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Research article

Generalization of the interval TOR method for solving interval linear systems

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Abstract: Iterative methods for solving interval linear systems were presented in this paper. By generalizing interval diagonal matrices to interval band matrices, a generalization of the interval two-parameter overrelaxation method (GITOR) was introduced, and the convergence analysis of the proposed method was discussed. Specifically, if the coefficient matrices of the system are strictly diagonally dominant (SDD) matrices, interval M-matrices, or interval H-matrices, the proposed method converges under any initial approximation. Furthermore, an upper bound on the spectral radius of the iterative matrices was provided for interval strictly diagonally dominant matrices. Finally, numerical examples for each type of mentioned interval matrix were studied, demonstrating the efficiency of the proposed method.

Keywords: interval linear systems; iterative method; interval band matrices; generalized interval TOR method; convergence analysis
Mathematics Subject Classification: 15A30, 65G30, 65G40, 65H10

1. Introduction

Many practical problems, such as uncertainty in engineering or design, global optimization, and mathematical programming problems, can be reduced to solving interval linear systems (see [1–5]). Due to the uncertainty of the data, the enclosure of the solution set plays a key role. However, solving the best enclosure of the solution set is an NP-hard problem [6], so researchers have worked to develop less costly methods to enclose the solution set rather than pursuing the optimal solution.

Various methods have been proposed to find an outer approximation of the solution set, such as the interval Jacobi method, the interval Gauss-Seidel method, the Krawczyk method, the Bauer-Skeel method, and the Hansen-Bliek-Rohn method (see [3, 7–11]). In [3, 12–16], the authors demonstrated that the above methods may not produce the optimal enclosure. The aim of this paper is to develop the iterative method and its convergence to solve interval linear systems with uncertain coefficients.

Consider the interval linear system

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \tag{1.1}$$

where $\boldsymbol{A}$ and $\boldsymbol{b}$ are the $n \times n$ real interval matrix and the $n \times 1$ real interval vector, respectively. The corresponding solution set is defined as

$$\Sigma(\boldsymbol{A}, \boldsymbol{b}) := \{x \in \mathbb{R}^n; \exists A \in \boldsymbol{A}\ \exists b \in \boldsymbol{b} : Ax = b\}.$$

The interval hull of the solution set $\Sigma(\boldsymbol{A}, \boldsymbol{b})$ is denoted by

$$\square\Sigma := [inf(\Sigma(\boldsymbol{A}, \boldsymbol{b})), sup(\Sigma(\boldsymbol{A}, \boldsymbol{b}))],$$

that is, the smallest interval enclosure of $\Sigma(\boldsymbol{A}, \boldsymbol{b})$ with respect to inclusion.

In recent years, J. Chakravarty and M. Saha et al. in [17] and [18] have studied the convergence of the generalized interval Jacobi, Gauss-Seidel, successive overrelaxation (SOR), and accelerated overrelaxation (AOR) methods for interval linear systems. In the paper, the convergence of the generalized two-parameter overrelaxation (TOR) method will be studied, when the coefficient matrices of interval linear systems are interval strictly diagonally dominant matrices, interval M-matrices, or interval H-matrices. Some numerical examples are then given to demonstrate the convergence results obtained.

The structure of the paper is as follows: Section 2 provides notation and basic definitions related to interval analysis and defines the various classes of interval matrices considered. A generalization of the interval TOR method is presented in Section 3, along with a discussion of the convergence analysis for the various classes of interval coefficient matrices. Section 4 presents numerical experiments on the proposed method. Concluding remarks are provided in Section 5.

2. Notation and preliminaries

In this section, some definitions and notations are given. Intervals will be represented in boldface type. Additionally, the lower and upper bounds of the intervals will be indicated by underline and overline, respectively. We now list the symbols that are used often in the paper.

(i) $\mathbb{R}$: The set of real numbers.

(ii) $\mathbb{R}^n$: The set of n-dimensional real column vectors.

(iii) $\mathbb{R}^{m\times n}$: The set of $m \times n$ real matrices.

(iv) $\boldsymbol{x} = [\underline{x}, \overline{x}]$: Interval vector.

(v) $\boldsymbol{A} = [\underline{A}, \overline{A}]$: Interval matrix.

(vi) $\boldsymbol{x}_i$: i-th entry of an interval vector $\boldsymbol{x}$.

(vii) $\boldsymbol{A}_{ij}$: (i, j)-th entry of an interval matrix $\boldsymbol{A}$.

(viii) $\overline{A}$: Infimum of interval matrix $\boldsymbol{A}$, $\underline{A}$: Supremum of interval matrix $\boldsymbol{A}$.

(ix) $\Sigma(\boldsymbol{A}, \boldsymbol{b})$: The solution set of $Ax = b$ where $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$.

(x) $\square\Sigma$: The interval hull of the solution set $\Sigma(\boldsymbol{A}, \boldsymbol{b})$.

(xi) $\boldsymbol{A}^{-1}$: The inverse of interval matrix $\boldsymbol{A}$.

(xii) $\mathbb{IR}$: The set of real intervals.

(xiii) $\mathbb{IR}^n$: The set of n-dimensional real column vectors.

(xiv) $\mathbb{IR}^{m\times n}$: The set of $m \times n$ real interval matrices.

(xv) $|\boldsymbol{A}|$: The magnitude of interval matrix $\boldsymbol{A}$.

(xvi) $\langle \boldsymbol{A} \rangle$: The comparison matrix of interval matrix $\boldsymbol{A}$.

(xvii) $\sigma(A)$: The spectrum of a square matrix A, that is, $\sigma(A)$ is the set of all eigenvalues of A.

(xviii) $\rho(A)$: Spectral radius of square matrix A, $\rho(A) = \max_{\lambda\in\sigma(A)} |\lambda|$.

(xix) $\boldsymbol{R}_m = (\boldsymbol{R}_{ij})$, $\boldsymbol{E}_m = (\boldsymbol{E}_{ij})$, $\boldsymbol{F}_m = (\boldsymbol{F}_{ij})$, $\boldsymbol{U}_m = (\boldsymbol{U}_{ij})$,
$\tilde{R}_i = \sum\limits_{\substack{j=1 \\ j\neq i}}^{n} |\boldsymbol{R}_{ij}|$, $\tilde{E}_i = \sum\limits_{\substack{j=1 \\ j\neq i}}^{n} |\boldsymbol{E}_{ij}|$, $\tilde{F}_i = \sum\limits_{\substack{j=1 \\ j\neq i}}^{n} |\boldsymbol{F}_{ij}|$, $\tilde{U}_i = \sum\limits_{\substack{j=1 \\ j\neq i}}^{n} |\boldsymbol{U}_{ij}|$.

Next, we present the basic definitions and review the key properties of the interval matrices studied in this paper. The interval matrix $\boldsymbol{A}$ is defined as

$$\boldsymbol{A} := (\boldsymbol{A}_{ij}) = \left[\underline{A}, \overline{A}\right] = \left\{A \in \mathbb{R}^{m\times n} : \underline{A} \leq A \leq \overline{A}\right\},$$

where $\underline{A}, \overline{A} \in \mathbb{R}^{m\times n}$ are given, and $\underline{A} \leq \overline{A}$. The magnitude of $\boldsymbol{A} \in \mathbb{IR}^{m\times n}$ is defined as $|\boldsymbol{A}| = \max_{A\in\boldsymbol{A}} |A| = \max(|\underline{A}|, |\overline{A}|)$. The comparison matrix of $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ is the matrix $\langle \boldsymbol{A} \rangle \in \mathbb{R}^{n\times n}$ with entries

$$\langle \boldsymbol{A} \rangle_{ii} = \min\{|A_{ii}| : A_{ii} \in \boldsymbol{A}_{ii}\}, \quad i = 1, \ldots, n.$$
$$\langle \boldsymbol{A} \rangle_{ij} = -|\boldsymbol{A}_{ij}|, \text{ for } i \neq j.$$

Definition 2.1. [3, 19] For any real intervals $\boldsymbol{x} = [\underline{x}, \overline{x}], \boldsymbol{y} = [\underline{y}, \overline{y}]$, interval addition, subtraction, and multiplication are defined as

(i) $\boldsymbol{x} + \boldsymbol{y} = [\underline{x} + \underline{y}, \overline{x} + \overline{y}]$;

(ii) $\boldsymbol{x} - \boldsymbol{y} = [\underline{x} - \overline{y}, \overline{x} - \underline{y}]$;

(iii) The interval multiplication $\boldsymbol{xy}$ is displayed in the following table:

$*$	$\boldsymbol{y} \geq 0$	$\boldsymbol{y} \ni 0$	$\boldsymbol{y} \leq 0$
$\boldsymbol{x} \geq 0$	$[\underline{x}\underline{y}, \overline{x}\overline{y}]$	$[\overline{x}\underline{y}, \overline{x}\overline{y}]$	$[\overline{x}\underline{y}, \underline{x}\overline{y}]$
$\boldsymbol{x} \ni 0$	$[\underline{x}\overline{y}, \overline{x}\overline{y}]$	$[\min\{\underline{x}\overline{y}, \overline{x}\underline{y}\}, \max\{\underline{x}\underline{y}, \overline{x}\overline{y}\}]$	$[\overline{x}\underline{y}, \underline{x}\underline{y}]$
$\boldsymbol{x} \leq 0$	$[\underline{x}\overline{y}, \overline{x}\underline{y}]$	$[\underline{x}\overline{y}, \underline{x}\underline{y}]$	$[\overline{x}\overline{y}, \underline{x}\underline{y}]$

Definition 2.2. [3, 19] If $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{IR}^{n\times n}$, addition, subtraction, and multiplication for interval matrices are defined as

(i) $\boldsymbol{A} \pm \boldsymbol{B} = \square\{A \pm B : A \in \boldsymbol{A}, B \in \boldsymbol{B}\}$.
If $\boldsymbol{A} = [\underline{A}, \overline{A}], \boldsymbol{B} = [\underline{B}, \overline{B}]$, then

$$\boldsymbol{A} + \boldsymbol{B} = [\underline{A} + \underline{B}, \overline{A} + \overline{B}] \text{ and } \boldsymbol{A} - \boldsymbol{B} = [\underline{A} - \overline{B}, \overline{A} - \underline{B}].$$

(ii) $\boldsymbol{AB} = \square\{AB : A \in \boldsymbol{A}, B \in \boldsymbol{B}\}$.
If $\boldsymbol{A} = (\boldsymbol{A}_{ij})$ and $\boldsymbol{B} = (\boldsymbol{B}_{ij})$, then $(\boldsymbol{AB})_{ij} = \sum_{k=1}^{n} \boldsymbol{A}_{ik}\boldsymbol{B}_{kj}$.

Proposition 2.3. *[1, 19] For $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{IR}^{n\times n}$ and $\boldsymbol{C} \in \mathbb{IR}^{n\times p}$, the following properties hold:*

(i) $\langle \boldsymbol{A} \rangle = \langle \tilde{A} \rangle$, *for some* $\tilde{A} \in \boldsymbol{A}$;

(ii) $|\boldsymbol{AB}| \le |\boldsymbol{A}||\boldsymbol{B}|$;

(iii) $\langle \boldsymbol{A} \pm \boldsymbol{B} \rangle \ge \langle \boldsymbol{A} \rangle - |\boldsymbol{B}|$;

(iv) $|\boldsymbol{A}| - |\boldsymbol{B}| \le |\boldsymbol{A} \pm \boldsymbol{B}| \le |\boldsymbol{A}| + |\boldsymbol{B}|$;

(v) $|\boldsymbol{AC}| \ge \langle \boldsymbol{A} \rangle |\boldsymbol{C}|$.

Definition 2.4. [19] A matrix $A \in \mathbb{R}^{n\times n}$ is said to be a Z-matrix if off-diagonal entries A are non-positive. A Z-matrix is called an M-matrix if it can be written as $A = sI - B$, where $s > \rho(B)$, the spectral radius of B. From now on, we simply denote the non-singular M-matrix by M-matrix.

Theorem 2.5. *[3, 19] Let $A \in \mathbb{R}^{n\times n}$ be a Z-matrix. Then the following statements are equivalent:*

(i) A is an M-matrix;

(ii) $A^{-1} \ge 0$;

(iii) There exists $u > 0$ *such that* $Au > 0$.

Definition 2.6. [3] An interval M-matrix is a square interval matrix $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ such that $\boldsymbol{A}_{ik} \le 0$, that is, every element in $\boldsymbol{A}_{ik}$ is non-positive, for all $i \ne k$ and $\boldsymbol{A}u > 0$ for some positive vector $u \in \mathbb{R}^n$.

Theorem 2.7. *[3, 19] Let $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{IR}^{n\times n}$. Then the following statements hold:*

(i) If $\boldsymbol{A}$ is an interval M-matrix and $\boldsymbol{B} \subseteq \boldsymbol{A}$, then $\boldsymbol{B}$ is an interval M-matrix. In particular, each $\tilde{A} \in \boldsymbol{A}$ is an M-matrix.

(ii) $\boldsymbol{A} = [\underline{A}, \overline{A}]$ *is an interval M-matrix if and only if* $\underline{A}$ *and* $\overline{A}$ *are M-matrices.*

(iii) Every interval M-matrix $\boldsymbol{A} = [\underline{A}, \overline{A}]$ *is regular with* $\boldsymbol{A}^{-1} = [\overline{A}^{-1}, \underline{A}^{-1}] \ge 0$ *and* $|\boldsymbol{A}^{-1}| = \langle \boldsymbol{A} \rangle^{-1}$.

Definition 2.8. [3, 19] An interval matrix $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ is called an interval H-matrix if its comparison matrix $\langle \boldsymbol{A} \rangle$ is an M-matrix. Equivalently, we say that $\boldsymbol{A}$ is an interval H-matrix if and only if $\langle \boldsymbol{A} \rangle u > 0$ for some $u > 0$.

Theorem 2.9. *[3] For an interval matrix $\boldsymbol{A}$, we have that:*

(i) If $\boldsymbol{A}$ is an interval regular (lower/upper) triangular matrix, then $\boldsymbol{A}$ is an interval H-matrix.

(ii) If $\boldsymbol{A}$ is an interval H-matrix, then $|\boldsymbol{A}^{-1}| \leq \langle \boldsymbol{A} \rangle^{-1}$. The equality holds if $\boldsymbol{A}$ is an interval M-matrix.

Definition 2.10. [3, 19] Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ and $0 \notin \boldsymbol{A}_{ii}$ for all i. Then $\boldsymbol{A}$ is called an interval strictly diagonally dominant (SDD) matrix if its comparison matrix $\langle \boldsymbol{A} \rangle$ is strictly diagonally dominant, that is, if $\langle \boldsymbol{A}_{ii} \rangle > \sum_{j\neq i} |\boldsymbol{A}_{ij}|$, for $i \neq j$.

Definition 2.11. [19] A splitting of a matrix $A \in \mathbb{R}^{n\times n}$ is defined as $A = M - N$, with non-singular M. A splitting $A = M - N$ of the matrix A is called

(i) regular if $M^{-1} \geq 0$ and $N \geq 0$.

(ii) weak regular if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$.

(iii) M-splitting if M is an M-matrix and $N \geq 0$.

Proposition 2.12. *[19] If $A, B \in \mathbb{R}^{n\times n}$, and $|A| \leq B$, then $\rho(A) \leq \rho(B)$.*

Proposition 2.13. *[3, 19] If $A \in \mathbb{R}^{n\times n}$ is a non-negative matrix, $x \geq 0$, and $x \neq 0$ such that $Ax \geq \alpha x$, for some $\alpha \in \mathbb{R}$, then $\rho(A) \geq \alpha$.*

Theorem 2.14. *[19] Let A be an M-matrix and let $A = M - N$ be a regular or weak regular splitting of A, and then $\rho(M^{-1}N) < 1$.*

Theorem 2.15. *[19] Let A be an M-matrix and $B \geq 0$. Then $A - B$ is an M-matrix if and only if $\rho(A^{-1}B) < 1$.*

Theorem 2.16. *[19] Let $A = M - N$ be an M-splitting of A. Then $\rho(M^{-1}N) < 1$ if and only if A is a non-singular M-matrix.*

Theorem 2.17. *[3] If $A \in \mathbb{R}^{n\times n}$ is a non-negative matrix, then the spectral radius $\rho(A)$ is an eigenvalue of A, and there is a real, non-negative eigenvector $x \neq 0$ with $Ax = \rho(A)x$.*

Theorem 2.18. *[3, 16] Let $\boldsymbol{S}, \boldsymbol{T} : \mathbb{IR}^n \to \mathbb{IR}^n$ be sublinear maps satisfying $\rho(|\boldsymbol{S}||\boldsymbol{T}|) < 1$. Then, for every $\boldsymbol{x} \in \mathbb{IR}^n$, the following statements hold:*

(i) The equation

$$\boldsymbol{y} = \boldsymbol{S}(\boldsymbol{x} + \boldsymbol{T}\boldsymbol{y})$$

has a unique solution $\boldsymbol{y} \in \mathbb{IR}^n$.

(ii) For all starting vectors $\boldsymbol{y}_0 \in \mathbb{IR}^n$, the iteration

$$\boldsymbol{y}^{l+1} := \boldsymbol{S}(\boldsymbol{x} + \boldsymbol{T}\boldsymbol{y}^l)$$

converges to the solution $\boldsymbol{y}$ of the equation $\boldsymbol{y} = \boldsymbol{S}(\boldsymbol{x} + \boldsymbol{T}\boldsymbol{y})$.

(iii) If $\boldsymbol{y}^1 \subseteq \boldsymbol{y}^0$, then for all $i \geq 1$,

$$\boldsymbol{y} \subseteq \boldsymbol{y}^i \subseteq \boldsymbol{y}^{i-1} \cdots \subseteq \boldsymbol{y}^0.$$

(iv) If $\boldsymbol{y}^0 \subseteq \boldsymbol{y}^1$, then for all $i \geq 1$,

$$\boldsymbol{y}^0 \subseteq \cdots \subseteq \boldsymbol{y}^{i-1} \subseteq \boldsymbol{y}^i \subseteq \boldsymbol{y}.$$

3. Generalized interval TOR method

In 1983, J. Kuang ([20]) proposed the two-parameter overrelaxation method (TOR) and discussed the convergence under the assumption that the coefficient matrices are Hermitian positive definite matrices and L-matrices. In [21], Zeng considered the convergence of the TOR method when the coefficient matrices are symmetric positive definite matrices, H-matrices, L-matrices, and irreducibly diagonally dominant matrices. In [22], Martins et al. introduced the interval two-parameter overrelaxation method (ITOR). When the coefficient matrices belong to interval strictly diagonally dominant matrices or interval H-matrices, some convergence conditions were obtained for interval linear systems.

In this section, we introduce a generalization of the interval two-parameter overrelaxation method (GITOR). By restricting the possible values of the parameters α and β, we discuss the convergence properties of the GITOR method for interval strictly diagonally dominant (SDD) matrices, interval M-matrices, and interval H-matrices. We also give an upper bound on the spectral radius of iterative matrices for interval SDD-matrices. It is also shown that the spectral radius of iterative matrices of the ITOR method is smaller than that of the GITOR method, and that the spectral radius of iterative matrices decreases with increasing bandwidth.

Let $\boldsymbol{A} = [\underline{a_{ij}}, \overline{a_{ij}}] \in \mathbb{IR}^{n\times n}$. Consider an interval band matrix $\boldsymbol{T}_m = [\underline{t_{ij}}, \overline{t_{ij}}]$ of $2m+1$ bandwidth, which is characterized as

$$\boldsymbol{T}_{ij} = \begin{cases} [\underline{a_{ij}}, \overline{a_{ij}}], & \text{if } |i-j| \le m; \\ 0, & \text{elsewhere.} \end{cases}$$

For $1 \le m < n$, the coefficient matrix $\boldsymbol{A}$ can be decomposed as

$$\boldsymbol{A} = \boldsymbol{T}_m - \boldsymbol{E}_m - \boldsymbol{F}_m - \boldsymbol{U}_m,$$

where $-\boldsymbol{U}_m$ is a strictly upper triangular matrix, and $-\boldsymbol{E}_m$ and $-\boldsymbol{F}_m$ are strictly lower triangular matrices. For all $i > j$, either $\boldsymbol{E}_{ij} = -\boldsymbol{A}_{ij}$ and $\boldsymbol{F}_{ij} = 0$, or $\boldsymbol{E}_{ij} = 0$ and $\boldsymbol{F}_{ij} = -\boldsymbol{A}_{ij}$. Moreover, the interval matrices $\boldsymbol{T}_m$, $\boldsymbol{E}_m + \boldsymbol{F}_m$, and $\boldsymbol{U}_m$ can be expressed as

$$\boldsymbol{T}_m = \begin{bmatrix} [\underline{a_{11}}, \overline{a_{11}}] & \cdots & [\underline{a_{1,m+1}}, \overline{a_{1,m+1}}] & & 0 \\ \vdots & \ddots & & \ddots & \\ [\underline{a_{m+1,1}}, \overline{a_{m+1,1}}] & & & & [\underline{a_{n-m,n}}, \overline{a_{n-m,n}}] \\ & \ddots & & \ddots & \\ 0 \quad [\underline{a_{n,n-m}}, \overline{a_{n,n-m}}] & & [\underline{a_{n,n}}, \overline{a_{n,n}}] & & \end{bmatrix},$$

$$\boldsymbol{E}_m + \boldsymbol{F}_m = \begin{bmatrix} 0 & \cdots & & 0 \\ -[\underline{a_{m+2,1}}, \overline{a_{m+2,1}}] & & & \\ \vdots & \ddots & & \vdots \\ -[\underline{a_{n,1}}, \overline{a_{n,1}}] & \cdots & -[\underline{a_{n-m-1,n}}, \overline{a_{n-m-1,n}}] & \end{bmatrix},$$

$$\boldsymbol{U}_m = \begin{bmatrix} 0 & -[\underline{a_{1,m+2}}, \overline{a_{1,m+2}}] & \cdots & -[\underline{a_{1,n}}, \overline{a_{1,n}}] \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & -[\underline{a_{n-m-1,n}}, \overline{a_{n-m-1,n}}] \end{bmatrix}.$$

The GITOR iterative scheme to solve (1.1) is given as

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{L}_{\alpha,\beta}\boldsymbol{x}^{(k)} + \boldsymbol{f}, \tag{3.1}$$

where $\boldsymbol{L}_{\alpha,\beta} = (2\boldsymbol{T}_m - \alpha\boldsymbol{E}_m - \beta\boldsymbol{F}_m)^{-1}[(2-\alpha-\beta)\boldsymbol{T}_m + (\alpha+\beta)\boldsymbol{U}_m + \alpha\boldsymbol{F}_m + \beta\boldsymbol{E}_m]$ and $\boldsymbol{f} = (\alpha+\beta)(2\boldsymbol{T}_m - \alpha\boldsymbol{E}_m - \beta\boldsymbol{F}_m)^{-1}\boldsymbol{b}$. It may be noticed that the scheme (3.1) corresponds to the following splitting of $(\alpha+\beta)\boldsymbol{A}$:

$$(\alpha+\beta)\boldsymbol{A} = (2\boldsymbol{T}_m - \alpha\boldsymbol{E}_m - \beta\boldsymbol{F}_m) - [(2-\alpha-\beta)\boldsymbol{T}_m + (\alpha+\beta)\boldsymbol{U}_m + \alpha\boldsymbol{F}_m + \beta\boldsymbol{E}_m].$$

Several common iterative methods can be used as special cases:

(i) For $\boldsymbol{F}_m = 0, \alpha = 0, \beta = 2$, (3.1) is the generalized interval Jacobi method (GIJ), where $\boldsymbol{L}_{0,2} = \boldsymbol{T}_m^{-1}(\boldsymbol{U}_m + \boldsymbol{E}_m)$ and $\boldsymbol{f} = \boldsymbol{T}_m^{-1}\boldsymbol{b}$;

(ii) For $\boldsymbol{F}_m = 0, \alpha = 2, \beta = 0$, (3.1) is the generalized interval Gauss-Seidel method (GIGS), where $\boldsymbol{L}_{2,0} = (\boldsymbol{T}_m - \boldsymbol{E}_m)^{-1}\boldsymbol{U}_m$ and $\boldsymbol{f} = (\boldsymbol{T}_m - \boldsymbol{E}_m)^{-1}\boldsymbol{b}$;

(iii) For $\boldsymbol{F}_m = 0, \alpha = 2\omega, \beta = 0$, (3.1) is the generalized interval SOR method (GISOR), where $\boldsymbol{L}_{2\omega,0} = (\boldsymbol{T}_m - \omega\boldsymbol{E}_m)^{-1}[(1-\omega)\boldsymbol{T}_m + \omega\boldsymbol{U}_m]$ and $\boldsymbol{f} = \omega(\boldsymbol{T}_m - \omega\boldsymbol{E}_m)^{-1}\boldsymbol{b}$;

(iv) For $\boldsymbol{F}_m = 0, \alpha = 2\gamma, \alpha+\beta = 2\omega$, (3.1) is the generalized interval AOR method (GIAOR), where $\boldsymbol{L}_{2\gamma,2(\omega-\gamma)} = (\boldsymbol{T}_m - \gamma\boldsymbol{E}_m)^{-1}[(1-\omega)\boldsymbol{T}_m + \omega\boldsymbol{U}_m + (\omega-\gamma)\boldsymbol{E}_m]$ and $\boldsymbol{f} = \omega(\boldsymbol{T}_m - \gamma\boldsymbol{E}_m)^{-1}\boldsymbol{b}$.

Next, we analyze the convergence of the GITOR method for interval coefficient matrices of various types.

3.1. On the convergence of the GITOR method

In this section, we consider the splitting of $(\alpha+\beta)\boldsymbol{A}$ as

$$(\alpha+\beta)\boldsymbol{A} = \boldsymbol{M}_m - \boldsymbol{N}_m,$$

with $\boldsymbol{M}_m := 2\boldsymbol{T}_m - \alpha\boldsymbol{E}_m - \beta\boldsymbol{F}_m$ and $\boldsymbol{N}_m := (2-\alpha-\beta)\boldsymbol{T}_m + (\alpha+\beta)\boldsymbol{U}_m + \alpha\boldsymbol{F}_m + \beta\boldsymbol{E}_m$.

Let $\tilde{\boldsymbol{L}}_{\alpha,\beta} = |\boldsymbol{M}_m^{-1}||\boldsymbol{N}_m|$. It is known that the GITOR method converges if $\rho(\tilde{\boldsymbol{L}}_{\alpha,\beta}) < 1$ due to Theorem 2.18. Since computing the inverse of the interval matrix is an NP-hard problem, we use the matrix $\tilde{\boldsymbol{L}}_m = \langle \boldsymbol{M}_m \rangle^{-1}|\boldsymbol{N}_m|$ to analyze the convergence of the GITOR method. The following Theorem 3.1 provides a relation between the spectral radius of the iteration matrix $\tilde{\boldsymbol{L}}_{\alpha,\beta} = |\boldsymbol{M}_m^{-1}||\boldsymbol{N}_m|$ with $\tilde{\boldsymbol{L}}_m = \langle \boldsymbol{M}_m \rangle^{-1}|\boldsymbol{N}_m|$.

Theorem 3.1. *Let $\boldsymbol{A}$ be an interval H-matrix (or interval M-matrix), and then*

(i) $\tilde{\boldsymbol{L}}_{\alpha,\beta} \le \tilde{\boldsymbol{L}}_m$ (the equality holds if $\boldsymbol{A}$ is an interval M-matrix),

(ii) $\rho(\tilde{\boldsymbol{L}}_{\alpha,\beta}) \le \rho(\tilde{\boldsymbol{L}}_m)$.

Proof. Let $\boldsymbol{A}$ be an interval H-matrix. Then $\boldsymbol{M}_m$ is also an interval H-matrix, and from Theorem 2.9 we have that

$$\tilde{\boldsymbol{L}}_{\alpha,\beta} = |\boldsymbol{M}_m^{-1}||\boldsymbol{N}_m| \le \langle \boldsymbol{M}_m \rangle^{-1}|\boldsymbol{N}_m| = \tilde{\boldsymbol{L}}_m.$$

By Theorem 2.12, $\rho(\tilde{\boldsymbol{L}}_{\alpha,\beta}) \le \rho(\tilde{\boldsymbol{L}}_m)$ holds. □

The following Theorem 3.2 gives an upper bound of the spectral radius $\rho(\tilde{L}_m)$, and solves interval linear systems for interval strictly diagonally dominant (SDD) matrices.

Theorem 3.2. *Let $\boldsymbol{A}$ be an interval SDD-matrix with constant diagonals of order n. If $|\beta|\tilde{F}_i + |\alpha|\tilde{E}_i < 2\langle \boldsymbol{T}\rangle_i$, then*

$$\rho(\tilde{L}_m) \leq \max_i \frac{|2-\alpha-\beta||\boldsymbol{T}|_i + |\alpha+\beta|\tilde{U}_i + |\alpha|\tilde{F}_i + |\beta|\tilde{E}_i}{2\langle \boldsymbol{T}\rangle_i - |\beta|\tilde{F}_i - |\alpha|\tilde{E}_i}.$$

Proof. Let λ be an eigenvalue $\rho(\tilde{\boldsymbol{L}}_m)$. Choose $x \neq 0 \in \mathbb{R}^n$ such that

$$\begin{aligned}
&\tilde{L}_m x = \lambda x\\
&\Rightarrow \langle 2\boldsymbol{T}_m - \alpha\boldsymbol{E}_m - \beta\boldsymbol{F}_m\rangle^{-1}|(2-\alpha-\beta)\boldsymbol{T}_m + (\alpha+\beta)\boldsymbol{U}_m + \alpha\boldsymbol{F}_m + \beta\boldsymbol{E}_m|x = \lambda x\\
&\Rightarrow [2\lambda\langle \boldsymbol{T}_m\rangle - |2-\alpha-\beta||\boldsymbol{T}_m| - |\alpha+\beta||\boldsymbol{U}_m| - (|\alpha| + \lambda|\beta|)|\boldsymbol{F}_m| - (\lambda|\alpha| + |\beta|)|\boldsymbol{E}_m|]\, x = 0\\
&\Rightarrow \left[|D| - \left(\frac{2\lambda + |2-\alpha-\beta|}{2\lambda - |2-\alpha-\beta|}\right)|\boldsymbol{R}_m| - \left(\frac{|\alpha+\beta|}{2\lambda - |2-\alpha-\beta|}\right)|\boldsymbol{U}_m|\right.\\
&\qquad \left. - \left(\frac{|\alpha| + \lambda|\beta|}{2\lambda - |2-\alpha-\beta|}\right)|\boldsymbol{F}_m| - \left(\frac{\lambda|\alpha| + |\beta|}{2\lambda - |2-\alpha-\beta|}\right)|\boldsymbol{E}_m|\right] x = 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
Q = |D| &- \left(\frac{2\lambda + |2-\alpha-\beta|}{2\lambda - |2-\alpha-\beta|}\right)|\boldsymbol{R}_m| - \left(\frac{|\alpha+\beta|}{2\lambda - |2-\alpha-\beta|}\right)|\boldsymbol{U}_m|\\
&- \left(\frac{|\alpha| + \lambda|\beta|}{2\lambda - |2-\alpha-\beta|}\right)|\boldsymbol{F}_m| - \left(\frac{\lambda|\alpha| + |\beta|}{2\lambda - |2-\alpha-\beta|}\right)|\boldsymbol{E}_m|
\end{aligned}$$

is singular which implies that Q is not an SDD-matrix. There exists an $i \in N$ such that

$$\begin{aligned}
|D_{ii}| \leq &\left|\frac{2\lambda + |2-\alpha-\beta|}{2\lambda - |2-\alpha-\beta|}\right||\tilde{R}_i| + \left|\frac{|\alpha+\beta|}{2\lambda - |2-\alpha-\beta|}\right||\tilde{U}_i|\\
&+ \left|\frac{|\alpha| + \lambda|\beta|}{2\lambda - |2-\alpha-\beta|}\right||\tilde{F}_i| + \left|\frac{\lambda|\alpha| + |\beta|}{2\lambda - |2-\alpha-\beta|}\right||\tilde{E}_i|.
\end{aligned}$$

This is simplified to

$$|\lambda| \leq \max_i \frac{|2-\alpha-\beta|(|D_{ii}| + \tilde{R}_i) + |\alpha+\beta|\tilde{U}_i + |\alpha|\tilde{F}_i + |\beta|\tilde{E}_i}{2(|D_{ii}| - \tilde{R}_i) - |\beta|\tilde{F}_i - |\alpha|\tilde{E}_i},$$

that is,

$$\rho(\tilde{L}_m) \leq \max_i \frac{|2-\alpha-\beta||\boldsymbol{T}|_i + |\alpha+\beta|\tilde{U}_i + |\alpha|\tilde{F}_i + |\beta|\tilde{E}_i}{2\langle \boldsymbol{T}\rangle_i - |\beta|\tilde{F}_i - |\alpha|\tilde{E}_i}.$$

□

Corollary 3.3. *Let $\boldsymbol{A}$ be an interval SDD-matrix with constant diagonals of order n. If $\widehat{R}_i = D_{ii}^{-1}\widehat{R}_i, \widehat{U}_i = D_{ii}^{-1}\widehat{U}_i, \widehat{E}_i = D_{ii}^{-1}\widehat{E}_i, \widehat{F}_i = D_{ii}^{-1}\widehat{F}_i$, then*

$$\rho(\tilde{L}_m) \leq \max_i \frac{|2-\alpha-\beta| + |\alpha+\beta|\widehat{U}_i + |\alpha|\widehat{F}_i + |\beta|\widehat{E}_i + |2-\alpha-\beta|\widehat{R}_i}{2 - |\beta|\widehat{F}_i - |\alpha|\widehat{E}_i - 2\widehat{R}_i}.$$

Proof. By Theorem 3.2 this conclusion is evident. □

Theorem 3.4. *Let $\boldsymbol{A}$ be an interval SDD-matrix with constant diagonals of order n, and*

$$\alpha > 0, \beta > 0, 0 < \alpha + \beta \leq \frac{4}{1 + max(\widehat{U_i} + \widehat{F_i} + \widehat{E_i} + \widehat{R_i})}.$$

Then the GITOR method for solving (1.1) converges for any initial approximation.

Proof. We distinguish two cases:

(i) $0 < \alpha + \beta \leq 2$.
Since $\boldsymbol{A}$ is an interval SDD-matrix, we have that $D^{-1}\boldsymbol{A}$ is also an interval SDD-matrix. Let λ be an eigenvalue $\rho(\tilde{\boldsymbol{L}}_m)$ and $|\lambda| \geq 1$.

$$\begin{aligned}
&\tilde{L}_m x = \lambda x \\
&\Rightarrow \langle 2\boldsymbol{T}_m - \alpha \boldsymbol{E}_m - \beta \boldsymbol{F}_m \rangle^{-1} |(2 - \alpha - \beta)\boldsymbol{T}_m + (\alpha + \beta)\boldsymbol{U}_m + \alpha \boldsymbol{F}_m + \beta \boldsymbol{E}_m| x = \lambda x \\
&\Rightarrow Q \text{ is singular.}
\end{aligned}$$

Next, we analyze that Q is an SDD-matrix, which leads to a contradiction that Q is singular. We just need to prove that

$$\left|\frac{2\lambda + 2 - \alpha - \beta}{2\lambda - 2 + \alpha + \beta}\right| < 1, \left|\frac{\alpha + \beta}{2\lambda - 2 + \alpha + \beta}\right| < 1,$$

$$\left|\frac{\alpha + \lambda\beta}{2\lambda - 2 + \alpha + \beta}\right| < 1, \left|\frac{\lambda\alpha + \beta}{2\lambda - 2 + \alpha + \beta}\right| < 1.$$

$|2\lambda + 2 - \alpha - \beta| < |2\lambda - 2 + \alpha + \beta|$ holds, for $\lambda < -1$;
$|\alpha + \beta| < |2\lambda - 2 + \alpha + \beta|$ is obvious.
Write $\lambda = re^{i\theta}$, $r \geq 1$. Then

$$\begin{aligned}
&- |\lambda\beta + \alpha|^2 + |2\lambda - 2 + \alpha + \beta|^2 \\
&= -r^2\beta^2 - 2r\beta\alpha \cos\theta - \alpha^2 + 4r^2 - 4r(2 - \alpha - \beta)\cos\theta + (2 - \alpha - \beta)^2 \\
&\geq -r^2\beta^2 - 2r\beta\alpha - \alpha^2 + 4r^2 - 4r(2 - \alpha - \beta) + (2 - \alpha - \beta)^2 \\
&= -(r\beta + \alpha)^2 + [2r - (2 - \alpha - \beta)]^2 \\
&= (r - 1)(2 - \beta)[2(r - 1 + \alpha) + \beta(1 + r)] > 0.
\end{aligned}$$

Similarly, it can be obtained that $-|\beta + \lambda\alpha|^2 + |2\lambda - 2 + \alpha + \beta|^2 > 0$.
That is, $\rho(\tilde{\boldsymbol{L}}_{\alpha,\beta}) \leq \rho(\tilde{\boldsymbol{L}}_m) < 1$.

(ii) $\alpha + \beta > 2$.
By assuming that we have

$$\alpha + \beta < \frac{4}{1 + max(\widehat{U_i} + \widehat{F_i} + \widehat{E_i} + \widehat{R_i})} < \frac{4}{1 + \widehat{U_i} + \widehat{F_i} + \widehat{E_i} + \widehat{R_i}}, \text{ for all } i,$$

that is,

$$
\begin{aligned}
&(\alpha+\beta)(1+\widehat{U}_i+\widehat{F}_i+\widehat{E}_i+\widehat{R}_i)<4\\
&\Rightarrow (\alpha+\beta-2)+(\alpha+\beta)\widehat{U}_i+\alpha\widehat{F}_i+\beta\widehat{E}_i+(\alpha+\beta-2)\widehat{R}_i<2-\beta\widehat{F}_i-\alpha\widehat{E}_i-2\widehat{R}_i\\
&\Rightarrow \max_i\left\{\frac{|2-\alpha-\beta|+|\alpha+\beta|\widehat{U}_i+|\alpha|\widehat{F}_i+|\beta|\widehat{E}_i+|2-\alpha-\beta|\widehat{R}_i}{2-|\beta|\widehat{F}_i-|\alpha|\widehat{E}_i-2\widehat{R}_i}\right\}<1\\
&\text{where } \widehat{F}_i\neq 0, \widehat{E}_i\neq 0, |\beta|<\frac{1}{\widehat{F}_i}, |\alpha|<\frac{1}{\widehat{E}_i}.
\end{aligned}
$$

Hence $\rho(\tilde{L}_{\alpha,\beta})<1$ by Theorem 3.2.

□

The following Theorem 3.5 presents a convergence analysis of the proposed method for interval M-matrices.

Theorem 3.5. *Let $\boldsymbol{A}$ be an interval M-matrix, and $\alpha,\beta>0$, $\alpha+\beta\leq 2$. Then the GITOR method for solving (1.1) converges for any initial approximation.*

Proof. Since $\boldsymbol{A}$ is an interval M-matrix, we have that $\boldsymbol{M}_m$ is also an interval M-matrix.

$$\langle \boldsymbol{M}_m\rangle = 2\langle \boldsymbol{T}_m\rangle - \alpha|\boldsymbol{E}_m| - \beta|\boldsymbol{F}_m|,$$

$$|\boldsymbol{N}_m| = (2-\alpha-\beta)|\boldsymbol{T}_m| + (\alpha+\beta)|\boldsymbol{U}_m| + \alpha|\boldsymbol{F}_m| + \beta|\boldsymbol{E}_m|,$$

so $\langle \boldsymbol{M}_m\rangle$ is an M-matrix, that is, $\langle \boldsymbol{M}_m\rangle^{-1}>0$. Since $|\boldsymbol{N}_m|>0$, it can be seen from Definition 2.11 that $\langle \boldsymbol{M}_m\rangle - |\boldsymbol{N}_m|$ is a regular splitting. Theorem 2.14 implies that $\rho(\langle \boldsymbol{M}_m\rangle^{-1}|\boldsymbol{N}_m|)<1$. By Theorem 2.9 we have $|\boldsymbol{M}_m^{-1}| = \langle \boldsymbol{M}_m\rangle^{-1}$. Hence, $\rho(|\boldsymbol{M}_m^{-1}||N_m|) = \rho(\langle \boldsymbol{M}_m\rangle^{-1}|\boldsymbol{N}_m|)<1$. □

The following Theorem 3.6 concerns a specific instance of interval M-matrices, comparing the spectral radii of the GITOR iterative matrices for different bandwidths.

Theorem 3.6. *Let $\boldsymbol{A}\in\mathbb{IR}^{n\times n}$ be an interval M-matrix, and $0<\alpha,\beta\leq 2$, $\alpha+\beta=2$, and the interval matrices $\boldsymbol{E}_m$ and $\boldsymbol{E}_p$ ($\boldsymbol{F}_m$ and $\boldsymbol{F}_p$) have the same elements but different dimensions. If $\tilde{\boldsymbol{L}}_k = \langle \boldsymbol{M}_k\rangle^{-1}|\boldsymbol{N}_k|$, for $k=m,p$, then $\rho(\tilde{L}_m)\leq\rho(\tilde{L}_p)<1$, for $m\geq p\geq 1$.*

Proof. Let $\boldsymbol{A}$ be an interval M-matrix, and then $\langle \boldsymbol{M}_m\rangle$ is an M-matrix, and $\tilde{\boldsymbol{L}}_p$ is a non-negative matrix. By Theorem 2.17 we can choose an eigenvector $x\geq 0$ and $x\neq 0$ associated with the eigenvalue $\lambda=\rho(\tilde{\boldsymbol{L}}_p)$. Then $\tilde{\boldsymbol{L}}_p x=\lambda x$, which implies $(|\boldsymbol{N}_p|-\lambda\langle \boldsymbol{M}_p\rangle)x=0$. Since $\langle \boldsymbol{M}_m\rangle^{-1}\geq 0$, and $|\boldsymbol{N}_m|>0$, it follows from Definition 2.11 that $\langle \boldsymbol{M}_m\rangle - |\boldsymbol{N}_m|$ is a regular splitting. Theorem 2.14 implies that $0\leq\lambda=\rho(\tilde{\boldsymbol{L}}_p)<1$, for some p. Let $\boldsymbol{T}_p=\boldsymbol{D}+\boldsymbol{R}_p$, $\boldsymbol{T}_m=\boldsymbol{D}+\boldsymbol{R}_m$, and then

$$|\boldsymbol{A}| = |\boldsymbol{D}|+|\boldsymbol{R}_p|+|\boldsymbol{E}_p|+|\boldsymbol{F}_p|+|\boldsymbol{U}_p| = |\boldsymbol{D}|+|\boldsymbol{R}_m|+|\boldsymbol{E}_m|+|\boldsymbol{F}_m|+|\boldsymbol{U}_m|,$$

which implies that $\tilde{R}+\tilde{E}+\tilde{F}+\tilde{U}=0$, where $\tilde{R}=|R_p|-|R_m|\le 0$, $\tilde{E}=|E_p|-|E_m|\ge 0$, $\tilde{F}=|F_p|-|F_m|\ge 0$, $\tilde{U}=|U_p|-|U_m|\ge 0$. Then

$$\begin{aligned}
&\tilde{L}_m x-\lambda x\\
&=\langle 2\boldsymbol{T}_m-\alpha\boldsymbol{E}_m-\beta\boldsymbol{F}_m\rangle^{-1}\left(|2\boldsymbol{U}_m+\beta\boldsymbol{E}_m+\alpha\boldsymbol{F}_m|x-\lambda\langle 2\boldsymbol{T}_m-\alpha\boldsymbol{E}_m-\beta\boldsymbol{F}_m\rangle x\right)\\
&=\langle\boldsymbol{M}_m\rangle^{-1}\left[2(|\boldsymbol{U}_p|-\tilde{U})+(\beta+\lambda\alpha)(|\boldsymbol{E}_p|-\tilde{E})+(\alpha+\lambda\beta)(|\boldsymbol{F}_p|-\tilde{F})-2\lambda(\langle\boldsymbol{D}\rangle-(|\boldsymbol{R}_p|-\tilde{R}))\right]x\\
&=\langle\boldsymbol{M}_m\rangle^{-1}\left\{|\boldsymbol{N}_p|-\lambda\langle\boldsymbol{M}_p\rangle-\left[2\tilde{U}+(\beta+\lambda\alpha)\tilde{E}+(\alpha+\lambda\beta)\tilde{F}+2\lambda\tilde{R}\right]\right\}x\\
&=\langle\boldsymbol{M}_m\rangle^{-1}\left[2(\tilde{R}+\tilde{E}+\tilde{F})-(\beta+\lambda\alpha)\tilde{E}-(\alpha+\lambda\beta)\tilde{F}-2\lambda\tilde{R}\right]x\\
&=\langle\boldsymbol{M}_m\rangle^{-1}\left[(1-\lambda)(2\tilde{R}+\alpha\tilde{E}+2\tilde{F}-\alpha\tilde{F})\right]x\le 0,
\end{aligned}$$

that is, $\tilde{L}_m x\le\lambda x$. By Theorem 2.13, $\rho(\tilde{L}_m)\le\lambda=\rho(\tilde{L}_p)<1$. □

The next example illustrates the result of Theorem 3.6.

Example 3.7. Consider the example of an interval M-matrix

$$\boldsymbol{A}=\begin{pmatrix}3&[-1,0]&[-1,0]&[-1,0]&[-1,0]\\ [-1,0]&4&[-1,0]&[-1,0]&[-1,0]\\ [-1,0]&[-1,0]&5&[-1,0]&[-1,0]\\ [-1,0]&[-1,0]&[-1,0]&6&[-1,0]\\ [-1,0]&[-1,0]&[-1,0]&[-1,0]&7\end{pmatrix}.$$

If $\alpha=1.3,\beta=0.7$, then $\rho(\tilde{L})=0.6976$. For $p=1,m=2$, we get $\rho(\tilde{L}_1)=0.6417<1$ and $\rho(\tilde{L}_2)=0.5547<1$, that is, $\rho(\tilde{L}_2)<\rho(\tilde{L}_1)<\rho(\tilde{L})<1$, which shows that the above results hold.

The following Theorem 3.8 compares the spectral radius of the iterative matrices between the GITOR and ITOR methods.

Theorem 3.8. *Let $\boldsymbol{A}\in\mathbb{IR}^{n\times n}$ be an interval M-matrix, and $m\ge 1$. If $0<\alpha,\beta\le 1$, $\alpha+\beta\le 1$, then $\rho(\tilde{L})\le\rho(\tilde{L}_m)$.*

Proof. Let $\tilde{\boldsymbol{L}}=\langle\boldsymbol{M}\rangle^{-1}|\boldsymbol{N}|$, where $\boldsymbol{M}:=2\boldsymbol{D}-\alpha\boldsymbol{E}-\beta\boldsymbol{F}$ and $\boldsymbol{N}:=(2-\alpha-\beta)\boldsymbol{D}+(\alpha+\beta)\boldsymbol{U}+\alpha\boldsymbol{F}+\beta\boldsymbol{E}$, $\boldsymbol{T}_m=\boldsymbol{D}+\boldsymbol{R}_m$, $\boldsymbol{R}_m=\boldsymbol{R}_m^E+\boldsymbol{R}_m^F+\boldsymbol{R}_m^U$.

Since $\boldsymbol{A}$ is an interval M-matrix, we have that $\langle\boldsymbol{M}\rangle$ is an M-matrix, and $\tilde{\boldsymbol{L}}$ is a non-negative matrix. By Theorem 2.17 we have that $\exists x>0$ associated with the eigenvalue $\lambda=\rho(\tilde{\boldsymbol{L}})$, such that $\tilde{\boldsymbol{L}}x=\lambda x$, that is, $\lambda\langle\boldsymbol{M}\rangle x=|\boldsymbol{N}|x$.

Since $\langle\boldsymbol{T}_m\rangle=\langle\boldsymbol{D}\rangle-|\boldsymbol{R}_m|$, $|\boldsymbol{T}|=|D|+|\boldsymbol{R}_m^E|+|\boldsymbol{R}_m^F+|\boldsymbol{R}_m^U|$, $|\boldsymbol{E}|=|\boldsymbol{E}_m|+|\boldsymbol{R}_m^E|$, $|\boldsymbol{F}|=|\boldsymbol{F}_m|+|\boldsymbol{R}_m^F|$, $|\boldsymbol{U}|=|\boldsymbol{U}_m|+|\boldsymbol{R}_m^U|$, we have

$$\begin{aligned}
\langle\boldsymbol{M}_m\rangle&=2\langle\boldsymbol{T}_m\rangle-|\alpha||\boldsymbol{E}_m|-|\beta||\boldsymbol{F}_m|\\
&=\langle\boldsymbol{M}\rangle-2|\boldsymbol{R}_m^U|-(2-\alpha)|\boldsymbol{R}_m^E|-(2-\beta)|\boldsymbol{R}_m^F|.
\end{aligned}$$

Hence $\langle\boldsymbol{M}_m\rangle\le\langle\boldsymbol{M}\rangle$, that is, $\langle\boldsymbol{M}\rangle^{-1}\le\langle\boldsymbol{M}_m\rangle^{-1}$. Then

$$\begin{aligned}
|\boldsymbol{N}_m|&=|2-\alpha-\beta||\boldsymbol{T}_m|+|\alpha+\beta||\boldsymbol{U}_m|+|\alpha||\boldsymbol{F}_m|+|\beta||\boldsymbol{E}_m|\\
&=|\boldsymbol{N}|+(2-2\alpha-2\beta)|\boldsymbol{R}_m^U|+(2-\alpha-2\beta)|\boldsymbol{R}_m^E|+(2-2\alpha-\beta)|\boldsymbol{R}_m^F|,
\end{aligned}$$

that is, $|\boldsymbol{N}|\le|\boldsymbol{N}_m|$. Hence, $\langle\boldsymbol{M}\rangle^{-1}|\boldsymbol{N}|\le\langle\boldsymbol{M}_m\rangle^{-1}|\boldsymbol{N}_m|$, and by Theorem 2.12 we have that $\rho(\tilde{L})\le\rho(\tilde{L}_m)$. □

Example 3.9. Consider the example of the interval M-matrix mentioned in Example 3.7. If $\alpha = 0.5, \beta = 0.4$, then $\rho(\tilde{L}) = 0.9107$. For $p = 1, m = 2$, we get $\rho(\tilde{L}_1) = 1.5710$ and $\rho(\tilde{L}_2) = 2.7891$, that is, $\rho(\tilde{L}) < \rho(\tilde{L}_1), \rho(\tilde{L}) < \rho(\tilde{L}_2)$, which shows that the above results hold.

Finally, we analyze the convergence of the proposed method for an interval H-matrix.

Theorem 3.10. *Let $\boldsymbol{A}$ be an interval H-matrix. If $\alpha, \beta \geq 0$, and $0 < \alpha + \beta \leq 2$, then the GITOR method for solving (1.1) converges for any initial approximation.*

Proof. Since $\boldsymbol{A}$ is an interval H-matrix, we have that $\langle \boldsymbol{A} \rangle$ is an M-matrix, and $\langle \boldsymbol{M}_m \rangle$ is also an M-matrix. By Definition 2.8, $\langle \boldsymbol{M}_m \rangle$ is an H-matrix. According to Theorem 2.9, we have $|\boldsymbol{M}_m^{-1}| \leq \langle \boldsymbol{M}_m \rangle^{-1}$. Consequently, $\langle \boldsymbol{M}_m \rangle - |\boldsymbol{N}_m|$ is a regular splitting. By Theorem 2.14, it follows that $\rho(\langle \boldsymbol{M}_m \rangle^{-1} |\boldsymbol{N}_m|) < 1$. Therefore, $\rho(|\boldsymbol{M}_m^{-1}||\boldsymbol{N}_m|) \leq \rho(\langle \boldsymbol{M}_m \rangle^{-1} |\boldsymbol{N}_m|) < 1$. □

4. Numerical illustration

In this section, numerical examples are provided to compare the convergence of the GITOR method with the GIAOR, GISOR, GIGS, and GIJ methods for various types of interval matrices. Specifically, numerical examples for interval SDD-matrices, interval M-matrices, and interval H-matrices will be considered. All computations were performed on a computer equipped with an Intel(R) Core(TM) i5-10210U CPU @ 1.60GHz 2.11 GHz processor and 16GB of RAM, and were implemented and tested using MATLAB (2022b) with the interval toolbox INTLAB v12 [23]. The result intervals shown below are accurate to four rounded digits, and the stopping criterion is chosen as $\|qdist(\boldsymbol{x}^{(k+1)}, \boldsymbol{x}^{(k)})\| \leq 10^{-6}$, where $qdist(\boldsymbol{x}, \boldsymbol{y}) := \max\{|\underline{x} - \underline{y}|, |\overline{x} - \overline{y}|\}$ denotes the distance metric between the intervals $\boldsymbol{x} = [\underline{x}, \overline{x}]$ and $\boldsymbol{y} = [\underline{y}, \overline{y}]$. In the case of $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{IR}^n$, then $qdist(\boldsymbol{x}, \boldsymbol{y}) = [qdist(\boldsymbol{x}_1, \boldsymbol{y}_1), \ldots, qdist(\boldsymbol{x}_n, \boldsymbol{y}_n)] \in \mathbb{R}^n$, where $\boldsymbol{x}_i$ represents the i-th entry of the interval vector $\boldsymbol{x}$.

We next give examples of various types of interval coefficient matrices and compare our proposed method with the GIAOR, GISOR, GIGS, and GIJ methods in terms of the number of iterations, time (in seconds), and $r_k = \|qdist(\boldsymbol{x}^{(k+1)}, \boldsymbol{x})\|$, where $\boldsymbol{x}$ is the enclosure obtained using verifylss.

Example 4.1. Consider the interval linear system (1.1) for which the coefficient interval matrix is an interval SDD-matrix with a constant diagonal, as in

$$\boldsymbol{A} = \begin{pmatrix} 4 & [-1,1] & 0 & [-1,1] & [-1,1] \\ [-1,1] & 4 & [-1,1] & [-1,1] & 0 \\ [-1,1] & 0 & 4 & [-1,1] & [-1,1] \\ [-1,1] & [-1,1] & 0 & 4 & [-1,1] \\ [-1,1] & [-1,1] & [-1,1] & 0 & 4 \end{pmatrix} \text{ and } \boldsymbol{b} = \begin{pmatrix} [-2,-1] \\ [-2,-1] \\ [1,2] \\ [4,5] \\ [3,4] \end{pmatrix}.$$

It can be seen that $\boldsymbol{A}$ is not only an interval SDD-matrix but also an interval H-matrix. Then the function verifylss from the package INTLAB yields the enclosure

$$\boldsymbol{x} = \begin{pmatrix} [-2.8291, 2.0791] \\ [-2.7752, 2.0252] \\ [-2.0925, 2.8425] \\ [-1.1791, 3.4291] \\ [-1.3618, 3.1118] \end{pmatrix}.$$

Take the initial guess $\boldsymbol{x}_0 = ([-4,4],[-4,4],[-4,4],[-4,4],[-4,4])^T$. For $m = 3$, the GITOR method converges after 9 iterations, while the GIAOR, GISOR, GIGS, and GIJ methods converge after 10, 9, 9, and 14 iterations, respectively. The solution set enclosures of the above methods are as follows in Table 1:

Table 1. Enclosure of the solution set for the interval SDD-matrix with $m = 3$.

GITOR ($\alpha = 1.9, \beta = 0.1$)	GIAOR ($\omega = 1, \sigma = 0.9$)	GISOR ($\omega = 0.9$)
$\boldsymbol{x} = \begin{pmatrix} [-2.8290, 2.0790] \\ [-2.7751, 2.0251] \\ [-2.0924, 2.8424] \\ [-1.1790, 3.4290] \\ [-1.3616, 3.1116] \end{pmatrix}$	$\boldsymbol{x} = \begin{pmatrix} [-2.8290, 2.0790] \\ [-2.7751, 2.0251] \\ [-2.0924, 2.8424] \\ [-1.1790, 3.4290] \\ [-1.3617, 3.1117] \end{pmatrix}$	$\boldsymbol{x} = \begin{pmatrix} [-2.8291, 2.0791] \\ [-2.7753, 2.0253] \\ [-2.0926, 2.8426] \\ [-1.1791, 3.4291] \\ [-1.3618, 3.1118] \end{pmatrix}$
GIGS	GIJ	
$\boldsymbol{x} = \begin{pmatrix} [-2.8290, 2.0790] \\ [-2.7751, 2.0251] \\ [-2.0924, 2.8424] \\ [-1.1790, 3.4290] \\ [-1.3617, 3.1117] \end{pmatrix}$	$\boldsymbol{x} = \begin{pmatrix} [-2.8290, 2.0790] \\ [-2.7752, 2.0252] \\ [-2.0925, 2.8425] \\ [-1.1790, 3.4290] \\ [-1.3617, 3.1117] \end{pmatrix}$	

From Table 1, it can be seen that in the case of the SDD-interval linear system, the enclosure of the solution set obtained by the GITOR method is the tightest under the bandwidth of $m = 3$. Table 2 compares the GITOR, GIAOR, GISOR, GIGS, and GIJ methods for different bandwidths (taking $m = 0, 1, 2, 3$) with an initial guess of $\boldsymbol{x}_0 = ([-4,4],[-4,4],[-4,4],[-4,4],[-4,4])^T$.

Table 2. Performance comparison of iterative methods for the interval SDD-matrix with $m = 0, 1, 2, 3$.

m	Iterative method	No. of iterations	r_k	Time in seconds
0	ITOR	28	3.6766×10^{-4}	0.0897
	IAOR	30	3.6811×10^{-4}	0.0783
	ISOR	27	3.6787×10^{-4}	0.0654
	IGS	27	3.6787×10^{-4}	0.0555
	IJ	47	3.6693×10^{-4}	0.3794
1	GITOR	17	3.3103×10^{-4}	0.0561
	GIAOR	18	3.3789×10^{-4}	0.0597
	GISOR	16	1.9362×10^{-4}	0.0640
	GIGS	16	3.2242×10^{-4}	0.0427
	GIJ	34	3.6420×10^{-4}	0.0741
2	GITOR	13	2.0163×10^{-4}	0.0456
	GIAOR	13	1.9413×10^{-4}	0.0443
	GISOR	12	8.6896×10^{-5}	0.0419
	GIGS	12	2.1368×10^{-4}	0.0324
	GIJ	23	3.7416×10^{-5}	0.0535
3	GITOR	9	2.2981×10^{-4}	0.0343
	GIAOR	10	2.0759×10^{-4}	0.0410
	GISOR	9	1.7419×10^{-4}	0.0329
	GIGS	9	1.8889×10^{-4}	0.0292
	GIJ	14	1.4363×10^{-4}	0.0603

From Table 2, it can be seen that for the interval SDD-matrix, the number of iterations, r_k, and time (in seconds) of the GITOR method decreases as the bandwidth m increases.

We now describe the change in the residual r_k with respect to the iteration for the SDD-interval linear system. Figure 1 plots the residual against the iteration for the interval SDD-matrix with bandwidths $m = 0, 1, 2, 3$, respectively.

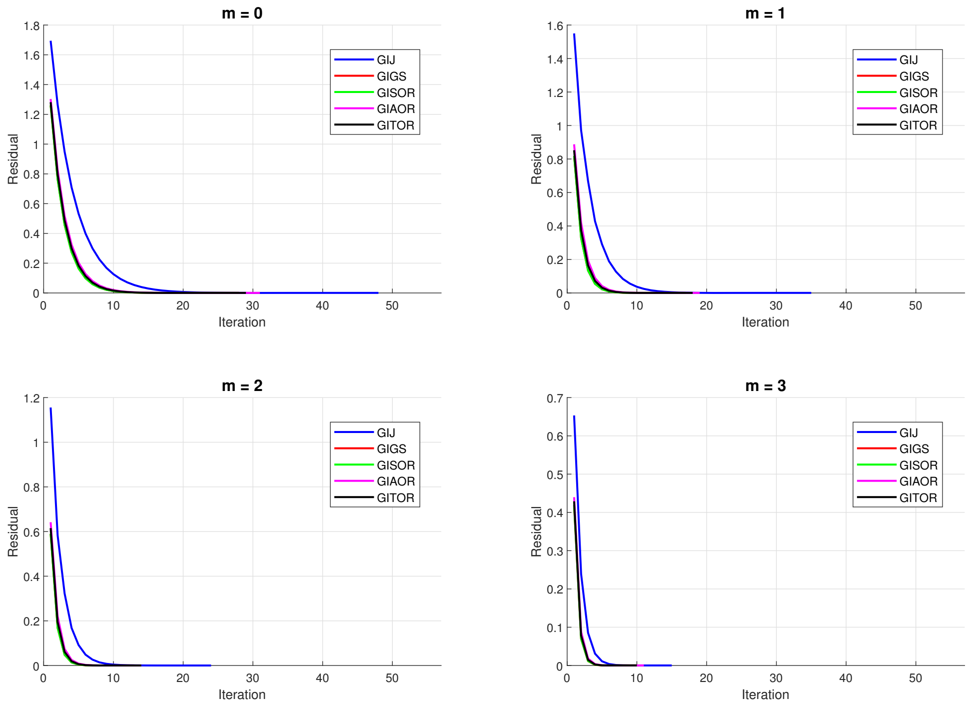

Figure 1. Residual vs. iteration plot for the interval SDD-matrix with $m = 0, 1, 2, 3$.

Figure 1 shows that the rate of residual convergence accelerates significantly as the matrix bandwidth increases. Specifically, the convergence performance of the GITOR iterative method is significantly better than that of the ITOR method for this strictly diagonally dominated (SDD) interval linear system. Among the five generalized iterative methods (GITOR, GIAOR, GISOR, GIGS, GIJ) compared, the GITOR method has the best residual descent properties.

Example 4.2. Consider the interval linear system (1.1) for which the coefficient interval matrix is an interval M-matrix, as in

$$\boldsymbol{A} = \begin{pmatrix} 3 & [-1,0] & [-1,0] & [-1,0] & [-1,0] \\ [-1,0] & 4 & [-1,0] & [-1,0] & [-1,0] \\ [-1,0] & [-1,0] & 5 & [-1,0] & [-1,0] \\ [-1,0] & [-1,0] & [-1,0] & 6 & [-1,0] \\ [-1,0] & [-1,0] & [-1,0] & [-1,0] & 7 \end{pmatrix} \text{ and } \boldsymbol{b} = \begin{pmatrix} [1,2] \\ [1,2] \\ [1,2] \\ [-1,1] \\ [1,2] \end{pmatrix}.$$

Then the function verifylss from the package INTLAB yields the enclosure

$$\boldsymbol{x} = ([-2.4237, 4.0212], [-1.9746, 3.2171], [-1.4825, 2.6810], [-1.7564, 2.1551], [-1.2652, 2.0107])^T.$$

Take the initial guess $\boldsymbol{x}_0 = ([-3,5], [-3,5], [-3,5], [-3,5], [-3,5])^T$. For $m = 3$, the GITOR method converges after 11 iterations, while the GIAOR, GISOR, GIGS, and GIJ methods converge after 12, 11, 11, and 18 iterations, respectively. The solution set enclosures of the above methods are as follows in Table 3:

Table 3. Enclosure of the solution set for the interval M-matrix with $m = 3$.

GITOR ($\alpha = 1.9, \beta = 0.1$)	GIAOR ($\omega = 1, \sigma = 0.9$)	GISOR ($\omega = 0.9$)
$\boldsymbol{x} = \begin{pmatrix} [-2.4232, 4.0208] \\ [-1.9742, 3.2167] \\ [-1.4821, 2.6805] \\ [-1.7561, 2.1547] \\ [-1.2649, 2.0104] \end{pmatrix}$	$\boldsymbol{x} = \begin{pmatrix} [-2.4232, 4.0208] \\ [-1.9742, 3.2167] \\ [-1.4821, 2.6805] \\ [-1.7561, 2.1547] \\ [-1.2649, 2.0104] \end{pmatrix}$	$\boldsymbol{x} = \begin{pmatrix} [-2.4236, 4.0211] \\ [-1.9745, 3.2170] \\ [-1.4824, 2.6808] \\ [-1.7563, 2.1550] \\ [-1.2651, 2.0106] \end{pmatrix}$
GIGS	GIJ	
$\boldsymbol{x} = \begin{pmatrix} [-2.4232, 4.0208] \\ [-1.9742, 3.2167] \\ [-1.4821, 2.6805] \\ [-1.7561, 2.1547] \\ [-1.2649, 2.0104] \end{pmatrix}$	$\boldsymbol{x} = \begin{pmatrix} [-2.4232, 4.0207] \\ [-1.9741, 3.2166] \\ [-1.4821, 2.6805] \\ [-1.7561, 2.1547] \\ [-1.2649, 2.0104] \end{pmatrix}$	

In this example of an M-linear system, we can see that the GIJ method produces the most compact set of solutions for $m = 3$. Table 4 compares the GITOR, GIAOR, GISOR, GIGS, and GIJ methods for different bandwidths (taking $m = 0, 1, 2, 3$) with an initial guess of $\boldsymbol{x}_0 = ([-3, 5], [-3, 5], [-3, 5], [-3, 5], [-3, 5])^T$.

Table 4. Performance comparison of iterative methods for the interval M-matrix with $m = 0, 1, 2, 3$.

m	Iterative method	No. of iterations	r_k	Time in seconds
0	ITOR	51	1.1×10^{-3}	0.1281
	IAOR	53	1.1×10^{-3}	0.1203
	ISOR	49	1.1×10^{-3}	0.1122
	IGS	49	1.1×10^{-3}	0.0814
	IJ	89	1.1×10^{-3}	0.1508
1	GITOR	33	1.1×10^{-3}	0.0884
	GIAOR	35	1.1×10^{-3}	0.0788
	GISOR	32	7.9904×10^{-4}	0.0703
	GIGS	32	1.1×10^{-3}	0.0604
	GIJ	58	1.1×10^{-3}	0.1343
2	GITOR	21	1.1×10^{-3}	0.0521
	GIAOR	22	1.1×10^{-3}	0.0564
	GISOR	20	5.7872×10^{-4}	0.0516
	GIGS	20	1.1×10^{-3}	0.0497
	GIJ	34	1.0×10^{-3}	0.0643
3	GITOR	11	8.9016×10^{-4}	0.0366
	GIAOR	12	8.9073×10^{-4}	0.0356
	GISOR	11	2.7698×10^{-4}	0.0330
	GIGS	11	8.8996×10^{-4}	0.0267
	GIJ	18	1.0×10^{-3}	0.0353

From Table 4, it can be seen that for the interval M-matrix, the number of iterations, r_k, and time (in seconds) of the GITOR method decreases as the bandwidth m increases.

We now describe the change in the residual r_k with respect to the iteration for the M-interval linear system. Figure 2 plots the residual against the iteration for the interval M-matrix with bandwidths $m = 0, 1, 2, 3$, respectively.

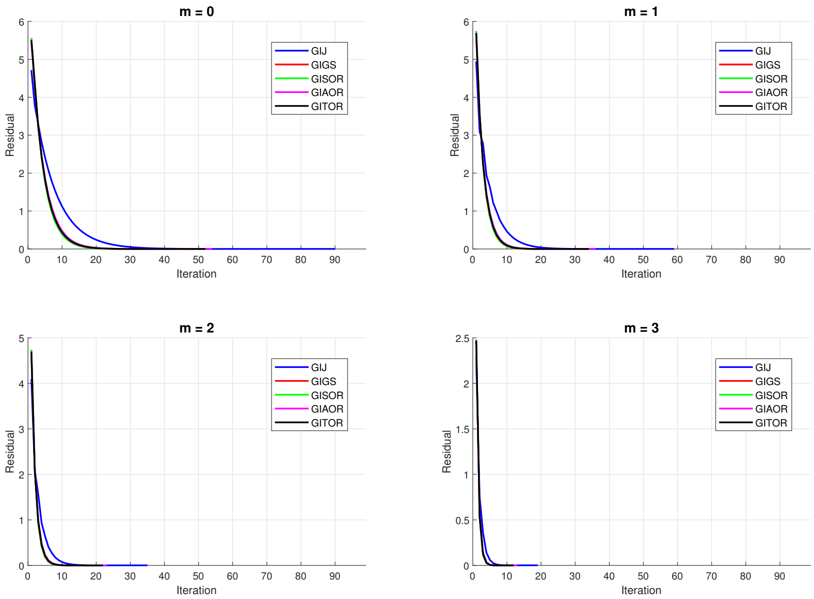

Figure 2. Residual vs. iteration plot for the interval M-matrix with $m = 0, 1, 2, 3$.

Figure 2 shows that the rate of residual convergence accelerates dramatically as the matrix bandwidth increases. Specifically, for this M-interval linear system, the GITOR iterative method outperforms the ITOR method in terms of convergence. Among the five generalized iterative methods analyzed (GITOR, GIAOR, GISOR, GIGS, and GIJ), the GITOR method has better residual descent features.

Example 4.3. Consider the interval linear system (1.1) for which the coefficient interval matrix is an interval H-matrix, as in

$$\boldsymbol{A} = \begin{pmatrix} [4,5] & [0,0] & [-1,0] & [-1,0] \\ [0,1] & [3,5] & [-1,1] & [-1,1] \\ [-1,1] & [0,0] & [4,5] & [-4,4] \\ [0,1] & [0,1] & [0,0] & [5,5] \end{pmatrix} \text{ and } \boldsymbol{b} = \begin{pmatrix} [-1,0.5] \\ [0.2,0.3] \\ [0.1,0.2] \\ [-0.3,-0.2] \end{pmatrix}.$$

It can be seen that $\boldsymbol{A}$ is an interval H-matrix, but not an interval SDD-matrix. Then the function verifylss from the package INTLAB yields the enclosure

$$\boldsymbol{x} = \begin{pmatrix} [-0.3662, 0.2511] \\ [-0.2573, 0.3967] \\ [-0.2660, 0.3327] \\ [-0.1912, 0.0887] \end{pmatrix}.$$

Take the initial guess $\boldsymbol{x}_0 = ([-3,5],[-3,5],[-3,5],[-3,5])^T$. For $m = 2$, the GITOR method converges after 7 iterations, while the GIAOR, GISOR, GIGS, and GIJ methods converge after 8, 7, 7, and 12 iterations, respectively. The solution set enclosures of the above methods are as follows in Table 5:

Table 5. Enclosure of the solution set for the interval H-matrix with $m = 2$.

GITOR ($\alpha = 1.9, \beta = 0.1$)	GIAOR ($\omega = 1, \sigma = 0.9$)	GISOR ($\omega = 0.9$)
$\boldsymbol{x} = \begin{pmatrix} [-0.3829, 0.2678] \\ [-0.2801, 0.4195] \\ [-0.2927, 0.3593] \\ [-0.2017, 0.0992] \end{pmatrix}$	$\boldsymbol{x} = \begin{pmatrix} [-0.3826, 0.2675] \\ [-0.2795, 0.4189] \\ [-0.2918, 0.3584] \\ [-0.2014, 0.0990] \end{pmatrix}$	$\boldsymbol{x} = \begin{pmatrix} [-0.3832, 0.2681] \\ [-0.2808, 0.4202] \\ [-0.2936, 0.3602] \\ [-0.2019, 0.0995] \end{pmatrix}$
GIGS	GIJ	
$\boldsymbol{x} = \begin{pmatrix} [-0.3831, 0.2681] \\ [-0.2808, 0.4202] \\ [-0.2936, 0.3602] \\ [-0.2019, 0.0995] \end{pmatrix}$	$\boldsymbol{x} = \begin{pmatrix} [-0.3842, 0.2691] \\ [-0.2790, 0.4184] \\ [-0.2904, 0.3571] \\ [-0.2091, 0.1067] \end{pmatrix}$	

In this example of an H-linear system, we can see that the GIAOR method produces the most compact set of solutions for $m = 2$. Table 6 compares the GITOR, GIAOR, GISOR, GIGS and GIJ methods for different bandwidths (taking $m = 0, 1, 2, 3$) with an initial guess of $\boldsymbol{x}_0 = ([-3, 5], [-3, 5], [-3, 5], [-3, 5])^T$.

Table 6. Performance comparison of iterative methods for the interval H-matrix with $m = 0, 1, 2$.

m	Iterative method	No. of iterations	r_k	Time in seconds
0	ITOR	20	1.76×10^{-2}	0.0543
	IAOR	21	1.94×10^{-2}	0.0535
	ISOR	19	1.58×10^{-2}	0.0476
	IGS	19	1.58×10^{-2}	0.0359
	IJ	31	3.55×10^{-2}	0.0751
1	GITOR	12	2.96×10^{-2}	0.0375
	GIAOR	13	2.99×10^{-2}	0.0332
	GISOR	11	2.94×10^{-2}	0.0349
	GIGS	11	2.94×10^{-2}	0.0323
	GIJ	22	3.55×10^{-2}	0.0420
2	GITOR	7	4.02×10^{-2}	0.0226
	GIAOR	8	3.91×10^{-2}	0.0272
	GISOR	7	4.14×10^{-2}	0.0321
	GIGS	7	4.14×10^{-2}	0.0244
	GIJ	12	4.14×10^{-2}	0.0604

From Table 6, it can be seen that for the interval H-matrix, the number of iterations, r_k and time (in second) of GITOR decreases as the bandwidth m increases.

We now describe the change in the residual r_k with respect to the iteration for the H-interval linear system. Figure 3 plots the residual against the iteration for the interval H-matrix with bandwidths

$m = 0, 1, 2$, respectively.

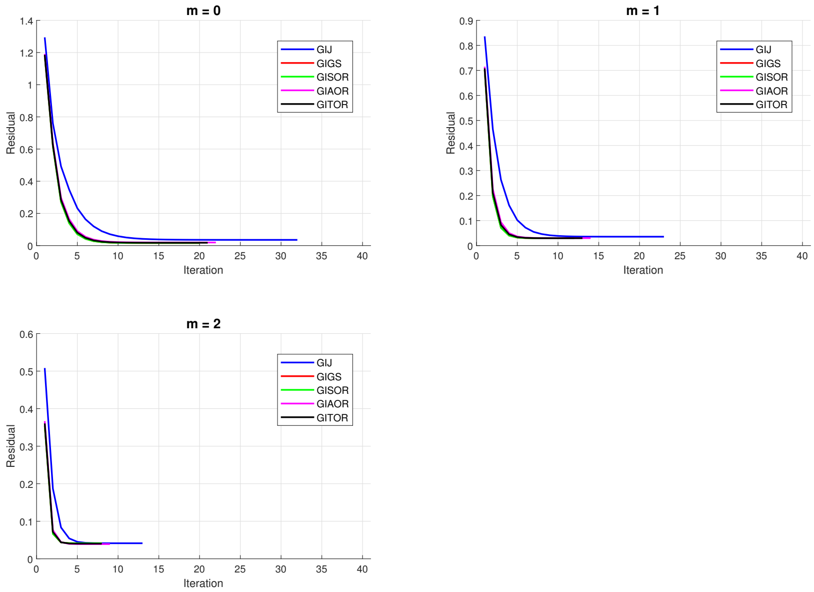

Figure 3. Residual vs. iteration plot for the interval H-matrix with $m = 0, 1, 2$.

From Figure 3, the residuals converge significantly faster as the matrix bandwidth increases. Specifically, for this H-linear system, the convergence of the GITOR method is superior to the ITOR method. Among the five generalized iterations analyzed (GITOR, GIAOR, GISOR, GIGS, and GIJ), the GITOR method has a superior rate of residual decline.

5. Conclusions

In this paper, we introduce the GITOR method, which is a generalization of the ITOR method. We discuss the convergence properties of the GITOR method for interval strictly diagonally dominant (SDD) matrices, interval M-matrices, and interval H-matrices by limiting the possible values of the parameters α and β. Furthermore, for an interval SDD-matrix, we provide upper bounds on the spectral radius of the iterative matrices. Moreover, we find a relationship not only between the bandwidth and the spectral radius of the iterative matrix, but also a link between the spectral radius of the iterative matrices of the GITOR and ITOR methods. We provide numerical examples of different kinds of interval matrices to compare the proposed method with some classical interval iterative methods. The GITOR method converges for interval SDD-matrices, M-matrices, and H-matrices with constant diagonals, as demonstrated by the numerical examples. For the different matrices above, the convergence speed of the GITOR method is significantly better than that of the ITOR method. In addition, through a comprehensive comparison of five generalized iterative methods (GITOR, GIAOR, GISOR, GIGS, and GIJ), the GITOR method always has the best residual reduction properties.

Author contributions

Lingjian Pu: Writing—original draft, methodology, conceptualization, validation, investigation, writing—review and editing; Yan Zhu and Shiliang Wu: Writing—review and editing, supervision, validation, methodology. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. G. Alefeld, J. Herzberger, *Introduction to interval computation*, Academic Press, 1983. https://doi.org/10.1016/C2009-0-21898-8
2. D. Wang, T. Li, P. Deng, Z. Luo, P. Zhang, K. Liu, et al., DNSRF: Deep network-based semi-NMF representation framework, *ACM Trans. Intell. Syst. Technol.*, **15** (2024), 1–20. http://dx.doi.org/10.1145/3670408
3. A. Neumaier, *Interval methods for systems of equations*, Cambridge University Press, 1991. https://doi.org/10.1017/CBO9780511526473
4. R. S. Varga, *Matrix iterative analysis*, Berlin, Heidelberg: Springer, 2000. https://doi.org/10.1007/978-3-642-05156-2
5. Q. R. Lian, Z. Y. Jin, K. Nickel, The SOR method for solving linear interval equations: A method for solving the eigenproblem of symmetric interval matrix, *Inst. für Angew. Math.*, **87** (1987), 17–18.
6. M. Fiedler, J. Nedoma, J. Ramík, J. Rohn, K. Zimmermann, *Linear optimization problems with inexact data*, New York: Springer, 2006. https://doi.org/10.1007/0-387-32698-7
7. J. Chakravarty, A. Athikho, M. Saha, Convergence of interval AOR method for linear interval equations, *Numer. Algebra Control Optim.*, **12** (2022), 293–308. http://dx.doi.org/10.3934/naco.2021006

8. A. Neumaier, A simple derivation of the Hansen-Bliek-Rohn-Ning-Kearfott enclosure for linear interval equations, *Reliab. Comput.*, **5** (1999), 131–136. http://dx.doi.org/10.1023/A:1009997221089
9. S. Ning, R. B. Kearfott, A comparison of some methods for solving linear interval equations, *SIAM J. Numer. Anal.*, **34** (1997), 1289–1305. http://dx.doi.org/10.1137/s0036142994270995
10. D. W. Chang, Convergence analysis of the parallel multisplitting TOR method, *J. Comput. Appl. Math.*, **72** (1996), 169–177. http://dx.doi.org/10.1016/0377-0427(95)00270-7
11. D. W. Chang, The parallel multisplitting TOR (MTOR) method for linear systems, *Comput. Math. Appl.*, **41** (2001), 215–227. http://dx.doi.org/10.1016/s0898-1221(01)85017-3
12. J. Kuang, J. Ji, A survey of AOR and TOR methods, *J. Comput. Appl. Math.*, **24** (1988), 3–12. http://dx.doi.org/10.1016/0377-0427(88)90340-8
13. L. T. Zhang, T. Z. Huang, S. H. Cheng, T. X. Gu, Y. P. Wang, A note on parallel multisplitting TOR method for H-matrices, *Int. J. Comput. Math.*, **88** (2011), 501–507. http://dx.doi.org/10.1080/00207160903501917
14. M. Hladik, I. Skalna, Relations between various methods for solving linear interval and parametric equations, *Linear Algebra Appl.*, **574** (2019), 1–21. http://dx.doi.org/10.1016/j.laa.2019.03.019
15. M. Hladik, New operator and method for solving real preconditioned interval linear equations, *SIAM J. Numer. Anal.*, **52** (2014), 194–206. http://dx.doi.org/10.1137/130914358
16. A. Neumaier, Further results on linear interval equations, *Linear Algebra Appl.*, **87** (1987), 155–179. http://dx.doi.org/10.1016/0024-3795(87)90164-9
17. J. Chakravarty, M. Saha, Generalized interval AOR method for solving interval linear equations, *Math. Found. Comput.*, **8** (2025), 16–35. http://dx.doi.org/10.3934/mfc.2023035
18. M. Saha, J. Chakravarty, Convergence of generalized SOR, Jacobi and Gauss–Seidel methods for linear systems, *Int. J. Appl. Comput. Math.*, **6** (2020), 77. http://dx.doi.org/10.1007/s40819-020-00830-5
19. A. Neumaier, New techniques for the analysis of linear interval equations, *Linear Algebra Appl.*, **58** (1984), 273–325. http://dx.doi.org/10.1016/0024-3795(84)90217-9
20. J. Kuang, On the two-parameter overrelaxation method for numerical solution of large linear systems, *J. Shanghai Norm. Univ.*, **4** (1983), 1–11.
21. W P. Zeng, On the convergence of the TOR iterative method, *Numer. Math. J. Chinese Univ.*, **8** (1986), 65–71.
22. M. M. Martins, D. J. Evans, M. E. Trigo, Convergence of the interval and point TOR method, *Int. J. Comput. Math.*, **80** (2003), 1227–1241. http://dx.doi.org/10.1080/0020716031000162417 9
23. S. M. Rump, INTLAB-INTerval LABoratory, In: *Developments in reliable computing*, Dordrecht: Springer, 1999, 77–104. https://doi.org/10.1007/978-94-017-1247-7_7