}(X) + \pi_{21}) \le \rho_{g_1}(X),$$
$$\text{CP2}: \alpha_2\rho_{g_2}(X - f_{12}(X) + \pi_{12}) + (1-\alpha_2)\rho_{g_2}(X - f_{22}(X) + \pi_{22}) \le \rho_{g_2}(X),$$
$$\text{CP3}: \alpha_3\rho_{g_1}(X - f_{11}(X) + \pi_{11}) + (1-\alpha_3)\rho_{g_1}(X - f_{21}(X) + \pi_{21}) \le \rho_{g_1}(X - f_{12}(X) + \pi_{12}),$$
$$\text{CP4}: \alpha_4\rho_{g_1}(X - f_{11}(X) + \pi_{11}) + (1-\alpha_4)\rho_{g_1}(X - f_{21}(X) + \pi_{21}) \le \rho_{g_1}(X - f_{22}(X) + \pi_{22}),$$
$$\text{CP5}: \alpha_5\rho_{g_2}(X - f_{12}(X) + \pi_{12}) + (1-\alpha_5)\rho_{g_2}(X - f_{22}(X) + \pi_{22}) \le \rho_{g_2}(X - f_{11}(X) + \pi_{11}),$$
$$\text{CP6}: \alpha_6\rho_{g_2}(X - f_{12}(X) + \pi_{12}) + (1-\alpha_6)\rho_{g_2}(X - f_{22}(X) + \pi_{22}) \le \rho_{g_2}(X - f_{21}(X) + \pi_{21}),$$

where $\lambda, \alpha_i \in (0,1), i = 1,2,3,4,5,6$. According to the comonotonic additivity and translation invariance of distortion risk measures, we further obtain

$$\max_{\{(\pi_{11},f_{11}),(\pi_{12},f_{12}),(\pi_{21},f_{21}),(\pi_{22},f_{22})\}\in\mathcal{M}} \lambda P_1 + (1-\lambda)P_2, \tag{2.10}$$
$$\text{IR11}: \pi_{11} \le \rho_{g_1}(f_{11}(X)),$$
$$\text{IR21}: \pi_{21} \le \rho_{g_1}(f_{21}(X)),$$
$$\text{IR12}: \pi_{12} \le \rho_{g_2}(f_{12}(X)),$$
$$\text{IR22}: \pi_{22} \le \rho_{g_2}(f_{22}(X)),$$
$$\text{IC11}: \begin{cases} \pi_{12} - \rho_{g_2}(f_{12}(X)) \le \pi_{11} - \rho_{g_2}(f_{11}(X)), \\ \pi_{22} - \rho_{g_2}(f_{22}(X)) \le \pi_{11} - \rho_{g_2}(f_{11}(X)), \end{cases}$$
$$\text{IC12}: \begin{cases} \pi_{11} - \rho_{g_1}(f_{11}(X)) \le \pi_{12} - \rho_{g_1}(f_{12}(X)), \\ \pi_{21} - \rho_{g_1}(f_{21}(X)) \le \pi_{12} - \rho_{g_1}(f_{12}(X)), \end{cases}$$
$$\text{IC21}: \begin{cases} \pi_{12} - \rho_{g_2}(f_{12}(X)) \le \pi_{21} - \rho_{g_2}(f_{21}(X)), \\ \pi_{22} - \rho_{g_2}(f_{22}(X)) \le \pi_{21} - \rho_{g_2}(f_{21}(X)), \end{cases}$$
$$\text{IC22}: \begin{cases} \pi_{11} - \rho_{g_1}(f_{11}(X)) \le \pi_{22} - \rho_{g_1}(f_{22}(X)), \\ \pi_{21} - \rho_{g_1}(f_{21}(X)) \le \pi_{22} - \rho_{g_1}(f_{22}(X)), \end{cases}$$
$$\text{CP1}: \alpha_1\pi_{11} + (1-\alpha_1)\pi_{21} \le \alpha_1\rho_{g_1}(f_{11}(X)) + (1-\alpha_1)\rho_{g_1}(f_{21}(X)),$$
$$\text{CP2}: \alpha_2\pi_{12} + (1-\alpha_2)\pi_{22} \le \alpha_2\rho_{g_1}(f_{12}(X)) + (1-\alpha_2)\rho_{g_1}(f_{22}(X)),$$
$$\text{CP3}: \alpha_3\pi_{11} + (1-\alpha_3)\pi_{21} - \alpha_3\rho_{g_1}(f_{11}(X)) - (1-\alpha_3)\rho_{g_1}(f_{21}(X)) \le \pi_{12} - \rho_{g_1}(f_{12}(X)),$$
$$\text{CP4}: \alpha_4\pi_{11} + (1-\alpha_4)\pi_{21} - \alpha_4\rho_{g_1}(f_{11}(X)) - (1-\alpha_4)\rho_{g_1}(f_{21}(X)) \le \pi_{22} - \rho_{g_1}(f_{22}(X)),$$
$$\text{CP5}: \alpha_5\pi_{12} + (1-\alpha_5)\pi_{22} - \alpha_5\rho_{g_2}(f_{12}(X)) - (1-\alpha_5)\rho_{g_2}(f_{22}(X)) \le \pi_{11} - \rho_{g_2}(f_{11}(X)),$$
$$\text{CP6}: \alpha_6\pi_{12} + (1-\alpha_6)\pi_{22} - \alpha_6\rho_{g_2}(f_{12}(X)) - (1-\alpha_6)\rho_{g_2}(f_{22}(X)) \le \pi_{21} - \rho_{g_2}(f_{21}(X)).$$

Under Assumption 2.2, Problem 2.10 transforms into

$$\max_{\{\pi_{\cdot 1},\pi_{\cdot 2},f_{11},f_{12},f_{21},f_{22}\}\in\mathcal{M}} \lambda P_1 + (1-\lambda)P_2, \tag{2.11}$$
$$\text{IR1}: \pi_{\cdot 1} \le \rho_{g_1}(f_{\cdot 1}(X)),$$
$$\text{IR2}: \pi_{\cdot 2} \le \rho_{g_2}(f_{\cdot 2}(X)),$$

$$\begin{aligned}
&\text{IC12}: \pi_{\cdot 1} \le \pi_{\cdot 2} + \rho_{g_1}(f_{\cdot 1}(X)) - \rho_{g_1}(f_{12}(X)),\\
&\text{IC22}: \pi_{\cdot 1} \le \pi_{\cdot 2} + \rho_{g_1}(f_{\cdot 1}(X)) - \rho_{g_1}(f_{22}(X)),\\
&\text{IC11}: \pi_{\cdot 2} \le \pi_{\cdot 1} + \rho_{g_2}(f_{\cdot 2}(X)) - \rho_{g_2}(f_{11}(X)),\\
&\text{IC12}: \pi_{\cdot 2} \le \pi_{\cdot 1} + \rho_{g_2}(f_{\cdot 2}(X)) - \rho_{g_2}(f_{21}(X)).
\end{aligned}$$

Remark 2.3. *It is noteworthy that, under Assumption* 2.2*, all* CP *conditions degenerate into their corresponding* IR *or* IC *conditions. For example,* CP1 *is formulated based on the preferences derived from* IR11 *and* IR21 *in Problem* 2.11*, and degenerate into* IR1 *in Problem* 2.11 *under Assumption* 2.2*.*

3. The optimal reinsurance menus

In this section, we derive the optimal reinsurance menus in each case through classified analysis. We first discuss the existence and uniqueness of an optimal solution.

Existence. Inspired by the work of Chi et al. [27], we present a proof for the existence of an optimal solution as follows. The objective function in Problem 2.11 is defined as P. Define $\mathfrak{M} = \sup_{(\pi_{\cdot 1},\pi_{\cdot 2},f_{11},f_{12},f_{21},f_{22})\in\mathcal{M}} P$. By the definition of the supremum, there exists a family of sequences $(\pi_{\cdot 1n},\pi_{\cdot 2n},f_{11n},f_{12n},f_{21n},f_{22n})_{n\ge 1} \subseteq \mathcal{M}$ such that

$$\mathfrak{M} = \lim_{n\to\infty} P(\pi_{\cdot 1n},\pi_{\cdot 2n},f_{11n},f_{12n},f_{21n},f_{22n}).$$

The boundedness of $(f_{ijn})_{n\ge 1}$ by M and $(\pi_{\cdot jn})_{n\ge 1}$ by ω prevents the objective function from reaching infinity. Consequently, there exists some constant L such that

$$\mathfrak{M} = \lim_{n\to\infty} P(\pi_{\cdot 1n},\pi_{\cdot 2n},f_{11n},f_{12n},f_{21n},f_{22n}) \le L.$$

The Arzelà-Ascoli theorem (Theorem 7.25) [28] shows that there exist subsequences $(\pi_{\cdot 1n_k},\pi_{\cdot 2n_k},f_{11n_k},f_{12n_k},f_{21n_k},f_{22n_k})_{k\ge 1}$ that uniformly converge to $(\pi^*_{\cdot 1},\pi^*_{\cdot 2},f^*_{11},f^*_{12},f^*_{21},f^*_{22})$ on $\mathcal{M}$. A direct verification shows that $(\pi^*_{\cdot 1},\pi^*_{\cdot 2},f^*_{11},f^*_{12},f^*_{21},f^*_{22}) \in \mathcal{M}$. Moreover, $P(\pi_{\cdot 1n_k},\pi_{\cdot 2n_k},f_{11n_k},f_{12n_k},f_{21n_k},f_{22n_k}) \to P(\pi^*_{\cdot 1},\pi^*_{\cdot 2},f^*_{11},f^*_{12},f^*_{21},f^*_{22})$ a.s. Using Fatou's lemma, we obtain

$$\begin{aligned}
L - P(\pi^*_{\cdot 1},\pi^*_{\cdot 2},f^*_{11},f^*_{12},f^*_{21},f^*_{22}) &= \mathbb{E}\Big[\lim_{k\to\infty}\Big(L - \big[\lambda\left(\pi_{\cdot 1n_k} - f_{11n_k}(X)\right)\mathbf{1}_{\{i=1,j=1\}} + \lambda\left(\pi_{\cdot 2n_k} - f_{12n_k}(X)\right)\mathbf{1}_{\{i=1,j=2\}}\\
&\quad + (1-\lambda)\left(\pi_{\cdot 1n_k} - f_{21n_k}(X)\right)\mathbf{1}_{\{i=2,j=1\}} + (1-\lambda)\left(\pi_{\cdot 2n_k} - f_{22n_k}(X)\right)\mathbf{1}_{\{i=2,j=2\}}\big]\Big)\Big]\\
&\le \liminf_{k\to\infty}[L - P(\pi_{\cdot 1n_k},\pi_{\cdot 2n_k},f_{11n_k},f_{12n_k},f_{21n_k},f_{22n_k})]\\
&= L - \mathfrak{M},
\end{aligned}$$

which implies $\mathfrak{M} \le P(\pi^*_{\cdot 1},\pi^*_{\cdot 2},f^*_{11},f^*_{12},f^*_{21},f^*_{22})$. Additionally, the definition of $\mathfrak{M}$ implies that $\mathfrak{M} \ge P(\pi^*_{\cdot 1},\pi^*_{\cdot 2},f^*_{11},f^*_{12},f^*_{21},f^*_{22})$, so we have $\mathfrak{M} = P(\pi^*_{\cdot 1},\pi^*_{\cdot 2},f^*_{11},f^*_{12},f^*_{21},f^*_{22})$. Therefore, $(\pi^*_{\cdot 1},\pi^*_{\cdot 2},f^*_{11},f^*_{12},f^*_{21},f^*_{22})$ is a solution to Problem 2.11.

Uniqueness. Whether all Pareto-optimal reinsurance menus can be obtained through this method warrants careful investigation. In general, the answer is negative. However, when the set $S := \{(P_1,P_2) : \{(\pi_{11},f_{11}),(\pi_{12},f_{12}),(\pi_{21},f_{21}),(\pi_{22},f_{22})\} \in \mathcal{M}\}$ is convex, this method offers all Pareto-optimal solutions [26].

Notably, all conditions in Problem 2.11 can be interpreted as imposing an upper bound on a specific type of premium. By classifying premium types with constrained upper bounds across all conditions, Problem 2.11 can be reformulated as

$$\max_{\{\pi_{\cdot 1},\pi_{\cdot 2},f_{11},f_{12},f_{21},f_{22}\}\in\mathcal{M}} \lambda P_1 + (1-\lambda)P_2, \tag{3.1}$$

$$\pi_{\cdot 1} \le \min\left\{\rho_{g_1}(f_{\cdot 1}), \pi_{\cdot 2} + \rho_{g_1}(f_{\cdot 1}) - \rho_{g_1}(f_{12}), \pi_{\cdot 2} + \rho_{g_1}(f_{\cdot 1}) - \rho_{g_1}(f_{22})\right\} \ (\textit{through}\ \text{IR1},\ \text{IC12},\ \textit{and}\ \text{IC22}),$$

$$\pi_{\cdot 2} \le \min\left\{\rho_{g_2}(f_{\cdot 2}), \pi_{\cdot 1} + \rho_{g_2}(f_{\cdot 2}) - \rho_{g_2}(f_{11}), \pi_{\cdot 1} + \rho_{g_2}(f_{\cdot 2}) - \rho_{g_2}(f_{21})\right\} \ (\textit{through}\ \text{IR2},\ \text{IC11},\ \textit{and}\ \text{IC21}).$$

From (2.4), (2.5), and Problem 3.1, $\pi_{\cdot 1}$ and $\pi_{\cdot 2}$ must be set as large as possible to maximize the objective function. Since $\pi_{\cdot 1}$ and $\pi_{\cdot 2}$ are constrained by the conditions in (3.1), the objective function attains its maximum when $\pi_{\cdot 1} = \min\left\{\rho_{g_1}(f_{\cdot 1}), \pi_{\cdot 2} + \rho_{g_1}(f_{\cdot 1}) - \rho_{g_1}(f_{12}), \pi_{\cdot 2} + \rho_{g_1}(f_{\cdot 1}) - \rho_{g_1}(f_{22})\right\}$ and $\pi_{\cdot 2} = \min\left\{\rho_{g_2}(f_{\cdot 2}), \pi_{\cdot 1} + \rho_{g_2}(f_{\cdot 2}) - \rho_{g_2}(f_{11}), \pi_{\cdot 1} + \rho_{g_2}(f_{\cdot 2}) - \rho_{g_2}(f_{21})\right\}$. Hence, the following cases naturally arise depending on which condition yields the minimum: (1) IR1 and IR2; (2) IR1 and IC11; (3) IR1 and IC21; (4) IC12 and IR2; (5) IC22 and IR2; (6) IC12 and IC11; (7) IC12 and IC21; (8) IC22 and IC11; (9) IC22 and IC21.

However, Cases 6–9 do not impose upper bounds on $\pi_{\cdot 1}$ and $\pi_{\cdot 2}$. Consequently, reinsurers can increase both $\pi_{\cdot 1}$ and $\pi_{\cdot 2}$ until either IR1 or IR2 becomes binding. This implies that Cases 6–9 are effectively nested within Cases 1–5. Given this observation, Problem 3.1 can be decomposed into the following five cases.

Case 1. (IR1 *and* IR2)

$$\max_{\{f_{11},f_{12},f_{21},f_{22}\}\in\mathcal{F}} \lambda\left[pm\left(\rho_{g_1}(f_{\cdot 1}(X)) - \mathbb{E}[f_{11}(X)]\right) + (1-p)\,n\left(\rho_{g_2}(f_{\cdot 2}(X)) - \mathbb{E}[f_{12}(X)]\right)\right] + (1-\lambda)$$
$$\left[p(1-m)\left(\rho_{g_1}(f_{\cdot 1}(X)) - \mathbb{E}[f_{21}(X)]\right) + (1-p)(1-n)\left(\rho_{g_2}(f_{\cdot 2}(X)) - \mathbb{E}[f_{22}(X)]\right)\right], \tag{3.2}$$

$$\text{IC12}: \rho_{g_1}(f_{12}(X)) \le \rho_{g_2}(f_{\cdot 2}(X)),$$
$$\text{IC22}: \rho_{g_1}(f_{22}(X)) \le \rho_{g_2}(f_{\cdot 2}(X)),$$
$$\text{IC11}: \rho_{g_2}(f_{11}(X)) \le \rho_{g_1}(f_{\cdot 1}(X)),$$
$$\text{IC21}: \rho_{g_2}(f_{21}(X)) \le \rho_{g_1}(f_{\cdot 1}(X)).$$

Case 2. (IR1 *and* IC11)

$$\max_{\{f_{11},f_{12},f_{21},f_{22}\}\in\mathcal{F}} \lambda\left[pm\left(\rho_{g_1}(f_{\cdot 1}(X)) - \mathbb{E}[f_{11}(X)]\right) + (1-p)\,n\left(\rho_{g_1}(f_{\cdot 1}(X)) + \rho_{g_2}(f_{\cdot 2}(X)) - \rho_{g_2}(f_{11}(X))\right.\right.$$
$$\left. - \mathbb{E}[f_{12}(X)])\right] + (1-\lambda)\left[p(1-m)\left(\rho_{g_1}(f_{\cdot 1}(X)) - \mathbb{E}[f_{21}(X)]\right) + (1-p)(1-n)\left(\rho_{g_1}(f_{\cdot 1}(X))\right.\right.$$
$$\left.\left. + \rho_{g_2}(f_{\cdot 2}(X)) - \rho_{g_2}(f_{11}(X)) - \mathbb{E}[f_{22}(X)]\right)\right], \tag{3.3}$$

$$\text{IR2}: \rho_{g_1}(f_{\cdot 1}(X)) \le \rho_{g_2}(f_{11}(X)),$$
$$\text{IC12}: \rho_{g_1}(f_{12}(X)) \le \rho_{g_2}(f_{\cdot 2}(X)) + \rho_{g_1}(f_{\cdot 1}(X)) - \rho_{g_2}(f_{11}(X)),$$
$$\text{IC22}: \rho_{g_1}(f_{22}(X)) \le \rho_{g_2}(f_{\cdot 2}(X)) + \rho_{g_1}(f_{\cdot 1}(X)) - \rho_{g_2}(f_{11}(X)),$$
$$\text{IC21}: \rho_{g_2}(f_{21}(X)) \le \rho_{g_2}(f_{11}(X)).$$

Case 3. (IR1 *and* IC21)

$$\max_{\{f_{11},f_{12},f_{21},f_{22}\}\in\mathcal{F}} \lambda\left[pm\left(\rho_{g_1}(f_{\cdot 1}(X)) - \mathbb{E}[f_{11}(X)]\right) + (1-p)\,n\left(\rho_{g_1}(f_{\cdot 1}(X)) + \rho_{g_2}(f_{\cdot 2}(X)) - \rho_{g_2}(f_{21}(X))\right.\right.$$

$$
\begin{aligned}
&- \mathbb{E}\left[f_{12}(X)\right])] + (1-\lambda)\Big[p\,(1-m)\Big(\rho_{g_1}\left(f_{\cdot 1}(X)\right) - \mathbb{E}\left[f_{21}(X)\right]\Big) + (1-p)\,(1-n)\Big(\rho_{g_1}\left(f_{\cdot 1}(X)\right) \\
&+ \rho_{g_2}\left(f_{\cdot 2}(X)\right) - \rho_{g_2}\left(f_{21}(X)\right) - \mathbb{E}\left[f_{22}(X)\right]\Big)\Big], \qquad (3.4)
\end{aligned}
$$

$$
\begin{aligned}
\text{IR2}&: \rho_{g_1}(f_{\cdot 1}(X)) \leq \rho_{g_2}(f_{21}(X)), \\
\text{IC12}&: \rho_{g_1}(f_{12}(X)) \leq \rho_{g_2}(f_{\cdot 2}(X)) + \rho_{g_1}(f_{\cdot 1}(X)) - \rho_{g_2}(f_{21}(X)), \\
\text{IC22}&: \rho_{g_1}(f_{22}(X)) \leq \rho_{g_2}(f_{\cdot 2}(X)) + \rho_{g_1}(f_{\cdot 1}(X)) - \rho_{g_2}(f_{21}(X)), \\
\text{IC11}&: \rho_{g_2}(f_{11}(X)) \leq \rho_{g_2}(f_{21}(X)).
\end{aligned}
$$

Case 4. (IC12 *and* IR2)

$$
\begin{aligned}
\max_{\{f_{11}, f_{12}, f_{21}, f_{22}\} \in \mathcal{F}} &\lambda\Big[pm\Big(\rho_{g_2}\left(f_{\cdot 2}(X)\right) + \rho_{g_1}\left(f_{\cdot 1}(X)\right) - \rho_{g_1}\left(f_{12}(X)\right) - \mathbb{E}\left[f_{11}(X)\right]\Big) + (1-p)\,n\Big(\rho_{g_2}\left(f_{\cdot 2}(X)\right) \\
&- \mathbb{E}\left[f_{12}(X)\right])] + (1-\lambda)\Big[p\,(1-m)\Big(\rho_{g_2}\left(f_{\cdot 2}(X)\right) + \rho_{g_1}\left(f_{\cdot 1}(X)\right) - \rho_{g_1}\left(f_{12}(X)\right) - \mathbb{E}\left[f_{21}(X)\right]\Big) \\
&+ (1-p)\,(1-n)\Big(\rho_{g_2}\left(f_{\cdot 2}(X)\right) - \mathbb{E}\left[f_{22}(X)\right]\Big)\Big], \qquad (3.5)
\end{aligned}
$$

$$
\begin{aligned}
\text{IR1}&: \rho_{g_2}(f_{\cdot 2}(X)) \leq \rho_{g_1}(f_{12}(X)), \\
\text{IC22}&: \rho_{g_1}(f_{22}(X)) \leq \rho_{g_1}(f_{12}(X)), \\
\text{IC11}&: \rho_{g_2}(f_{11}(X)) \leq \rho_{g_1}(f_{\cdot 1}(X)) + \rho_{g_2}(f_{\cdot 2}(X)) - \rho_{g_1}(f_{12}(X)), \\
\text{IC21}&: \rho_{g_2}(f_{21}(X)) \leq \rho_{g_1}(f_{\cdot 1}(X)) + \rho_{g_2}(f_{\cdot 2}(X)) - \rho_{g_1}(f_{12}(X)).
\end{aligned}
$$

Case 5. (IC22 *and* IR2)

$$
\begin{aligned}
\max_{\{f_{11}, f_{12}, f_{21}, f_{22}\} \in \mathcal{F}} &\lambda\Big[pm\Big(\rho_{g_2}\left(f_{\cdot 2}(X)\right) + \rho_{g_1}\left(f_{\cdot 1}(X)\right) - \rho_{g_1}\left(f_{22}(X)\right) - \mathbb{E}\left[f_{11}(X)\right]\Big) + (1-p)\,n\Big(\rho_{g_2}\left(f_{\cdot 2}(X)\right) \\
&- \mathbb{E}\left[f_{12}(X)\right])] + (1-\lambda)\Big[p\,(1-m)\Big(\rho_{g_2}\left(f_{\cdot 2}(X)\right) + \rho_{g_1}\left(f_{\cdot 1}(X)\right) - \rho_{g_1}\left(f_{22}(X)\right) - \mathbb{E}\left[f_{21}(X)\right]\Big) \\
&+ (1-p)\,(1-n)\Big(\rho_{g_2}\left(f_{\cdot 2}(X)\right) - \mathbb{E}\left[f_{22}(X)\right]\Big)\Big], \qquad (3.6)
\end{aligned}
$$

$$
\begin{aligned}
\text{IR1}&: \rho_{g_2}(f_{\cdot 2}(X)) \leq \rho_{g_1}(f_{22}(X)), \\
\text{IC12}&: \rho_{g_1}(f_{12}(X)) \leq \rho_{g_1}(f_{22}(X)), \\
\text{IC11}&: \rho_{g_2}(f_{11}(X)) \leq \rho_{g_1}(f_{\cdot 1}(X)) + \rho_{g_2}(f_{\cdot 2}(X)) - \rho_{g_1}(f_{22}(X)), \\
\text{IC21}&: \rho_{g_2}(f_{21}(X)) \leq \rho_{g_1}(f_{\cdot 1}(X)) + \rho_{g_2}(f_{\cdot 2}(X)) - \rho_{g_1}(f_{22}(X)).
\end{aligned}
$$

Combining Definition 2.1 and (2.3), we observe that

$$
\begin{aligned}
\mathbb{E}\left[f_{ij}(X)\right] &= \int_0^M S_X(z) h_{ij}(z) \mathrm{d}z, \\
\rho_{g_j}\left(f_{ij}(X)\right) &= \int_0^M g_j\left(S_X(z)\right) h_{ij}(z) \mathrm{d}z, \\
\rho_{g_j}\left(f_{\cdot j}(X)\right) &:= \int_0^M g_j\left(S_X(z)\right) h_{\cdot j}(z) \mathrm{d}z,
\end{aligned}
$$

where h_{ij} and $h_{\cdot j}$ are the MIFs of the indemnity functions f_{ij} and $f_{\cdot j}$, respectively. Coupled with Assumption 2.2, Cases 1–5 can then be reformulated in the following integral form.

Case 1. (IR1 *and* IR2)

$$\max_{\{h_{11},h_{12},h_{21},h_{22}\}\in\mathcal{H}} \lambda\left[pm\left(\int_0^M g_1\left(S_X(z)\right)h_{11}(z)\mathrm{d}z - \int_0^M S_X(z)h_{11}(z)\mathrm{d}z\right) + (1-p)\,n\left(\int_0^M g_2\left(S_X(z)\right)h_{12}(z)\mathrm{d}z \right.\right.$$
$$\left.\left. - \int_0^M S_X(z)h_{12}(z)\mathrm{d}z\right)\right] + (1-\lambda)\left[p\,(1-m)\left(\int_0^M g_1\left(S_X(z)\right)h_{21}(z)\mathrm{d}z - \int_0^M S_X(z)h_{21}(z)\mathrm{d}z\right)\right.$$
$$\left. + (1-p)\,(1-n)\left(\int_0^M g_2\left(S_X(z)\right)h_{22}(z)\mathrm{d}z - \int_0^M S_X(z)h_{22}(z)\mathrm{d}z\right)\right], \tag{3.7}$$

$$\begin{aligned}
\text{IC12}:&\int_0^M g_1\left(S_X(z)\right)h_{12}(z)\mathrm{d}z \le \int_0^M g_2\left(S_X(z)\right)h_{12}(z)\mathrm{d}z,\\
\text{IC22}:&\int_0^M g_1\left(S_X(z)\right)h_{22}(z)\mathrm{d}z \le \int_0^M g_2\left(S_X(z)\right)h_{22}(z)\mathrm{d}z,\\
\text{IC11}:&\int_0^M g_2\left(S_X(z)\right)h_{11}(z)\mathrm{d}z \le \int_0^M g_1\left(S_X(z)\right)h_{11}(z)\mathrm{d}z,\\
\text{IC21}:&\int_0^M g_2\left(S_X(z)\right)h_{21}(z)\mathrm{d}z \le \int_0^M g_1\left(S_X(z)\right)h_{21}(z)\mathrm{d}z.
\end{aligned}$$

Case 2. (IR1 *and* IC11)

$$\max_{\{h_{11},h_{12},h_{21},h_{22}\}\in\mathcal{H}} \lambda\left[pm\left(\int_0^M g_1\left(S_X(z)\right)h_{11}(z)\mathrm{d}z - \int_0^M S_X(z)h_{11}(z)\mathrm{d}z\right) + (1-p)\,n\left(\int_0^M g_1\left(S_X(z)\right)h_{11}(z)\mathrm{d}z \right.\right.$$
$$\left.\left. + \int_0^M g_2\left(S_X(z)\right)h_{12}(z) - \int_0^M g_2\left(S_X(z)\right)h_{11}(z)\mathrm{d}z - \int_0^M S_X(z)h_{12}(z)\mathrm{d}z\right)\right] + (1-\lambda)\,[p$$
$$(1-m)\left(\int_0^M g_1\left(S_X(z)\right)h_{21}(z)\mathrm{d}z - \int_0^M S_X(z)h_{21}(z)\mathrm{d}z\right) + (1-p)\,(1-n)\left(\int_0^M g_1\left(S_X(z)\right)\right.$$
$$\left.\left. h_{11}(z)\mathrm{d}z + \int_0^M g_2\left(S_X(z)\right)h_{22}(z)\mathrm{d}z - \int_0^M g_2\left(S_X(z)\right)h_{11}(z)\mathrm{d}z - \int_0^M S_X(z)h_{22}(z)\mathrm{d}z\right)\right], \tag{3.8}$$

$$\begin{aligned}
\text{IR2}:&\int_0^M \left[g_2\left(S_X(z)\right) - g_1\left(S_X(z)\right)\right]h_{11}(z)\mathrm{d}z \ge 0,\\
\text{IC12}:&\int_0^M \left[g_1\left(S_X(z)\right) - g_2\left(S_X(z)\right)\right]h_{11}(z)\mathrm{d}z + \int_0^M \left[g_2\left(S_X(z)\right) - g_1\left(S_X(z)\right)\right]h_{12}(z)\mathrm{d}z \ge 0,\\
\text{IC22}:&\int_0^M \left[g_1\left(S_X(z)\right) - g_2\left(S_X(z)\right)\right]h_{11}(z)\mathrm{d}z + \int_0^M \left[g_2\left(S_X(z)\right) - g_1\left(S_X(z)\right)\right]h_{22}(z)\mathrm{d}z \ge 0,\\
\text{IC21}:&\int_0^M g_2\left(S_X(z)\right)\left[h_{11}(z) - h_{21}(z)\right]\mathrm{d}z \ge 0.
\end{aligned}$$

Case 3. (IR1 *and* IC21)

$$\max_{\{h_{11},h_{12},h_{21},h_{22}\}\in\mathcal{H}} \lambda\left[pm\left(\int_0^M g_1\left(S_X(z)\right)h_{11}(z)\mathrm{d}z - \int_0^M S_X(z)h_{11}(z)\mathrm{d}z\right) + (1-p)\,n\left(\int_0^M g_1\left(S_X(z)\right)h_{21}(z)\mathrm{d}z \right.\right.$$
$$\left.\left. + \int_0^M g_2\left(S_X(z)\right)h_{12}(z) - \int_0^M g_2\left(S_X(z)\right)h_{21}(z)\mathrm{d}z\mathrm{d}z - \int_0^M S_X(z)h_{12}(z)\mathrm{d}z\right)\right] + (1-\lambda)\,[p$$

$$(1-m)\left(\int_0^M g_1\left(S_X(z)\right)h_{21}(z)\mathrm{d}z-\int_0^M S_X(z)h_{21}(z)\mathrm{d}z\right)+(1-p)(1-n)\left(\int_0^M g_1\left(S_X(z)\right)\right.$$
$$\left.\left.h_{21}(z)\mathrm{d}z+\int_0^M g_2\left(S_X(z)\right)h_{22}(z)\mathrm{d}z-\int_0^M g_2\left(S_X(z)\right)h_{21}(z)\mathrm{d}z-\int_0^M S_X(z)h_{22}(z)\mathrm{d}z\right)\right], \tag{3.9}$$

$$\begin{aligned}
\text{IR2}:&\int_0^M\left[g_2\left(S_X(z)\right)-g_1\left(S_X(z)\right)\right]h_{21}(z)\mathrm{d}z\geq 0,\\
\text{IC12}:&\int_0^M\left[g_1\left(S_X(z)\right)-g_2\left(S_X(z)\right)\right]h_{21}(z)\mathrm{d}z+\int_0^M\left[g_2\left(S_X(z)\right)-g_1\left(S_X(z)\right)\right]h_{12}(z)\mathrm{d}z\geq 0,\\
\text{IC22}:&\int_0^M\left[g_1\left(S_X(z)\right)-g_2\left(S_X(z)\right)\right]h_{21}(z)\mathrm{d}z+\int_0^M\left[g_2\left(S_X(z)\right)-g_1\left(S_X(z)\right)\right]h_{22}(z)\mathrm{d}z\geq 0,\\
\text{IC11}:&\int_0^M g_2\left(S_X(z)\right)\left[h_{21}(z)-h_{11}(z)\right]\mathrm{d}z\geq 0.
\end{aligned}$$

Case 4. (IC12 *and* IR2)

$$\max_{\{h_{11},h_{12},h_{21},h_{22}\}\in\mathcal{H}}\lambda\left[pm\left(\int_0^M g_2\left(S_X(z)\right)h_{12}(z)\mathrm{d}z+\int_0^M g_1\left(S_X(z)\right)h_{11}(z)\mathrm{d}z-\int_0^M g_1\left(S_X(z)\right)h_{12}(z)\mathrm{d}z\right.\right.$$
$$\left.\left.-\int_0^M S_X(z)h_{11}(z)\mathrm{d}z\right)+(1-p)\,n\left(\int_0^M g_2\left(S_X(z)\right)h_{12}(z)-\int_0^M S_X(z)h_{12}(z)\mathrm{d}z\right)\right]+(1-\lambda)$$
$$\left[p(1-m)\left(\int_0^M g_2\left(S_X(z)\right)h_{12}(z)\mathrm{d}z+\int_0^M g_1\left(S_X(z)\right)h_{21}(z)\mathrm{d}z-\int_0^M g_1\left(S_X(z)\right)h_{12}(z)\mathrm{d}z\right.\right.$$
$$\left.\left.-\int_0^M S_X(z)h_{21}(z)\mathrm{d}z\right)+(1-p)(1-n)\left(\int_0^M g_2\left(S_X(z)\right)h_{22}(z)\mathrm{d}z-\int_0^M S_X(z)h_{22}(z)\mathrm{d}z\right)\right], \tag{3.10}$$

$$\begin{aligned}
\text{IR1}:&\int_0^M\left[g_1\left(S_X(z)\right)-g_2\left(S_X(z)\right)\right]h_{12}(z)\mathrm{d}z\geq 0,\\
\text{IC22}:&\int_0^M g_1\left(S_X(z)\right)\left[h_{12}(z)-h_{22}(z)\right]\mathrm{d}z\geq 0,\\
\text{IC11}:&\int_0^M\left[g_2\left(S_X(z)\right)-g_1\left(S_X(z)\right)\right]h_{12}(z)\mathrm{d}z+\int_0^M\left[g_1\left(S_X(z)\right)-g_2\left(S_X(z)\right)\right]h_{11}(z)\mathrm{d}z\geq 0,\\
\text{IC21}:&\int_0^M\left[g_2\left(S_X(z)\right)-g_1\left(S_X(z)\right)\right]h_{12}(z)\mathrm{d}z+\int_0^M\left[g_1\left(S_X(z)\right)-g_2\left(S_X(z)\right)\right]h_{21}(z)\mathrm{d}z\geq 0.
\end{aligned}$$

Case 5. (IC22 *and* IR2)

$$\max_{\{h_{11},h_{12},h_{21},h_{22}\}\in\mathcal{H}}\lambda\left[pm\left(\int_0^M g_2\left(S_X(z)\right)h_{22}(z)\mathrm{d}z+\int_0^M g_1\left(S_X(z)\right)h_{11}(z)\mathrm{d}z-\int_0^M g_1\left(S_X(z)\right)h_{22}(z)\mathrm{d}z\right.\right.$$
$$\left.\left.-\int_0^M S_X(z)h_{11}(z)\mathrm{d}z\right)+(1-p)\,n\left(\int_0^M g_2\left(S_X(z)\right)h_{12}(z)-\int_0^M S_X(z)h_{12}(z)\mathrm{d}z\right)\right]+(1-\lambda)$$
$$\left[p(1-m)\left(\int_0^M g_2\left(S_X(z)\right)h_{22}(z)\mathrm{d}z+\int_0^M g_1\left(S_X(z)\right)h_{21}(z)\mathrm{d}z-\int_0^M g_1\left(S_X(z)\right)h_{22}(z)\mathrm{d}z\right.\right.$$

$$-\int_0^M S_X(z)h_{21}(z)\mathrm{d}z\Big) + (1-p)(1-n)\left(\int_0^M g_2(S_X(z))\,h_{22}(z)\mathrm{d}z - \int_0^M S_X(z)h_{22}(z)\mathrm{d}z\right)\Big], \tag{3.11}$$

$$\begin{aligned}
\mathrm{IR1}: &\int_0^M \left[g_1(S_X(z)) - g_2(S_X(z))\right] h_{22}(z)\mathrm{d}z \geq 0,\\
\mathrm{IC12}: &\int_0^M g_1(S_X(z))\left[h_{22}(z) - h_{12}(z)\right]\mathrm{d}z \geq 0,\\
\mathrm{IC11}: &\int_0^M \left[g_2(S_X(z)) - g_1(S_X(z))\right] h_{22}(z)\mathrm{d}z + \int_0^M \left[g_1(S_X(z)) - g_2(S_X(z))\right] h_{11}(z)\mathrm{d}z \geq 0,\\
\mathrm{IC21}: &\int_0^M \left[g_2(S_X(z)) - g_1(S_X(z))\right] h_{22}(z)\mathrm{d}z + \int_0^M \left[g_1(S_X(z)) - g_2(S_X(z))\right] h_{21}(z)\mathrm{d}z \geq 0.
\end{aligned}$$

By employing the Lagrange dual approach [29], we derive the optimal MIFs, which then enables us to obtain the optimal reinsurance menus as presented in the following theorem.

Theorem 3.1. *Under Assumptions* 2.1 *and* 2.2*, we derive the optimal reinsurance menus in each case.*

Case 1. *Let*

$$\begin{aligned}
\psi_{11}^{\{1\}}\left(z,\theta_1^{\{1\}}\right) &= \lambda pm\left[g_1(S_X(z)) - S_X(z)\right] + \theta_1^{\{1\}}\left[g_1(S_X(z)) - g_2(S_X(z))\right],\\
\psi_{12}^{\{1\}}\left(z,\theta_2^{\{1\}}\right) &= \lambda(1-p)n\left[g_2(S_X(z)) - S_X(z)\right] + \theta_2^{\{1\}}\left[g_2(S_X(z)) - g_1(S_X(z))\right],\\
\psi_{21}^{\{1\}}\left(z,\theta_3^{\{1\}}\right) &= (1-\lambda)p(1-m)\left[g_1(S_X(z)) - S_X(z)\right] + \theta_3^{\{1\}}\left[g_1(S_X(z)) - g_2(S_X(z))\right],\\
\psi_{22}^{\{1\}}\left(z,\theta_4^{\{1\}}\right) &= (1-\lambda)(1-p)(1-n)\left[g_2(S_X(z)) - S_X(z)\right] + \theta_4^{\{1\}}\left[g_2(S_X(z)) - g_1(S_X(z))\right],
\end{aligned}$$

for the Lagrangian coefficients $\theta_1^{\{1\}}, \theta_2^{\{1\}}, \theta_3^{\{1\}}, \theta_4^{\{1\}} \geq 0$. *By employing the Lagrange dual approach, the optimal MIFs in Case* 1 *are given by*

$$h_{11}^{\{1\}*}\left(z,\theta_1^{\{1\}}\right) = \begin{cases} 1, & \psi_{11}^{\{1\}}(S_X(z)) > 0\\ \varphi_1^{\{1\}}(z), & \psi_{11}^{\{1\}}(S_X(z)) = 0\\ 0, & \psi_{11}^{\{1\}}(S_X(z)) < 0 \end{cases} \qquad h_{12}^{\{1\}*}\left(z,\theta_2^{\{1\}}\right) = \begin{cases} 1, & \psi_{12}^{\{1\}}(S_X(z)) > 0\\ \varphi_2^{\{1\}}(z), & \psi_{12}^{\{1\}}(S_X(z)) = 0\\ 0, & \psi_{12}^{\{1\}}(S_X(z)) < 0 \end{cases}$$

$$h_{21}^{\{1\}*}\left(z,\theta_3^{\{1\}}\right) = \begin{cases} 1, & \psi_{21}^{\{1\}}(S_X(z)) > 0\\ \varphi_3^{\{1\}}(z), & \psi_{21}^{\{1\}}(S_X(z)) = 0\\ 0, & \psi_{21}^{\{1\}}(S_X(z)) < 0 \end{cases} \qquad h_{22}^{\{1\}*}\left(z,\theta_4^{\{1\}}\right) = \begin{cases} 1, & \psi_{22}^{\{1\}}(S_X(z)) > 0\\ \varphi_4^{\{1\}}(z), & \psi_{22}^{\{1\}}(S_X(z)) = 0\\ 0, & \psi_{22}^{\{1\}}(S_X(z)) < 0 \end{cases} \tag{3.12}$$

where $\varphi_1^{\{1\}}(z), \varphi_2^{\{1\}}(z), \varphi_3^{\{1\}}(z)$, $\varphi_4^{\{1\}}(z)$ *are any Lebesgue integrable functions between* 0 *and* 1. $h_{11}^{\{1\}*}\left(z,\theta_1^{\{1\}}\right), h_{12}^{\{1\}*}\left(z,\theta_2^{\{1\}}\right), h_{21}^{\{1\}*}\left(z,\theta_3^{\{1\}}\right)$, $h_{22}^{\{1\}*}\left(z,\theta_4^{\{1\}}\right)$ *satisfy the original conditions (see Appendix A).*

Then the optimal indemnity functions are given by $f_{11}^{\{1\}*}(x) = \int_0^x h_{11}^{\{1\}*}\left(z,\theta_1^{\{1\}*}\right)\mathrm{d}z$, $f_{12}^{\{1\}*}(x) = \int_0^x h_{12}^{\{1\}*}\left(z,\theta_2^{\{1\}*}\right)\mathrm{d}z$, $f_{21}^{\{1\}*}(x) = \int_0^x h_{21}^{\{1\}*}\left(z,\theta_3^{\{1\}*}\right)\mathrm{d}z$, $f_{22}^{\{1\}*}(x) = \int_0^x h_{22}^{\{1\}*}\left(z,\theta_4^{\{1\}*}\right)\mathrm{d}z$, *where* $\theta_1^{\{1\}*}$, $\theta_2^{\{1\}*}$, $\theta_3^{\{1\}*}$, $\theta_4^{\{1\}*}$ *satisfy the slackness conditions (see Appendix A). The optimal premiums are given by* $\pi_{\cdot 1}^{\{1\}*} = \rho_{g_1}\left(f_{\cdot 1}^{\{1\}*}(X)\right)$ *and* $\pi_{\cdot 2}^{\{1\}*} = \rho_{g_2}\left(f_{\cdot 2}^{\{1\}*}(X)\right)$. *Therefore, we obtain the optimal reinsurance menu in Case* 1*:* $\left\{\left(\pi_{\cdot 1}^{\{1\}*}, f_{11}^{\{1\}*}(X)\right), \left(\pi_{\cdot 2}^{\{1\}*}, f_{12}^{\{1\}*}(X)\right), \left(\pi_{\cdot 1}^{\{1\}*}, f_{21}^{\{1\}*}(X)\right), \left(\pi_{\cdot 2}^{\{1\}*}, f_{22}^{\{1\}*}(X)\right)\right\}$.

Case 2. *Let*

$$\begin{aligned}\psi_{11}^{\{2\}}\left(z,\theta_i^{\{2\}}\right) &= \left[\lambda pm+\lambda(1-p)n+(1-\lambda)(1-p)(1-n)-\theta_1^{\{2\}}+\theta_2^{\{2\}}+\theta_3^{\{2\}}\right]g_1\left(S_X(z)\right)-\lambda pmS_X(z)\\&\quad+\left[-\lambda(1-p)n-(1-\lambda)(1-p)(1-n)+\theta_1^{\{2\}}-\theta_2^{\{2\}}-\theta_3^{\{2\}}+\theta_4^{\{2\}}\right]g_2\left(S_X(z)\right),\\ \psi_{12}^{\{2\}}\left(z,\theta_i^{\{2\}}\right) &= \left[\lambda(1-p)n+\theta_2^{\{2\}}\right]g_2\left(S_X(z)\right)-\theta_2^{\{2\}}g_1\left(S_X(z)\right)-\lambda(1-p)nS_X(z),\\ \psi_{21}^{\{2\}}\left(z,\theta_i^{\{2\}}\right) &= (1-\lambda)p(1-m)g_1\left(S_X(z)\right)-\theta_4^{\{2\}}g_2\left(S_X(z)\right)-(1-\lambda)p(1-m)S_X(z),\\ \psi_{22}^{\{2\}}\left(z,\theta_i^{\{2\}}\right) &= \left[(1-\lambda)(1-p)(1-n)+\theta_3^{\{2\}}\right]g_2\left(S_X(z)\right)-\theta_3^{\{2\}}g_1\left(S_X(z)\right)-(1-\lambda)(1-p)(1-n)S_X(z),\end{aligned}$$

for the Lagrangian coefficients $\theta_i \geq 0, i=1,2,3,4$. *By employing the Lagrange dual approach, the optimal MIFs in Case* 2 *are given by*

$$h_{11}^{\{2\}*}\left(z,\theta_i^{\{2\}}\right)=\begin{cases}1, & \psi_{11}^{\{2\}}\left(z,\theta_i^{\{2\}}\right)>0\\ \varphi_1^{\{2\}}(z), & \psi_{11}^{\{2\}}\left(z,\theta_i^{\{2\}}\right)=0\\ 0, & \psi_{11}^{\{2\}}\left(z,\theta_i^{\{2\}}\right)<0\end{cases}\qquad h_{12}^{\{2\}*}\left(z,\theta_i^{\{2\}}\right)=\begin{cases}1, & \psi_{12}^{\{2\}}\left(z,\theta_i^{\{2\}}\right)>0\\ \varphi_2^{\{2\}}(z), & \psi_{12}^{\{2\}}\left(z,\theta_i^{\{2\}}\right)=0\\ 0, & \psi_{12}^{\{2\}}\left(z,\theta_i^{\{2\}}\right)<0\end{cases}$$

$$h_{21}^{\{2\}*}\left(z,\theta_i^{\{2\}}\right)=\begin{cases}1, & \psi_{21}^{\{2\}}\left(z,\theta_i^{\{2\}}\right)>0\\ \varphi_3^{\{2\}}(z), & \psi_{21}^{\{2\}}\left(z,\theta_i^{\{2\}}\right)=0\\ 0, & \psi_{21}^{\{2\}}\left(z,\theta_i^{\{2\}}\right)<0\end{cases}\qquad h_{22}^{\{2\}*}\left(z,\theta_i^{\{2\}}\right)=\begin{cases}1, & \psi_{22}^{\{2\}}\left(z,\theta_i^{\{2\}}\right)>0\\ \varphi_4^{\{2\}}(z), & \psi_{22}^{\{2\}}\left(z,\theta_i^{\{2\}}\right)=0\\ 0, & \psi_{22}^{\{2\}}\left(z,\theta_i^{\{2\}}\right)<0\end{cases}\tag{3.13}$$

where $\varphi_1^{\{2\}}(z), \varphi_2^{\{2\}}(z), \varphi_3^{\{2\}}(z)$, $\varphi_4^{\{2\}}(z)$ *are any Lebesgue integrable functions between* 0 *and* 1. $h_{11}^{\{2\}*}\left(z,\theta_i^{\{2\}}\right), h_{12}^{\{2\}*}\left(z,\theta_i^{\{2\}}\right), h_{21}^{\{2\}*}\left(z,\theta_i^{\{2\}}\right)$, $h_{22}^{\{2\}*}\left(z,\theta_i^{\{2\}}\right)$ *satisfy the original conditions (see Appendix A).*

Then the optimal indemnity functions are given by $f_{11}^{\{2\}*}(x)=\int_0^x h_{11}^{\{2\}*}\left(z,\theta_i^{\{2\}*}\right)\mathrm{d}z$, $f_{12}^{\{2\}*}(x)=\int_0^x h_{12}^{\{2\}*}\left(z,\theta_i^{\{2\}*}\right)\mathrm{d}z$, $f_{21}^{\{2\}*}(x)=\int_0^x h_{21}^{\{2\}*}\left(z,\theta_i^{\{2\}*}\right)\mathrm{d}z$, $f_{22}^{\{2\}*}(x)=\int_0^x h_{22}^{\{2\}*}\left(z,\theta_i^{\{2\}*}\right)\mathrm{d}z$, *where* $\theta_1^{\{2\}*}$, $\theta_2^{\{2\}*}$, $\theta_3^{\{2\}*}$, $\theta_4^{\{2\}*}$ *satisfy the slackness conditions (see Appendix A). The optimal premiums are given by* $\pi_{\cdot 1}^{\{2\}*}=\rho_{g_1}\left(f_{\cdot 1}^{\{2\}*}(X)\right)$ *and* $\pi_{\cdot 2}^{\{2\}*}=\rho_{g_2}\left(f_{\cdot 2}^{\{2\}*}(X)\right)$. *Therefore, we obtain the optimal reinsurance menu in Case* 2*:* $\left\{\left(\pi_{\cdot 1}^{\{2\}*}, f_{11}^{\{2\}*}(X)\right), \left(\pi_{\cdot 2}^{\{2\}*}, f_{12}^{\{2\}*}(X)\right), \left(\pi_{\cdot 1}^{\{2\}*}, f_{21}^{\{2\}*}(X)\right), \left(\pi_{\cdot 2}^{\{2\}*}, f_{22}^{\{2\}*}(X)\right)\right\}$.

Case 3. *Let*

$$\begin{aligned}\psi_{11}^{\{3\}}\left(z,\theta_i^{\{3\}}\right) &= \lambda pm\left[g_1\left(S_X(z)\right)-S_X(z)\right]-\theta_4^{\{3\}}g_2\left(S_X(z)\right),\\ \psi_{12}^{\{3\}}\left(z,\theta_i^{\{3\}}\right) &= \left[\lambda(1-p)n+\theta_2^{\{3\}}\right]g_2\left(S_X(z)\right)-\theta_2^{\{3\}}g_1\left(S_X(z)\right)-\lambda(1-p)nS_X(z),\\ \psi_{21}^{\{3\}}\left(z,\theta_i^{\{3\}}\right) &= \left[\lambda(1-p)n+(1-\lambda)p(1-m)+(1-\lambda)(1-p)(1-n)-\theta_1^{\{3\}}+\theta_2^{\{3\}}+\theta_3^{\{3\}}\right]g_1\left(S_X(z)\right)\\&\quad+\left[-\lambda(1-p)n-(1-\lambda)(1-p)(1-n)+\theta_1^{\{3\}}-\theta_2^{\{3\}}-\theta_3^{\{3\}}+\theta_4^{\{3\}}\right]g_2\left(S_X(z)\right)-(1-\lambda)p\\&\quad(1-m)S_X(z),\\ \psi_{22}^{\{3\}}\left(z,\theta_i^{\{3\}}\right) &= \left[(1-\lambda)(1-p)(1-n)+\theta_3^{\{3\}}\right]g_2\left(S_X(z)\right)-\theta_3^{\{3\}}g_1\left(S_X(z)\right)-(1-\lambda)(1-p)(1-n)S_X(z),\end{aligned}$$

for the Lagrangian coefficients $\theta_i^{\{3\}} \geq 0, i=1,2,3,4$. *By employing the Lagrange dual approach, the optimal MIFs in Case* 3 *are given by*

$$h_{11}^{\{3\}*}\left(z,\theta_i^{\{3\}}\right)=\begin{cases}1, & \psi_{11}^{\{3\}}\left(z,\theta_i^{\{3\}}\right)>0\\ \varphi_1^{\{3\}}(z), & \psi_{11}^{\{3\}}\left(z,\theta_i^{\{3\}}\right)=0\\ 0, & \psi_{11}^{\{3\}}\left(z,\theta_i^{\{3\}}\right)<0\end{cases}\qquad h_{12}^{\{3\}*}\left(z,\theta_i^{\{3\}}\right)=\begin{cases}1, & \psi_{12}^{\{3\}}\left(z,\theta_i^{\{3\}}\right)>0\\ \varphi_2^{\{3\}}(z), & \psi_{12}^{\{3\}}\left(z,\theta_i^{\{3\}}\right)=0\\ 0, & \psi_{12}^{\{3\}}\left(z,\theta_i^{\{3\}}\right)<0\end{cases}$$

$$h_{21}^{\{3\}*}\left(z,\theta_i^{\{3\}}\right)=\begin{cases}1, & \psi_{21}^{\{3\}}\left(z,\theta_i^{\{3\}}\right)>0\\ \varphi_3^{\{3\}}(z), & \psi_{21}^{\{3\}}\left(z,\theta_i^{\{3\}}\right)=0\\ 0, & \psi_{21}^{\{3\}}\left(z,\theta_i^{\{3\}}\right)<0\end{cases}\qquad h_{22}^{\{3\}*}\left(z,\theta_i^{\{3\}}\right)=\begin{cases}1, & \psi_{22}^{\{3\}}\left(z,\theta_i^{\{3\}}\right)>0\\ \varphi_4^{\{3\}}(z), & \psi_{22}^{\{3\}}\left(z,\theta_i^{\{3\}}\right)=0\\ 0, & \psi_{22}^{\{3\}}\left(z,\theta_i^{\{3\}}\right)<0\end{cases}\tag{3.14}$$

where $\varphi_1^{\{3\}}(z),\varphi_2^{\{3\}}(z),\varphi_3^{\{3\}}(z)$, $\varphi_4^{\{3\}}(z)$ *are any Lebesgue integrable functions between* 0 *and* 1. $h_{11}^{\{3\}*}\left(z,\theta_i^{\{3\}}\right),h_{12}^{\{3\}*}\left(z,\theta_i^{\{3\}}\right),h_{21}^{\{3\}*}\left(z,\theta_i^{\{3\}}\right)$, $h_{22}^{\{3\}*}\left(z,\theta_i^{\{3\}}\right)$ *satisfy the original conditions (see Appendix A).*

Then the optimal indemnity functions are given by $f_{11}^{\{3\}*}(x)=\int_0^x h_{11}^{\{3\}*}\left(z,\theta_i^{\{3\}*}\right)\mathrm{d}z$, $f_{12}^{\{3\}*}(x)=\int_0^x h_{12}^{\{3\}*}\left(z,\theta_i^{\{3\}*}\right)\mathrm{d}z$, $f_{21}^{\{3\}*}(x)=\int_0^x h_{21}^{\{3\}*}\left(z,\theta_i^{\{3\}*}\right)\mathrm{d}z$, $f_{22}^{\{3\}*}(x)=\int_0^x h_{22}^{\{3\}*}\left(z,\theta_i^{\{3\}*}\right)\mathrm{d}z$, *where* $\theta_1^{\{3\}*}$, $\theta_2^{\{3\}*}$, $\theta_3^{\{3\}*}$, $\theta_4^{\{3\}*}$ *satisfy the slackness conditions (see Appendix A). The optimal premiums are given by* $\pi_{\cdot 1}^{\{3\}*}=\rho_{g_1}\left(f_{\cdot 1}^{\{3\}*}(X)\right)$ *and* $\pi_{\cdot 2}^{\{3\}*}=\rho_{g_2}\left(f_{\cdot 2}^{\{3\}*}(X)\right)$. *Therefore, we obtain the optimal reinsurance menu in Case* 3*:* $\left\{\left(\pi_{\cdot 1}^{\{3\}*},f_{11}^{\{3\}*}(X)\right),\left(\pi_{\cdot 2}^{\{3\}*},f_{12}^{\{3\}*}(X)\right),\left(\pi_{\cdot 1}^{\{3\}*},f_{21}^{\{3\}*}(X)\right),\left(\pi_{\cdot 2}^{\{3\}*},f_{22}^{\{3\}*}(X)\right)\right\}$.

Case 4. *Let*

$$\begin{aligned}\psi_{11}^{\{4\}}\left(z,\theta_i^{\{4\}}\right)&=\left(\lambda pm+\theta_3^{\{4\}}\right)g_1\left(S_X(z)\right)-\theta_3^{\{4\}}g_2\left(S_X(z)\right)-\lambda pmS_X(z),\\ \psi_{12}^{\{4\}}\left(z,\theta_i^{\{4\}}\right)&=\left[-\lambda pm-(1-\lambda)p(1-m)+\theta_1^{\{4\}}+\theta_2^{\{4\}}-\theta_3^{\{4\}}-\theta_4^{\{4\}}\right]g_1\left(S_X(z)\right)+\left[\lambda pm+\lambda(1-p)n\right.\\ &\quad\left.+(1-\lambda)p(1-m)-\theta_1^{\{4\}}+\theta_3^{\{4\}}+\theta_4^{\{4\}}\right]g_2\left(S_X(z)\right)-\lambda(1-p)nS_X(z),\\ \psi_{21}^{\{4\}}\left(z,\theta_i^{\{4\}}\right)&=\left[(1-\lambda)p(1-m)+\theta_4^{\{4\}}\right]g_1\left(S_X(z)\right)-\theta_4^{\{4\}}g_2\left(S_X(z)\right)-(1-\lambda)p(1-m)S_X(z),\\ \psi_{22}^{\{4\}}\left(z,\theta_i^{\{4\}}\right)&=-\theta_2^{\{4\}}g_1\left(S_X(z)\right)+(1-\lambda)(1-p)(1-n)\left[g_2\left(S_X(z)\right)-S_X(z)\right],\end{aligned}$$

for the Lagrangian coefficients $\theta_i^{\{4\}}\geq 0, i=1,2,3,4$. *By employing the Lagrange dual approach, the optimal MIFs in Case* 4 *are given by*

$$h_{11}^{\{4\}*}\left(z,\theta_i^{\{4\}}\right)=\begin{cases}1, & \psi_{11}^{\{4\}}\left(z,\theta_i^{\{4\}}\right)>0\\ \varphi_1^{\{4\}}(z), & \psi_{11}^{\{4\}}\left(z,\theta_i^{\{4\}}\right)=0\\ 0, & \psi_{11}^{\{4\}}\left(z,\theta_i^{\{4\}}\right)<0\end{cases}\qquad h_{12}^{\{4\}*}\left(z,\theta_i^{\{4\}}\right)=\begin{cases}1, & \psi_{12}^{\{4\}}\left(z,\theta_i^{\{4\}}\right)>0\\ \varphi_2^{\{4\}}(z), & \psi_{12}^{\{4\}}\left(z,\theta_i^{\{4\}}\right)=0\\ 0, & \psi_{12}^{\{4\}}\left(z,\theta_i^{\{4\}}\right)<0\end{cases}$$

$$h_{21}^{\{4\}*}\left(z,\theta_i^{\{4\}}\right)=\begin{cases}1, & \psi_{21}^{\{4\}}\left(z,\theta_i^{\{4\}}\right)>0\\ \varphi_3^{\{4\}}(z), & \psi_{21}^{\{4\}}\left(z,\theta_i^{\{4\}}\right)=0\\ 0, & \psi_{21}^{\{4\}}\left(z,\theta_i^{\{4\}}\right)<0\end{cases}\qquad h_{22}^{\{4\}*}\left(z,\theta_i^{\{4\}}\right)=\begin{cases}1, & \psi_{22}^{\{4\}}\left(z,\theta_i^{\{4\}}\right)>0\\ \varphi_4^{\{4\}}(z), & \psi_{22}^{\{4\}}\left(z,\theta_i^{\{4\}}\right)=0\\ 0, & \psi_{22}^{\{4\}}\left(z,\theta_i^{\{4\}}\right)<0\end{cases}\tag{3.15}$$

where $\varphi_1^{\{4\}}(z),\varphi_2^{\{4\}}(z),\varphi_3^{\{4\}}(z)$, $\varphi_4^{\{4\}}(z)$ *are any Lebesgue integrable functions between* 0 *and* 1. $h_{11}^{\{4\}*}\left(z,\theta_i^{\{4\}}\right),h_{12}^{\{4\}*}\left(z,\theta_i^{\{4\}}\right),h_{21}^{\{4\}*}\left(z,\theta_i^{\{4\}}\right)$, $h_{22}^{\{4\}*}\left(z,\theta_i^{\{4\}}\right)$ *satisfy the original conditions (see Appendix A).*

Then the optimal indemnity functions are given by $f_{11}^{\{4\}*}(x)=\int_0^x h_{11}^{\{4\}*}\left(z,\theta_i^{\{4\}*}\right)\mathrm{d}z$, $f_{12}^{\{4\}*}(x)=\int_0^x h_{12}^{\{4\}*}\left(z,\theta_i^{\{4\}*}\right)\mathrm{d}z$, $f_{21}^{\{4\}*}(x)=\int_0^x h_{21}^{\{4\}*}\left(z,\theta_i^{\{4\}*}\right)\mathrm{d}z$, $f_{22}^{\{4\}*}(x)=\int_0^x h_{22}^{\{4\}*}\left(z,\theta_i^{\{4\}*}\right)\mathrm{d}z$, *where* $\theta_1^{\{4\}*}$, $\theta_2^{\{4\}*}$, $\theta_3^{\{4\}*}$, $\theta_4^{\{4\}*}$ *satisfy the slackness conditions (see Appendix A). The optimal premiums are given by* $\pi_{\cdot 1}^{\{4\}*}=\rho_{g_1}\left(f_{\cdot 1}^{\{4\}*}(X)\right)$ *and* $\pi_{\cdot 2}^{\{4\}*}=\rho_{g_2}\left(f_{\cdot 2}^{\{4\}*}(X)\right)$. *Therefore, we obtain the optimal reinsurance menu in Case* 4*:* $\left\{\left(\pi_{\cdot 1}^{\{4\}*},f_{11}^{\{4\}*}(X)\right),\left(\pi_{\cdot 2}^{\{4\}*},f_{12}^{\{4\}*}(X)\right),\left(\pi_{\cdot 1}^{\{4\}*},f_{21}^{\{4\}*}(X)\right),\left(\pi_{\cdot 2}^{\{4\}*},f_{22}^{\{4\}*}(X)\right)\right\}$.

Case 5. *Let*

$$\psi_{11}^{\{5\}}\left(z,\theta_i^{\{5\}}\right)=\left(\lambda pm+\theta_3^{\{5\}}\right)g_1\left(S_X(z)\right)-\theta_3^{\{5\}}g_2\left(S_X(z)\right)-\lambda pmS_X(z),$$

$$\psi_{12}^{\{5\}}\left(z,\theta_i^{\{5\}}\right)=-\theta_2^{\{5\}}g_1\left(S_X(z)\right)+\lambda\left(1-p\right)n\left[g_2\left(S_X(z)\right)-S_X(z)\right],$$
$$\psi_{21}^{\{5\}}\left(z,\theta_i^{\{5\}}\right)=\left[\left(1-\lambda\right)p\left(1-m\right)+\theta_4^{\{5\}}\right]g_1\left(S_X(z)\right)-\theta_4^{\{5\}}g_2\left(S_X(z)\right)-\left(1-\lambda\right)p\left(1-m\right)S_X(z),$$
$$\begin{aligned}\psi_{22}^{\{5\}}\left(z,\theta_i^{\{5\}}\right)=&\left[-\lambda pm-\left(1-\lambda\right)p\left(1-m\right)+\theta_1^{\{5\}}+\theta_2^{\{5\}}-\theta_3^{\{5\}}-\theta_4^{\{5\}}\right]g_1\left(S_X(z)\right)+\left[\lambda pm+\left(1-\lambda\right)p\left(1-m\right)\right.\\&\left.+\left(1-\lambda\right)\left(1-p\right)\left(1-n\right)-\theta_1^{\{5\}}+\theta_3^{\{5\}}+\theta_4^{\{5\}}\right]g_2\left(S_X(z)\right)-\left(1-\lambda\right)\left(1-p\right)\left(1-n\right)S_X(z),\end{aligned}$$

for the Lagrangian coefficients $\theta_i^{\{5\}}\geq 0, i=1,2,3,4$. *By employing the Lagrange dual approach, the optimal MIFs in Case* 5 *are given by*

$$h_{11}^{\{5\}*}\left(z,\theta_i^{\{5\}}\right)=\begin{cases}1, & \psi_{11}^{\{5\}}\left(z,\theta_i^{\{5\}}\right)>0\\ \varphi_1^{\{5\}}(z), & \psi_{11}^{\{5\}}\left(z,\theta_i^{\{5\}}\right)=0\\ 0, & \psi_{11}^{\{5\}}\left(z,\theta_i^{\{5\}}\right)<0\end{cases}\qquad h_{12}^{\{5\}*}\left(z,\theta_i^{\{5\}}\right)=\begin{cases}1, & \psi_{12}^{\{5\}}\left(z,\theta_i^{\{5\}}\right)>0\\ \varphi_2^{\{5\}}(z), & \psi_{12}^{\{5\}}\left(z,\theta_i^{\{5\}}\right)=0\\ 0, & \psi_{12}^{\{5\}}\left(z,\theta_i^{\{5\}}\right)<0\end{cases}$$
$$h_{21}^{\{5\}*}\left(z,\theta_i^{\{5\}}\right)=\begin{cases}1, & \psi_{21}^{\{5\}}\left(z,\theta_i^{\{5\}}\right)>0\\ \varphi_3^{\{5\}}(z), & \psi_{21}^{\{5\}}\left(z,\theta_i^{\{5\}}\right)=0\\ 0, & \psi_{21}^{\{5\}}\left(z,\theta_i^{\{5\}}\right)<0\end{cases}\qquad h_{22}^{\{5\}*}\left(z,\theta_i^{\{5\}}\right)=\begin{cases}1, & \psi_{22}^{\{5\}}\left(z,\theta_i^{\{5\}}\right)>0\\ \varphi_4^{\{5\}}(z), & \psi_{22}^{\{5\}}\left(z,\theta_i^{\{5\}}\right)=0\\ 0, & \psi_{22}^{\{5\}}\left(z,\theta_i^{\{5\}}\right)<0\end{cases}\tag{3.16}$$

where $\varphi_1^{\{5\}}(z),\varphi_2^{\{5\}}(z),\varphi_3^{\{5\}}(z)$, $\varphi_4^{\{5\}}(z)$ *are any Lebesgue integrable functions between* 0 *and* 1. $h_{11}^{\{5\}*}\left(z,\theta_i^{\{5\}}\right),h_{12}^{\{5\}*}\left(z,\theta_i^{\{5\}}\right),h_{21}^{\{5\}*}\left(z,\theta_i^{\{5\}}\right)$, $h_{22}^{\{5\}*}\left(z,\theta_i^{\{5\}}\right)$ *satisfy the original conditions (see Appendix A).*

Then the optimal indemnity functions are given by $f_{11}^{\{5\}*}(x)=\int_0^x h_{11}^{\{5\}*}\left(z,\theta_i^{\{5\}*}\right)\mathrm{d}z$, $f_{12}^{\{5\}*}(x)=\int_0^x h_{12}^{\{5\}*}\left(z,\theta_i^{\{5\}*}\right)\mathrm{d}z$, $f_{21}^{\{5\}*}(x)=\int_0^x h_{21}^{\{5\}*}\left(z,\theta_i^{\{5\}*}\right)\mathrm{d}z$, $f_{22}^{\{5\}*}(x)=\int_0^x h_{22}^{\{5\}*}\left(z,\theta_i^{\{5\}*}\right)\mathrm{d}z$, *where* $\theta_1^{\{5\}*}$, $\theta_2^{\{5\}*}$, $\theta_3^{\{5\}*}$, $\theta_4^{\{5\}*}$ *satisfy the slackness conditions (see Appendix A). The optimal premiums are given by* $\pi_{\cdot 1}^{\{5\}*}=\rho_{g_1}\left(f_{\cdot 1}^{\{5\}*}(X)\right)$ *and* $\pi_{\cdot 2}^{\{5\}*}=\rho_{g_2}\left(f_{\cdot 2}^{\{5\}*}(X)\right)$. *Therefore, we obtain the optimal reinsurance menu in Case* 5*:* $\left\{\left(\pi_{\cdot 1}^{\{5\}*},f_{11}^{\{5\}*}(X)\right),\left(\pi_{\cdot 2}^{\{5\}*},f_{12}^{\{5\}*}(X)\right),\left(\pi_{\cdot 1}^{\{5\}*},f_{21}^{\{5\}*}(X)\right),\left(\pi_{\cdot 2}^{\{5\}*},f_{22}^{\{5\}*}(X)\right)\right\}$.

Remark 3.1. *Case* 1 *can be decomposed into the following four sub-problems.*

$$\begin{cases}\max\limits_{h_{11}\in\mathcal{H}}\int_0^M\lambda pm\left(g_1\left(S_X(z)\right)-S_X(z)\right)h_{11}(z)\mathrm{d}z\\ \text{IC11}:\int_0^M\left[g_1\left(S_X(z)\right)-g_2\left(S_X(z)\right)\right]h_{11}(z)\mathrm{d}z\geq 0\end{cases}$$
$$\begin{cases}\max\limits_{h_{12}\in\mathcal{H}}\int_0^M\lambda\left(1-p\right)n\left(g_2\left(S_X(z)\right)-S_X(z)\right)h_{12}(z)\mathrm{d}z\\ \text{IC12}:\int_0^M\left[g_2\left(S_X(z)\right)-g_1\left(S_X(z)\right)\right]h_{12}(z)\mathrm{d}z\geq 0\end{cases}$$
$$\begin{cases}\max\limits_{h_{21}\in\mathcal{H}}\int_0^M\left(1-\lambda\right)p\left(1-m\right)\left(g_1\left(S_X(z)\right)-S_X(z)\right)h_{21}(z)\mathrm{d}z\\ \text{IC21}:\int_0^M\left[g_1\left(S_X(z)\right)-g_2\left(S_X(z)\right)\right]h_{21}(z)\mathrm{d}z\geq 0\end{cases}$$
$$\begin{cases}\max\limits_{h_{22}\in\mathcal{H}}\int_0^M\left(1-\lambda\right)\left(1-p\right)\left(1-n\right)\left(g_2\left(S_X(z)\right)-S_X(z)\right)h_{22}(z)\mathrm{d}z\\ \text{IC12}:\int_0^M\left[g_2\left(S_X(z)\right)-g_1\left(S_X(z)\right)\right]h_{22}(z)\mathrm{d}z\geq 0\end{cases}$$

We solve the four sub-problems sequentially via the Lagrange dual approach to obtain the optimal MIFs $h_{11}^{\{1\}*}(z,\theta_1^{\{1\}*})$, $h_{12}^{\{1\}*}(z,\theta_2^{\{1\}*})$, $h_{21}^{\{1\}*}(z,\theta_3^{\{1\}*})$, *and* $h_{22}^{\{1\}*}(z,\theta_4^{\{1\}*})$ *in (3.12).*

4. Numerical examples

This section demonstrates an application of Theorem 3.1 when VaR is used as the risk measure by insurers. Under this framework, we derive the optimal reinsurance menus in each case, and by comparing the optimized values of the objective function across all cases, we identify which particular case yields the globally optimal reinsurance. Furthermore, we compare exponential and Pareto distributions with identical expected losses to study tail risk effects.

According to Definition 2.2 and Assumption 2.1, the distortion functions of the type 1 and type 2 insurers are given by $g_1(S_X(z)) = \mathbf{1}_{\{1-\alpha_1<S_X(z)\leq 1\}}$ and $g_2(S_X(z)) = \mathbf{1}_{\{1-\alpha_2<S_X(z)\leq 1\}}$ for $0 \leq \alpha_1 \leq \alpha_2 \leq 1$.

Theorem 4.1. *Under Assumption* 2.1 *and* 2.2*, and the VaR risk measure, Cases* 1–3 *provide the globally optimal reinsurance menu* $\left\{\left(\pi^*_{\cdot 1}, f^*_{11}(X)\right), \left(\pi^*_{\cdot 2}, f^*_{12}(X)\right), \left(\pi^*_{\cdot 1}, f^*_{21}(X)\right), \left(\pi^*_{\cdot 2}, f^*_{22}(X)\right)\right\}$ *as follows.*

$$\pi^*_{\cdot 1} = S_X^{-1}(1-\alpha_1),\ f^*_{11}(X) = f^*_{21}(X) = \min\left\{x, S_X^{-1}(1-\alpha_1)\right\},$$
$$\pi^*_{\cdot 2} = S_X^{-1}(1-\alpha_2),\ f^*_{12}(X) = f^*_{22}(X) = \min\left\{x, S_X^{-1}(1-\alpha_2)\right\}.$$

Proof. See Appendix B for details. □

Remark 4.1. *According to Theorem* 4.1 *and the condition* $0 \leq \alpha_1 \leq \alpha_2 \leq 1$*, type* 2 *insurers, who have greater risk aversion, tend to cede larger risk exposures and consequently pay higher premiums.*

Under the VaR risk measure with $\alpha_1 = 0.95$ and $\alpha_2 = 0.99$, we further assume $X \sim Exp(\delta)$ with $\delta = 1$. From Theorem 4.1, we have $\pi^*_{\cdot 1} = 3.00$, $f^*_{11}(X) = f^*_{21}(X) = \min\{x, 3.00\}$, $\pi^*_{\cdot 2} = 4.61$, $f^*_{12}(X) = f^*_{22}(X) = \min\{x, 4.61\}$, $\mathbb{E}\left[f^*_{11}(X)\right] = \mathbb{E}\left[f^*_{21}(X)\right] = 0.95$, $\mathbb{E}\left[f^*_{12}(X)\right] = \mathbb{E}\left[f^*_{22}(X)\right] = 0.99$. Then we obtain the maximum value of the objective function in Problem 3.1.

$$P^* = \lambda\left[2.05pm + 3.62(1-p)n\right] + (1-\lambda)\left[2.05p(1-m) + 3.62(1-p)(1-n)\right]. \tag{4.1}$$

Remark 4.2. *Analysis of* (4.1) *reveals the following optimization principles: When* $\lambda > \frac{1}{2}$*, the objective function increases with respect to m and n; when* $\lambda < \frac{1}{2}$*, the function decreases with respect to m and n. Correspondingly, for* $m, n > \frac{1}{2}$*, the objective function increases with respect to* λ*, whereas for* $m, n < \frac{1}{2}$*, the function decreases with respect to* λ*. This finding demonstrates that in the reinsurance market established in our study, a reinsurer's bargaining power corresponds to its market share. For instance, when* $\lambda > \frac{1}{2}$*, indicating that Reinsurer* 1 *possesses greater bargaining power than Reinsurer* 2*, the objective function value increases with higher values of m and n, which represent Reinsurer* 1*'s market shares in the respective submarkets.*

When $\lambda = \frac{1}{2}$*, indicating balanced market influence between the two reinsurers,* (4.1) *simplifies to* $P^* = 1.81 - 0.785p$ *(see Figure* 1*), where* P^* *becomes independent of the market share parameters m and n. Notably,* $P^* = 1.81 - 0.785p$ *remains valid when market shares are equally distributed (*$m = n = \frac{1}{2}$*), where* P^* *shows independence from the bargaining power parameter* λ *instead. Under these parametric conditions, a relatively small value of p corresponds to a larger* P^**, indicating that type* 2 *insurers account for a greater proportion than type* 1 *insurers. This occurs because* $\pi^*_{\cdot 2} - \mathbb{E}\left[f^*_{12}(X)\right] = \pi^*_{\cdot 2} - \mathbb{E}\left[f^*_{22}(X)\right] \geq \pi^*_{\cdot 1} - \mathbb{E}\left[f^*_{11}(X)\right] = \pi^*_{\cdot 1} - \mathbb{E}\left[f^*_{21}(X)\right]$*, meaning that reinsurers can derive greater profits from the policies of type* 2 *insurers.*

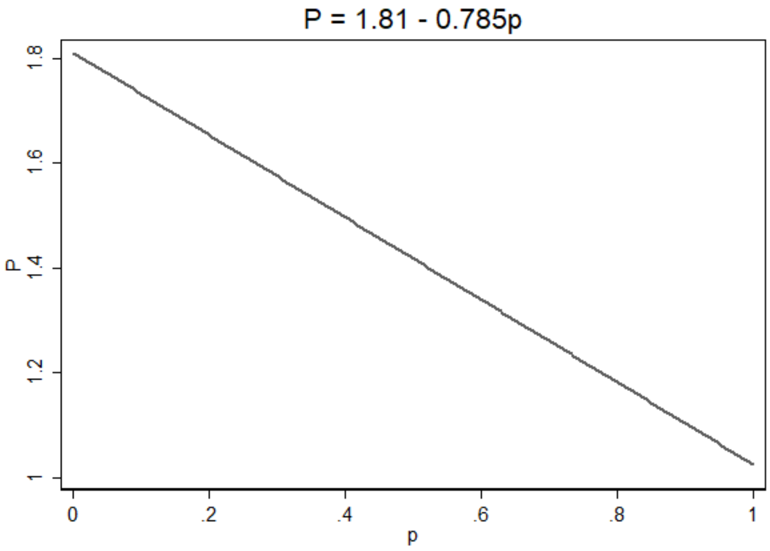

Figure 1. Maximum of the objective function under VaR and exponential distribution.

Under the VaR risk measure with $\alpha_1 = 0.95$ and $\alpha_2 = 0.99$, we further assume X follows a Pareto distribution with

$$F_X(x) = 1 - \left(\frac{k}{k+x}\right)^{\xi}, \; k = 1, \; \xi = 2, \; x \geq 0. \tag{4.2}$$

From Theorem 4.1, we have $\pi^*_{\cdot 1} = 3.47$, $f^*_{11}(X) = f^*_{21}(X) = \min\{x, 3.47\}$, $\pi^*_{\cdot 2} = 9.00$, $f^*_{12}(X) = f^*_{22}(X) = \min\{x, 9.00\}$, $\mathbb{E}\left[f^*_{11}(X)\right] = \mathbb{E}\left[f^*_{21}(X)\right] = 1.00$, $\mathbb{E}\left[f^*_{12}(X)\right] = \mathbb{E}\left[f^*_{22}(X)\right] = 0.99$. Then we obtain the maximum value of the objective function in Problem 3.1.

$$P^* = \lambda\left[141.04pm + 924.15(1-p)n\right] + (1-\lambda)\left[141.04p(1-m) + 924.15(1-p)(1-n)\right]. \tag{4.3}$$

Compared to the light-tailed exponential distribution, despite having the same expected risk loss, the Pareto distribution demands higher premiums at the same confidence level due to its pronounced heavy-tailed characteristics (see Figure 2). This difference fully demonstrates reinsurers' more conservative pricing strategy for extreme tail risks (such as catastrophic events).

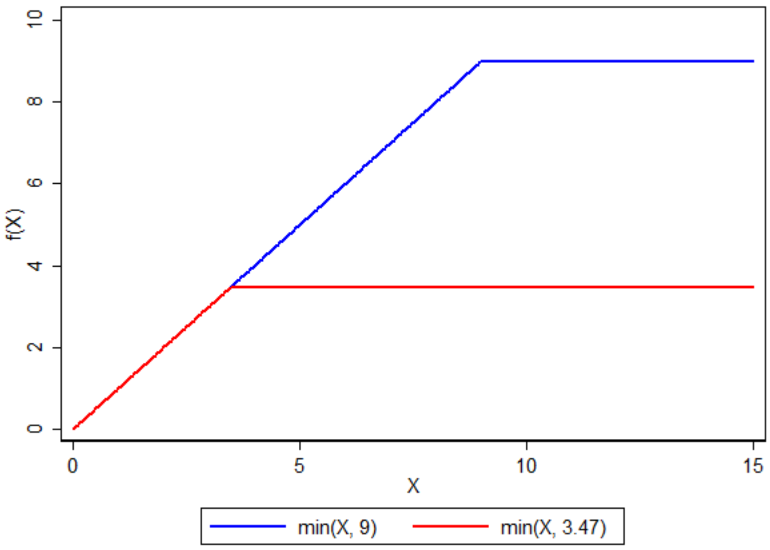

Figure 2. Optimal indemnity functions under VaR and Pareto distribution.

5. Conclusions

We study the optimal reinsurance design in a duopolistic market comprising two types of insurers and two reinsurers under asymmetric information, where reinsurers cannot directly observe insurers' risk types. Assume that type 2 insurers have the same risk distribution as type 1 insurers but are more risk-averse. We model reinsurers as risk-neutral agents who aim to maximize their expected net profit, subject to individual rationality, incentive compatibility, and convex preference constraints. Under Assumption 2.2, all convex preference conditions degenerate into their corresponding IR or IC conditions. We introduce the principle of Pareto optimality to formulate the objective function in a multi-agent setting. Applying the Lagrange dual approach, we derive optimal reinsurance menus for all cases. Under the VaR risk measure, Cases 1–3 provide the globally optimal reinsurance menu with its closed-form solution. The optimal solution under VaR reveals that type 2 insurers, who are characterized by greater risk aversion, tend to cede larger risk exposures and consequently pay higher premiums. Furthermore, we compare exponential and Pareto distributions with identical expected losses to study tail risk effects. The comparison reveals that the heavy-tailed Pareto distribution requires higher premiums, demonstrating reinsurers' conservative pricing strategy for extreme risks such as catastrophic events. The finding provides practical guidance for reinsurers' risk pricing strategies and insurers' premium expectations regarding tail-dependent risks.

Our study is subject to certain limitations. First, we assume identical risk distributions for both insurer types, which may not reflect reality. Second, Assumption 2.1 requires type 2 insurers to be always more risk-averse than type 1 insurers, ignoring potential risk-specific variations. Third, the perfect premium alignment between reinsurers under Assumption 2.2 is difficult to sustain in practice. Future research could incorporate differentiated risk distributions, context-dependent risk aversion, and more competitive pricing models.

Author contributions

Haonan Ma: Conceptualization, methodology, formal Analysis, writing – original draft, writing – review and editing; Ying Fang: Conceptualization, methodology, writing – review and editing.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

References

1. K. Borch, The safety loading of reinsurance premiums, *Scand. Actuar. J.,* **43** (1960), 163–184. https://doi.org/10.2307/334458
2. K. Arrow, Uncertainty and the welfare economics of medical care, *Amer. Econ. Rev.,* **53** (1963), 941–973.
3. H. Bühlmann, An economic premium principle, *ASTIN Bull.,* **11** (1980), 52–60. https://doi.org/10.1017/S0515036100006619
4. J. Cai, C. Lemieux, F. Liu, Optimal reinsurance from the perspectives of both an insurer and a reinsurer, *ASTIN Bull.,* **46** (2016), 815–849. https://doi.org/10.1017/asb.2015.23
5. W. Jiang, H. Hong, J. Ren, On Pareto-optimal reinsurance with constraints under distortion risk measures, *Eur. Actuar. J.,* **8** (2018), 215–243. https://doi.org/10.1007/s13385-017-0163-1
6. T. J. Boonen, W. Jiang, Pareto-optimal insurance under robust distortion risk measures, *EJOR,* **324** (2025), 690–705. https://doi.org/10.1016/j.insmatheco.2012.12.001
7. W. Jiang, Pareto-optimal insurance under heterogeneous beliefs and incentive compatibility, *Scand. Actuar. J.,* **9** (2022), 775–793. https://doi.org/10.1080/03461238.2022.2028185
8. F. Chang, Y. Fang, Pareto-optimal reinsurance under Vajda condition and heterogenous beliefs, *Commun. Stat. Theor. Meth.,* (2025), 1–28. https://doi.org/10.1080/03610926.2025.2485340
9. R. Arnott, J. Stiglitz, Equilibrium in competitive insurance markets: an essay on the economics of imperfect information, *Uncert. Econ.,* (1978), 257–280. https://doi.org/10.1016/B978-0-12-214850-7.50024-3
10. J. E. Stiglitz, Monopoly, non-linear pricing and imperfect information: the insurance market, *Rev. Econ. Stud.,* **44** (1977), 407–430. https://doi.org/10.2307/2296899
11. K. C. Cheung, S. C. P. Yam, F. L. Yuen, Reinsurance contract design with adverse selection, *Scand. Actuar. J.,* **9** (2019), 784–798. https://doi.org/10.1080/03461238.2019.1616323
12. K. C. Cheung, S. C. P. Yam, F. L. Yuen, Y. Zhang, Concave distortion risk minimizing reinsurance design under adverse selection, *Insur. Math. Econ.,* **91** (2020), 155–165. https://doi.org/10.1016/j.insmatheco.2020.02.001

13. T. J. Boonen, Y. Zhang, Optimal reinsurance design with distortion risk measures and asymmetric information, *ASTIN Bull.,* **51** (2021), 607–629. https://doi.org/10.1017/asb.2021.8

14. Z. Liang, J. Zou, W. Jiang, Revisiting the optimal insurance design under adverse selection: Distortion risk measures and tail-risk overestimation, *Insur. Math. Econ.,* **104** (2022), 200–221. https://doi.org/10.1016/j.insmatheco.2022.03.002

15. M. E. Yaari, The dual theory of choice under risk, *Econometrica,* **55** (1987), 95–115. https://doi.org/10.2307/1911158

16. S. Wang, Premium Calculation by Transforming the Layer Premium Density, *ASTIN Bull.,* **26** (1996), 71–92. https://doi.org/10.2143/AST.26.1.563234

17. J. Belles-Sampera, M. Guillén, M. Santolino, Beyond value-at-risk: GlueVaR distortion risk measures, *Risk Anal.,* **34** (2014), 121–134. https://doi.org/10.1111/risa.12080

18. M. Santolino, J. Belles-Sampera, M. Guillén I Estany, J. M. Sarabia, An examination of the tail contribution to distortion risk measures, *J. Risk,* **23** (2019).

19. C. Bernard, S. M. Pesenti, S. Vanduffel, Robust distortion risk measures, *Math. Finance,* **34** (2023), 774–818. https://doi.org/10.1111/mafi.12414

20. A. V. Asimit, T. J. Boonen, Y. Chi, W. F. Chong, Risk sharing with multiple indemnity environments, *Eur. J. Oper. Res.,* **295** (2021), 587–603. https://doi.org/10.1016/j.ejor.2021.03.012

21. T. J. Boonen, K. Tan, S. Zhuang, Optimal reinsurance with multiple reinsurers: Competitive pricing and coalition stability, *Insur. Math. Econ.,* **101** (2021), 302–319. https://doi.org/10.1016/j.insmatheco.2021.08.005

22. G. Huberman, D. Mayers, J. R. Smith, Optimal insurance policy indemnity schedules, *Bell J. Econ.,* **14** (1983), 415–426. https://doi.org/10.2307/3003643

23. A. Nekvinda, L. Zajíček, A simple proof of the Rademacher theorem, *Čas. Pest. Mat.,* **4** (1988), 337–341.

24. H. Assa, On optimal reinsurance policy with distortion risk measures and premiums, *Insur. Math. Econ.,* **61** (2015), 70–75. https://doi.org/10.1016/j.insmatheco.2014.11.007

25. L. Yin, Game-theoretic analysis of commodity price convergence, *Shandong Econ.,* **1** (2004), 21–23.

26. H. Gerber, *An introduction to mathematical risk theory*, 1 Eds., Philadelphia: Huebner Foundation for Insurance Education, 1979. http://dx.doi.org/10.1016/B978-0-12-775850-3.50017-0

27. Y. Chi, W. Wei, Optimal insurance with background risk: An analysis of general dependence structures, *Financ. Stoch.,* **4** (2020), 903–937. https://doi.org/10.1007/s00780-020-00429-0

28. W. Rudin, *Principles of mathematical analysis*, 3 Eds., New York: McGraw-Hill, 1976.

29. S. Zhuang, C. Weng, K. Tan, H. Assa, Marginal indemnification function formulation for optimal reinsurance, *Insur. Math. Econ.,* **67** (2016), 65–76. https://doi.org/10.1016/j.insmatheco.2015.12.003

Appendix A. Conditions in each case of Theorem 3.1

Case 1. *The original conditions*

$$
\begin{aligned}
\text{IC11}: &\int_0^M \left[g_1\left(S_X(z)\right) - g_2\left(S_X(z)\right)\right] h_{11}^{\{1\}*}(z, \theta_1^{\{1\}}) \mathrm{d}z \geq 0,\\
\text{IC12}: &\int_0^M \left[g_2\left(S_X(z)\right) - g_1\left(S_X(z)\right)\right] h_{12}^{\{1\}*}(z, \theta_2^{\{1\}}) \mathrm{d}z \geq 0,\\
\text{IC21}: &\int_0^M \left[g_1\left(S_X(z)\right) - g_2\left(S_X(z)\right)\right] h_{21}^{\{1\}*}(z, \theta_3^{\{1\}}) \mathrm{d}z \geq 0,\\
\text{IC22}: &\int_0^M \left[g_2\left(S_X(z)\right) - g_1\left(S_X(z)\right)\right] h_{22}^{\{1\}*}(z, \theta_4^{\{1\}}) \mathrm{d}z \geq 0.
\end{aligned}
$$

The slackness conditions

$$
\begin{aligned}
&\theta_1^{\{1\}*} \int_0^M \left[g_1\left(S_X(z)\right) - g_2\left(S_X(z)\right)\right] h_{11}^{\{1\}*}(z, \theta_1^{\{1\}*}) \mathrm{d}z = 0,\\
&\theta_2^{\{1\}*} \int_0^M \left[g_2\left(S_X(z)\right) - g_1\left(S_X(z)\right)\right] h_{12}^{\{1\}*}(z, \theta_2^{\{1\}*}) \mathrm{d}z = 0,\\
&\theta_3^{\{1\}*} \int_0^M \left[g_1\left(S_X(z)\right) - g_2\left(S_X(z)\right)\right] h_{21}^{\{1\}*}(z, \theta_3^{\{1\}*}) \mathrm{d}z = 0,\\
&\theta_4^{\{1\}*} \int_0^M \left[g_2\left(S_X(z)\right) - g_1\left(S_X(z)\right)\right] h_{22}^{\{1\}*}(z, \theta_4^{\{1\}*}) \mathrm{d}z = 0.
\end{aligned}
$$

Case 2. *The original conditions*

$$
\begin{aligned}
&\text{IR2}: \int_0^M \left[g_2\left(S_X(z)\right) - g_1\left(S_X(z)\right)\right] h_{11}^{\{2\}*}\left(z, \theta_i^{\{2\}}\right) \mathrm{d}z \geq 0,\\
&\text{IC12}: \int_0^M \left[g_1\left(S_X(z)\right) - g_2\left(S_X(z)\right)\right] h_{11}^{\{2\}*}\left(z, \theta_i^{\{2\}}\right) \mathrm{d}z + \int_0^M \left[g_2\left(S_X(z)\right) - g_1\left(S_X(z)\right)\right] h_{12}^{\{2\}*}\left(z, \theta_i^{\{2\}}\right) \mathrm{d}z \geq 0,\\
&\text{IC22}: \int_0^M \left[g_1\left(S_X(z)\right) - g_2\left(S_X(z)\right)\right] h_{11}^{\{2\}*}\left(z, \theta_i^{\{2\}}\right) \mathrm{d}z + \int_0^M \left[g_2\left(S_X(z)\right) - g_1\left(S_X(z)\right)\right] h_{22}^{\{2\}*}\left(z, \theta_i^{\{2\}}\right) \mathrm{d}z \geq 0,\\
&\text{IC21}: \int_0^M g_2\left(S_X(z)\right) \left[h_{11}^{\{2\}*}\left(z, \theta_i^{\{2\}}\right) - h_{21}^{\{2\}*}\left(z, \theta_i^{\{2\}}\right)\right] \mathrm{d}z \geq 0.
\end{aligned}
$$

The slackness conditions

$$
\begin{aligned}
&\theta_1^{\{2\}*} \int_0^M \left[g_2\left(S_X(z)\right) - g_1\left(S_X(z)\right)\right] h_{11}^{\{2\}*}\left(z, \theta_i^{\{2\}*}\right) \mathrm{d}z = 0,\\
&\theta_2^{\{2\}*} \left[\int_0^M \left[g_1\left(S_X(z)\right) - g_2\left(S_X(z)\right)\right] h_{11}^{\{2\}*}\left(z, \theta_i^{\{2\}*}\right) \mathrm{d}z + \int_0^M \left[g_2\left(S_X(z)\right) - g_1\left(S_X(z)\right)\right] h_{12}^{\{2\}*}\left(z, \theta_i^{\{2\}*}\right) \mathrm{d}z\right] = 0,\\
&\theta_3^{\{2\}*} \left[\int_0^M \left[g_1\left(S_X(z)\right) - g_2\left(S_X(z)\right)\right] h_{11}^{\{2\}*}\left(z, \theta_i^{\{2\}*}\right) \mathrm{d}z + \int_0^M \left[g_2\left(S_X(z)\right) - g_1\left(S_X(z)\right)\right] h_{22}^{\{2\}*}\left(z, \theta_i^{\{2\}*}\right) \mathrm{d}z\right] = 0,\\
&\theta_4^{\{2\}*} \int_0^M g_2\left(S_X(z)\right) \left[h_{11}^{\{2\}*}\left(z, \theta_i^{\{2\}*}\right) - h_{21}^{\{2\}*}\left(z, \theta_i^{\{2\}*}\right)\right] \mathrm{d}z = 0.
\end{aligned}
$$

Case 3. *The original conditions*

$$\text{IR2}: \int_0^M \left[g_2\left(S_X(z)\right) - g_1\left(S_X(z)\right)\right] h_{21}^{\{3\}*}\left(z, \theta_i^{\{3\}}\right) \mathrm{d}z \geq 0,$$

$$\text{IC12}: \int_0^M \left[g_1\left(S_X(z)\right) - g_2\left(S_X(z)\right)\right] h_{21}^{\{3\}*}\left(z, \theta_i^{\{3\}}\right) \mathrm{d}z + \int_0^M \left[g_2\left(S_X(z)\right) - g_1\left(S_X(z)\right)\right] h_{12}^{\{3\}*}\left(z, \theta_i^{\{3\}}\right) \mathrm{d}z \geq 0,$$

$$\text{IC22}: \int_0^M \left[g_1\left(S_X(z)\right) - g_2\left(S_X(z)\right)\right] h_{21}^{\{3\}*}\left(z, \theta_i^{\{3\}}\right) \mathrm{d}z + \int_0^M \left[g_2\left(S_X(z)\right) - g_1\left(S_X(z)\right)\right] h_{22}^{\{3\}*}\left(z, \theta_i^{\{3\}}\right) \mathrm{d}z \geq 0,$$

$$\text{IC11}: \int_0^M g_2\left(S_X(z)\right) \left[h_{21}^{\{3\}*}\left(z, \theta_i^{\{3\}}\right) - h_{11}^{\{3\}*}\left(z, \theta_i^{\{3\}}\right)\right] \mathrm{d}z \geq 0.$$

The slackness conditions

$$\theta_1^{\{3\}*} \int_0^M \left[g_2\left(S_X(z)\right) - g_1\left(S_X(z)\right)\right] h_{21}^{\{3\}*}\left(z, \theta_i^{\{3\}*}\right) \mathrm{d}z = 0,$$

$$\theta_2^{\{3\}*} \left[\int_0^M \left[g_1\left(S_X(z)\right) - g_2\left(S_X(z)\right)\right] h_{21}^{\{3\}*}\left(z, \theta_i^{\{3\}*}\right) \mathrm{d}z + \int_0^M \left[g_2\left(S_X(z)\right) - g_1\left(S_X(z)\right)\right] h_{12}^{\{3\}*}\left(z, \theta_i^{\{3\}*}\right) \mathrm{d}z\right] = 0,$$

$$\theta_3^{\{3\}*} \left[\int_0^M \left[g_1\left(S_X(z)\right) - g_2\left(S_X(z)\right)\right] h_{21}^{\{3\}*}\left(z, \theta_i^{\{3\}*}\right) \mathrm{d}z + \int_0^M \left[g_2\left(S_X(z)\right) - g_1\left(S_X(z)\right)\right] h_{22}^{\{3\}*}\left(z, \theta_i^{\{3\}*}\right) \mathrm{d}z\right] = 0,$$

$$\theta_4^{\{3\}*} \int_0^M g_2\left(S_X(z)\right) \left[h_{21}^{\{3\}*}\left(z, \theta_i^{\{3\}*}\right) - h_{11}^{\{3\}*}\left(z, \theta_i^{\{3\}*}\right)\right] \mathrm{d}z = 0.$$

Case 4. *The original conditions*

$$\text{IR1}: \int_0^M \left[g_1\left(S_X(z)\right) - g_2\left(S_X(z)\right)\right] h_{12}^{\{4\}*}\left(z, \theta_i^{\{4\}}\right) \mathrm{d}z \geq 0,$$

$$\text{IC22}: \int_0^M g_1\left(S_X(z)\right) \left[h_{12}^{\{4\}*}\left(z, \theta_i^{\{4\}}\right) - h_{22}^{\{4\}*}\left(z, \theta_i^{\{4\}}\right)\right] \mathrm{d}z \geq 0,$$

$$\text{IC11}: \int_0^M \left[g_2\left(S_X(z)\right) - g_1\left(S_X(z)\right)\right] h_{12}^{\{4\}*}\left(z, \theta_i^{\{4\}}\right) \mathrm{d}z + \int_0^M \left[g_1\left(S_X(z)\right) - g_2\left(S_X(z)\right)\right] h_{11}^{\{4\}*}\left(z, \theta_i^{\{4\}}\right) \mathrm{d}z \geq 0,$$

$$\text{IC21}: \int_0^M \left[g_2\left(S_X(z)\right) - g_1\left(S_X(z)\right)\right] h_{12}^{\{4\}*}\left(z, \theta_i^{\{4\}}\right) \mathrm{d}z + \int_0^M \left[g_1\left(S_X(z)\right) - g_2\left(S_X(z)\right)\right] h_{21}^{\{4\}*}\left(z, \theta_i^{\{4\}}\right) \mathrm{d}z \geq 0.$$

The slackness conditions

$$\theta_1^{\{4\}*} \int_0^M \left[g_1\left(S_X(z)\right) - g_2\left(S_X(z)\right)\right] h_{12}^{\{4\}*}\left(z, \theta_i^{\{4\}*}\right) \mathrm{d}z = 0,$$

$$\theta_2^{\{4\}*} \int_0^M g_1\left(S_X(z)\right) \left[h_{12}^{\{4\}*}(z, \theta_2^{\{4\}*}) - h_{22}^{\{4\}*}\left(z, \theta_i^{\{4\}*}\right)\right] \mathrm{d}z = 0,$$

$$\theta_3^{\{4\}*} \left[\int_0^M \left[g_2\left(S_X(z)\right) - g_1\left(S_X(z)\right)\right] h_{12}^{\{4\}*}\left(z, \theta_i^{\{4\}*}\right) \mathrm{d}z + \int_0^M \left[g_1\left(S_X(z)\right) - g_2\left(S_X(z)\right)\right] h_{11}^{\{4\}*}\left(z, \theta_i^{\{4\}*}\right) \mathrm{d}z\right] = 0,$$

$$\theta_4^{\{4\}*} \left[\int_0^M \left[g_2\left(S_X(z)\right) - g_1\left(S_X(z)\right)\right] h_{12}^{\{4\}*}\left(z, \theta_i^{\{4\}*}\right) \mathrm{d}z + \int_0^M \left[g_1\left(S_X(z)\right) - g_2\left(S_X(z)\right)\right] h_{21}^{\{4\}*}\left(z, \theta_i^{\{4\}*}\right) \mathrm{d}z\right] = 0.$$

Case 5. *The original conditions*

$$
\begin{aligned}
&\text{IR1}: \int_0^M \left[g_1\left(S_X(z)\right) - g_2\left(S_X(z)\right)\right] h_{22}^{\{5\}*}\left(z, \theta_i^{\{5\}}\right) \mathrm{d}z \geq 0,\\
&\text{IC12}: \int_0^M g_1\left(S_X(z)\right)\left[h_{22}^{\{5\}*}\left(z, \theta_i^{\{5\}}\right) - h_{12}^{\{5\}*}\left(z, \theta_i^{\{5\}}\right)\right] \mathrm{d}z \geq 0,\\
&\text{IC11}: \int_0^M \left[g_2\left(S_X(z)\right) - g_1\left(S_X(z)\right)\right] h_{22}^{\{5\}*}\left(z, \theta_i^{\{5\}}\right) \mathrm{d}z + \int_0^M \left[g_1\left(S_X(z)\right) - g_2\left(S_X(z)\right)\right] h_{11}^{\{5\}*}\left(z, \theta_i^{\{5\}}\right) \mathrm{d}z \geq 0,\\
&\text{IC21}: \int_0^M \left[g_2\left(S_X(z)\right) - g_1\left(S_X(z)\right)\right] h_{22}^{\{5\}*}\left(z, \theta_i^{\{5\}}\right) \mathrm{d}z + \int_0^M \left[g_1\left(S_X(z)\right) - g_2\left(S_X(z)\right)\right] h_{21}^{\{5\}*}\left(z, \theta_i^{\{5\}}\right) \mathrm{d}z \geq 0.
\end{aligned}
$$

The slackness conditions

$$
\begin{aligned}
&\theta_1^{\{5\}*} \int_0^M \left[g_1\left(S_X(z)\right) - g_2\left(S_X(z)\right)\right] h_{22}^{\{5\}*}(z, \theta_i^{\{5\}*}) \mathrm{d}z = 0,\\
&\theta_2^{\{5\}*} \int_0^M g_1\left(S_X(z)\right)\left[h_{22}^{\{5\}*}(z, \theta_i^{\{5\}*}) - h_{12}^{\{5\}*}(z, \theta_i^{\{5\}*})\right] \mathrm{d}z = 0,\\
&\theta_3^{\{5\}*}\left[\int_0^M \left[g_2\left(S_X(z)\right) - g_1\left(S_X(z)\right)\right] h_{22}^{\{5\}*}(z, \theta_i^{\{5\}*}) \mathrm{d}z + \int_0^M \left[g_1\left(S_X(z)\right) - g_2\left(S_X(z)\right)\right] h_{11}^{\{5\}*}(z, \theta_i^{\{5\}*}) \mathrm{d}z\right] = 0,\\
&\theta_4^{\{5\}*}\left[\int_0^M \left[g_2\left(S_X(z)\right) - g_1\left(S_X(z)\right)\right] h_{22}^{\{5\}*}(z, \theta_i^{\{5\}*}) \mathrm{d}z + \int_0^M \left[g_1\left(S_X(z)\right) - g_2\left(S_X(z)\right)\right] h_{21}^{\{5\}*}(z, \theta_i^{\{5\}*}) \mathrm{d}z\right] = 0.
\end{aligned}
$$

Appendix B. Proof of Theorem 4.1

Case 1. *Let*

$$
\psi_{11}^{\{1\}}\left(z, \theta_1^{\{1\}}\right) = \begin{cases} -\lambda pmS_X(z), & 0 < S_X(z) \leq 1 - \alpha_2 \\ -\lambda pmS_X(z) - \theta_1^{\{1\}}, & 1 - \alpha_2 < S_X(z) \leq 1 - \alpha_1 \\ \lambda pm\left[1 - S_X(z)\right], & 1 - \alpha_1 < S_X(z) < 1 \\ 0, & S_X(z) = 0, 1 \end{cases}
$$

$$
\psi_{12}^{\{1\}}\left(z, \theta_2^{\{1\}}\right) = \begin{cases} -\lambda\left(1 - p\right) nS_X(z), & 0 < S_X(z) \leq 1 - \alpha_2 \\ \lambda\left(1 - p\right) n\left[1 - S_X(z)\right] + \theta_2^{\{1\}}, & 1 - \alpha_2 < S_X(z) \leq 1 - \alpha_1 \\ \lambda\left(1 - p\right) n\left[1 - S_X(z)\right], & 1 - \alpha_1 < S_X(z) < 1 \\ 0, & S_X(z) = 0, 1 \end{cases}
$$

$$
\psi_{21}^{\{1\}}\left(z, \theta_3^{\{1\}}\right) = \begin{cases} -\left(1 - \lambda\right) p\left(1 - m\right) S_X(z), & 0 < S_X(z) \leq 1 - \alpha_2 \\ -\left(1 - \lambda\right) p\left(1 - m\right) S_X(z) - \theta_3^{\{1\}}, & 1 - \alpha_2 < S_X(z) \leq 1 - \alpha_1 \\ \left(1 - \lambda\right) p\left(1 - m\right)\left[1 - S_X(z)\right], & 1 - \alpha_1 < S_X(z) < 1 \\ 0, & S_X(z) = 0, 1 \end{cases}
$$

$$\psi_{22}^{\{1\}}\left(z,\theta_4^{\{1\}}\right)=\begin{cases}-(1-\lambda)(1-p)(1-n)S_X(z), & 0<S_X(z)\le 1-\alpha_2\\ (1-\lambda)(1-p)(1-n)\left[1-S_X(z)\right]+\theta_4^{\{1\}}, & 1-\alpha_2<S_X(z)\le 1-\alpha_1\\ (1-\lambda)(1-p)(1-n)\left[1-S_X(z)\right], & 1-\alpha_1<S_X(z)<1\\ 0, & S_X(z)=0,1\end{cases}$$

for the Lagrangian coefficients $\theta_1^{\{1\}},\theta_2^{\{1\}},\theta_3^{\{1\}},\theta_4^{\{1\}}\ge 0$ *and they must satisfy the slackness conditions in Case* 1*. The optimal MIFs take the same form as in* (3.12)*, and can be verified to satisfy the original conditions in Case* 1*. Then we obtain the optimal indemnity functions:*

$$\begin{aligned}
f_{11}^{\{1\}*}(x)&=\int_0^x h_{11}^{\{1\}*}\left(z,\theta_1^{\{1\}*}\right)\mathrm{d}z=\int_0^x \mathbf{1}_{\left\{\psi_{11}^{\{1\}}\left(z,\theta_1^{\{1\}*}\right)>0\right\}}\mathrm{d}z=\min\left\{x,S_X^{-1}(1-\alpha_1)\right\},\\
f_{12}^{\{1\}*}(x)&=\int_0^x h_{12}^{\{1\}*}\left(z,\theta_2^{\{1\}*}\right)\mathrm{d}z=\int_0^x \mathbf{1}_{\left\{\psi_{12}^{\{1\}}\left(z,\theta_2^{\{1\}*}\right)>0\right\}}\mathrm{d}z=\min\left\{x,S_X^{-1}(1-\alpha_2)\right\},\\
f_{21}^{\{1\}*}(x)&=\int_0^x h_{21}^{\{1\}*}\left(z,\theta_3^{\{1\}*}\right)\mathrm{d}z=\int_0^x \mathbf{1}_{\left\{\psi_{21}^{\{1\}}\left(z,\theta_3^{\{1\}*}\right)>0\right\}}\mathrm{d}z=\min\left\{x,S_X^{-1}(1-\alpha_1)\right\},\\
f_{22}^{\{1\}*}(x)&=\int_0^x h_{22}^{\{1\}*}\left(z,\theta_4^{\{1\}*}\right)\mathrm{d}z=\int_0^x \mathbf{1}_{\left\{\psi_{22}^{\{1\}}\left(z,\theta_4^{\{1\}*}\right)>0\right\}}\mathrm{d}z=\min\left\{x,S_X^{-1}(1-\alpha_2)\right\}.
\end{aligned}$$

Under Case 1*, where* IR1 *and* IR2 *are binding, we obtain the optimal premiums:*

$$\begin{aligned}
\pi_{\cdot 1}^{\{1\}*}&=\rho_{g_1}\left(f_{\cdot 1}^{\{1\}*}(X)\right)=S_X^{-1}(1-\alpha_1),\\
\pi_{\cdot 2}^{\{1\}*}&=\rho_{g_2}\left(f_{\cdot 2}^{\{1\}*}(X)\right)=S_X^{-1}(1-\alpha_2).
\end{aligned}$$

Let

$$Y=\begin{cases}X, & 0\le x\le S_X^{-1}(1-\alpha_1)\\ 0, & otherwise\end{cases}\quad and\quad K=\begin{cases}X, & 0\le x\le S_X^{-1}(1-\alpha_2)\\ 0, & otherwise\end{cases},$$

and then we have

$$\begin{aligned}
\mathbb{E}\left[f_{11}^{\{1\}*}(X)\right]&=\int_0^M S_X(z)h_{11}^{\{1\}*}(z,\theta_1^{\{1\}*})\mathrm{d}z=\int_0^M S_X(z)\cdot\mathbf{1}_{\left\{\psi_{11}^{\{1\}}(z,\theta_1^{\{1\}*})>0\right\}}\mathrm{d}z=\int_0^{S_X^{-1}(1-\alpha_1)}S_X(z)\mathrm{d}z=\mathbb{E}[Y],\\
\mathbb{E}\left[f_{12}^{\{1\}*}(X)\right]&=\int_0^M S_X(z)h_{12}^{\{1\}*}(z,\theta_2^{\{1\}*})\mathrm{d}z=\int_0^M S_X(z)\cdot\mathbf{1}_{\left\{\psi_{12}^{\{1\}}(z,\theta_2^{\{1\}*})>0\right\}}\mathrm{d}z=\int_0^{S_X^{-1}(1-\alpha_2)}S_X(z)\mathrm{d}z=\mathbb{E}[K],\\
\mathbb{E}\left[f_{21}^{\{1\}*}(X)\right]&=\int_0^M S_X(z)h_{21}^{\{1\}*}(z,\theta_3^{\{1\}*})\mathrm{d}z=\int_0^M S_X(z)\cdot\mathbf{1}_{\left\{\psi_{21}^{\{1\}}(z,\theta_3^{\{1\}*})>0\right\}}\mathrm{d}z=\int_0^{S_X^{-1}(1-\alpha_1)}S_X(z)\mathrm{d}z=\mathbb{E}[Y],\\
\mathbb{E}\left[f_{22}^{\{1\}*}(X)\right]&=\int_0^M S_X(z)h_{22}^{\{1\}*}(z,\theta_4^{\{1\}*})\mathrm{d}z=\int_0^M S_X(z)\cdot\mathbf{1}_{\left\{\psi_{22}^{\{1\}}(z,\theta_4^{\{1\}*})>0\right\}}\mathrm{d}z=\int_0^{S_X^{-1}(1-\alpha_2)}S_X(z)\mathrm{d}z=\mathbb{E}[K].
\end{aligned}$$

So we obtain the maximum of the objective function in Case 1.

$$\begin{aligned}
P^{\{1\}*}&=\lambda P_1^{\{1\}*}+(1-\lambda)P_2^{\{1\}*}\\
&=p\left[\lambda m+(1-\lambda)(1-m)\right]\left[S_X^{-1}(1-\alpha_1)-\mathbb{E}[Y]\right]+(1-p)\left[\lambda n+(1-\lambda)(1-n)\right]\Big[S_X^{-1}(1-\alpha_2)\\
&\quad-\mathbb{E}[K]\Big]
\end{aligned}\tag{A.1}$$

Next, we show that the original conditions in Cases 2–5 can be relaxed. We begin by solving the corresponding problems without these constraints and then prove that the resulting optimal reinsurance strategies naturally satisfy the original conditions.

Case 2. *We first solve Problem* 3.8 *without the original conditions, which means* $\theta_1^{\{2\}} = \theta_2^{\{2\}} = \theta_3^{\{2\}} = \theta_4^{\{2\}} = 0$*. Then we have*

$$\psi_{11}^{\{2\}}(z) = \begin{cases} -\lambda pmS_X(z), & 0 < S_X(z) \leq 1-\alpha_2 \\ -\lambda pmS_X(z) - \lambda(1-p)n - (1-\lambda)(1-p)(1-n), & 1-\alpha_2 < S_X(z) \leq 1-\alpha_1 \\ \lambda pm[1-S_X(z)], & 1-\alpha_1 < S_X(z) < 1 \\ 0, & S_X(z) = 0,1 \end{cases}$$

$$\psi_{12}^{\{2\}}(z) = \begin{cases} -\lambda(1-p)nS_X(z), & 0 < S_X(z) \leq 1-\alpha_2 \\ \lambda(1-p)n[1-S_X(z)], & 1-\alpha_2 < S_X(z) < 1 \\ 0, & S_X(z) = 0,1 \end{cases}$$

$$\psi_{21}^{\{2\}}(z) = \begin{cases} -(1-\lambda)p(1-m)S_X(z), & 0 < S_X(z) \leq 1-\alpha_1 \\ (1-\lambda)p(1-m)[1-S_X(z)], & 1-\alpha_1 < S_X(z) < 1 \\ 0, & S_X(z) = 0,1 \end{cases}$$

$$\psi_{22}^{\{2\}}(z) = \begin{cases} -(1-\lambda)(1-p)(1-n)S_X(z), & 0 < S_X(z) \leq 1-\alpha_2 \\ (1-\lambda)(1-p)(1-n)[1-S_X(z)], & 1-\alpha_2 < S_X(z) < 1 \\ 0, & S_X(z) = 0,1 \end{cases}$$

The optimal MIFs take the same form as in 3.13*, and can be verified to satisfy the original conditions in Case* 2*. Therefore, these conditions can be removed. Then we obtain*

$$f_{11}^{\{2\}*}(x) = \int_0^x h_{11}^{\{2\}*}(z)\mathrm{d}z = \int_0^x \mathbf{1}_{\{\psi_{11}^{\{2\}}(z)>0\}}\mathrm{d}z = \min\left\{x, S_X^{-1}(1-\alpha_1)\right\},$$
$$f_{12}^{\{2\}*}(x) = \int_0^x h_{12}^{\{2\}*}(z)\mathrm{d}z = \int_0^x \mathbf{1}_{\{\psi_{12}^{\{2\}}(z)>0\}}\mathrm{d}z = \min\left\{x, S_X^{-1}(1-\alpha_2)\right\},$$
$$f_{21}^{\{2\}*}(x) = \int_0^x h_{21}^{\{2\}*}(z)\mathrm{d}z = \int_0^x \mathbf{1}_{\{\psi_{21}^{\{2\}}(z)>0\}}\mathrm{d}z = \min\left\{x, S_X^{-1}(1-\alpha_1)\right\},$$
$$f_{22}^{\{2\}*}(x) = \int_0^x h_{22}^{\{2\}*}(z)\mathrm{d}z = \int_0^x \mathbf{1}_{\{\psi_{22}^{\{2\}}(z)>0\}}\mathrm{d}z = \min\left\{x, S_X^{-1}(1-\alpha_2)\right\},$$
$$\mathbb{E}\left[f_{11}^{\{2\}*}(X)\right] = \int_0^M S_X(z)h_{11}^{\{2\}*}(z)\mathrm{d}z = \int_0^M S_X(z)\cdot\mathbf{1}_{\{\psi_{11}^{\{2\}}(z)>0\}}\mathrm{d}z = \int_0^{S_X^{-1}(1-\alpha_1)} S_X(z)\mathrm{d}z = \mathbb{E}[Y],$$
$$\mathbb{E}\left[f_{12}^{\{2\}*}(X)\right] = \int_0^M S_X(z)h_{12}^{\{2\}*}(z)\mathrm{d}z = \int_0^M S_X(z)\cdot\mathbf{1}_{\{\psi_{12}^{\{2\}}(z)>0\}}\mathrm{d}z = \int_0^{S_X^{-1}(1-\alpha_2)} S_X(z)\mathrm{d}z = \mathbb{E}[K],$$
$$\mathbb{E}\left[f_{21}^{\{2\}*}(X)\right] = \int_0^M S_X(z)h_{21}^{\{2\}*}(z)\mathrm{d}z = \int_0^M S_X(z)\cdot\mathbf{1}_{\{\psi_{21}^{\{2\}}(z)>0\}}\mathrm{d}z = \int_0^{S_X^{-1}(1-\alpha_1)} S_X(z)\mathrm{d}z = \mathbb{E}[Y],$$
$$\mathbb{E}\left[f_{22}^{\{2\}*}(X)\right] = \int_0^M S_X(z)h_{22}^{\{2\}*}(z)\mathrm{d}z = \int_0^M S_X(z)\cdot\mathbf{1}_{\{\psi_{22}^{\{2\}}(z)>0\}}\mathrm{d}z = \int_0^{S_X^{-1}(1-\alpha_2)} S_X(z)\mathrm{d}z = \mathbb{E}[K].$$

Under Case 2*, where* IR1 *and* IC11 *are binding, we obtain the optimal premiums:*

$$\pi_{\cdot 1}^{\{2\}*} = \rho_{g_1}\left(f_{\cdot 1}^{\{2\}*}(X)\right) = S_X^{-1}(1-\alpha_1),$$
$$\pi_{\cdot 2}^{\{2\}*} = \pi_{\cdot 1}^{\{2\}*} + \rho_{g_2}\left(f_{\cdot 2}^{\{2\}*}(X)\right) - \rho_{g_2}\left(f_{11}^{\{2\}*}(X)\right) = S_X^{-1}(1-\alpha_2).$$

So we obtain the maximum of the objective function in Case 2*.*

$$\begin{aligned}P^{\{2\}*} &= \lambda P_1^{\{2\}*} + (1-\lambda) P_2^{\{2\}*}\\ &= p\left[\lambda m + (1-\lambda)(1-m)\right]\left[S_X^{-1}(1-\alpha_1) - \mathbb{E}[Y]\right] + (1-p)\left[\lambda n + (1-\lambda)(1-n)\right]\left[S_X^{-1}(1-\alpha_2)\right.\\ &\quad \left. - \mathbb{E}[K]\right] \end{aligned} \tag{A.2}$$

Case 3. *We first solve Problem* 3.9 *without the original conditions, which means* $\theta_1^{\{3\}} = \theta_2^{\{3\}} = \theta_3^{\{3\}} = \theta_4^{\{3\}} = 0$*. Then we have*

$$\psi_{11}^{\{3\}}(z) = \begin{cases} -\lambda pm S_X(z), & 0 < S_X(z) \le 1-\alpha_1\\ \lambda pm\left[1 - S_X(z)\right], & 1-\alpha_1 < S_X(z) < 1\\ 0, & S_X(z) = 0,1 \end{cases}$$

$$\psi_{12}^{\{3\}}(z) = \begin{cases} -\lambda(1-p) n S_X(z), & 0 < S_X(z) \le 1-\alpha_2\\ \lambda(1-p) n\left[1 - S_X(z)\right], & 1-\alpha_2 < S_X(z) < 1\\ 0, & S_X(z) = 0,1 \end{cases}$$

$$\psi_{21}^{\{3\}}(z) = \begin{cases} -(1-\lambda) p(1-m) S_X(z), & 0 < S_X(z) \le 1-\alpha_2\\ -\lambda(1-p)n - (1-\lambda)(1-p)(1-n) - (1-\lambda)p(1-m) S_X(z), & 1-\alpha_2 < S_X(z) \le 1-\alpha_1\\ (1-\lambda)p(1-m)\left[1 - S_X(z)\right], & 1-\alpha_1 < S_X(z) < 1\\ 0, & S_X(z) = 0,1 \end{cases}$$

$$\psi_{22}^{\{3\}}(z) = \begin{cases} -(1-\lambda)(1-p)(1-n) S_X(z), & 0 < S_X(z) \le 1-\alpha_2\\ (1-\lambda)(1-p)(1-n)\left[1 - S_X(z)\right], & 1-\alpha_2 < S_X(z) < 1\\ 0, & S_X(z) = 0,1 \end{cases}$$

The optimal MIFs take the same form as in 3.14*, and can be verified to satisfy the original conditions in Case* 3*. Therefore, these conditions can be removed. Then we obtain*

$$f_{11}^{\{3\}*}(x) = \int_0^x h_{11}^{\{3\}*}(z)\mathrm{d}z = \int_0^x \mathbf{1}_{\left\{\psi_{11}^{\{3\}}(z)>0\right\}}\mathrm{d}z = \min\left\{x, S_X^{-1}(1-\alpha_1)\right\},$$
$$f_{12}^{\{3\}*}(x) = \int_0^x h_{12}^{\{3\}*}(z)\mathrm{d}z = \int_0^x \mathbf{1}_{\left\{\psi_{12}^{\{3\}}(z)>0\right\}}\mathrm{d}z = \min\left\{x, S_X^{-1}(1-\alpha_2)\right\},$$
$$f_{21}^{\{3\}*}(x) = \int_0^x h_{21}^{\{3\}*}(z)\mathrm{d}z = \int_0^x \mathbf{1}_{\left\{\psi_{21}^{\{3\}}(z)>0\right\}}\mathrm{d}z = \min\left\{x, S_X^{-1}(1-\alpha_1)\right\},$$
$$f_{22}^{\{3\}*}(x) = \int_0^x h_{22}^{\{3\}*}(z)\mathrm{d}z = \int_0^x \mathbf{1}_{\left\{\psi_{22}^{\{3\}}(z)>0\right\}}\mathrm{d}z = \min\left\{x, S_X^{-1}(1-\alpha_2)\right\},$$
$$\mathbb{E}\left[f_{11}^{\{3\}*}(X)\right] = \int_0^M S_X(z) h_{11}^{\{3\}*}(z)\mathrm{d}z = \int_0^M S_X(z)\cdot \mathbf{1}_{\left\{\psi_{11}^{\{3\}}(z)>0\right\}}\mathrm{d}z = \int_0^{S_X^{-1}(1-\alpha_1)} S_X(z)\mathrm{d}z = \mathbb{E}[Y],$$

$$\mathbb{E}\left[f_{12}^{\{3\}*}(X)\right] = \int_0^M S_X(z)h_{12}^{\{3\}*}(z)\mathrm{d}z = \int_0^M S_X(z)\cdot \mathbf{1}_{\left\{\psi_{12}^{\{3\}}(z)>0\right\}}\mathrm{d}z = \int_0^{S_X^{-1}(1-\alpha_2)} S_X(z)\mathrm{d}z = \mathbb{E}[K],$$
$$\mathbb{E}\left[f_{21}^{\{3\}*}(X)\right] = \int_0^M S_X(z)h_{21}^{\{3\}*}(z)\mathrm{d}z = \int_0^M S_X(z)\cdot \mathbf{1}_{\left\{\psi_{21}^{\{3\}}(z)>0\right\}}\mathrm{d}z = \int_0^{S_X^{-1}(1-\alpha_1)} S_X(z)\mathrm{d}z = \mathbb{E}[Y],$$
$$\mathbb{E}\left[f_{22}^{\{3\}*}(X)\right] = \int_0^M S_X(z)h_{22}^{\{3\}*}(z)\mathrm{d}z = \int_0^M S_X(z)\cdot \mathbf{1}_{\left\{\psi_{22}^{\{3\}}(z)>0\right\}}\mathrm{d}z = \int_0^{S_X^{-1}(1-\alpha_2)} S_X(z)\mathrm{d}z = \mathbb{E}[K].$$

Under Case 3*, where* IR1 *and* IC21 *are binding, we obtain the optimal premiums:*

$$\pi_{\cdot 1}^{\{3\}*} = \rho_{g_1}\left(f_{\cdot 1}^{\{3\}*}(X)\right) = S_X^{-1}(1-\alpha_1),$$
$$\pi_{\cdot 2}^{\{3\}*} = \pi_{\cdot 1}^{\{3\}*} + \rho_{g_2}\left(f_{\cdot 2}^{\{3\}*}(X)\right) - \rho_{g_2}\left(f_{21}^{\{3\}*}(X)\right) = S_X^{-1}(1-\alpha_2).$$

So we obtain the maximum of the objective function in Case 3*.*

$$\begin{aligned} P^{\{3\}*} &= \lambda P_1^{\{3\}*} + (1-\lambda)P_2^{\{3\}*} \\ &= p\left[\lambda m + (1-\lambda)(1-m)\right]\left[S_X^{-1}(1-\alpha_1) - \mathbb{E}[Y]\right] + (1-p)\left[\lambda n + (1-\lambda)(1-n)\right]\left[S_X^{-1}(1-\alpha_2)\right. \\ &\quad \left. - \mathbb{E}[K]\right] \end{aligned} \tag{A.3}$$

Case 4. *We first solve Problem 3.10 without the original conditions, which means* $\theta_1^{\{4\}} = \theta_2^{\{4\}} = \theta_3^{\{4\}} = \theta_4^{\{4\}} = 0$*. Then we have*

$$\psi_{11}^{\{4\}}(z) = \begin{cases} -\lambda pmS_X(z), & 0 < S_X(z) \le 1-\alpha_1 \\ \lambda pm\left[1-S_X(z)\right], & 1-\alpha_1 < S_X(z) < 1 \\ 0, & S_X(z) = 0, 1 \end{cases}$$

$$\psi_{12}^{\{4\}}(z) = \begin{cases} -\lambda(1-p)nS_X(z), & 0 < S_X(z) \le 1-\alpha_2 \\ \lambda pm + \lambda(1-p)n\left[1-S_X(z)\right] + (1-\lambda)p(1-m), & 1-\alpha_2 < S_X(z) \le 1-\alpha_1 \\ \lambda(1-p)n\left[1-S_X(z)\right], & 1-\alpha_1 < S_X(z) < 1 \\ 0, & S_X(z) = 0, 1 \end{cases}$$

$$\psi_{21}^{\{4\}}(z) = \begin{cases} -(1-\lambda)p(1-m)S_X(z), & 0 < S_X(z) \le 1-\alpha_1 \\ (1-\lambda)p(1-m)\left[1-S_X(z)\right], & 1-\alpha_1 < S_X(z) < 1 \\ 0, & S_X(z) = 0, 1 \end{cases}$$

$$\psi_{22}^{\{4\}}(z) = \begin{cases} -(1-\lambda)(1-p)(1-n)S_X(z), & 0 < S_X(z) \le 1-\alpha_2 \\ (1-\lambda)(1-p)(1-n)\left[1-S_X(z)\right], & 1-\alpha_2 < S_X(z) < 1 \\ 0, & S_X(z) = 0, 1 \end{cases}$$

The optimal MIFs take the same form as in (3.15)*. Substituting them into* IR1*, we have*

$$\int_0^M \left[g_1(S_X(z)) - g_2(S_X(z))\right]h_{12}^{\{4\}*}(z)\mathrm{d}z = S_X^{-1}(1-\alpha_1) - S_X^{-1}(1-\alpha_2) \le 0$$

Therefore, IR1 *holds if and only if* $\alpha_1 = \alpha_2$*, meaning that both types of insurers have identical risk preferences. Then we have*

$$f_{11}^{\{4\}*}(x) = f_{12}^{\{4\}*}(x) = f_{21}^{\{4\}*}(x) = f_{22}^{\{4\}*}(x) = \min\left\{x, S_X^{-1}(1-\alpha_2)\right\},$$
$$\mathbb{E}\left[f_{11}^{\{4\}*}(X)\right] = \mathbb{E}\left[f_{12}^{\{4\}*}(X)\right] = \mathbb{E}\left[f_{21}^{\{4\}*}(X)\right] = \mathbb{E}\left[f_{22}^{\{4\}*}(X)\right] = \mathbb{E}[K].$$

Under Case 4*, where* IC12 *and* IR2 *are binding, we obtain the optimal premiums:*

$$\pi_{\cdot 1}^{\{4\}*} = \pi_{\cdot 2}^{\{4\}*} + \rho_{g_1}\left(f_{\cdot 1}^{\{4\}*}(X)\right) - \rho_{g_1}\left(f_{12}^{\{4\}*}(X)\right) = S_X^{-1}(1-\alpha_2),$$
$$\pi_{\cdot 2}^{\{4\}*} = \rho_{g_2}\left(f_{\cdot 2}^{\{4\}*}(X)\right) = S_X^{-1}(1-\alpha_2).$$

So we obtain the maximum of the objective function in Case 4*.*

$$\begin{aligned} P^{\{4\}*} &= \lambda P_1^{\{4\}*} + (1-\lambda) P_2^{\{4\}*} \\ &= \left[p\left[\lambda m + (1-\lambda)(1-m)\right] + (1-p)\left[\lambda n + (1-\lambda)(1-n)\right]\right]\left[S_X^{-1}(1-\alpha_2) - \mathbb{E}[K]\right] \end{aligned} \tag{A.4}$$

Case 5. *We first solve Problem* 3.11 *without the original conditions, which means* $\theta_1^{\{5\}} = \theta_2^{\{5\}} = \theta_3^{\{5\}} = \theta_4^{\{5\}} = 0$*. Then we have*

$$\psi_{11}^{\{5\}}(z) = \begin{cases} -\lambda p m S_X(z), & 0 < S_X(z) \le 1-\alpha_1 \\ \lambda p m\left[1 - S_X(z)\right], & 1-\alpha_1 < S_X(z) < 1 \\ 0, & S_X(z) = 0, 1 \end{cases}$$

$$\psi_{12}^{\{5\}}(z) = \begin{cases} -\lambda(1-p) n S_X(z), & 0 < S_X(z) \le 1-\alpha_2 \\ \lambda(1-p) n\left[1 - S_X(z)\right], & 1-\alpha_2 < S_X(z) < 1 \\ 0, & S_X(z) = 0, 1 \end{cases}$$

$$\psi_{21}^{\{5\}}(z) = \begin{cases} -(1-\lambda) p(1-m) S_X(z), & 0 < S_X(z) \le 1-\alpha_1 \\ (1-\lambda) p(1-m)\left[1 - S_X(z)\right], & 1-\alpha_1 < S_X(z) < 1 \\ 0, & S_X(z) = 0, 1 \end{cases}$$

$$\psi_{22}^{\{5\}}(z) = \begin{cases} -(1-\lambda)(1-p)(1-n) S_X(z), & 0 < S_X(z) \le 1-\alpha_2 \\ \lambda p m + (1-\lambda) p(1-m) + (1-\lambda)(1-p)(1-n)\left[1 - S_X(z)\right], & 1-\alpha_2 < S_X(z) \le \alpha_1 \\ (1-\lambda)(1-p)(1-n)\left[1 - S_X(z)\right], & 1-\alpha_1 < S_X(z) < 1 \\ 0, & S_X(z) = 0, 1 \end{cases}$$

The optimal MIFs take the same form as in (3.16)*. Substituting them into* IR1*, we have*

$$\int_0^M \left[g_1(S_X(z)) - g_2(S_X(z))\right] h_{22}^{\{5\}*}(z)\mathrm{d}z = S_X^{-1}(1-\alpha_1) - S_X^{-1}(1-\alpha_2) \le 0$$

Therefore, IR1 *holds if and only if* $\alpha_1 = \alpha_2$*, meaning that both types of insurers have identical risk preferences. Then we have*

$$f_{11}^{\{5\}*}(x) = f_{12}^{\{5\}*}(x) = f_{21}^{\{5\}*}(x) = f_{22}^{\{5\}*}(x) = \min\left\{x, S_X^{-1}(1-\alpha_2)\right\},$$

$$\mathbb{E}\left[f_{11}^{\{5\}*}(X)\right] = \mathbb{E}\left[f_{12}^{\{5\}*}(X)\right] = \mathbb{E}\left[f_{21}^{\{5\}*}(X)\right] = \mathbb{E}\left[f_{22}^{\{5\}*}(X)\right] = \mathbb{E}\left[K\right].$$

Under Case 5*, where* IC22 *and* IR2 *are binding, we obtain the optimal premiums:*

$$\begin{aligned}
\pi_{\cdot 1}^{\{5\}*} &= \pi_{\cdot 2}^{\{5\}*} + \rho_{g_1}\left(f_{\cdot 1}^{\{5\}*}(X)\right) - \rho_{g_1}\left(f_{22}^{\{5\}*}(X)\right) = S_X^{-1}(1-\alpha_2),\\
\pi_{\cdot 2}^{\{5\}*} &= \rho_{g_2}\left(f_{\cdot 2}^{\{5\}*}(X)\right) = S_X^{-1}(1-\alpha_2).
\end{aligned}$$

So we obtain the maximum of the objective function in Case 5*.*

$$\begin{aligned}
P^{\{5\}*} &= \lambda P_1^{\{5\}*} + (1-\lambda) P_2^{\{5\}*}\\
&= \left[p\left[\lambda m + (1-\lambda)(1-m)\right] + (1-p)\left[\lambda n + (1-\lambda)(1-n)\right]\right]\left[S_X^{-1}(1-\alpha_2) - \mathbb{E}\left[K\right]\right] \qquad \text{(A.5)}
\end{aligned}$$

The comparison of (A.1)–(A.5) shows that $P^{\{1\}*} = P^{\{2\}*} = P^{\{3\}*}$ and $P^{\{4\}*} = P^{\{5\}*}$. We observe that the objective function maximum in Cases 4–5 are the actually special instance of Cases 1–3 when $\alpha_1 = \alpha_2$, so we have

$$P^{\{1\}*} = P^{\{2\}*} = P^{\{3\}*} \geq P^{\{4\}*} = P^{\{5\}*}.$$

Therefore, Cases 1–3 yield the globally optimal reinsurance menu under the VaR risk measure.