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Research article

Almost periodic dynamics of fractional-order stochastic Hopfield neural networks with time-varying delays

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Abstract: Although fractional-order stochastic neural networks exhibit rich dynamics, there are currently no research results on their almost periodic dynamics in the sense of distribution. This paper investigated a class of fractional-order stochastic Hopfield neural networks with time-varying delays. By employing the Banach fixed point theorem and inequality techniques, we first established sufficient criteria for the existence and uniqueness of almost periodic solutions in distribution for the considered model. Subsequently, through the application of a generalized Gronwall inequality, the finite-time stability in the mean square sense of this unique almost periodic solution in distribution was rigorously demonstrated. The results obtained in this study are entirely novel. To validate the theoretical findings, a numerical example with explicit parameters was constructed for simulation verification, where computational results exhibit a high degree of consistency with theoretical derivations.

Keywords: fractional-order stochastic Hopfield neural networks; almost periodic solution in distribution; finite-time stability
Mathematics Subject Classification: 34K14, 34K37, 92B20

1. Introduction

Under the continuous advancement of modern science and engineering technology, neural networks have remained at the forefront of academic research as powerful tools for investigating complex system behaviors. Conventional integer-order neural network models face limitations in precisely characterizing the dynamic properties of systems with memory and hereditary features due to inherent constraints in their theoretical frameworks. The emergence of fractional-order neural networks, through the organic integration of fractional calculus and neural networks, has achieved substantial advancements beyond traditional models. The distinctive "infinite memory" property of fractional calculus enables neural networks to effectively capture historical information along the temporal dimension, thereby accurately describing system memory effects and hereditary characteristics [1–3].

Utilizing classical fractional derivative definitions such as Caputo and Riemann-Liouville, fractional-order neural networks significantly enhance the modeling capability for complex systems, offering superior solutions across diverse research domains.

From the perspective of dynamical characteristics, fractional-order neural networks exhibit rich and scientifically valuable behavioral patterns. The discovery of complex dynamic properties such as chaos [4], bifurcations [5, 6], Mittag-Leffler stability [7], and synchronization [8, 9] not only enriches nonlinear system theory but also opens new avenues for exploring intrinsic system mechanisms. In practical applications, fractional-order neural networks have demonstrated remarkable achievements in multiple critical domains. For system identification and control [10–12], they enable precise system modeling and efficient control implementation. In optimization and signal processing [13–15], they effectively enhance algorithmic performance and signal quality. Within communication and security fields [16–18], these networks play pivotal roles in ensuring reliable and secure information transmission.

Particularly, fractional-order Hopfield neural networks significantly enhance the modeling capability for complex dynamical behaviors such as bifurcation and chaos through the introduction of fractional-order derivatives. Stability analysis of such systems primarily relies on tools including fractional-order Lyapunov theory and Mittag-Leffler functions. Existing research has extensively investigated critical directions such as bifurcation characteristics analysis [19], stability [20–23], and almost periodic oscillation [24].

However, just like biological population systems [25–28], practical neural network systems–whether biological neural networks or engineering control systems–universally contain stochastic perturbations and time delays that cannot be neglected. Stochastic perturbations encompass factors such as environmental noise and parameter fluctuations, for example, the stochastic opening/closing of ion channels in biological neurons or electromagnetic interference in industrial systems. Neglecting these stochastic elements during modeling may lead to significant discrepancies between theoretical predictions and actual system dynamics. Time delays also exert profound influences; transmission delays in network control and synaptic signal propagation latencies may induce system oscillations or even chaotic behaviors. Without appropriate modeling, potential system risks cannot be effectively predicted.

Stochastic perturbation models facilitate stability analysis in noisy environments (such as the attractiveness of equilibrium points), finding critical applications in areas like financial fluctuation simulations and sensor noise processing. Understanding delay effects aids controller optimization, such as designing predictive compensators for robots to prevent control latency failures, thereby enhancing reliability in spacecraft attitude control systems.

Compared to idealized models neglecting perturbations and delays, models incorporating stochastic and delayed elements possess greater generality. Investigating their stochastic stability and delay-dependent characteristics constitutes a crucial step toward advancing neural network dynamics theory.

Despite the theoretical and practical significance of fractional-order stochastic neural network models with time delays, current research on their dynamical properties remains relatively limited, primarily focusing on stability and synchronization studies [29–33]. As an essential dynamic behavior of neural networks, almost periodic oscillations have not yet been investigated in the context of fractional-order stochastic neural networks. Also, finite-time stability addresses the gap in traditional stability theories for rapid-response scenarios by providing "time-quantifiable" convergence guarantees. Its theoretical significance lies in enhancing the dynamical analysis framework of fractional-order neural networks, while practical implications extend to improving control precision, communication security, hardware energy efficiency, and other critical domains.

Motivated by the above observations, this paper focuses on the following class of stochastic fractional-order Hopfield neural networks:

$$\begin{aligned}{}^{c}D_t^{\kappa}x_i(t) = &- a_i x_i(t) + \sum_{j=1}^{n} b_{ij}(t) f_j(x_j(t)) + \sum_{j=1}^{n} c_{ij}(t) g_j(x_j(t-\tau_{ij}(t))) + J_i(t) \\ &+ \sum_{j=1}^{n} d_{ij}(t) h_{ij}(x_j(t-\varsigma_{ij}(t))) \frac{dB_j(t)}{dt}, \quad \frac{1}{2} < \kappa \le 1, \quad t \ge t_0 \ge 0,\end{aligned} \tag{1.1}$$

where $i, j \in \{1, 2, \cdots, n\} := \mathcal{J}$, n is the number of units of neurons; $x_i(t) \in \mathbb{R}$ represents the state of the ith neuron at time t; t_0 is the initial time; ${}^{c}D_t^{\kappa}x_i(t)$ represents the Caputo derivative of order κ for $x_i(t)$; $a_i \in \mathbb{R}^+$ represents the self-feedback connection weight; $b_{ij}(t) \in \mathbb{R}$, $d_{ij}(t) \in \mathbb{R}$, and $c_{ij}(t) \in \mathbb{R}$ are the connection weights and delay connection weights from the ith neuron to the jth neuron at time t, respectively; $J_i(t)$ represents the external input on the ith unit at time t; f_j and $g_j : \mathbb{R} \to \mathbb{R}$ represent the activation functions; $\tau_{ij}(t) \ge 0$ and $\varsigma_{ij}(t) \ge 0$ represent the transmission delays; $B(t) = (B_1(t), B_2(t), \cdots, B_n(t))^T$ is the n-dimensional Brownian motion defined on a complete probability space; and $h_{ij} : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function.

This study aims to rigorously investigate the existence of almost periodic solutions in the distributional sense and finite-time stability properties, seeking to address this theoretical void. The findings are expected to provide novel perspectives and methodologies for fractional-order stochastic neural network dynamics, while also establishing a more robust theoretical foundation for complex system analysis and control.

The organizational structure of this paper is as follows: Section 2 presents essential definitions, mathematical preliminaries, and foundational lemmas. In Section 3, we rigorously establish the existence of almost periodic solutions in distribution for system (1.1) and prove their finite-time stochastic stability through analytical methods. Section 4 validates the theoretical findings via numerical simulations with concrete parameter configurations. The study concludes with a synthesis of key contributions and implications in Section 5.

2. Preliminaries

If $(\mathbb{V}, d)$ is a separable and complete metric space, then we let $\mathfrak{B}(\mathbb{V})$ be the σ-algebra of Borel sets of $\mathbb{V}$ and $\mathcal{P}(\mathbb{V})$ be the set of all probability measures on $\mathfrak{B}(\mathbb{V})$. We denote by $\mathbb{BC}(\mathbb{V})$ the space of all bounded and continuous functions $g : \mathbb{V} \to \mathbb{R}$ with $\|g\|_S := \sup_{x\in\mathbb{V}} |g(x)|$.

For $g \in \mathbb{BC}(\mathbb{V}), \mu, \nu \in \mathcal{P}(\mathbb{V})$, we define

$$\|g\|_L = \sup_{a\neq b} \frac{|\, g(a) - g(b)\, |}{d(a, b)}, \ \|g\|_B = \max\{\|g\|_S, \|g\|_L\}, \ d_{\mathcal{P}}(\mu, \nu) := \sup_{\|g\|_B\le 1} \left| \int_{\mathbb{V}} g d\mu - \int_{\mathbb{V}} g\nu \right|.$$

Then, metric $d_{\mathcal{P}}$ is a complete metric on $\mathfrak{B}(\mathbb{V})$.

For a random variable $X : (\Omega, \mathcal{F}, P) \to \mathbb{X}$, we define by $\mu(X) := P \circ X^{-1}$ its law and by $E(X)$ its expectation.

For $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, we denote $|x|_0 = \max_{i \in \mathcal{J}}\{|x_i|\}$. Let $\mathcal{L}^2(\Omega, \mathbb{R}^n)$ be the space of all $\mathbb{R}^n$-valued random variables X such that $E\left(|X|_0^2\right) = \int_\Omega |X|_0^2 dP < \infty$.

Definition 2.1. *[34] A continuous mapping $f : \mathbb{R} \to \mathbb{R}^n$ is called almost periodic if for every $\varepsilon > 0$, there is a corresponding positive number $\ell = \ell(\varepsilon)$ such that within every interval of length $\ell(\varepsilon)$, there exists at least one translation number $\tau = \tau(\varepsilon)$ satisfying*

$$\sup_{t \in \mathbb{R}} |f(t + \tau) - f(t)|_0 < \varepsilon.$$

The complete set of such almost periodic functions is denoted by $AP(\mathbb{R}, \mathbb{R}^n)$.

Definition 2.2. *A stochastic process $X : \mathbb{R} \to \mathcal{L}^2(\Omega, \mathbb{R}^n)$ is called $\mathcal{L}^2$-continuous, if for any $\varepsilon > 0$ and every $s \in \mathbb{R}$, there is a $\delta > 0$ such that for all $t \in \mathbb{R}$ with $|t - s| < \delta$, it holds*

$$E|X(t) - X(s)|_0^2 < \varepsilon.$$

It is called $\mathcal{L}^2$-bounded if $\sup_{t \in \mathbb{R}} E|X(t)|_0^2 < \infty$.

Definition 2.3. *[35] A stochastic process $X : \mathbb{R} \to \mathcal{L}^2(\Omega, \mathbb{R}^n)$ is said to be almost periodic in distribution if for every $\varepsilon > 0$, there exists a real number $\ell = \ell(\varepsilon)$, for each interval with length $\ell(\varepsilon)$, there is a number $\tau = \tau(\varepsilon)$ in this interval such that for all $t \in \mathbb{R}$, it holds that*

$$d_{\mathcal{P}}(\mu(X(t + \tau)), \mu(X(t))) < \varepsilon.$$

Definition 2.4. *[36] The α-order integral of the function $f \in \mathcal{L}^1([t_0, T], \mathbb{R})$ is defined as*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{f(s)}{(t - s)^{1-\alpha}} ds, \ \ t \in [t_0, T],$$

where $\alpha > 0$, $\Gamma(\alpha)$ denotes the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$, t_0 and T may take $-\infty$ and $+\infty$ as their values.

Definition 2.5. *[36] The Caputo derivative of order α for function $f \in C^{n-1}([t_0, T], \mathbb{R})$ and $f^{(n)} \in \mathcal{L}^1([t_0, T], \mathbb{R})$ is defined by*

$${}^c D_t^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \int_{t_0}^{t} \frac{f^{(n)}(s)}{(t - s)^{\alpha+1-n}} ds = I^{n-\alpha} f^{(n)}(t), \ \ t \in [t_0, T], \ \ 0 \le n - 1 < \alpha < n,$$

where n is a nature number, t_0 and T may take $-\infty$ and $+\infty$ as their values.

Definition 2.6. *[37] A solution $y(t)$ of system (1.1) with initial value χ is called mean-square finite-time stable with respect to parameters $\{\delta, \varepsilon, S\}$, where $\delta < \varepsilon$, $\delta, \varepsilon \in \mathbb{R}^+$, and $S > t_0 \ge 0$, if for any solution $x(t)$ with initial value φ satisfying*

$$E|\chi - \varphi|_1^2 < \delta \quad \text{with} \quad \mathbb{E}|\chi - \varphi|_1^2 := \sup_{s \in [t_0 - \zeta, t_0]} \mathbb{E}|\chi(s) - \varphi(s)|_0^2,$$

then it holds that

$$E|y(t) - x(t)|_0^2 < \varepsilon \quad \forall t \in [t_0, S].$$

Lemma 2.1. *[38] Let $g, a : [t_0, T] \to \mathbb{R}^+$ be two continuous functions and $a(t)$ be a nondecreasing function. If*

$$g(t) \le a(t) + b_1(t)\int_{t_0}^{t}(t-s)^{\beta_1 - 1}g(s)ds + b_2(t)\int_{t_0}^{t}(t-s)^{\beta_2 - 1}g(s)ds,$$

where b_1, b_2 are nonnegative and monotonic functions on $[t_0, T]$, and β_1, β_2 are positive constants, then it holds that

$$g(t) \le a(t)[E_{\beta_1}(b_1(t)\Gamma(\beta_1)t^{\beta_1}) + E_{\beta_2}(b_2(t)\Gamma(\beta_2)t^{\beta_2}) - 1],$$

where Γ is the Gamma function, $E_{\beta_i}, i = 1, 2$ are the Mittag-Leffler functions given by the series $E_{\beta_i}(z) = \sum_{k=0}^{n} \frac{z^k}{\Gamma(k\beta_i + 1)}$.

Definition 2.7. *An $\mathcal{F}_t$-progressively measurable stochastic process $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))$ is called a solution to system (1.1), if $x(t)$ fulfills the equation:*

$$\begin{aligned} x_i(t) =& U_i(t-t_0)x_i(t_0) + \int_{t_0}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s) \\ &\times\Big[\sum_{j=1}^{n} b_{ij}(s)f_j(x_j(s)) + \sum_{j=1}^{n} c_{ij}(s)g_j(x_j(s-\tau_{ij}(s))) + J_i(s)\Big]ds \\ &+ \int_{t_0}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\sum_{j=1}^{n} d_{ij}(t)h_{ij}(x_j(s-\varsigma_{ij}(s)))dB_j(s), \quad t \ge t_0, \end{aligned} \tag{2.1}$$

in which

$$U_i(t) = \int_0^{\infty}\zeta_\kappa(\gamma)e^{-a_i t^\kappa \gamma}d\gamma, \quad \upsilon_i(t) = \kappa\int_0^{\infty}\gamma\zeta_\kappa(\gamma)e^{-a_i t^\kappa\gamma}d\gamma,$$

$$\zeta_\kappa(\gamma) = \frac{1}{\kappa}\gamma^{-1-\frac{1}{\kappa}}\xi_\kappa(\gamma^{\frac{-1}{\kappa}}) \ge 0, \;\; \zeta_\kappa(\gamma) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\gamma^{-n\kappa-1}\frac{\Gamma(n\kappa+1)}{n!}\sin(n\pi\kappa).$$

According to [39, 40], the function ζ_κ has the following properties:

$$\zeta_\kappa(\gamma) \ge 0, \;\; \gamma \in (0,\infty), \;\; \int_0^{\infty}\zeta_\kappa(\gamma)d\gamma = 1, \;\; \int_0^{\infty}\gamma^{\eta}\zeta_\kappa(\gamma)d\gamma = \frac{\Gamma(1+\eta)}{\Gamma(1+\kappa\eta)}, \;\; 0 \le \eta \le 1.$$

Taking $t_0 \to -\infty$ in (2.1), we get

$$\begin{aligned} x_i(t) =& \int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\Big[\sum_{j=1}^{n} b_{ij}(s)f_j(x_j(s)) + \sum_{j=1}^{n} c_{ij}(s)g_j(x_j(s-\tau_{ij}(s))) + J_i(s)\Big]ds \\ &+ \int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\sum_{j=1}^{n} d_{ij}(s)h_{ij}(x_j(s-\varsigma_{ij}(s)))dB_j(s), \quad t \in \mathbb{R}, i \in \mathcal{J}, \end{aligned} \tag{2.2}$$

which is also a solution of system (1.1).

In this paper, we will use the following notations:

$$b_{ij}^+ = \sup_{t\in\mathbb{R}}|b_{ij}(t)|, \quad c_{ij}^+ = \sup_{t\in\mathbb{R}}|c_{ij}(t)|, \quad d_{ij}^+ = \sup_{t\in\mathbb{R}}|d_{ij}(t)|,$$

$$\tau_{ij}^+ = \sup_{t\in\mathbb{R}}\tau_{ij}(t), \quad \varsigma_{ij}^+ = \sup_{t\in\mathbb{R}}\varsigma_{ij}(t), \quad \zeta = \max_{i,j\in\mathcal{J}}\{\tau_{ij}^+, \varsigma_{ij}^+\}.$$

To obtain our results, we need the following assumptions:

(H_1) For $i, j \in \mathcal{J}$, $b_{ij}, d_{ij}, c_{ij}, u_i \in AP(\mathbb{R}, \mathbb{R})$, $\tau_{ij}, \varsigma_{ij} \in AP(\mathbb{R}, \mathbb{R}^+)$.

(H_2) For $i, j \in \mathcal{J}$, $f_j, g_j, h_{ij} \in C(\mathbb{R}, \mathbb{R})$, and there are constants $F_j^L, G_j^L, H_{ij}^L > 0$ such that for all $x, y \in \mathbb{R}$,

$$|f_j(x) - f_j(y)| \le F_j^L\, |x - y|, \;\; |g_j(x) - g_j(y)| \le G_j^L|x - y|, \;\; |h_{ij}(x) - h_{ij}(y)| \le H_{ij}^L|x - y|,$$

moreover $f_j(0) = g_j(0) = h_{ij}(0) = 0$.

(H_3) The two inequalities below hold.

$$\begin{aligned}\rho := \max_{i\in\mathcal{J}}\Big\{&\frac{3}{(a_i)^2}\Big[\sum_{j=1}^{n}(b_{ij}^+)^2\sum_{j=1}^{n}(F_j^L)^2 + \sum_{j=1}^{n}(c_{ij}^+)^2\sum_{j=1}^{n}(G_j^L)^2\Big] \\ &+ \frac{3\Gamma(2-\frac{1}{\kappa})\Gamma(\frac{1}{\kappa})}{(2a_i)^{2-\frac{1}{\kappa}}\Gamma(\kappa)\Gamma(2-\kappa)}\sum_{j=1}^{n}(d_{ij}^+)^2\sum_{j=1}^{n}(H_{ij}^L)^2\Big\} < \frac{1}{4}, \\ Q := \max_{i\in\mathcal{J}}\Big\{&\frac{7}{(a_i)^2}\sum_{j=1}^{n}(b_{ij}^+)^2\sum_{j=1}^{n}(F_j^L)^2 + \frac{14}{(a_i)^2}\sum_{j=1}^{n}(c_{ij}^+)^2\sum_{j=1}^{n}(G_j^L)^2 \\ &+ \frac{14\Gamma(2-\frac{1}{\kappa})\Gamma(\frac{1}{\kappa})}{(2a_i)^{2-\frac{1}{\kappa}}\Gamma(\kappa)\Gamma(2-\kappa)}\sum_{j=1}^{n}(d_{ij}^+)^2\sum_{j=1}^{n}(H_{ij}^L)^2\Big\} < 1.\end{aligned}$$

Through a simple calculation, one can easily show the following lemma.

Lemma 2.2. *If* $\kappa \in (\frac{1}{2}, 1), a_i > 0, i \in \mathcal{J}$, *then*

$$\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)ds = \frac{1}{a_i} \quad \text{and} \quad \int_0^{\infty} z^{1-\frac{1}{\kappa}}e^{-2a_i z}dz = \frac{\Gamma(2-\frac{1}{\kappa})}{(2a_i)^{2-\frac{1}{\kappa}}},$$

where ψ_1 *is the same as in Definition 2.7.*

3. Main results

We denote by $\mathbb{B}$ the space of all $\mathcal{L}^2$-bounded and $\mathcal{L}^2$-uniformly continuous functions from $\mathbb{R}$ to $\mathcal{L}^2(\Omega, \mathbb{R}^n)$. Then, $\mathbb{B}$ with the norm $\|\phi\|_{\mathbb{B}} = \sup_{t\in\mathbb{R}}\{E|\phi(t)|_0^2\}^{\frac{1}{2}}$ is a Banach space, where $|\phi(t)|_0 = \max_{i\in\mathcal{J}}\{|\phi_i(t)|\}$ and $\phi(t) = (\phi_1(t), \phi_2(t), \ldots, \phi_n(t)) \in \mathbb{B}$.

Let $\phi^0 = (\phi_1^0, \phi_2^0, \cdots, \phi_n^0)$, where $\phi_i^0(t) = \int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)J_i(s)ds, t \in \mathbb{R}$ and take a positive constant $\hbar$ such that $\|\phi^0\|_{\mathbb{B}} \le \hbar$. Then, we have the following existence result.

Theorem 3.1. *Assume that* (H_1)–(H_3) *hold, then system* (1.1) *has a unique almost periodic solution in distribution belonging to* $\mathbb{B}_0 = \{\phi|\phi \in \mathbb{B}, \|\phi - \phi^0\|_{\mathbb{B}} \le \hbar\}$.

Proof. Consider a nonlinear operator $\Psi : \mathbb{B} \longrightarrow \mathbb{B}$ defined by setting

$$(\phi_1, \phi_2, \cdots, \phi_n) \mapsto (x_1^\phi, x_2^\phi, \cdots, x_n^\phi),$$

for $(\phi_1, \phi_2, \cdots, \phi_n) \in \mathbb{B}$, in which

$$x_i^\phi(t) = \int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\Big[\sum_{j=1}^{n}b_{ij}(s)f_j(\phi_j(s)) + \sum_{j=1}^{n}c_{ij}(s)g_j(\phi_j(s-\tau_{ij}(s))) + J_i(s)\Big]ds$$

$$+\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\sum_{j=1}^{n}d_{ij}(s)h_{ij}(\phi_j(s-\varsigma_{ij}(s)))dB_j(s),\ \ t\in\mathbb{R}, i\in\mathcal{J}. \tag{3.1}$$

First of all, we will show that Ψ is a self-mapping from $\mathbb{B}_0$ to $\mathbb{B}_0$. In fact, for each $\phi\in\mathbb{B}_0$, it holds that

$$\|\phi\|_{\mathbb{B}}\le\|\phi^0\|_{\mathbb{B}}+\|\phi-\phi^0\|_{\mathbb{B}}\le 2\hbar,$$

as a consequence,

$$\begin{aligned}\|\Psi\phi-\phi^0\|_{\mathbb{B}}^2\le&3\sup_{t\in\mathbb{R}}\max_{i\in\mathcal{J}}\Big\{E\Big|\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\sum_{j=1}^{n}b_{ij}(s)f_j(\phi_j(s))ds\Big|^2\Big\}\\&+3\sup_{t\in\mathbb{R}}\max_{i\in\mathcal{J}}\Big\{E\Big|\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\sum_{j=1}^{n}c_{ij}(s)g_j(\phi_j(s-\tau_{ij}(s)))ds\Big|^2\Big\}\\&+3\sup_{t\in\mathbb{R}}\max_{i\in\mathcal{J}}\Big\{E\Big|\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\sum_{j=1}^{n}d_{ij}(s)h_{ij}(\phi_j(s-\varsigma_{ij}(s)))dB_j(s)\Big|^2\Big\}\\:=&A_1+A_2+A_3.\end{aligned}\tag{3.2}$$

By the Cauchy-Schwarz inequality and Lemma 2.2, we can deduce that

$$\begin{aligned}A_1\le&3\sup_{t\in\mathbb{R}}\max_{i\in\mathcal{J}}\Big\{E\Big|\Big(\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)ds\Big)\\&\times\Big[\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\Big(\sum_{j=1}^{n}b_{ij}(s)f_j(\phi_j(s))\Big)^2ds\Big]\Big|\Big\}\\\le&\sup_{t\in\mathbb{R}}\max_{i\in\mathcal{J}}\Big\{\frac{3}{a_i}E\Big[\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\sum_{j=1}^{n}\big|b_{ij}(s)\big|^2\sum_{j=1}^{n}(F_j^L)^2\big|\phi_j(s)\big|^2ds\Big]\Big\}\\\le&\max_{i\in\mathcal{J}}\Big\{\frac{3}{(a_i)^2}\sum_{j=1}^{n}(b_{ij}^+)^2\sum_{j=1}^{n}(F_j^L)^2\Big\}\big\|\phi\big\|_{\mathbb{B}}^2.\end{aligned}\tag{3.3}$$

Similarly, we can get

$$A_2\le\max_{i\in\mathcal{J}}\Big\{\frac{3}{(a_i)^2}\sum_{j=1}^{n}(c_{ij}^+)^2\sum_{j=1}^{n}(G_j^L)^2\Big\}\big\|\phi\big\|_{\mathbb{B}}^2.\tag{3.4}$$

Moreover, from the Itô isometry, the Cauchy-Schwarz inequality and Lemma 2.2, it follows that

$$\begin{aligned}A_3=&3\sup_{t\in\mathbb{R}}\max_{i\in\mathcal{J}}\Big\{E\Big[\int_{-\infty}^{t}(t-s)^{2(\kappa-1)}\Big[\kappa\int_0^\infty\gamma\zeta_\kappa(\gamma)e^{-a_i(t-s)^\kappa\gamma}d\gamma\Big]^2\\&\times\Big|\sum_{j=1}^{n}d_{ij}(s)h_{ij}(\phi_j(s-\varsigma_{ij}(s)))\Big|^2ds\Big]\Big\}\\\le&\max_{i\in\mathcal{J}}\Big\{3\kappa^2\sum_{j=1}^{n}(d_{ij}^+)^2\sum_{j=1}^{n}(H_{ij}^L)^2\int_0^\infty\gamma\zeta_\kappa(\gamma)d\gamma\end{aligned}$$

$$\times \int_{-\infty}^{t} (t-s)^{2(\kappa-1)} \int_0^\infty \gamma\zeta_\kappa(\gamma) e^{-2a_i(t-s)^\kappa\gamma} d\gamma ds\Big\}\big\|\phi\big\|_{\mathbb{B}}^2$$
$$\leq \max_{i\in\mathcal{J}}\Big\{\frac{3\kappa^2}{\Gamma(\kappa+1)}\sum_{j=1}^n (d_{ij}^+)^2 \sum_{j=1}^n (H_{ij}^L)^2$$
$$\times \int_{-\infty}^{t} (t-s)^{2(\kappa-1)} \int_0^\infty \gamma\zeta_\kappa(\gamma) e^{-2a_i(t-s)^\kappa\gamma} d\gamma ds\Big\}\big\|\phi\big\|_{\mathbb{B}}^2$$
$$\leq \max_{i\in\mathcal{J}}\Big\{\frac{3\kappa}{\Gamma(\kappa+1)}\sum_{j=1}^n (d_{ij}^+)^2 \sum_{j=1}^n (H_{ij}^L)^2 \int_0^\infty \gamma^{\frac{1}{\kappa}-1}\zeta_\kappa(\gamma)d\gamma \int_0^\infty z^{1-\frac{1}{\kappa}} e^{-2a_i z} dz\Big\}\big\|\phi\big\|_{\mathbb{B}}^2$$
$$\leq \max_{i\in\mathcal{J}}\Big\{\frac{3\Gamma(2-\frac{1}{\kappa})\Gamma(\frac{1}{\kappa})}{(2a_i)^{2-\frac{1}{\kappa}}\Gamma(\kappa)\Gamma(2-\kappa)}\sum_{j=1}^n (d_{ij}^+)^2 \sum_{j=1}^n (H_{ij}^L)^2\Big\}\big\|\phi\big\|_{\mathbb{B}}^2. \tag{3.5}$$

Substituting (3.3)–(3.5) into (3.2), we obtain that

$$\begin{aligned}\|\Psi\phi-\phi^0\|_{\mathbb{B}}^2 \leq& \max_{i\in\mathcal{J}}\Big\{\frac{3}{(a_i)^2}\Big[\sum_{j=1}^n (b_{ij}^+)^2 \sum_{j=1}^n (F_j^L)^2 + \sum_{j=1}^n (c_{ij}^+)^2 \sum_{j=1}^n (G_j^L)^2\Big] \\ &+ \frac{3\Gamma(2-\frac{1}{\kappa})\Gamma(\frac{1}{\kappa})}{(2a_i)^{2-\frac{1}{\kappa}}\Gamma(\kappa)\Gamma(2-\kappa)}\sum_{j=1}^n (d_{ij}^+)^2 \sum_{j=1}^n (H_{ij}^L)^2\Big\}\big\|\phi\big\|_{\mathbb{B}}^2 \\ \leq& \frac{1}{4}(2\hbar)^2 = \hbar^2,\end{aligned}$$

which implies that

$$\|\Psi\phi-\phi^0\|_{\mathbb{B}} \leq \hbar.$$

Since $\phi\in\mathbb{B}_0$, ϕ is $\mathcal{L}^2$-uniformly continuous. In addition, by (H_1) and the fact that almost periodic functions are uniformly continuous, for any $\varepsilon>0$, there exists $\delta>0$ with $\delta<\varepsilon$ such that for all $t\in\mathbb{R}$ and $h\in(-\delta,\delta)$, the following holds

$$E|\phi(s+h)-\phi(s)|_0^2 < \varepsilon,$$
$$|b_{ij}(s+h)-b_{ij}(t)|<\varepsilon, \quad |c_{ij}(s+h)-c_{ij}(s)|<\varepsilon,$$
$$|d_{ij}(s+h)-d_{ij}(s)|<\varepsilon, \quad |u_i(s+h)-J_i(s)|<\varepsilon,$$
$$|\tau_{ij}(s+h)-\tau_{ij}(s)|<\varepsilon=\delta, \quad |\varsigma_{ij}(s+h)-\varsigma_{ij}(s)|<\varepsilon=\delta.$$

Consequently, we can get

$$E|\phi(s-\tau_{ij}(s+h))-\phi(s-\tau_{ij}(s))|_0^2<\varepsilon, \quad E|\phi(s-\varsigma_{ij}(s+h))-\phi(s-\varsigma_{ij}(s))|_0^2<\varepsilon.$$

Then,

$$\begin{aligned}&E\big|\Psi\phi(t+h)-\Psi\phi(t)\big|_0^2 \\ =&\max_{i\in\mathcal{J}}\Big\{E\Big|\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\Big[\sum_{j=1}^n b_{ij}(s+h)f_j(\phi_j(s+h))\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{n} c_{ij}(s+h)g_j(\phi_j(s+h-\tau_{ij}(s+h)))+u_i(s+h)\Big]ds\\
&+\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\sum_{j=1}^{n} d_{ij}(s+h)h_{ij}(\phi_j(s+h-\varsigma_{ij}(s+h)))d\hat{B}_j(s)\\
&-\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\Big[\sum_{j=1}^{n} b_{ij}(s)f_j(\phi_j(s))+\sum_{j=1}^{n} c_{ij}(s)g_j(\phi_j(s-\tau_{ij}(s)))+J_i(s)\Big]ds\\
&-\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\sum_{j=1}^{n} d_{ij}(s)h_{ij}(\phi_j(s-\varsigma_{ij}(s)))d\hat{B}_j(s)\Big|^2\Big\}\\
\leq& 4\max_{i\in\mathcal{J}}\Big\{E\Big|\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\Big[\sum_{j=1}^{n} b_{ij}(s+h)f_j(\phi_j(s+h))-\sum_{j=1}^{n} b_{ij}(s)f_j(\phi_j(s))\Big]ds\Big|^2\Big\}\\
&+4\max_{i\in\mathcal{J}}\Big\{E\Big|\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\Big[\sum_{j=1}^{n} c_{ij}(s+h)g_j(\phi_j(s+h-\tau_{ij}(s+h)))\\
&-\sum_{j=1}^{n} c_{ij}(s)g_j(\phi_j(s-\tau_{ij}(s)))\Big]ds\Big|^2\Big\}\\
&+4\max_{i\in\mathcal{J}}\Big\{E\Big|\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)[u_i(s+h)-J_i(s)]ds\Big|^2\Big\}\\
&+4\max_{i\in\mathcal{J}}\Big\{E\Big|\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\Big[\sum_{j=1}^{n} d_{ij}(s+h)h_{ij}(\phi_j(s+h-\varsigma_{ij}(s+h)))\\
&-\sum_{j=1}^{n} d_{ij}(s)h_{ij}(\phi_j(s-\varsigma_{ij}(s)))\Big]dB_j(s)\Big|^2\Big\}\\
:=&A_4+A_5+A_6+A_7, \qquad (3.6)
\end{aligned}
$$

where $\hat{B}_j(t)=B_j(t+h)-B_j(t)$ is a Brownian motion that has the same distribution as $B_j(t)$.

In the same way as the method used to estimate (3.3), we infer that

$$
\begin{aligned}
A_4 \leq& 8\max_{i\in\mathcal{J}}\Big\{\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)ds\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\\
&\times E\Big|\sum_{j=1}^{n} b_{ij}(s+h)[f_j(\phi_j(s+h))-f_j(\phi_j(s))]\Big|^2 ds\Big\}\\
&+8\max_{i\in\mathcal{J}}\Big\{\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)ds\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\\
&\times E\Big[\sum_{j=1}^{n}\big|b_{ij}(s+h)-b_{ij}(s)\big|\big|f_j(\phi_j(s))\big|\Big]^2 ds\Big\}\\
\leq& 8\max_{i\in\mathcal{J}}\Big\{\frac{1}{a_i}\sum_{j=1}^{n}(b_{ij}^{+})^2\sum_{j=1}^{n}(F_j^L)^2\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)E|\phi(s+h)-\phi(s)|_0^2 ds\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+8\max_{i\in\mathcal{J}}\Big\{\frac{n}{a_i}\sum_{j=1}^{n}(F_j^L)^2\varepsilon^2\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)E|\phi(s)|_0^2ds\Big\}\\
&\le 8\max_{i\in\mathcal{J}}\Big\{\frac{\varepsilon}{a_i}\sum_{j=1}^{n}(b_{ij}^+)^2\sum_{j=1}^{n}(F_j^L)^2\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)ds\Big\}\\
&+8\max_{i\in\mathcal{J}}\Big\{\frac{n}{a_i}\sum_{j=1}^{n}(F_j^L)^2\varepsilon^2\|\phi\|_{\mathbb{B}}^2\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)ds\Big\}\\
&\le\max_{i\in\mathcal{J}}\Big\{\frac{8}{(a_i)^2}\sum_{j=1}^{n}(b_{ij}^+)^2\sum_{j=1}^{n}(F_j^L)^2\varepsilon+\frac{8n}{(a_i)^2}\sum_{j=1}^{n}(F_j^L)^2\varepsilon^2(2\hbar)^2\Big\},
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
A_5 \le\,&16\max_{i\in\mathcal{J}}\Big\{\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)ds\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\\
&\times E\Big|\sum_{j=1}^{n}c_{ij}(s+h)[g_j(\phi_j(s+h-\tau_{ij}(s+h)))-g_j(\phi_j(s-\tau_{ij}(s+h)))]\Big|^2ds\Big\}\\
&+16\max_{i\in\mathcal{J}}\Big\{\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)ds\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\\
&\times E\Big|\sum_{j=1}^{n}c_{ij}(s+h)[g_j(\phi_j(s-\tau_{ij}(s+h)))-g_j(\phi_j(s-\tau_{ij}(s)))]\Big|^2ds\Big\}\\
&+8\max_{i\in\mathcal{J}}\Big\{\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)ds\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\\
&\times E\Big[\sum_{j=1}^{n}\big|c_{ij}(s+h)-c_{ij}(s)\big|\big|g_j(\phi_j(s-\tau_{ij}(s)))\big|\Big]^2ds\Big\}\\
\le\,&16\max_{i\in\mathcal{J}}\Big\{\frac{1}{a_i}\sum_{j=1}^{n}(c_{ij}^+)^2\sum_{j=1}^{n}(G_j^L)^2\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\\
&\times E|\phi(s+h-\tau_{ij}(s+h))-\phi(s-\tau_{ij}(s+h))|_0^2ds\Big\}\\
&+16\max_{i\in\mathcal{J}}\Big\{\frac{1}{a_i}\sum_{j=1}^{n}(c_{ij}^+)^2\sum_{j=1}^{n}(G_j^L)^2\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\\
&\times E|\phi(s-\tau_{ij}(s+h))-\phi(s-\tau_{ij}(s))|_0^2ds\Big\}\\
&+8\max_{i\in\mathcal{J}}\Big\{\frac{n}{a_i}\sum_{j=1}^{n}(G_j^L)^2\varepsilon^2\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)E|\phi(s-\tau_{ij}(s))|_0^2ds\Big\}\\
\le\,&16\max_{i\in\mathcal{J}}\Big\{\frac{2}{a_i}\sum_{j=1}^{n}(c_{ij}^+)^2\sum_{j=1}^{n}(G_j^L)^2\varepsilon\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)ds\Big\}\\
&+8\max_{i\in\mathcal{J}}\Big\{\frac{n}{a_i}\sum_{j=1}^{n}(G_j^L)^2\varepsilon^2\|\phi\|_{\mathbb{B}}^2\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)ds\Big\}
\end{aligned}
$$

$$\le \max_{i\in\mathcal{J}}\Big\{\frac{32}{(a_i)^2}\sum_{j=1}^{n}(c_{ij}^{+})^2\sum_{j=1}^{n}(G_j^L)^2\varepsilon+\frac{8n}{(a_i)^2}\sum_{j=1}^{n}(G_j^L)^2\varepsilon^2(2\hbar)^2\Big\} \tag{3.8}$$

and

$$\begin{aligned} A_6 \le& 4\max_{i\in\mathcal{J}}\Big\{\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)ds\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)E|u_i(s+h)-J_i(s)|^2ds\Big\}\\ \le& \max_{i\in\mathcal{J}}\Big\{\frac{4\varepsilon^2}{(a_i)^2}\Big\}. \end{aligned} \tag{3.9}$$

Moreover, in the same way as the method used to estimate (3.5), we deduce that

$$\begin{aligned} A_7 \le& 16\max_{i\in\mathcal{J}}\Big\{E\Big|\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\sum_{j=1}^{n}d_{ij}(s+h)\\ &\times[h_{ij}(\phi_j(s+h-\varsigma_{ij}(s+h)))-h_{ij}(\phi_j(s-\varsigma_{ij}(s+h)))]dB_j(s)\Big|^2\Big\}\\ &+16\max_{i\in\mathcal{J}}\Big\{E\Big|\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\sum_{j=1}^{n}d_{ij}(s+h)\\ &\times[h_{ij}(\phi_j(s-\varsigma_{ij}(s+h)))-h_{ij}(\phi_j(s-\varsigma_{ij}(s)))]dB_j(s)\Big|^2\Big\}\\ &+8\max_{i\in\mathcal{J}}\Big\{E\Big|\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\sum_{j=1}^{n}[d_{ij}(s+h)-d_{ij}(s)]\\ &\times h_{ij}(\phi_j(s-\varsigma_{ij}(s)))dB_j(s)\Big|^2\Big\}\\ \le& 16\max_{i\in\mathcal{J}}\Big\{\int_{-\infty}^{t}(t-s)^{2(\kappa-1)}\psi_i^2(t-s)\Big|\sum_{j=1}^{n}d_{ij}(s+h)\\ &\times[h_{ij}(\phi_j(s+h-\varsigma_{ij}(s+h)))-h_{ij}(\phi_j(s-\varsigma_{ij}(s+h)))]\Big|^2ds\Big\}\\ &+16\max_{i\in\mathcal{J}}\Big\{\int_{-\infty}^{t}(t-s)^{2(\kappa-1)}\psi_i^2(t-s)E\Big|\sum_{j=1}^{n}d_{ij}(s+h)\\ &\times[h_{ij}(\phi_j(s-\varsigma_{ij}(s+h)))-h_{ij}(\phi_j(s-\varsigma_{ij}(s)))]\Big|^2ds\Big\}\\ &+8\max_{i\in\mathcal{J}}\Big\{\int_{-\infty}^{t}(t-s)^{2(\kappa-1)}\psi_i^2(t-s)\\ &\times E\Big|\sum_{j=1}^{n}|d_{ij}(s+h)-d_{ij}(s)|h_{ij}(\phi_j(s-\varsigma_{ij}(s)))\Big|^2ds\Big\}\\ \le& 16\max_{i\in\mathcal{J}}\Big\{\sum_{j=1}^{n}(d_{ij}^{+})^2\sum_{j=1}^{n}(H_{ij}^L)^2\int_{-\infty}^{t}(t-s)^{2(\kappa-1)}\psi_i^2(t-s)\\ &\times E|\phi(s+h-\varsigma_{ij}(s+h))-\phi(s-\varsigma_{ij}(s+h))|_0^2ds\Big\} \end{aligned}$$

$$
\begin{aligned}
&+ 16\max_{i\in\mathcal{J}}\Big\{\sum_{j=1}^{n}(d_{ij}^{+})^2\sum_{j=1}^{n}(H_{ij}^{L})^2\int_{-\infty}^{t}(t-s)^{2(\kappa-1)}\psi_i^2(t-s)\\
&\times E|\phi(s-\varsigma_{ij}(s+h))-\phi(s-\varsigma_{ij}(s))|_0^2 ds\Big\}\\
&+ 8\max_{i\in\mathcal{J}}\Big\{n\varepsilon^2\sum_{j=1}^{n}(H_{ij}^{L})^2\int_{-\infty}^{t}(t-s)^{2(\kappa-1)}\psi_i^2(t-s)E|\phi(s-\varsigma_{ij}(s))|_0^2 ds\Big\}\\
\le& 16\max_{i\in\mathcal{J}}\Big\{2\sum_{j=1}^{n}(d_{ij}^{+})^2\sum_{j=1}^{n}(H_{ij}^{L})^2\varepsilon\int_{-\infty}^{t}(t-s)^{2(\kappa-1)}\psi_i^2(t-s)ds\Big\}\\
&+ 8\max_{i\in\mathcal{J}}\Big\{n\varepsilon^2\sum_{j=1}^{n}(H_{ij}^{L})^2\|\phi\|_{\mathbb{B}}^2\int_{-\infty}^{t}(t-s)^{2(\kappa-1)}\psi_i^2(t-s)ds\Big\}\\
\le& \max_{i\in\mathcal{J}}\Big\{\frac{32\Gamma(2-\frac{1}{\kappa})\Gamma(\frac{1}{\kappa})}{(2a_i)^{2-\frac{1}{\kappa}}\Gamma(\kappa)\Gamma(2-\kappa)}\sum_{j=1}^{n}(d_{ij}^{+})^2\sum_{j=1}^{n}(H_{ij}^{L})^2\varepsilon\\
&+ \frac{8\Gamma(2-\frac{1}{\kappa})\Gamma(\frac{1}{\kappa})}{(2a_i)^{2-\frac{1}{\kappa}}\Gamma(\kappa)\Gamma(2-\kappa)}n\varepsilon^2(2\hbar)^2\Big\}. \qquad (3.10)
\end{aligned}
$$

Substituting (3.7)–(3.10) into (3.6) yields that

$$
\begin{aligned}
E\Big|\Psi\phi(t+h)-\Psi\phi(t)\Big|_0^2 \le& \max_{i\in\mathcal{J}}\Big\{\frac{8}{(a_i)^2}\Big[\sum_{j=1}^{n}(b_{ij}^{+})^2\sum_{j=1}^{n}(F_j^{L})^2+n\sum_{j=1}^{n}(F_j^{L})^2(2\hbar)^2\varepsilon\\
&+4\sum_{j=1}^{n}(c_{ij}^{+})^2\sum_{j=1}^{n}(G_j^{L})^2+n\sum_{j=1}^{n}(G_j^{L})^2(2\hbar)^2\varepsilon+\frac{1}{2}\varepsilon\Big]\\
&+\frac{8\Gamma(2-\frac{1}{\kappa})\Gamma(\frac{1}{\kappa})}{(2a_i)^{2-\frac{1}{\kappa}}\Gamma(\kappa)\Gamma(2-\kappa)}\Big[4\sum_{j=1}^{n}(d_{ij}^{+})^2\sum_{j=1}^{n}(H_{ij}^{L})^2+n2\hbar^2\varepsilon\Big]\Big\}\varepsilon,
\end{aligned}
$$

which indicates that $\Psi\phi$ is $\mathcal{L}^2$-uniformly continuous. Therefore, we arrive at $\Psi\mathbb{B}_0 \subset \mathbb{B}_0$.

Then, we will prove that Ψ is a contraction mapping. For any $\mu, \nu \in \mathbb{B}_0$, we have

$$
\begin{aligned}
&\|\Psi\mu-\Psi\nu\|_{\mathbb{B}}^2\\
\le& 3\sup_{t\in\mathbb{R}}\max_{i\in\mathcal{J}}\Big\{E\Big|\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\sum_{j=1}^{n}b_{ij}(s)(f_j(\mu_j(s))-f_j(\nu_j(s)))ds\Big|^2\Big\}\\
&+3\sup_{t\in\mathbb{R}}\max_{i\in\mathcal{J}}\Big\{E\Big|\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\sum_{j=1}^{n}c_{ij}(s)(g_j(\mu_j(s-\tau_{ij}(s)))\\
&-g_j(\nu_j(s-\tau_{ij}(s))))ds\Big|^2\Big\}+3\sup_{t\in\mathbb{R}}\max_{i\in\mathcal{J}}\Big\{E\Big|\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\\
&\times\sum_{j=1}^{n}d_{ij}(s)(h_{ij}(\mu_j(s-\varsigma_{ij}(s)))-h_{ij}(\nu_j(s-\varsigma_{ij}(s))))dB_j(s)\Big|^2\Big\}\\
\le& \max_{i\in\mathcal{J}}\Big\{\frac{3}{(a_i)^2}\Big[\sum_{j=1}^{n}(b_{ij}^{+})^2\sum_{j=1}^{n}(F_j^{L})^2+\sum_{j=1}^{n}(c_{ij}^{+})^2\sum_{j=1}^{n}(G_j^{L})^2\Big]
\end{aligned}
$$

$$+\frac{3\Gamma(2-\frac{1}{\kappa})\Gamma(\frac{1}{\kappa})}{(2a_i)^{2-\frac{1}{\kappa}}\Gamma(\kappa)\Gamma(2-\kappa)}\sum_{j=1}^{n}(d_{ij}^{+})^2\sum_{j=1}^{n}(H_{ij}^{L})^2\Big\}\|\mu-\nu\|_{\mathbb{B}}^2.$$

Consequently, in view of (H_3), we obtain that

$$\|\Psi\mu-\Psi\nu\|_{\mathbb{B}}\leq\frac{1}{2}\|\mu-\nu\|_{\mathbb{B}},$$

which means that Ψ is a contraction mapping. Thus, Ψ has a unique fixed point x in $\mathbb{B}_0$, i.e., system (1.1) has a unique solution x in $\mathbb{B}_0$.

Finally, we will show that $x=(x_1,x_2,\cdots x_n)$ is almost periodic in distribution. Due to the $\mathcal{L}^2$-uniform continuity of x, for any $\varepsilon>0$, there exists $\delta>0$ with $\delta<\varepsilon$ such that for all $t\in\mathbb{R}$ and $|h|<\delta$, we have $E|x(t+h)-x(t)|_n^2<\varepsilon$. For the $\delta>0$ above, in view of (H_1), there exists a positive number ℓ such that every interval of length ℓ includes a number τ such that for $i,j\in\mathcal{J}$,

$$|\tau_{ij}(t+\tau)-\tau_{ij}(t)|<\delta,\quad |\varsigma_{ij}(t+\tau)-\varsigma_{ij}(t)|<\delta,\quad |b_{ij}(t+\tau)-b_{ij}(t)|^2<\delta,$$

$$|u_i(t+\tau)-J_i(t)|^2<\delta,\quad |c_{ij}(t+\tau)-c_{ij}(t)|^2<\delta,\quad |d_{ij}(t+\tau)-d_{ij}(t)|^2<\delta.$$

Hence, we have

$$E|x(t-\tau_{ij}(t+\tau))-x(t-\tau_{ij}(t))|_0^2<\varepsilon,\quad E|x(t-\varsigma_{ij}(t+\tau))-x(t-\varsigma_{ij}(t))|_0^2<\varepsilon.$$

According to (2.2), we can get

$$\begin{aligned}x_i(t+\tau)=&\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\Big[\sum_{j=1}^{n}b_{ij}(s+\tau)f_j(x_j(s+\tau))+\sum_{j=1}^{n}c_{ij}(s+\tau)\\&\times g_j(x_j(s+\tau-\tau_{ij}(s+\tau)))+u_i(s+\tau)\Big]ds+\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\\&\times\sum_{j=1}^{n}d_{ij}(s+\tau)h_{ij}(x_j(s+\tau-\varsigma_{ij}(s+\tau)))d[B_j(s+\tau)-B_j(\tau)],\end{aligned}$$

where $i,j\in\mathcal{J}$, $B_j(s+\tau)-B_j(\tau)$ is a Brownian motion with the same distribution as $B_j(s)$.

Let us consider the process

$$\begin{aligned}x_i(t+\tau)=&\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\Big[\sum_{j=1}^{n}b_{ij}(s+\tau)f_j(x_j(s+\tau))\\&+\sum_{j=1}^{n}c_{ij}(s+\tau)g_j(x_j(s+\tau-\tau_{ij}(s+\tau)))+u_i(s+\tau)\Big]ds\\&+\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\sum_{j=1}^{n}d_{ij}(s+\tau)h_{ij}(x_j(s+\tau-\varsigma_{ij}(s+\tau)))dB_j(s).\end{aligned}$$

Then, we can get

$$E|x(t+\tau)-x(t)|_0^2$$

$$
\begin{aligned}
&\le 7\max_{i\in\mathcal{J}}\left\{E\left|\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\sum_{j=1}^{n}b_{ij}(s+\tau)[f_j(x_j(s+\tau))-f_j(x_j(s))]ds\right|^2\right\}\\
&\quad+7\max_{i\in\mathcal{J}}\left\{E\left|\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\sum_{j=1}^{n}[b_{ij}(s+\tau)-b_{ij}(s)]f_j(x_j(s))ds\right|^2\right\}\\
&\quad+7\max_{i\in\mathcal{J}}\left\{E\left|\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\sum_{j=1}^{n}c_{ij}(s+\tau)\right.\right.\\
&\quad\left.\left.\times[g_j(x_j(s+\tau-\tau_{ij}(s+\tau)))-g_j(x_j(s-\tau_{ij}(s)))]ds\right|^2\right\}\\
&\quad+7\max_{i\in\mathcal{J}}\left\{E\left|\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\sum_{j=1}^{n}[c_{ij}(s+\tau)-c_{ij}(s)]g_j(x_j(s-\tau_{ij}(s)))ds\right|^2\right\}\\
&\quad+7\max_{i\in\mathcal{J}}\left\{E\left|\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)[u_i(s+\tau)-J_i(s)]ds\right|^2\right\}\\
&\quad+7\max_{i\in\mathcal{J}}\left\{E\left|\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\right.\right.\\
&\quad\left.\left.\times\sum_{j=1}^{n}d_{ij}(s+\tau)[h_{ij}(x_j(s+\tau-\varsigma_{ij}(s+\tau)))-h_{ij}(x_j(s-\varsigma_{ij}(s)))]dB_j(s)\right|^2\right\}\\
&\quad+7\max_{i\in\mathcal{J}}\left\{E\left|\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\sum_{j=1}^{n}[d_{ij}(s+\tau)-d_{ij}(s)]h_{ij}(x_j(s-\varsigma_{ij}(s)))dB_j(s)\right|^2\right\}\\
&:=\sum_{i=1}^{7}\varrho_i(t).
\end{aligned}
\tag{3.11}
$$

Using the Cauchy-Schwarz inequality and Lemma 2.2, we can obtain

$$
\begin{aligned}
\varrho_1(t)\le&7\max_{i\in\mathcal{J}}\left\{\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)ds\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\right.\\
&\left.\times\sum_{j=1}^{n}(b_{ij}^{+})^2\sum_{j=1}^{n}(F_j^L)^2E\left|x(s+\tau)-x(s)\right|_0^2ds\right\}\\
\le&\max_{i\in\mathcal{J}}\left\{\frac{7}{(a_i)^2}\sum_{j=1}^{n}(b_{ij}^{+})^2\sum_{j=1}^{n}(F_j^L)^2\right\}\sup_{t\in\mathbb{R}}E\left|x(t+\tau)-x(t)\right|_0^2,
\end{aligned}
\tag{3.12}
$$

and

$$
\begin{aligned}
\varrho_2(t)\le&7\max_{i\in\mathcal{J}}\left\{\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)ds\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\right.\\
&\left.\times\sum_{j=1}^{n}\left|b_{ij}(s+\tau)-b_{ij}(s)\right|^2\sum_{j=1}^{n}(F_j^L)^2E\left|x(s)\right|_0^2ds\right\}\\
\le&\max_{i\in\mathcal{J}}\left\{\frac{7}{(a_i)^2}\sum_{j=1}^{n}(F_j^L)^2(2\hbar)^2n\varepsilon\right\}.
\end{aligned}
\tag{3.13}
$$

Similarly, we have

$$\varrho_4(t) \leq \max_{i\in\mathcal{J}}\Big\{\frac{7}{(a_i)^2}\sum_{j=1}^{n}(G_j^L)^2(2\hbar)^2 n\varepsilon\Big\}, \tag{3.14}$$

$$\varrho_5(t) \leq \max_{i\in\mathcal{J}}\Big\{\frac{7}{(a_i)^2}\varepsilon\Big\}, \tag{3.15}$$

and

$$\begin{aligned}\varrho_3(t) \leq& 14\max_{i\in\mathcal{J}}\Big\{E\Big|\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\sum_{j=1}^{n}c_{ij}(s+\tau)[g_j(x_j(s+\tau-\tau_{ij}(s+\tau)))\\ &-g_j(x_j(s-\tau_{ij}(s+\tau)))]ds\Big|^2\Big\}+14\max_{i\in\mathcal{J}}\Big\{E\Big|\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\\ &\times\sum_{j=1}^{n}c_{ij}(s+\tau)[g_j(x_j(s-\tau_{ij}(s+\tau)))-g_j(x_j(s-\tau_{ij}(s)))]ds\Big|^2\Big\}\\ \leq&\max_{i\in\mathcal{J}}\Big\{\frac{14}{a_i}\sum_{j=1}^{n}(c_{ij}^{+})^2\sum_{j=1}^{n}(G_j^L)^2\Big[\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\\ &\times E\Big|x(s+\tau-\tau_{ij}(s+\tau))-x(s-\tau_{ij}(s+\tau))\Big|_0^2 ds\\ &+\int_{-\infty}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)E\Big|x(s-\tau_{ij}(s+\tau))-x(s-\tau_{ij}(s))\Big|_0^2 ds\Big]\Big\}\\ \leq&\max_{i\in\mathcal{J}}\Big\{\frac{14}{(a_i)^2}\sum_{j=1}^{n}(c_{ij}^{+})^2\sum_{j=1}^{n}(G_j^L)^2\Big[\sup_{t\in\mathbb{R}}E\Big|x(t+\tau)-x(t)\Big|_0^2+\varepsilon\Big]\Big\}.\end{aligned} \tag{3.16}$$

Using the Itô isometry, we can get

$$\begin{aligned}\varrho_6(t) =& 7\max_{i\in\mathcal{J}}\Big\{E\Big|\int_{-\infty}^{t}(t-s)^{2(\kappa-1)}\psi_i^2(t-s)\Big[\sum_{j=1}^{n}d_{ij}(s+\tau)\\ &\times(h_{ij}(x_j(s+\tau-\varsigma_{ij}(s+\tau)))-h_{ij}(x_j(s-\varsigma_{ij}(s))))\Big]^2 ds\Big\}\\ \leq&\max_{i\in\mathcal{J}}\Big\{\frac{14\Gamma(2-\frac{1}{\kappa})\Gamma(\frac{1}{\kappa})}{(2a_i)^{2-\frac{1}{\kappa}}\Gamma(\kappa)\Gamma(2-\kappa)}\sum_{j=1}^{n}(d_{ij}^{+})^2\sum_{j=1}^{n}(H_{ij}^L)^2\Big[\sup_{t\in\mathbb{R}}E\Big|x(t+\tau)-x(t)\Big|_0^2+\varepsilon\Big]\Big\},\end{aligned} \tag{3.17}$$

and

$$\varrho_7(t) \leq \max_{i\in\mathcal{J}}\Big\{\frac{7\Gamma(2-\frac{1}{\kappa})\Gamma(\frac{1}{\kappa})}{(2a_i)^{2-\frac{1}{\kappa}}\Gamma(\kappa)\Gamma(2-\kappa)}\sum_{j=1}^{n}(H_{ij}^L)^2(2\hbar)^2 n\varepsilon\Big\}. \tag{3.18}$$

Putting (3.12)–(3.18) into (3.11) yields that

$$E|x(t+\tau)-x(t)|_0^2 \leq Q_0\varepsilon + Q\sup_{t\in\mathbb{R}}E\Big|x(t+\tau)-x(t)\Big|_0^2,$$

where

$$Q_0 = \max_{i\in\mathcal{J}}\Big\{\frac{7}{(a_i)^2}\Big[1+\sum_{j=1}^{n}(F_j^L)^2(2\hbar)^2 n+\sum_{j=1}^{n}(G_j^L)^2(2\hbar)^2 n+2\sum_{j=1}^{n}(c_{ij}^+)^2\sum_{j=1}^{n}(G_j^L)^2\Big]$$
$$+\frac{7\Gamma(2-\frac{1}{\kappa})\Gamma(\frac{1}{\kappa})}{(2a_i)^{2-\frac{1}{\kappa}}\Gamma(\kappa)\Gamma(2-\kappa)}\Big[2\sum_{j=1}^{n}(d_{ij}^+)\sum_{j=1}^{n}(H_{ij}^L)^2+\sum_{j=1}^{n}(H_{ij}^L)^2(2\hbar)^2 n\Big]\Big\}.$$

Consequently, we can get

$$E|x(t+\tau)-x(t)|_0^2 \le \sup_{t\in\mathbb{R}} E\Big|x(t+\tau)-x(t)\Big|_0^2 \le \frac{Q_0\varepsilon}{1-Q}. \tag{3.19}$$

Noting the fact that

$$d_{\mathcal{P}}(\mu(x(t+\sigma)),\mu(x(t))) \le \Big(E\|x(t+\sigma)-x(t)\|^2\Big)^{\frac{1}{2}}.$$

By (3.19), we see that $x(t)$ is almost periodic in distribution. This completes the proof. □

Our finite-time stability result is as follows:

Theorem 3.2. *Let* (H_1)–(H_3) *be fulfilled. Assume that*

$$4[E_\kappa(\Theta S^\kappa)+E_{2\kappa-1}(\Omega S^{2\kappa-1})-1] \le \frac{\varepsilon}{\delta}, \tag{3.20}$$

where

$$\Theta = \max_{i\in\mathcal{J}}\Big\{\frac{4}{a_i}\Big(\sum_{j=1}^{n}(b_{ij}^+)^2\sum_{j=1}^{n}(F_j^L)^2+\sum_{j=1}^{n}(c_{ij}^+)^2\sum_{j=1}^{n}(G_j^L)^2\Big)\Big\},$$
$$\Omega = \max_{i\in\mathcal{J}}\Big\{\frac{4\Gamma(2\kappa-1)}{\Gamma^2(\kappa)}\sum_{j=1}^{n}(d_{ij}^+)^2\sum_{j=1}^{n}(H_{ij}^L)^2\Big\}.$$

Then system (1.1) *has a unique almost solution in distribution that is mean square finite-time stable with respect to* $\{\delta,\varepsilon,S\}$.

Proof. Let $x=(x_1,x_2,\cdots,x_n)$ and $y=(y_1,y_2,\cdots,y_n)^T$ be the unique almost periodic solution in distribution of system (1.1) with initial value $\varphi=(\varphi_1,\varphi_2,\cdots,\varphi_n)$ and an arbitrary solution of (1.1) with initial value $\chi=(\chi_1,\chi_2,\cdots,\chi_n)$, respectively. Set $z=y-x$. Then by (1.1), for $t\ge t_0, i\in\mathcal{J}$, we can get

$$z_i(t) = U_i(t-t_0)z_i(t_0)+\int_{t_0}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\Big[\sum_{j=1}^{n}b_{ij}(s)[f_j(y_j(t))-f_j(x_j(t))]$$
$$+\sum_{j=1}^{n}c_{ij}(s)[g_j(y_j(t-\tau_{ij}(t)))-g_j(x_j(t-\tau_{ij}(t)))]\Big]ds$$
$$+\int_{t_0}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\sum_{j=1}^{n}d_{ij}(s)[h_{ij}(y_j(t-\varsigma_{ij}(t)))-h_{ij}(x_j(t-\varsigma_{ij}(t)))]dB_j(s).$$

Hence, for any $t \geq t_0$ and $i \in \mathcal{J}$, we infer that

$$\begin{aligned}
&E|z_i(t)|^2\\
\leq &4E|U_i(t-t_0)z_i(t_0)|^2 + 4E\bigg|\int_{t_0}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\sum_{j=1}^{n} b_{ij}(s)[f_j(y_j(s)) - f_j(x_j(s))]ds\bigg|^2\\
&+ 4E\bigg|\int_{t_0}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\sum_{j=1}^{n} c_{ij}(s)[g_j(y_j(s-\tau_{ij}(s))) - g_j(x_j(s-\tau_{ij}(s)))]ds\bigg|^2\\
&+ 4E\bigg|\int_{t_0}^{t}(t-s)^{\kappa-1}\upsilon_i(t-s)\sum_{j=1}^{n} d_{ij}(s)[h_{ij}(y_j(s-\varsigma_{ij}(s))) - h_{ij}(x_j(s-\varsigma_{ij}(s)))]dB_j(s)\bigg|^2\\
:= &\sum_{i=1}^{4}\Lambda_i(t). \qquad (3.21)
\end{aligned}$$

Therefore, we have

$$\Lambda_1(t) = 4E|U_i(t-t_0)z_i(t_0)|^2 \leq 4\Big(\int_0^{\infty}\zeta_\kappa(\gamma)d\gamma\Big)^2 E\big|\chi-\varphi\big|_1^2 \leq 4E\big|\chi-\varphi\big|_1^2. \qquad (3.22)$$

By the Cauchy-Schwarz inequality, we can deduce that

$$\begin{aligned}
\Lambda_2(t) \leq &4\Big[\kappa\int_{t_0}^{t}\int_0^{\infty}\gamma\zeta_\kappa(\gamma)(t-s)^{\kappa-1}e^{-a_i(t-s)^\kappa\gamma}d\gamma ds\Big]\\
&\times\frac{\kappa}{\Gamma(\kappa+1)}\sum_{j=1}^{n}(b_{ij}^{+})^2\sum_{j=1}^{n}(F_j^L)^2\int_{t_0}^{t}(t-s)^{\kappa-1}E\big|y_j(s)-x_j(s)\big|^2 ds\\
\leq &4\Big[\int_0^{\infty}\zeta_\kappa(\gamma)d\gamma\int_0^{\infty}e^{-a_i z}dz\Big]\frac{1}{\Gamma(\kappa)}\sum_{j=1}^{n}(b_{ij}^{+})^2\sum_{j=1}^{n}(F_j^L)^2\int_{t_0}^{t}(t-s)^{\kappa-1}E\big|z_j(s)\big|^2 ds.\\
\leq &\frac{4}{a_i\Gamma(\kappa)}\sum_{j=1}^{n}(b_{ij}^{+})^2\sum_{j=1}^{n}(F_j^L)^2\int_{t_0}^{t}(t-s)^{\kappa-1}E\big|z_j(s)\big|^2 ds, \qquad (3.23)
\end{aligned}$$

and

$$\Lambda_3(t) \leq \frac{4}{a_i\Gamma(\kappa)}\sum_{j=1}^{n}(c_{ij}^{+})^2\sum_{j=1}^{n}(G_j^L)^2\int_{t_0}^{t}(t-s)^{\kappa-1}E\big|z_j(s-\tau_{ij}(s))\big|^2 ds. \qquad (3.24)$$

In addition, based on the Itô isometry, we gain that

$$\begin{aligned}
\Lambda_4(t) = &4E\Big\{\int_{t_0}^{t}(t-s)^{2(\kappa-1)}\psi_i^2(t-s)\Big|\sum_{j=1}^{n} d_{ij}(s)[h_{ij}(y_j(s-\varsigma_{ij}(s))) - h_{ij}(x_j(s-\varsigma_{ij}(s)))]\Big|^2 ds\Big\}\\
\leq &\frac{4}{\Gamma^2(\kappa)}\sum_{j=1}^{n}(d_{ij}^{+})^2\sum_{j=1}^{n}(H_{ij}^L)^2\int_{t_0}^{t}(t-s)^{2(\kappa-1)}E\big|z_j(s-\varsigma_{ij}(s))\big|^2 ds. \qquad (3.25)
\end{aligned}$$

Let $\eta(t) = \max_{i\in\mathcal{J}}\{\eta_i(t)\}$, where $\eta_i(t) = \sup_{\theta\in[t_0-\zeta,t]} |z_i(\theta)|$. Then by substituting (3.22)-(3.25) into (3.21), we can get

$$
\begin{aligned}
&E\left|z_i(t)\right|^2 \\
\leq &4E\left|\chi-\varphi\right|_1^2 + \frac{4}{a_i\Gamma(\kappa)}\Big[\sum_{j=1}^n (b_{ij}^+)^2 \sum_{j=1}^n (F_j^L)^2 + \sum_{j=1}^n (c_{ij}^+)^2 \sum_{j=1}^n (G_j^L)^2\Big] \int_{t_0}^t (t-s)^{\kappa-1} E\left|\eta_j(s)\right|^2 ds \\
&+ \frac{4}{\Gamma^2(\kappa)} \sum_{j=1}^n (d_{ij}^+)^2 \sum_{j=1}^n (H_{ij}^L)^2 \int_{t_0}^t (t-s)^{2(\kappa-1)} E\left|\eta_j(s)\right|^2 ds, \ \ t\geq t_0, i\in\mathcal{J}.
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
&E\left|z(t)\right|_0^2 \\
\leq &4E\left|\chi-\varphi\right|_1^2 + \frac{\Theta}{\Gamma(\kappa)}\int_{t_0}^t (t-s)^{\kappa-1} E\left|\eta(s)\right|_0^2 ds + \frac{\Omega}{\Gamma(2\kappa-1)}\int_{t_0}^t (t-s)^{2(\kappa-1)} E\left|\eta(s)\right|_0^2 ds\Big\} \\
=:&U(t).
\end{aligned}
$$

Obviously, there holds $E\left|z(\theta)\right|_0^2 \leq U(t)$ for any $\theta\in[t_0,t]$. Therefore, for all $t\in[t_0,S]$,

$$
\begin{aligned}
E\left|\eta(t)\right|_0^2 &\leq \max\Big\{\max_{i\in\mathcal{J}}\Big\{\sup_{\theta\in[t_0-\zeta,t_0]} E\left|z_i(\theta)\right|^2\Big\}, \max_{i\in\mathcal{J}}\Big\{\sup_{\theta\in[t_0,t]} E\left|z_i(\theta)\right|^2\Big\}\Big\} \\
&\leq \max\{E\left|\chi-\varphi\right|_1^2, U(t)\} = U(t).
\end{aligned}
$$

By virtue of Lemma 2.1 and the definition of $U(t)$, for all $t\in[t_0,S]$, we can obtain

$$
E\left|z(t)\right|_0^2 \leq E\left|\eta(t)\right|_0^2 \ \leq 4[E_\kappa(\Theta S^\kappa) + E_{2\kappa-1}(\Omega S^{2\kappa-1}) - 1]E\left|\chi-\varphi\right|_1^2.
$$

Thus, if $E\left|\chi-\varphi\right|_1^2 < \delta$ and hypothesis (3.20) holds, we have $E\left|z(t)\right|_0^2 < \varepsilon$, for any $t\in[t_0,S]$. This ends the proof. □

4. An example

To verify our results, let us consider the following example:

Example 4.1. *In system* (1.1)*, let* $n = 2$*,* $t_0 = 0$*,* $a_1 = 1$ *and* $a_2 = 2$*, for* $i, j \in \{1,2\}$*, take*

$$f_j(x) = 0.03\sin x + 0.153\arctan x,$$

$$g_j(x) = 0.045\sin x - 0.0095\tanh x,$$

$$h_{ij}(x) = 0.06\arctan x + 0.0251\tanh x,$$

$$\tau_{11}(t) = \tau_{21}(t) = 0.25 + 0.1\cos\sqrt{3}t, \quad \tau_{12}(t) = \tau_{22}(t) = 0.3 + 0.14\sin t,$$

$$\varsigma_{11}(t) = \varsigma_{21}(t) = 0.2 + 0.05\sin t, \quad \varsigma_{12}(t) = \varsigma_{22}(t) = 0.1 + 0.06\cos t,$$

$$u_1(t) = 0.05\cos\sqrt{11}t + 0.11\sin t, \quad u_2(t) = 0.12\cos 3t + 0.09\sin\sqrt{5}t,$$
$$b_{11}(t) = 0.04\sin t + 0.05\sin\sqrt{3}t, \quad b_{12}(t) = 0.03\cos\sqrt{7}t + 0.1\sin t,$$
$$b_{21}(t) = 0.02 + 0.1\sin\sqrt{3}t, \quad b_{22}(t) = 0.3 + 0.02\sin\sqrt{5}t,$$
$$c_{11}(t) = 0.1 - 0.2\sin\sqrt{5}t, \quad c_{12}(t) = 0.04 - 0.1\cos\sqrt{3}t,$$
$$c_{21}(t) = 0.02\cos t + 0.3\sin\sqrt{7}t, \quad c_{22}(t) = 0.1\sin\sqrt{3}t - 0.2\sin t,$$
$$d_{11}(t) = 0.1\sin t + 0.04\sin\sqrt{5}t, \quad d_{12}(t) = 0.03 - 0.5\cos\sqrt{2}t,$$
$$d_{21}(t) = 0.1 + 0.4\sin t, \quad d_{22}(t) = 0.02\cos\sqrt{11}t + 0.3\cos t.$$

When $\kappa = 0.8$, *by a simple calculation, we can get*

$$L_j^f = 0.0453,\ L_j^g = 0.0545,\ L_{ij}^\delta = 0.0851,\ b_{11}^+ = 0.09,\ b_{12}^+ = 0.13,$$
$$b_{21}^+ = 0.12,\quad b_{22}^+ = 0.32,\ c_{11}^+ = 0.3,\ c_{12}^+ = 0.14,\ c_{21}^+ = 0.32,$$
$$c_{22}^+ = 0.3,\ d_{11}^+ = 0.14,\ d_{12}^+ = 0.53,\ d_{21}^+ = 0.5,\ d_{22}^+ = 0.32,$$
$$\rho = \max\{0.0103, 0.0081\} < \frac{1}{4}, \quad \text{and} \quad Q = \max\{0.0475, 0.0311\} < 1.$$

By Theorem 3.1, we see that the system (1.1) *has an almost periodic solution in distribution.*

Choosing $\delta = 0.1$, $\varepsilon = 0.5$ *and* $S = 10$, *then*

$$\Theta = 0.0032, \quad \Omega = 0.0224, \quad \Theta S^\kappa = 0.0205, \quad \Omega S^{2\kappa-1} = 0.0893,$$
$$4\delta[E_\kappa(\Theta S^\kappa) + E_{2\kappa-1}(\Omega S^{2\kappa-1}) - 1] \approx 0.452 < 0.5 = \varepsilon.$$

By Theorem 3.2, the almost periodic solution in distribution of system (1.1) *is mean square finite-time stable with respect to* $\{0.1, 0.5, 10\}$ *(see Figure 1).*

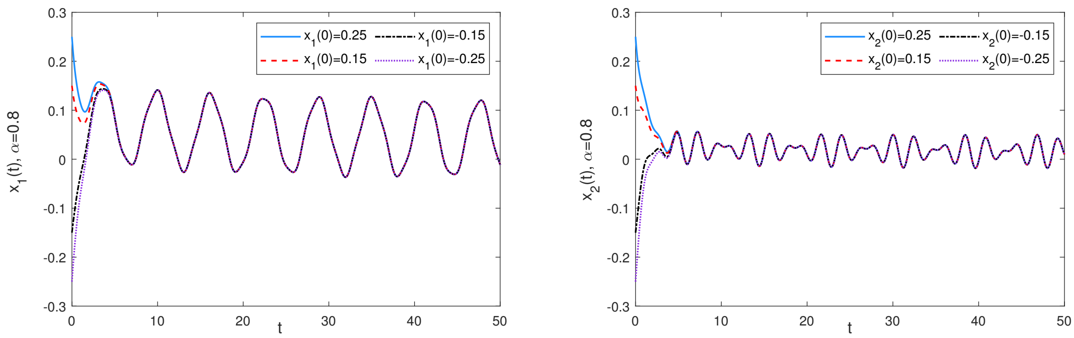

Figure 1. Curves of $x_1(t)$ and $x_2(t)$ for $\kappa = 0.8$ with the initial values $x_1(0) = x_2(0) = 0.25, 0.15, -0.15, -0.25$.

Similarly, when $\kappa = 0.6$, *choose* $\delta = 0.1$, $\varepsilon = 0.5$, *and* $S = 20$, *we have*

$$\rho = \max\{0.0212, 0.0123\} < \frac{1}{4}, \quad Q = \max\{0.0984, 0.0872\} < 1,$$
$$\Theta = 0.0032, \quad \Omega = 0.0423, \quad \Theta S^\kappa = 0.0196, \quad \Omega S^{2\kappa-1} = 0.0769,$$
$$4\delta[E_\kappa(\Theta S^\kappa) + E_{2\kappa-1}(\Omega S^{2\kappa-1}) - 1] \approx 0.445 < 0.5 = \varepsilon.$$

By Theorem 3.2, the almost periodic solution in distribution of system (1.1) *is mean square finite-time stable with respect to* $\{0.1, 0.5, 20\}$ *(see Figure 2).*

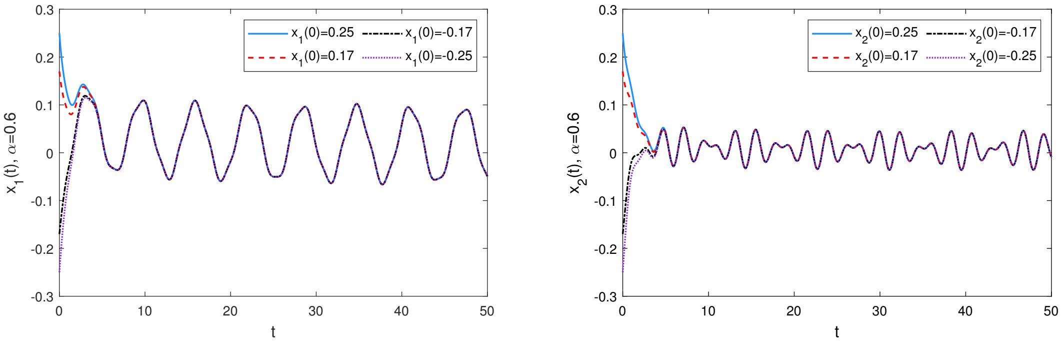

Figure 2. Curves of $x_1(t)$ and $x_2(t)$ for $\kappa = 0.6$ with the initial values $x_1(0) = x_2(0) = 0.25, 0.17, -0.17, -0.25$.

5. Conclusions

This study investigates the existence and uniqueness of almost periodic solutions in the distributional sense for fractional-order stochastic Hopfield neural networks with time-varying delays. It should be emphasized that this work constitutes the first systematic exploration of distributional almost periodic solutions in fractional-order stochastic neural networks, rendering the obtained results fundamentally novel. Furthermore, the proposed methodology demonstrates applicability to analyzing almost periodic solutions in other classes of fractional-order stochastic neural networks. The research significantly enriches the dynamical theory of fractional-order stochastic neural systems.

Author contributions

Yongkun Li: Conceptualization, writing-review and editing, supervision, funding acquisition; Binrong Peng: Writing-original draft, visualization. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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