Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(8): 18356–18380.
DOI: 10.3934/math.2025820
Received: 23 April 2025
Revised: 21 June 2025
Accepted: 25 July 2025
Published: 13 August 2025

Research article

Robust stability of switched interconnected systems with multiple time delays and switching uncertainties

Huanbin Xue and Xiaopeng Yang*

School of Mathematics and Statistics, Hanshan Normal University, Chaozhou 521041, China

* **Correspondence:** Email: 706697032@qq.com.

Abstract: This study investigates the problem of robust stability in a class of switched interconnected systems that involve unstable modes, time-varying and continuously distributed state delays, as well as uncertainties in switching signal and system parameters. Switching uncertainties lead to fluctuations in both the prescribed switching instants and the nominal switching sequence, which significantly affect system stability. To address these challenges, two novel concepts are introduced: the composite switching signal and the generalized nominal switching signal. Additionally, a new index, referred to as the generalized mode-changing rate, is proposed. By utilizing these concepts and the new index, the average dwell time approach and the vector Lyapunov function method are integrated to derive sufficient conditions for ensuring the robust exponential stability of the system. Finally, a numerical example is provided to illustrate the effectiveness and applicability of the proposed theory.

Keywords: robust stability; switched system; switching uncertainties; unstable modes; unbounded delay
Mathematics Subject Classification: 34D20

1. Introduction

In recent decades, switched systems have attracted significant attention in research due to their potentially wide applications in practice and theory. A typical switched system consists of a finite set of continuous-time or discrete-time subsystems, which are controlled by a switching signal. Extensive studies have been conducted on their stability and stabilization, leading to a wealth of theoretical results (see, e.g., [1–4], and references therein). The stability of a switched system depends not only on the stability of each individual mode (subsystem) but also on the characteristics of the switching signals. Even if all modes are stable, the system can become unstable under incorrect switching signals. Conversely, appropriate switching signals can stabilize a system composed entirely of unstable modes [1]. In early research on the stability and stabilization of switched systems, it was generally assumed that all subsystems were stable, and many meaningful results were obtained. However, in practical operations, systems are often influenced by various factors, such as uncertain disturbances and controller failures, which inevitably lead to the activation of unstable subsystems. As a result, switched nonlinear systems with unstable modes have attracted significant attention due to their theoretical challenges and practical applications. In recent years, numerous studies (e.g., those in references [5–11]) have investigated the stability and stabilization issues of switched systems with unstable modes. In [5], the average dwell time technique is applied to analyze switched systems consisting of both Hurwitz stable and unstable subsystems. A specific set of switching laws is proposed, which ensures the exponential stability of the entire system by enforcing particular activation time ratios between stable and unstable subsystems, thus providing a solid theoretical foundation for stability analysis. The work in [7] presents a general strategy for stabilizing switched nonlinear systems that feature partially unstable modes, eliminating the need for the conventional constant ratio condition (referred to as the μ-condition). The study in [8] investigates the input-to-state stability (ISS) of impulsive switched nonlinear systems with unstable subsystems. Sufficient conditions for ensuring ISS are derived using Lyapunov functions and the method of average impulsive switching intervals. In [9], the exponential stability and non-weighted L_2-gain performance of switched neutral systems with unstable subsystems are explored by employing multiple discontinuous Lyapunov functions and mode-dependent average dwell time strategies, and sufficient conditions are derived using the linear matrix inequality method. The study in [10] investigates the asymptotic stability of discrete-time switched systems that consist solely of unstable subsystems. Using Lyapunov functions and divergence time, the research derives linear matrix inequalities as stability conditions.

Time delays are prevalent and unavoidable in many practical systems. The existence of delays can have a significant impact on system performance and, in some instances, may even destroy system stability. Therefore, the influence of time delays on system performance should not be overlooked. In switched systems, the impact of time delays is not only manifested in the influence of past states on the system but also in their potential effect on the switching signals, which may result in asynchronous switching. Significant progress has been made in the analysis of stability and stabilization in switched systems with asynchronous switching. For instance, studies such as those in [12–16] provide valuable insights into this area.

In real-world applications, uncertainty in switching frequently arises due to challenges in accurately executing switching signals. These challenges stem from factors such as communication delays between the controller and the plant, environmental disturbances, and sensor noise. Despite its practical relevance, this issue has been rarely addressed in the literature. In [17], a stability analysis framework is presented for switched systems subject to temporary uncertainties in the switching signal, where dwell-time-based conditions are derived for discrete-time systems with input delays and continuous-time systems operating within digital control loops. In [18], the robust stability of quasi-periodic hybrid dynamical systems with polytopic uncertainties is investigated, where stability conditions are derived based on a repeated switching cycle. Another type of uncertainty arises in stochastic switched systems, where the transition rate is not precisely known [19]. In [20], an event-triggered resilient asynchronous estimation approach is proposed for stochastic Markovian jumping complex-valued networks with missing measurements, wherein a co-design control strategy integrating robust terms and dynamic triggering intervals is developed to guarantee mean-square stability of the augmented error system. In [21], a novel sliding mode control scheme is developed for the stabilization of delayed uncertain semi-Markovian jumping complex-valued networks, where generalized Dynkin's formula and Lyapunov stability theory are combined to ensure stochastic stability and finite-time convergence to the sliding mode surface. Asynchronous switching represents a specific form of disturbance within switched systems. Disturbances affect not only the system's operating parameters but also modify both the switching sequence and the timing of switching events. When uncertainty is introduced into switching signals, traditional robust or adaptive analysis methods become challenging to apply. To address these challenges, Yang et al. introduced novel indices, namely time-changing ratios and mode-changing ratios. Building on these concepts, the robustness of systems with disturbances in both switching sequences and switching instants was studied [22]. In contrast, Sun proposed a state-feedback path-wise switching signal for discrete-time switched linear systems, which guarantees exponential stability and robustness against switching perturbations [23]. Unlike the studies mentioned above, the disturbance considered in this paper not only affects the system's operating parameters but also alters its switching sequence and switching instants.

Switched interconnected systems, which consist of multiple subsystems that interact and are governed by switching signals, have attracted significant attention due to their broad range of applications in various industries and technologies, such as networked control systems, vehicle platooning and traffic control, and robotics and automated systems, as well as their importance in theoretical research [24–32]. The interdependence among the dynamics of subsystems means that the overall stability of the system is influenced not only by the stability of each subsystem but also, in certain cases, by the degree of interconnection between them. A conventional method for stability analysis involves defining the dynamics of each individual subsystem and then combining them to establish a stability criterion for the interconnection operator [33]. However, as the system's dimensionality increases, the complexity of this approach grows significantly. Specifically, the computational complexity involved in robust stability analysis expands dramatically as the system dimension rises [34]. Due to the coupling between subsystems, time delay effects, and the discrete and continuous dynamic characteristics, the dynamical behavior of switched-delay interconnected systems is highly complex. Even when assuming that all state variables are available for feedback control, achieving effective control remains challenging due to the critical role of information transfer between subsystems [35]. The inclusion of uncertain disturbances affecting switching signals further complicates the situation. Therefore, the study of robust stability in switched interconnected systems holds substantial theoretical and practical importance, which provides the motivation for the current research.

This paper focuses on switched interconnected systems characterized by time-varying and continuously distributed state delays, as well as parameter and switching uncertainties, and investigates the robust exponential stability of such systems without individual controllers for each mode. The system parameters are subject to uncertain disturbances and are confined within a specific range, with both the nominal switching sequence and the switching instants potentially changing. To the best of our knowledge, no prior research has explored this particular aspect.

The primary contributions of this paper are outlined as follows: (1) Two innovative concepts are introduced in the presence of switching uncertain disturbances: the composite switching signal and the generalized nominal switching signal. Additionally, a new index, referred to as the generalized mode-changing rate, is presented. These new indices effectively capture the differences between the perturbed and nominal switching signals. (2) A sufficient condition for robust exponential stability is derived using the average dwell time ratio and the vector Lyapunov function. This condition effectively addresses the limitations of conventional stability criteria, which fail when the dwell times of individual subsystems are excessively short or long. (3) Results derived from the linear matrix inequality approach generally involve manually identifying specific parameters, while the conditions outlined in this paper are presented in algebraic form, making them more straightforward and practical for real-world applications.

2. Model description and preliminaries

In this paper, we consider the following switched-delay interconnected uncertain system, which is composed of p modes described by:

$$
\begin{aligned}
\Pi_{\sigma(t)}:\ \dot{\boldsymbol{e}}_i(t) =&\ \boldsymbol{u}_i^{\sigma(t)}(t,\boldsymbol{e}_i(t)) + \sum_{j=1}^{q}\boldsymbol{C}_{ij}^{\sigma(t)}\boldsymbol{y}_{ij}^{\sigma(t)}(t,\boldsymbol{e}_j(t)) + \sum_{j=1}^{q}\boldsymbol{D}_{ij}^{\sigma(t)}\boldsymbol{y}_{ij}^{\sigma(t)}(t,\boldsymbol{e}_j(t-\tau_{ij}^{\sigma(t)}(t))) \\
&+ \sum_{j=1}^{q}\boldsymbol{G}_{ij}^{\sigma(t)}\int_{-\infty}^{t}\kappa_{ij}^{\sigma(t)}(t-s)\,\boldsymbol{y}_{ij}^{\sigma(t)}(t,\boldsymbol{e}_j(s))\,\mathrm{d}s, \\
\boldsymbol{e}_i(s) =&\ \boldsymbol{\phi}_i(s),\ s\in(-\infty,t_0],\ i=1,2,\ldots,q,
\end{aligned}
\tag{2.1}
$$

where q denotes the number of interconnected subsystems, and $\boldsymbol{e}_i(t)\in\mathbb{R}^n$ denotes the state vector of the i-th interconnected subsystem. Let Q denote the set $\{1,2,\ldots,q\}$ and N denote the set $\{1,2,\ldots,n\}$. The perturbed switching signal, denoted by $\sigma(t):\overline{\mathbb{R}}_+\to P=\{1,2,\ldots,p\}$, is a piecewise constant function of time that is right-continuous. For $l\in P$, let $\dot{\boldsymbol{e}}_i(t)=\boldsymbol{u}_i^l(t,\boldsymbol{e}_i(t))$ be the isolated subsystem, where $\boldsymbol{u}_i^l$ is mapping from $\overline{\mathbb{R}}_+\times\mathbb{R}^n$ to $\mathbb{R}^n$. $\boldsymbol{y}_{ij}^l\left(t,\boldsymbol{e}_j\right):\overline{\mathbb{R}}_+\times\mathbb{R}^n\to\mathbb{R}^n$ denotes the interconnection function between subsystems. The matrices $\boldsymbol{C}_{ij}^l$, $\boldsymbol{D}_{ij}^l$, and $\boldsymbol{G}_{ij}^l$ are incidence matrices of appropriate dimensions corresponding to mode Π_l. Consider the bounded function $\boldsymbol{\tau}_{ij}^l(t):\overline{\mathbb{R}}_+\to\overline{\mathbb{R}}_+^n$, which represents the state delays of the system. Let $\tau^l=\max_{i,j\in Q,k\in N}\{\sup_{t\geq t_0}\tau_{ij}^{l\cdot k}(t)\}$. The kernel function $\kappa_{ij}^l:\overline{\mathbb{R}}_+\to\overline{\mathbb{R}}_+$ is piecewise continuous on $\overline{\mathbb{R}}_+$ and satisfies the integral equation $\int_0^\infty e^{\gamma s}\kappa_{ij}^l(s)\,\mathrm{d}s=\vartheta_{ij}^l(\gamma)$, where $\vartheta_{ij}^l(\gamma)$ is continuous on $[0,\delta)$ for some $\delta>0$ and satisfies $\vartheta_{ij}^l(0)=1$. $\boldsymbol{\phi}_i\in C((-\infty,t_0],\mathbb{R}^n)$, where $C((-\infty,t_0],\mathbb{R}^n)$ represents the set of continuous functions mapping $(-\infty,t_0]$ to $\mathbb{R}^n$. Let $\|\boldsymbol{\phi}_i(s)\|_{\mathbb{C}_t}=\max_{j\in N}\{\sup_{s\in(-\infty,t]}|\phi_{ij}(s)|\}$.

Uncertainty of system parameters: Due to unavoidable factors such as modeling inaccuracies and external disturbances, uncertainties are inevitably present in system models. These uncertainties can significantly impact the dynamic performance of the systems. To effectively evaluate the robustness of the systems, a practical approach is assuming that system parameters lie within certain specified intervals. The interval matrix is defined as follows:

$$
\boldsymbol{C}_{ij}^{\mathrm{I}\cdot l}=\left\{\boldsymbol{C}_{ij}^l=\left(c_{km}^{ij\cdot l}\right):\underline{\boldsymbol{C}}_{ij}^l\leq\boldsymbol{C}_{ij}^l\leq\overline{\boldsymbol{C}}_{ij}^l,\text{i.e.,}\,\underline{c}_{km}^{ij\cdot l}\leq c_{km}^{ij\cdot l}\leq\overline{c}_{km}^{ij\cdot l},\ k,m\in N\right\},
$$

$$
\boldsymbol{D}_{ij}^{\mathrm{I}\cdot l}=\left\{\boldsymbol{D}_{ij}^l=\left(d_{km}^{ij\cdot l}\right):\underline{\boldsymbol{D}}_{ij}^l\leq\boldsymbol{D}_{ij}^l\leq\overline{\boldsymbol{D}}_{ij}^l,\text{i.e.,}\,\underline{d}_{km}^{ij\cdot l}\leq d_{km}^{ij\cdot l}\leq\overline{d}_{km}^{ij\cdot l},\ k,m\in N\right\},
$$

$$
\boldsymbol{G}_{ij}^{\mathrm{I}\cdot l}=\left\{\boldsymbol{G}_{ij}^l=\left(g_{km}^{ij\cdot l}\right):\underline{\boldsymbol{G}}_{ij}^l\leq\boldsymbol{G}_{ij}^l\leq\overline{\boldsymbol{G}}_{ij}^l,\text{i.e.,}\,\underline{g}_{km}^{ij\cdot l}\leq g_{km}^{ij\cdot l}\leq\overline{g}_{km}^{ij\cdot l},\ k,m\in N\right\},
$$

and assume that $\boldsymbol{C}_{ij}^l\in\boldsymbol{C}_{ij}^{\mathrm{I}\cdot l}$, $\boldsymbol{D}_{ij}^l\in\boldsymbol{D}_{ij}^{\mathrm{I}\cdot l}$, and $\boldsymbol{E}_{ij}^l\in\boldsymbol{E}_{ij}^{\mathrm{I}\cdot l}$. Define $\boldsymbol{C}_{ij}^{\circledast\cdot l}=\left(c_{km}^{ij\cdot l\circledast}\right)_{n\times n}$, $\boldsymbol{D}_{ij}^{\circledast\cdot l}=\left(d_{km}^{ij\cdot l\circledast}\right)_{n\times n}$, and $\boldsymbol{G}_{ij}^{\divideontimes\cdot l}=\left(g_{km}^{ij\cdot l\divideontimes}\right)_{n\times n}$, where $c_{km}^{ij\cdot l\divideontimes} = \max\left\{\left|\underline{c}_{km}^{ij\cdot l}\right|, \left|\overline{c}_{km}^{ij\cdot l}\right|\right\}$, $d_{km}^{ij\cdot l\divideontimes} = \max\left\{\left|\underline{d}_{km}^{ij\cdot l}\right|, \left|\overline{d}_{km}^{ij\cdot l}\right|\right\}$, and $g_{km}^{ij\cdot l\divideontimes} = \max\left\{\left|\underline{g}_{km}^{ij\cdot l}\right|, \left|\overline{g}_{km}^{ij\cdot l}\right|\right\}$.

For system (2.1), we assume that the state does not experience any jumps at switching times and that only a finite number of switchings can occur within any given finite interval. Additionally, $\boldsymbol{u}_i^l(\boldsymbol{0}) = \boldsymbol{0}$ and $\boldsymbol{y}_{ij}^l(\boldsymbol{0}) = \boldsymbol{0}$, for $l \in P$ and $i, j \in Q$, while satisfying the following conditions.

Assumption 1. *$\forall l \in P$ and $i, j \in Q$, the functions $\boldsymbol{y}_{ij}^l : \overline{\mathbb{R}}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ are globally Lipschitz continuous with corresponding Lipschitz constants $L_{ij}^l > 0$. Specifically, for all vectors $\boldsymbol{e}_1, \boldsymbol{e}_2 \in \mathbb{R}^n$, the following inequality holds:*

$$\left\|\boldsymbol{y}_{ij}^l(t, \boldsymbol{e}_1(t)) - \boldsymbol{y}_{ij}^l(t, \boldsymbol{e}_2(t))\right\| \leq L_{ij}^l \left\|\boldsymbol{e}_1(t) - \boldsymbol{e}_2(t)\right\|. \tag{2.2}$$

Assumption 2. *There exist continuously differentiable functions $V_i^l(t, \boldsymbol{e}_i) \in C^1$ and positive constants $\underline{a}_i^l$, $\overline{a}_i^l$, α_i^l, β_i^l, and γ_i^l, such that the following inequalities hold:*

$$\underline{a}_i^l \|\boldsymbol{e}_i(t)\|^2 \leq V_i^l(t, \boldsymbol{e}_i(t)) \leq \overline{a}_i^l \|\boldsymbol{e}_i(t)\|^2, \forall l \in P, \tag{2.3}$$

$$\left\|\frac{\partial V_i^k(t, \boldsymbol{e}_i(t))}{\partial \boldsymbol{e}_i(t)}\right\| \leq \gamma_i^l \|\boldsymbol{e}_i(t)\|, \forall l \in P, \tag{2.4}$$

$$\frac{\partial V_i^l(t, \boldsymbol{e}_i(t))}{\partial t} + \left(\frac{\partial V_i^l(t, \boldsymbol{e}_i(t))}{\partial \boldsymbol{e}_i(t)}\right)^{\mathrm{T}} \boldsymbol{u}_i^l(t, \boldsymbol{e}_i(t)) \leq -\alpha_i^l \|\boldsymbol{e}_i(t)\|^2, \forall l \in P_1, \tag{2.5}$$

$$\frac{\partial V_i^l(t, \boldsymbol{e}_i(t))}{\partial t} + \left(\frac{\partial V_i^l(t, \boldsymbol{e}_i(t))}{\partial \boldsymbol{e}_i(t)}\right)^{\mathrm{T}} \boldsymbol{u}_i^l(t, \boldsymbol{e}_i(t)) \leq \beta_i^l \|\boldsymbol{e}_i(t)\|^2, \forall l \in P_2, \tag{2.6}$$

where $P_1 \cup P_2 = P$ and $P_1 \cap P_2 = \emptyset$.

Assumption 2 is commonly used, as demonstrated in references such as [22, (3)–(6)], [36, Assumption A3], [37, Theorem 1], and [38, Assumption 1].

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ represent the sets of modes Π_p where $p \in P_1$ and Π_q where $q \in P_2$, respectively.

Uncertainty of switching signals: Let $\Sigma(t) : \overline{\mathbb{R}}_+ \to P$ be the nominal switching signal, which is a piecewise constant, right-continuous function and can be explicitly designed. Let $\Delta(t)$ denote the uncertainty in the switching signal, causing $\sigma(t)$ to deviate from $\Sigma(t)$, such that

$$\sigma(t) = \Sigma(t) + \Delta(t). \tag{2.7}$$

Let $\bar{t}_i$ represent the i-th switching instant of $\Sigma(t)$, where $i = 0, 1, \ldots$. For $t \in [\bar{t}_i, \bar{t}_{i+1})$, it follows that $\Sigma(t) = \bar{l}_i \in Q$. Define $\mathrm{N}_\Sigma(t_L, t_R)$ as the total number of switchings of $\Sigma(t)$ within the interval $[t_L, t_R]$, which can be decomposed into $\mathrm{N}_\Sigma^1(t_L, t_R)$ and $\mathrm{N}_\Sigma^2(t_L, t_R)$, such that:

$$\mathrm{N}_\Sigma(t_L, t_R) = \mathrm{N}_\Sigma^1(t_L, t_R) + \mathrm{N}_\Sigma^2(t_L, t_R).$$

Here, $\mathrm{N}_\Sigma^1(t_L, t_R)$ and $\mathrm{N}_\Sigma^2(t_L, t_R)$ represent the numbers of switchings for all modes belonging to $\mathcal{M}_1$ ($\forall l \in P_1, \Pi_l \in \mathcal{M}_1$) and $\mathcal{M}_2$ ($\forall l \in P_2, \Pi_l \in \mathcal{M}_2$), respectively, within the interval $[t_L, t_R]$.

Under the perturbed switching signal $\sigma(t)$, let $\hat{t}_i$ represent the i-th switching instant of system (2.1), where $i = 0, 1, \ldots$. For $t \in [\hat{t}_i, \hat{t}_{i+1})$, it follows that $\sigma(t) = l_i \in Q$. Define $\mathrm{N}_\sigma(t_L, t_R)$ as the total number of switchings of $\sigma(t)$ within the interval $[t_L, t_R]$, which can be decomposed into $\mathrm{N}^1_\sigma(t_L, t_R)$ and $\mathrm{N}^2_\sigma(t_L, t_R)$, such that:

$$\mathrm{N}_\sigma(t_L, t_R) = \mathrm{N}^1_\sigma(t_L, t_R) + \mathrm{N}^2_\sigma(t_L, t_R).$$

Here, $\mathrm{N}^1_\sigma(t_L, t_R)$ and $\mathrm{N}^2_\sigma(t_L, t_R)$ represent the numbers of switchings for all modes belonging to $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively, within the interval $[t_L, t_R]$.

Generalized nominal switching signal and composite switching signal: As previously discussed, the nominal switching signal $\Sigma(t)$ is designed in accordance with the operational objectives of the actual system. However, due to the influence of the switching uncertainty function $\Delta(t)$, the system is governed by the actual switching signal $\sigma(t)$. As a result, the switching instants and sequence of the nominal signal $\Sigma(t)$ may be altered, as illustrated in Figure 1(a) and 1(b). To assess the impact of the switching uncertainty function $\Delta(t)$ on the system's stability, reference [22] introduces the concept of time-varying ratios between the nominal and uncertain switching signals, under the assumption that the dwell times of both signals are proportional. In contrast, the present study proposes two novel concepts that eliminate the need for this assumption.

Definition 1. *Let the nominal switching signal $\Sigma(t)$ possess the switching instants set $\overline{T} = \{\bar{t}_0, \bar{t}_1, \ldots, \bar{t}_k, \ldots \mid 0 \le \bar{t}_0 < \bar{t}_1 < \ldots\}$, and the uncertain switching signal $\sigma(t)$ have the switching instants set $\hat{T} = \{\hat{t}_0, \hat{t}_1, \ldots, \hat{t}_k, \ldots \mid 0 \le \hat{t}_0 < \hat{t}_1 < \ldots\}$. Define the generalized switching time sequence as:*

$$T = \{t_0, t_1, \cdots \mid t_k \in \overline{T} \cup \hat{T},\ t_k < t_{k+1},\ k = 0, 1, \ldots\}.$$

We call the piecewise constant function

$$\overline{\Sigma}(t_k) = \begin{cases} \bar{t}_0 = \hat{t}_0, & k = 0, \\ \Sigma(t_k), & t_k \in \overline{T}, k \ge 1, \\ \Sigma(t_{k-1}), & t_k \in \hat{T}, k \ge 1, \end{cases}$$

the generalized nominal switching signal. Similarly, the composite switching signal is defined as

$$\overline{\sigma}(t_k) = \begin{cases} \bar{t}_0 = \hat{t}_0, & k = 0, \\ \sigma(t_{k-1}), & t_k \in \overline{T}, k \ge 1, \\ \sigma(t_k), & t_k \in \hat{T}, k \ge 1. \end{cases}$$

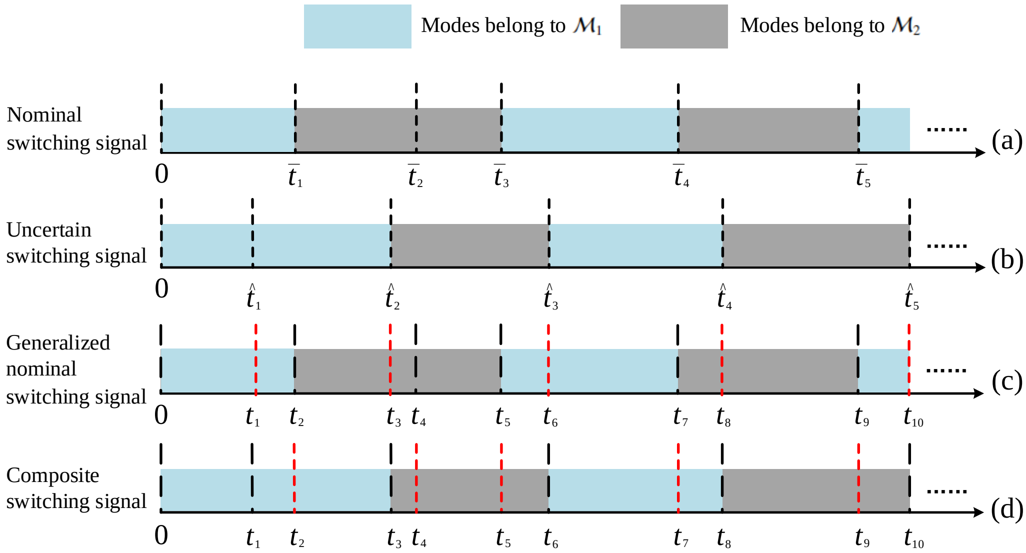

Figure 1. Nominal switching signal, uncertain switching signal, generalized nominal switching signal, and composite switching signal.

Remark 1. *As shown in Figure 1(c) and 1(d), the generalized nominal switching signal $\bar{\Sigma}(t)$ is constructed by integrating the switching instants $\hat{t}_k$ of the perturbed switching signal $\sigma(t)$ into the nominal switching signal $\Sigma(t)$. At these integrated switching moments $\hat{t}_k$, it holds that $\bar{\Sigma}(\hat{t}_k^-) = \bar{\Sigma}(\hat{t}_k^+)$ holds. In contrast, the composite switching signal is formed by incorporating the switching instants $\bar{t}_k$ of the nominal switching signal $\Sigma(t)$ into the perturbed switching signal $\sigma(t)$, such that $\bar{\sigma}(\bar{t}_k^-) = \bar{\sigma}(\bar{t}_k^+)$. In other words, neither $\bar{\Sigma}(t)$ nor $\bar{\sigma}(t)$ exhibit any mode switching at times $t = \hat{t}_k$ and $t = \bar{t}_k$, respectively.*

Similarly, the numbers of switchings for the generalized nominal switching signal $\bar{\Sigma}(t)$ and the composite switching signal $\bar{\sigma}(t)$ over the interval $[t_L, t_R]$ are denoted as $N_{\bar{\Sigma}}(t_L, t_R)$ and $N_{\bar{\sigma}}(t_L, t_R)$, respectively. In addition, under $\bar{\Sigma}(t)$ (resp., $\bar{\sigma}(t)$), the numbers of switchings corresponding to all modes $\Pi_l \in \mathcal{M}_1$ and $\Pi_j \in \mathcal{M}_2$ are denoted by $N^1_{\bar{\Sigma}}(t_L, t_R)$ and $N^2_{\bar{\Sigma}}(t_L, t_R)$ (resp., $N^1_{\bar{\sigma}}(t_L, t_R)$, and $N^2_{\bar{\sigma}}(t_L, t_R)$). For ease of reference in the following analysis, we introduce the following shorthand notation:

$$N_{\Sigma}(t) = N_{\Sigma}(t_0, t), \quad N^1_{\Sigma}(t) = N^1_{\Sigma}(t_0, t), \quad N^2_{\Sigma}(t) = N^2_{\Sigma}(t_0, t),$$

$$N_{\sigma}(t) = N_{\sigma}(t_0, t), \quad N^1_{\sigma}(t) = N^1_{\sigma}(t_0, t), \quad N^2_{\sigma}(t) = N^2_{\sigma}(t_0, t),$$

$$N_{\bar{\Sigma}}(t) = N_{\bar{\Sigma}}(t_0, t), \quad N^1_{\bar{\Sigma}}(t) = N^1_{\bar{\Sigma}}(t_0, t), \quad N^2_{\bar{\Sigma}}(t) = N^2_{\bar{\Sigma}}(t_0, t),$$

$$N_{\bar{\sigma}}(t) = N_{\bar{\sigma}}(t_0, t), \quad N^1_{\bar{\sigma}}(t) = N^1_{\bar{\sigma}}(t_0, t), \quad N^2_{\bar{\sigma}}(t) = N^2_{\bar{\sigma}}(t_0, t).$$

For nominal switching signal $\Sigma(t)$, let $\mathcal{R} \in [0, 1]$ be the upper bound of the ratio between the number of modes belonging to $\mathcal{M}_2$ and the total number of prescribed modes, i.e.,

$$\frac{N^2_{\Sigma}(t)}{N_{\Sigma}(t)} \leq \mathcal{R}. \tag{2.8}$$

Remark 2. *In many cases, modes $\Pi_l \in \mathcal{M}_2$ are unavoidable and must be activated. Therefore, the parameter $\mathcal{R}$ can be predetermined during the design of the nominal switching signal based on practical considerations. If $\mathcal{R} = 0$, it indicates that no modes $\Pi_l \in \mathcal{M}_2$ are scheduled for activation. Since $\Sigma(t)$ and $\overline{\Sigma}(t)$ essentially represent the same switching behavior and are both capable of stabilizing system* (2.1)*, then, provided that all stable modes share a common lower bound on their convergence rates and all unstable modes share a common upper bound on their divergence rates,* (2.8) *also implies that*

$$\frac{\mathrm{N}^2_{\overline{\Sigma}}(t)}{\mathrm{N}_{\overline{\Sigma}}(t)} \leq \mathcal{R}. \tag{2.9}$$

The switching uncertainty function $\Delta(t)$ causes a deviation between $\overline{\sigma}(t)$ and $\overline{\Sigma}(t)$, which results in the activation of a mode $\Pi_l \in \mathcal{M}_1$ (or $\Pi_l \in \mathcal{M}_2$) within the interval, while a mode $\Pi_k \in \mathcal{M}_2$ (or $\Pi_k \in \mathcal{M}_1$) may be activated instead. In the presence of the disturbance from the switching uncertainty function $\Delta(t)$, we introduce the following generalized mode switching ratios that describe the transition from $\overline{\Sigma}(t)$ to $\overline{\sigma}(t)$.

Definition 2. *By contrasting the generalized nominal switching signal $\overline{\Sigma}(t)$ with the composite switching signal $\overline{\sigma}(t)$. Let $\mathfrak{A}(t)$ (resp., $\mathfrak{B}(t)$) represent the number of modes Π_l, $l \in P_1$ (resp., Π_k, $k \in P_2$) in $\overline{\Sigma}(t)$ that are replaced by modes Π_k, $k \in P_2$ (resp., Π_l, $l \in P_1$) in $\overline{\sigma}(t)$ for $t \geq t_0$. If there exist constants $c \in [0, 1]$, $d \in [0, 1]$ such that:*

$$\mathfrak{A}(t) \leq c\mathrm{N}^1_{\overline{\Sigma}}(t), \tag{2.10}$$

$$\mathfrak{B}(t) \geq d\mathrm{N}^2_{\overline{\Sigma}}(t), \tag{2.11}$$

$$\frac{\mathrm{N}^1_{\overline{\sigma}}(t)}{\mathrm{N}_{\overline{\sigma}}(t)} \geq (1 - \mathcal{R})(1 - c) + \mathcal{R}d, \tag{2.12}$$

$$\frac{\mathrm{N}^2_{\overline{\sigma}}(t)}{\mathrm{N}_{\overline{\sigma}}(t)} \leq \mathcal{R}(1 - d) + (1 - \mathcal{R})c, \tag{2.13}$$

then $\{c, d\}$ are termed the generalized mode-changing ratios from $\overline{\Sigma}(t)$ to $\overline{\sigma}(t)$.

Definition 3 ([1,2])**.** *For system* (2.1)*, the equilibrium point $\boldsymbol{e}^* = \boldsymbol{0}$ is said to be robustly exponentially stable if there exist constants $\varepsilon > 0$ and $\kappa > 0$ such that, for $t \geq t_0$, the following condition holds:*

$$\|\boldsymbol{e}(t)\| \leq \kappa\|\boldsymbol{\Phi}\|_{t_0} \exp(-\varepsilon(t - t_0)),$$

where $\|\boldsymbol{\Phi}\|_{t_0}$ is defined as $\|\boldsymbol{\Phi}\|_{t_0} = \max_{i\in Q} \|\boldsymbol{\phi}_i(s)\|_{\mathbb{C}_{t_0}}$.

Lemma 1 ([39])**.** *Let $\boldsymbol{H}^I = \left[\underline{\boldsymbol{H}}, \overline{\boldsymbol{H}}\right] = \left\{\boldsymbol{H} = \left(h_{ij}\right)_{n\times n} : \underline{\boldsymbol{H}} \leq \boldsymbol{H} \leq \overline{\boldsymbol{H}}, \text{i.e.}, \underline{h}_{ij} \leq h_{ij} \leq \overline{h}_{ij}\right\}$ be an $n \times n$ interval matrix. $\boldsymbol{H}^* = \left(h^*_{ij}\right)_{n\times n}$, where $h^*_{ij} = \max_{i,j\in N} \left\{|\underline{h}_{ij}|, |\overline{h}_{ij}|\right\}$, then for any $\boldsymbol{H} \in \boldsymbol{H}^I$, it holds that*

$$\|\boldsymbol{H}\| \leq \|\boldsymbol{H}^*\|.$$

Remark 3. *Regarding the upper-bound norm estimation of the interval matrix, we can also refer to the following results:*

- *([39])* $\|\boldsymbol{H}\| \leq \left\|\boldsymbol{H}^{\star}\right\| + \|\boldsymbol{H}_{\star}\|$,
- *([40])* $\|\boldsymbol{H}\| \leq \sqrt{\|\boldsymbol{H}^{\star}\|^2 + \|\boldsymbol{H}_{\star}\|^2 + 2\left\|\boldsymbol{H}_{\star}^T\,|\boldsymbol{H}^{\star}|\right\|}$,
- *([41])* $\|\boldsymbol{H}\| \leq \sqrt{\left\||\boldsymbol{H}^{\star T}\boldsymbol{H}^{\star}| + 2\,|\boldsymbol{H}^{\star T}|\,\boldsymbol{H}_{\star} + \boldsymbol{H}_{\star}^T\boldsymbol{H}_{\star}\right\|}$,

where $\boldsymbol{H} \in \left[\underline{\boldsymbol{H}},\ \overline{\boldsymbol{H}}\right]$, $\boldsymbol{H}^{\star} = \left(\overline{\boldsymbol{H}} + \underline{\boldsymbol{H}}\right)/2$, *and* $\boldsymbol{H}_{\star} = \left(\overline{\boldsymbol{H}} - \underline{\boldsymbol{H}}\right)/2$.

Lemma 2 ([42])**.** *Let* $\boldsymbol{N}$ *be an* $n \times n$ *matrix whose off-diagonal elements are non-positive; then the following conditions are equivalent:*

- $\boldsymbol{N}$ *is a nonsingular M-matrix,*
- *There exists a positive vector* $\boldsymbol{\varrho}$ *such that* $\boldsymbol{N\varrho} > 0$.

3. Main result

3.1. Dynamical behavior analysis of modes

Theorem 1. *Consider the mode* $\Pi_l \in \mathcal{M}_1$ *described in Eq* (2.1)*, and assume that it satisfies conditions* (2.2)–(2.5)*. If* $\boldsymbol{Q}_l = (q_{ij}^l)$ *is a nonsingular* $n \times n$ *M-matrix, then the equilibrium of mode* Π_l *is robustly exponentially stable for all matrices* $\boldsymbol{C}_{ij}^l \in \boldsymbol{C}_{ij}^{l\cdot l}$, $\boldsymbol{D}_{ij}^l \in \boldsymbol{D}_{ij}^{l\cdot l}$, *and* $\boldsymbol{G}_{ij}^l \in \boldsymbol{G}_{ij}^{l\cdot l}$*. Where* $l \in P_1$ *and the entries of* $\boldsymbol{Q}_l$ *are defined by:*

$$
\begin{aligned}
q_{ii}^l =& \alpha_i^l \left(\overline{a}_i^l\right)^{-\frac{1}{2}} - \left(\underline{a}_i^l\right)^{-\frac{1}{2}} \gamma_i^l L_{ii}^l \left(\left\|\boldsymbol{C}_{ii}^{l\cdot *}\right\| + \left\|\boldsymbol{D}_{ii}^{l\cdot *}\right\| + \left\|\boldsymbol{G}_{ii}^{l\cdot *}\right\|\right),\\
q_{ij}^l =& -\left(\underline{a}_j^l\right)^{-\frac{1}{2}} \gamma_i^l L_{ij}^l \left(\left\|\boldsymbol{C}_{ij}^{l\cdot *}\right\| + \left\|\boldsymbol{D}_{ij}^{l\cdot *}\right\| + \left\|\boldsymbol{G}_{ij}^{l\cdot *}\right\|\right), i \neq j.
\end{aligned}
$$

Proof: $\forall\ l \in P_1$, let $W_i^l(t, \boldsymbol{e}_i(t)) = \exp\left(\lambda(t - t_0)\right) V_i^l(t, \boldsymbol{e}_i(t))$, where λ is yet to be determined. Calculating the upper right derivative $D^+W_i^l(t, \boldsymbol{e}_i(t))$ of $W_i^l(t, \boldsymbol{e}_i(t))$ along the trajectories of the mode Π_l, we can derive the following inequality:

$$
\begin{aligned}
D^+W_i^l(t, \boldsymbol{e}_i(t)) =& \lambda \exp\left(\lambda(t - t_0)\right) V_i^l(t, \boldsymbol{e}_i(t)) + \exp \lambda(t - t_0) \left\{ \frac{\partial V_i^l\left(t, \boldsymbol{e}_i\left(t\right)\right)}{\partial t} + \left(\frac{\partial V_i^l\left(t, \boldsymbol{e}_i\left(t\right)\right)}{\partial \boldsymbol{e}_i\left(t\right)}\right)^{\mathrm{T}} \right.\\
&\times \left[\boldsymbol{u}_i^l(t, \boldsymbol{e}_i(t)) + \sum_{j=1}^{q} \boldsymbol{C}_{ij}^l \boldsymbol{y}_{ij}^l(t, \boldsymbol{e}_j(t)) + \sum_{j=1}^{q} \boldsymbol{D}_{ij}^l \boldsymbol{y}_{ij}^l(t, \boldsymbol{e}_j(t - \tau_{ij}^l(t))) \right.\\
&\left.\left. + \sum_{j=1}^{q} \boldsymbol{G}_{ij}^l \int_{-\infty}^{t} \kappa_{ij}^l\left(t - s\right) \boldsymbol{y}_{ij}^l(t, \boldsymbol{e}_j\left(s\right))\, \mathrm{d}s \right] \right\}.
\end{aligned}
\tag{3.1}
$$

By applying the Schwartz inequality in conjunction with (2.2)–(2.5) and Lemma 1, we can obtain

$$
\begin{aligned}
& D^+W_i^l(t, \boldsymbol{e}_i(t))\\
\leq& \lambda \exp(\lambda(t - t_0)) \overline{a}_i^l \|\boldsymbol{e}_i\left(t\right)\|^2 - \exp(\lambda(t - t_0)) \alpha_i^l \|\boldsymbol{e}_i\left(t\right)\|^2\\
&+ \exp(\lambda(t - t_0)) \gamma_i^l \|\boldsymbol{e}_i\left(t\right)\| \left[\sum_{j=1}^{q} \|\boldsymbol{C}_{ij}^l\| \cdot \|\boldsymbol{y}_{ij}^l(t, \boldsymbol{e}_j(t))\| + \sum_{j=1}^{q} \|\boldsymbol{D}_{ij}^l\| \cdot \|\boldsymbol{y}_{ij}^l(t, \boldsymbol{e}_j(t - \tau_{ij}^l(t)))\| \right.
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{j=1}^{q} \|\boldsymbol{G}_{ij}^{l}\| \int_{-\infty}^{t} \kappa_{ij}^{l}(t-s) \|\boldsymbol{y}_{ij}^{l}(t, \boldsymbol{e}_j(s))\| \, \mathrm{ds} \Bigg] \\
\leq & \lambda \exp(\lambda(t-t_0)) \overline{a}_i^{l} \|\boldsymbol{e}_i(t)\|^2 - \exp(\lambda(t-t_0)) \alpha_i^{l} \|\boldsymbol{e}_i(t)\|^2 \\
&+ \exp(\lambda(t-t_0)) \gamma_i^{l} \|\boldsymbol{e}_i(t)\| \left[\sum_{j=1}^{q} \|\boldsymbol{C}_{ij}^{l}\| L_{ij}^{l} \|\boldsymbol{e}_j(t)\| + \sum_{j=1}^{q} \|\boldsymbol{D}_{ij}^{l}\| L_{ij}^{l} \|\boldsymbol{e}_j(t-\tau_{ij}^{l}(t))\| \right. \\
&\left. + \sum_{j=1}^{q} \|\boldsymbol{G}_{ij}^{l}\| L_{ij}^{l} \int_{-\infty}^{t} \kappa_{ij}^{l}(t-s) \|\boldsymbol{e}_j(s)\| \, \mathrm{ds} \right] \\
\leq & \exp(\lambda(t-t_0)/2) \|\boldsymbol{e}_i(t)\| \left[\left(\lambda \overline{a}_i^{l} - \alpha_i^{l} \right) \exp(\lambda(t-t_0)/2) \|\boldsymbol{e}_i(t)\| \right. \\
&+ \gamma_i^{l} \sum_{j=1}^{q} \|\boldsymbol{C}_{ij}^{l}\| L_{ij}^{l} \exp(\lambda(t-t_0)/2) \|\boldsymbol{e}_j(t)\| \\
&+ \exp(\lambda \tau^{l}/2) \gamma_i^{l} \sum_{j=1}^{q} \|\boldsymbol{D}_{ij}^{l}\| L_{ij}^{l} \exp(\lambda(t-t_0-\tau_{ij}^{l}(t))/2) \|\boldsymbol{e}_j(t-\tau_{ij}^{l}(t))\| \\
&\left. + \gamma_i^{l} \sum_{j=1}^{q} \|\boldsymbol{G}_{ij}^{l}\| L_{ij}^{l} \int_{-\infty}^{t} \kappa_{ij}^{l}(t-s) \exp(\lambda(t-s)/2) \exp(\lambda(s-t_0)/2) \|\boldsymbol{e}_j(s)\| \, \mathrm{ds} \right] \\
\leq & \sqrt{W_i^{l}(t, \boldsymbol{e}_i(t))} \left[\left(\lambda \overline{a}_i^{l} (\underline{a}_i^{l})^{-\frac{1}{2}} - \alpha_i^{l} (\overline{a}_i^{l})^{-\frac{1}{2}} \right) \sqrt{W_i^{l}(t, \boldsymbol{e}_i(t))} + \gamma_i^{l} \sum_{j=1}^{q} \|\boldsymbol{C}_{ij}^{l*}\| L_{ij}^{l} (\underline{a}_j^{l})^{-\frac{1}{2}} \sqrt{W_j^{l}(t, \boldsymbol{e}_j(t))} \right. \\
&+ \exp(\lambda \tau^{l}/2) \gamma_i^{l} \sum_{j=1}^{q} \|\boldsymbol{D}_{ij}^{l*}\| L_{ij}^{l} (\underline{a}_j^{l})^{-\frac{1}{2}} \sup_{t-\tau^{l} \leq s \leq t} \sqrt{W_j^{l}(s, \boldsymbol{e}_j(s))} \\
&\left. + \gamma_i^{l} \sum_{j=1}^{q} \|\boldsymbol{G}_{ij}^{l*}\| L_{ij}^{l} (\underline{a}_j^{l})^{-\frac{1}{2}} \int_{-\infty}^{t} \kappa_{ij}^{l}(t-s) \exp(\lambda(t-s)/2) \sqrt{W_j^{l}(s, \boldsymbol{e}_j(s))} \, \mathrm{ds} \right].
\end{aligned}
\tag{3.2}
$$

Given that $\forall l \in P_1$, $\boldsymbol{Q}_l$ is a nonsingular M-matrix, and based on Lemma 2, it follows that there exists a positive vector $\boldsymbol{R}_l = \left(r_1^{l}, r_2^{l}, \ldots, r_q^{l} \right)^{\mathrm{T}}$, for which the inequality

$$
-\alpha_i^{l} \left(\overline{a}_i^{l} \right)^{-\frac{1}{2}} r_i^{l} + \gamma_i^{l} \sum_{j=1}^{q} \left(\underline{a}_i^{l} \right)^{-\frac{1}{2}} L_{ij}^{l} \left(\left\| \boldsymbol{C}_{ij}^{l*} \right\| + \left\| \boldsymbol{D}_{ij}^{l*} \right\| + \left\| \boldsymbol{G}_{ij}^{l*} \right\| \right) r_j^{l} < 0, i \in Q, l \in P_1, \tag{3.3}
$$

holds.

Consider the function

$$
\begin{aligned}
\mathscr{F}_i^{l}(x) = & \left(\overline{a}_i^{l} (\underline{a}_i^{l})^{-\frac{1}{2}} x - \alpha_i^{l} (\overline{a}_i^{l})^{-\frac{1}{2}} \right) r_i^{l} + \gamma_i^{l} \sum_{j=1}^{q} \left(\underline{a}_i^{l} \right)^{-\frac{1}{2}} L_{ij}^{l} \left(\left\| \boldsymbol{C}_{ij}^{l*} \right\| \right. \\
&\left. + \exp(\tau^{l} x/2) \left\| \boldsymbol{D}_{ij}^{l*} \right\| + \vartheta_{ij}^{l}(x/2) \left\| \boldsymbol{G}_{ij}^{l*} \right\| \right) r_j^{l}, i \in Q, l \in P_1.
\end{aligned}
\tag{3.4}
$$

It is evident that $\mathscr{F}_i^{l}(x)$ is continuous and strictly increasing with respect to x. From (3.3), it follows that $\mathscr{F}_i^{l}(0) < 0$. Therefore, there exists a positive number ρ such that $\mathscr{F}_i^{l}(\lambda_i^{l}) < 0$, where $\lambda_i^{l} \in U(0, \rho)$.

Let $\lambda = \min_{i\in Q, l\in P}\{\lambda_i^l\}$, then

$$\mathscr{F}_i^l(\lambda) < 0. \tag{3.5}$$

Let $\zeta_0 = \zeta^2 = \overline{a}_{\max}^l \|\boldsymbol{\phi}_i(s)\|_{\mathbb{C}_{t_0}}^2 / R_{\min}^l$, where $\zeta > 0$, $R_i^l = (r_i^l)^2$, $R_{\min}^l = \min_{i\in Q}\{R_i^l\}$, and $\overline{a}_{\max}^l = \max_{i\in Q}\{\overline{a}_i^l\}$. Obviously, $\forall i \in Q$ and $l \in P_1$, when $s \in (-\infty, t_0]$,

$$W_i^l(s, \boldsymbol{e}_i(s)) = \exp(\lambda(s-t_0))\, V_i^l(s, \boldsymbol{e}_i(s)) \le \overline{a}_i^l \|\boldsymbol{\phi}_i(s)\|_{\mathbb{C}_{t_0}}^2 < R_i^l \overline{a}_{\max}^l \|\boldsymbol{\phi}_i(s)\|_{\mathbb{C}_{t_0}}^2 / R_{\min}^l = R_i^l \zeta_0. \tag{3.6}$$

The proof that follows will demonstrate that $\forall i \in Q$ and $l \in P_1$,

$$W_i^l(t, \boldsymbol{e}_i(t)) < R_i^l \zeta_0, t \ge t_0. \tag{3.7}$$

To prove inequality (3.7) for $t > t_0$, we proceed by contradiction. If (3.7) does not hold, then there must exist at least one index $i \in Q$ and a time $t' > t_0$ such that

$$W_i^l(t, \boldsymbol{e}_i(t)) < R_i^l \zeta_0, t < t', \tag{3.8}$$

$$W_i^l(t', \boldsymbol{e}_i(t')) = R_i^l \zeta_0, \tag{3.9}$$

$$D^+ W_i^l(t', \boldsymbol{e}_i(t')) \ge 0, \tag{3.10}$$

$$W_j^l(t, \boldsymbol{e}_j(t)) < R_j^l \zeta_0, t < t', j \in Q\backslash\{i\}. \tag{3.11}$$

However, from (3.2) and (3.5), it follows that

$$\begin{aligned}
& D^+ W_i^l(t', \boldsymbol{e}_i(t')) \\
\le & \sqrt{W_i^l(t', \boldsymbol{e}_i(t'))} \Bigg[\left(\lambda \overline{a}_i^l (\underline{a}_i^l)^{-\frac{1}{2}} - \alpha_i^l (\overline{a}_i^l)^{-\frac{1}{2}} \right) \sqrt{W_i^l(t', \boldsymbol{e}_i(t'))} + \gamma_i^l \sum_{j=1}^{q} \|\boldsymbol{C}_{ij}^{l*}\| L_{ij}^l (\underline{a}_j^l)^{-\frac{1}{2}} \sqrt{W_j^l(t', \boldsymbol{e}_j(t'))} \\
& + \exp(\lambda\tau^l/2)\gamma_i^l \sum_{j=1}^{q} \|\boldsymbol{D}_{ij}^{l*}\| L_{ij}^l (\underline{a}_j^l)^{-\frac{1}{2}} \sup_{t'-\tau^l \le s \le t'} \sqrt{W_j^l(s, \boldsymbol{e}_j(s))} \\
& + \gamma_i^l \sum_{j=1}^{q} \|\boldsymbol{G}_{ij}^{l*}\| L_{ij}^l (\underline{a}_j^l)^{-\frac{1}{2}} \int_{-\infty}^{t'} \kappa_{ij}^l (t'-s) \exp(\lambda(t'-s)/2) \sqrt{W_j^l(s, \boldsymbol{e}_j(s))}\, \mathrm{d}s \Bigg] \\
\le & \sqrt{W_i^l(t', \boldsymbol{e}_i(t'))} \Bigg[\left(\lambda \overline{a}_i^l (\underline{a}_i^l)^{-\frac{1}{2}} - \alpha_i^l (\overline{a}_i^l)^{-\frac{1}{2}} \right) r_i^l \zeta + \gamma_i^l \sum_{j=1}^{q} \|\boldsymbol{C}_{ij}^{l*}\| L_{ij}^l (\underline{a}_j^l)^{-\frac{1}{2}} r_j^l \zeta \\
& + \exp(\lambda\tau^l/2)\gamma_i^l \sum_{j=1}^{q} \|\boldsymbol{D}_{ij}^{l*}\| L_{ij}^l (\underline{a}_j^l)^{-\frac{1}{2}} r_j^l \zeta + \gamma_i^l \sum_{j=1}^{q} \|\boldsymbol{G}_{ij}^{l*}\| L_{ij}^l (\underline{a}_j^l)^{-\frac{1}{2}} \vartheta_{ij}^l(\lambda/2) r_j^l \zeta \Bigg] < 0.
\end{aligned} \tag{3.12}$$

This contradicts inequality (3.8). Therefore, for $t \ge t_0$, $\forall i \in Q$ and $l \in P_1$, inequality (3.7) holds. Combining (3.7) with (2.3), we can obtain the following inequality :

$$\exp(\lambda(t-t_0))\, \underline{a}_i^l \|\boldsymbol{e}_i(t)\|^2 \le \exp(\lambda(t-t_0))\, V_i^l(t, \boldsymbol{e}_i(t)) < R_i^l \zeta, \tag{3.13}$$

i.e.

$$\|\boldsymbol{e}_i(t)\| < \kappa_1^l \|\boldsymbol{\phi}_i(s)\|_{\mathbb{C}_{t_0}} \exp(-\lambda(t-t_0)/2), \tag{3.14}$$

where $\kappa_1^l = \sqrt{\frac{R_i^p \overline{a}_{\max}^l}{\underline{a}_i^l R_{\min}^l}}$. A simple calculation yields

$$\|\boldsymbol{e}(t)\| < \kappa_1 \|\boldsymbol{\Phi}\|_{t_0} \exp(-\lambda(t-t_0)/2), \forall l \in P_1, \tag{3.15}$$

where $\kappa_1 = \sqrt{q} \max_{l \in P_1} \{\kappa_1^l\}$. □

Remark 4. *Theorem 1 provides a sufficient, though not necessary, condition for the exponential stability of the interconnected system. As a result, models that do not meet the criteria outlined in Theorem 1 may still display stability. Instead of classifying models strictly as stable or unstable, we categorize them into two distinct groups: $\mathcal{M}_1$ and $\mathcal{M}_2$.*

In Theorem 1, it is established that if a mode $\Pi_l \in \mathcal{M}_2$ satisfies conditions (2.2)–(2.4) and (2.6), with all other conditions remaining unchanged, the states of these modes may experience exponential divergence. Based on inequality (3.2), it follows that for all $l \in P_2$,

$$\begin{aligned}
&D^+ W_i^l(t, \boldsymbol{e}_i(t)) \\
\leq &\sqrt{W_i^l(t, \boldsymbol{e}_i(t))} \Bigg[\left(\lambda \overline{a}_i^l (\underline{a}_i^l)^{-\frac{1}{2}} + \beta_i^l (\overline{a}_i^l)^{-\frac{1}{2}} \right) \sqrt{W_i^l(t, \boldsymbol{e}_i(t))} + \gamma_i^l \sum_{j=1}^{q} \|\boldsymbol{C}_{ij}^{l\cdot *}\| L_{ij}^l (\underline{a}_j^l)^{-\frac{1}{2}} \sqrt{W_j^l(t, \boldsymbol{e}_j(t))} \\
&+ \exp(\lambda \tau^l/2) \gamma_i^l \sum_{j=1}^{q} \|\boldsymbol{D}_{ij}^{l\cdot *}\| L_{ij}^l (\underline{a}_j^l)^{-\frac{1}{2}} \sqrt{W_j^l(t, \boldsymbol{e}_j(t-\tau_{ij}^l))} \\
&+ \gamma_i^l \sum_{j=1}^{q} \|\boldsymbol{G}_{ij}^{l\cdot *}\| L_{ij}^l (\underline{a}_j^l)^{-\frac{1}{2}} \int_{-\infty}^{t} \kappa_{ij}^l (t-s) \exp(\lambda(t-s)/2) \sqrt{W_j^l(s, \boldsymbol{e}_j(s))}\, \mathrm{ds} \Bigg].
\end{aligned} \tag{3.16}$$

By applying the Cauchy mean value theorem, it follows that there exists $\xi < t_0$, such that

$$\begin{aligned}
&D^+ W_i^l(t, \boldsymbol{e}_i(t)) \\
\leq &\Bigg[\left(\lambda \overline{a}_i^l (\underline{a}_i^l)^{-\frac{1}{2}} + \beta_i^l (\overline{a}_i^l)^{-\frac{1}{2}} \right) W_i^l(t, \boldsymbol{e}_i(t)) + \gamma_i^l \sum_{j=1}^{q} \|\boldsymbol{C}_{ij}^{l\cdot *}\| L_{ij}^l (\underline{a}_j^l)^{-\frac{1}{2}} \sqrt{W_i^l(t, \boldsymbol{e}_i(t))} \sqrt{W_j^l(t, \boldsymbol{e}_j(t))} \\
&+ \exp(\lambda \tau^l/2) \gamma_i^l \sum_{j=1}^{q} \|\boldsymbol{D}_{ij}^{l\cdot *}\| L_{ij}^l (\underline{a}_j^l)^{-\frac{1}{2}} \sqrt{W_i^l(t, \boldsymbol{e}_i(t))} \sqrt{W_j^l(t, \boldsymbol{e}_j(t-\tau_{ij}^l))} \\
&+ \gamma_i^l \sqrt{W_i^l(t, \boldsymbol{e}_i(t))} \sum_{j=1}^{q} \|\boldsymbol{G}_{ij}^{l\cdot *}\| L_{ij}^l (\underline{a}_j^l)^{-\frac{1}{2}} \sqrt{W_j^l(\xi, \boldsymbol{e}_j(\xi))} \vartheta_{ij}^l(\lambda/2) \Bigg] \\
\leq &\Bigg[\left(\lambda \overline{a}_i^l (\underline{a}_i^l)^{-\frac{1}{2}} + \beta_i^l (\overline{a}_i^l)^{-\frac{1}{2}} \right) W_i^l(t, \boldsymbol{e}_i(t)) + \gamma_i^l \sum_{j=1}^{q} \|\boldsymbol{C}_{ij}^{l\cdot *}\| L_{ij}^l (\underline{a}_j^l)^{-\frac{1}{2}} \frac{1}{2} \left(W_i^l(t, \boldsymbol{e}_i(t)) + W_j^l(t, \boldsymbol{e}_j(t)) \right) \\
&+ \exp(\lambda \tau^l/2) \gamma_i^l \sum_{j=1}^{q} \|\boldsymbol{D}_{ij}^{l\cdot *}\| L_{ij}^l (\underline{a}_j^l)^{-\frac{1}{2}} \frac{1}{2} \left(W_i^l(t, \boldsymbol{e}_i(t)) + W_j^l(t, \boldsymbol{e}_j(t-\tau_{ij}^l)) \right) \\
&+ \gamma_i^l \frac{1}{2} \left(W_i^l(t, \boldsymbol{e}_i(t)) + 1 \right) \sum_{j=1}^{q} \|\boldsymbol{G}_{ij}^{l\cdot *}\| L_{ij}^l (\underline{a}_j^l)^{-\frac{1}{2}} \sqrt{W_j^l(\xi, \boldsymbol{e}_j(\xi))} \vartheta_{ij}^l(\lambda/2) \Bigg].
\end{aligned} \tag{3.17}$$

Let $W^l(t, \boldsymbol{e}_i(t)) = \left(W_1^l(t, \boldsymbol{e}_i(t)), \dots, W_q^l(t, \boldsymbol{e}_i(t))\right)^{\mathrm{T}}$, the differential inequality (3.7) can be transformed into the following form

$$D^+ W^l(t, \boldsymbol{e}_i(t)) \leq C + \mathcal{U} W^l(t, \boldsymbol{e}_i(t)) + \mathcal{T} W^l(t, \boldsymbol{e}_i(t - \tau_i^l(t))), \tag{3.18}$$

where C, $\mathcal{U}$, and $\mathcal{T}$ are non-negative constant matrices with appropriate dimensions. Therefore, $\forall l \in P_2$, the states of mode Π_l may exhibit exponential divergence [43]. In light of this, we introduce the following assumption regarding modes $\Pi_l \in \mathcal{M}_2$.

Assumption 3. *$\forall\ \Pi_l \in \mathcal{M}_2$, there exists constants $\kappa_2^l > 1$ and $\bar{\lambda} > 0$ such that*

$$\|\boldsymbol{e}_i(t)\| < \kappa_2^l \|\boldsymbol{\phi}_i(s)\|_{\mathbb{C}_{t_0}} \exp(\bar{\lambda}(t - t_0)/2), i \in Q. \tag{3.19}$$

3.2. Stability under switching sequence uncertainties

This section addresses sequence uncertainties that satisfy (2.10) and (2.11) in Definition 2 and provides criteria for ensuring the robust exponential stability of system (2.1) by employing the average dwell time method.

Definition 4. *Let $T_1(t_L, t_R)$ (resp., $T_2(t_L, t_R)$) denote the total running time of all modes $\Pi_l \in \mathcal{M}_1$, where $l \in P_1$ (resp., $\Pi_l \in \mathcal{M}_2$, where $l \in P_2$), over the interval $[t_L, t_R]$. If there are positive constants N_1 and τ_1 (resp., N_2 and τ_2) such that*

$$\mathrm{N}_{\overline{\sigma}}^1(t_L, t_R) \leq \mathrm{N}_1 + \frac{T_1(t_L, t_R)}{\tau_1} \tag{3.20}$$

and

$$\mathrm{N}_{\overline{\sigma}}^2(t_L, t_R) \geq \mathrm{N}_2 + \frac{T_2(t_L, t_R)}{\tau_2} \tag{3.21}$$

hold, then τ_1 (resp., τ_2) is called the average dwell time of slow switching (resp., average dwell time of fast switching) under $\overline{\sigma}(t)$. Without loss of generality, throughout this paper, N_1 and N_2 are set to zero.

Theorem 2. *Consider a switched-delay interconnected uncertain system* (2.1) *where modes satisfy* (2.2)–(2.6)*, $\sigma(t)$ satisfies* (2.7)*, $\Sigma(t)$ satisfies* (2.8)*, and $\Delta(t)$ satisfies* (2.12)–(2.13)*. If $\forall l \in P_1$, matrix $Q_l = (q_{ij}^l)$ is a nonsingular M-matrix and*

$$\frac{\tau_1}{\tau_2} > \min\left\{\frac{\bar{\lambda} c}{\lambda(1-c)}, \frac{\lambda d}{\bar{\lambda}(1-d)}\right\}, \tag{3.22}$$

$$\ln \kappa + (1 - \mathcal{R})\varpi_1 + \mathcal{R}\varpi_2 = -\epsilon < 0, \tag{3.23}$$

then for $\boldsymbol{C}_{ij}^l \in \boldsymbol{C}_{ij}^{l \cdot l}$, $\boldsymbol{D}_{ij}^l \in \boldsymbol{D}_{ij}^{l \cdot l}$, and $\boldsymbol{G}_{ij}^l \in \boldsymbol{G}_{ij}^{l \cdot l}$, the switched-delay interconnected uncertain system (2.1) *is robustly exponentially stable. Where $\varpi_1 = \left[-\lambda(1-c)\tau_1 + \bar{\lambda} c \tau_2\right]$, $\varpi_2 = \left[-\lambda d \tau_1 + \bar{\lambda}(1-d)\tau_2\right]$ and*

$$\begin{aligned} q_{ii}^l =& \alpha_i^l \left(\overline{a}_i^l\right)^{-\frac{1}{2}} - \left(\underline{a}_i^l\right)^{-\frac{1}{2}} \gamma_i^l L_{ii}^l \left(\left\|\boldsymbol{C}_{ii}^{l \cdot *}\right\| + \left\|\boldsymbol{D}_{ii}^{l \cdot *}\right\| + \left\|\boldsymbol{G}_{ii}^{l \cdot *}\right\|\right), \\ q_{ij}^l =& -\left(\underline{a}_j^l\right)^{-\frac{1}{2}} \gamma_i^l L_{ij}^l \left(\left\|\boldsymbol{C}_{ij}^{l \cdot *}\right\| + \left\|\boldsymbol{D}_{ij}^{l \cdot *}\right\| + \left\|\boldsymbol{G}_{ij}^{l \cdot *}\right\|\right), i \neq j. \end{aligned}$$

Proof: Initially, we will analyze the system's behavior at each switching moment t_p $(p = 1, 2, \dots)$ under the switching signal $\overline{\sigma}(t)$. Under $\overline{\sigma}(t)$, let Δt_j^1 and Δt_j^2 represent the j-th activation durations for modes $\Pi_l \in \mathcal{M}_1$ and modes $\Pi_k \in \mathcal{M}_2$, respectively.

It follows from (3.14) and (3.19) that, for $t \in [t_p, t_{p+1})$,

$$\|\boldsymbol{e}_i(t)\| < \kappa_1^l \|\boldsymbol{e}_i(s)\|_{\mathbb{C}_{t_k}} \exp(-\lambda(t - t_k)/2), \text{if } l \in P_1 \tag{3.24}$$

or

$$\|\boldsymbol{e}_i(t)\| < \kappa_2^l \|\boldsymbol{e}_i(s)\|_{\mathbb{C}_{t_k}} \exp(\lambda_1(t - t_k)/2), \text{if } l \in P_2, \tag{3.25}$$

where $\|\boldsymbol{e}_i(s)\|_{\mathbb{C}_{t_k}} = \max_{j\in N}\{\sup_{s\in(-\infty,t_k]}|e_{ij}(s)|\}$. Considering the continuity of the system's state, we can derive the following from equations (3.24) and (3.25):

$$\begin{aligned}
\|\boldsymbol{e}_i(t)\| &< \kappa^{\mathrm{N}_{\overline{\sigma}}(t)} \exp\left(-\lambda T_1(t_0, t) + \bar{\lambda} T_2(t_0, t)\right) \|\boldsymbol{\phi}_i(s)\|_{\mathbb{C}_{t_0}} \\
&= \kappa^{\mathrm{N}_{\overline{\sigma}}(t)} \exp\left(-\lambda \sum\nolimits_{j=1}^{\mathrm{N}_{\overline{\sigma}}^1(t)} \Delta t_j^1 + \lambda_1 \sum\nolimits_{j=1}^{\mathrm{N}_{\overline{\sigma}}^2(t)} \Delta t_j^2\right) \|\boldsymbol{\phi}_i(s)\|_{\mathbb{C}_{t_0}},
\end{aligned} \tag{3.26}$$

where $\kappa = \max_{l\in P}\{\kappa_1^l, \kappa_2^l\}$.

By examining the composite switching signal $\overline{\sigma}(t)$ alongside the generalized nominal switching signal $\overline{\Sigma}(t)$, it is clear that in (3.26), each time interval Δt_j^1, where $j = 1, 2, \dots, \mathrm{N}_{\overline{\sigma}}^1(t)$, can be categorized into two possible cases, as detailed below:

Case 1: When $t \in \Delta t_j^1$, both $\overline{\sigma}(t) \in P_1$ and $\overline{\Sigma}(t) \in P_1$. Specifically, in the time intervals Δt_j^1, the modes $\Pi_l \in \mathcal{M}_1$ are set to be activated under $\overline{\Sigma}(t)$. In the presence of uncertainties, modes belonging to $\mathcal{M}_1$ continue to be activated under $\overline{\sigma}(t)$. It can be inferred from (2.12) that the number of these periods is no less than $(1 - \mathcal{R})(1 - c)\mathrm{N}_{\overline{\sigma}}(t)\tau_1$.

Case 2: When $t \in \Delta t_j^2$, $\overline{\Sigma}(t) \in P_2$, while $\overline{\sigma}(t) \in P_1$. Specifically, in the time intervals Δt_j^2, the modes $\Pi_l \in \mathcal{M}_2$ are set to be activated under $\overline{\Sigma}(t)$. However, due to switching uncertainties, the modes belonging to $\mathcal{M}_1$ replace these modes under $\overline{\sigma}(t)$. It can be inferred from (2.12) that the number of these periods is no less than $\mathcal{R}d\mathrm{N}_{\overline{\sigma}}(t)\tau_1$.

One further has that

$$T_1(t_0, t) \geq [(1 - \mathcal{R})(1 - c) + \mathcal{R}d]\, \mathrm{N}_{\overline{\sigma}}(t)\tau_1. \tag{3.27}$$

Similarly, for each time interval Δt_j^2, where $j = 1, 2, \dots, \mathrm{N}_{\overline{\sigma}}^2(t)$, Δt_j^2 can be classified into two possible cases, as detailed below:

Case 1′: When $t \in \Delta t_j^2$, both $\overline{\sigma}(t) \in P_2$ and $\overline{\Sigma}(t) \in P_2$. Specifically, in the time intervals Δt_j^2, the modes $\Pi_l \in \mathcal{M}_2$ are set to be activated under $\overline{\Sigma}(t)$. In the presence of uncertainties, modes belonging to $\mathcal{M}_2$ continue to be activated under $\overline{\sigma}(t)$. It can be inferred from (2.13) that the number of these periods is no more than $\mathcal{R}(1 - d)\mathrm{N}_{\overline{\sigma}}(t)\tau_2$.

Case 2′: When $t \in \Delta t_j^2$, $\overline{\Sigma}(t) \in P_2$, while $\overline{\sigma}(t) \in P_1$. Specifically, in the time intervals Δt_j^2, the modes $\Pi_l \in \mathcal{M}_2$ are set to be activated under $\overline{\Sigma}(t)$. However, due to switching uncertainties, the modes belonging to $\mathcal{M}_1$ replace these modes under $\overline{\sigma}(t)$. It can be inferred from (2.13) that the number of these periods is no more than $(1 - \mathcal{R})c\mathrm{N}_{\overline{\sigma}}(t)\tau_2$.

$$T_2(t_0, t) \leq [\mathcal{R}(1 - d) + (1 - \mathcal{R})c]\, \mathrm{N}_{\overline{\sigma}}(t)\tau_2. \tag{3.28}$$

Substituting (3.27) and (3.28) into (3.26) yields

$$\begin{aligned}
&\|\boldsymbol{e}_i(t)\| \\
<&\kappa^{\mathrm{N}_{\overline{\sigma}}(t)} \exp\left(-\lambda T_1(t_0,t) + \bar{\lambda} T_2(t_0,t)\right) \|\boldsymbol{\phi}_i(s)\|_{\mathbb{C}_{t_0}} \\
\leq&\exp\Big\{\mathrm{N}_{\overline{\sigma}}(t)\ln\kappa - \lambda\left[(1-\mathcal{R})(1-c)+\mathcal{R}d\right]\mathrm{N}_{\overline{\sigma}}(t)\tau_1 \\
&\quad + \bar{\lambda}\left[\mathcal{R}(1-d)+(1-\mathcal{R})c\right]\mathrm{N}_{\overline{\sigma}}(t)\tau_2\Big\}\|\boldsymbol{\phi}_i(s)\|_{\mathbb{C}_{t_0}} \\
=&\exp\Big\{\mathrm{N}_{\overline{\sigma}}(t)\Big\{\ln\kappa + (1-\mathcal{R})\left[-\lambda(1-c)\tau_1 + \bar{\lambda}c\tau_2\right] \\
&\quad + \mathcal{R}[-\lambda d\tau_1 + \bar{\lambda}(1-d)\tau_2]\Big\}\Big\}\|\boldsymbol{\phi}_i(s)\|_{\mathbb{C}_{t_0}} \\
=&\exp\Big\{\mathrm{N}_{\overline{\sigma}}(t)[\ln\kappa + (1-\mathcal{R})\varpi_1 + \mathcal{R}\varpi_2]\Big\}\|\boldsymbol{\phi}_i(s)\|_{\mathbb{C}_{t_0}} \\
\leq&\exp\left\{-\epsilon(t-t_0)/\tau\right\}\|\boldsymbol{\phi}_i(s)\|_{\mathbb{C}_{t_0}},
\end{aligned} \tag{3.29}$$

where $\tau = \max\{\tau_1, \tau_2\}$. Under condition (3.22), we examine the following three cases:

Case 1: $\frac{\tau_1}{\tau_2} > \frac{\bar{\lambda}c}{\lambda(1-c)}$. There is a small $\mathcal{R}$ satisfying condition (3.23).

Case 2: $\frac{\tau_1}{\tau_2} > \frac{\bar{\lambda}(1-d)}{\lambda d}$. There is a large $\mathcal{R}$ satisfying condition (3.23).

Case 3: $\frac{\tau_1}{\tau_2} > \max\left\{\frac{\bar{\lambda}c}{\lambda(1-c)}, \frac{\bar{\lambda}(1-d)}{\lambda d}\right\}$. In this case, selection of $\mathcal{R}$ can be made more flexible to satisfy condition (3.23).

Remark 5. *In equations* (3.14) *and* (3.19)*, the amplification factor κ_1^l, $l \in P_1$ (resp., κ_2^l, $l \in P_2$), determines the scaling of the initial condition but does not influence the exponential decay rate λ (resp., the exponential divergence rate $\bar{\lambda}$). Given the continuity of the system's state, the amplification factor at each switching instant can be fixed at a constant value of 1. As a result, inequality* (3.29) *can be rewritten as*

$$\begin{aligned}
\|\boldsymbol{e}_i(t)\| <&\kappa\exp\Big\{\mathrm{N}_{\overline{\sigma}}(t)[(1-\mathcal{R})\varpi_1 + \mathcal{R}\varpi_2]\Big\}\|\boldsymbol{\phi}_i(s)\|_{\mathbb{C}_{t_0}} \\
=&\kappa\exp\left\{-\psi(t-t_0)\right\}\|\boldsymbol{\phi}_i(s)\|_{\mathbb{C}_{t_0}}.
\end{aligned} \tag{3.30}$$

Here, $\kappa = \max_{l\in P}\{\kappa_1^l, \kappa_2^l\}$ and $\psi = [(1-\mathcal{R})\varpi_1 + \mathcal{R}\varpi_2]/\tau$.

Remark 6. *In the analysis of stability for switched systems with unstable modes, numerous studies impose constraints on the dwell time of the switching signal, typically given by $\Delta t_{\min} \leq t_{k+1} - t_k \leq \Delta t_{\max}$, where $k = 0, 1, \ldots$. The stability conditions in these cases are often dependent on $\Delta t_{\min}$ and/or $\Delta t_{\max}$. Consequently, if the dwell time for any subsystem is either too short or too long, the stability criteria may be violated, leading to overly conservative conditions. For example, in Theorem 2 of [44], if the dwell time of a single subsystem is too short, it becomes challenging to satisfy conditions (7) and (8). In this paper, we propose using the average dwell time ratio to assess system stability, thereby avoiding the issue of stability criteria failing due to excessively short or long dwell times for individual subsystems.*

Remark 7. *From the expressions of ϖ_1 and ϖ_2, it can be inferred that ϖ_1 is directly proportional to c, while ϖ_2 is inversely proportional to d. These relationships suggest that condition* (3.23) *is satisfied when c is small, which corresponds to a low switching rate from stable to unstable modes, and d is large, implying a high switching rate from unstable to stable modes.*

Remark 8. *As is well known, in the stability analysis of switched systems containing unstable subsystems, stability can be attained if the activation time of the stable subsystems is sufficiently prolonged. This ensures that the decay of the stable subsystems compensates for the divergence caused by the activation of the unstable subsystems [6–8]. When* $\Delta(t) \equiv 0$*, i,e.,* $\sigma(t) \equiv \Sigma(t)$*, condition* (3.23) *simplifies to*

$$-\ln\kappa + (1-\mathcal{R})\lambda\tau_1 - \mathcal{R}\bar{\lambda}\tau_2 > 0. \tag{3.31}$$

This result is consistent with the conclusions presented in [6–8]. However, when disturbances are present in the switching signal, this property becomes more subtle, as the stable modes may be replaced by unstable ones, and vice versa. Furthermore, both the ratios and activation times of the unstable and stable modes can be adjusted based on ϖ_1 *and* ϖ_2 *to satisfy condition* (3.23) *for system stability. For example, when d is sufficiently large, meaning that the ratio of change from unstable to stable modes is sufficiently large. It is possible to select a large value for* $\mathcal{R}$ *and set large (or small) prescribed activation times for unstable (or stable) modes to satisfy condition* (3.23)*. This corresponds to the case where* $\varpi_1 \geq 0$ *and* $\varpi_2 < 0$*.*

4. Numerical example

To illustrate the validity and practical relevance of the theoretical results, we present the following example.

Example 1. *Artificial neural networks, as typical examples of interconnected systems, have garnered considerable research attention due to their successful applications in diverse fields such as dynamic optimization, associative memory, and pattern recognition. The following presents a model of a two-neuron switched neural network with* 4 *modes [45, 46].*

$$\begin{aligned}\dot{e}_i(t) = &-u_i^l e_i(t) + \sum_{j=1}^{2} c_{ij}^l y_j^l(e_j(t)) + \sum_{j=1}^{2} d_{ij}^l y_j^l(e_j(t-\tau_{ij}^l(t))) \\ &+ \sum_{j=1}^{2} g_{ij}^l \int_{-\infty}^{t} \kappa_{ij}^l(t-s)\, y_j^l(e_j(s))\mathrm{d}s,\ i,j\in\{1,2\}.\end{aligned} \tag{4.1}$$

In the neural network (4.1), for $l \in \{1,2\}$, $y_j^l(e_j) = 0.5e_j + 0.5\sin e_j$, $\tau_{ij}^l = 1$; for $p \in \{3,4\}$, $y_j^p(e_j) = \tanh e_j$, $\tau_{ij}^p = \exp(t)/(1+\exp(t))$; for $q \in \{1,2,3,4\}$, $\kappa_{ij}^q(s) = \exp(-s)$; and

$$\boldsymbol{U}^{I\cdot 1} = \begin{bmatrix} [3.99\ 4.01] & 0 \\ 0 & [2.99\ 3.01] \end{bmatrix}, \boldsymbol{U}^{I\cdot 2} = \begin{bmatrix} [2.81\ 2.95] & 0 \\ 0 & [3.60\ 3.72] \end{bmatrix},$$

$$\boldsymbol{C}^{I\cdot 1} = \begin{bmatrix} [1.19\ 1.21] & [2.35\ 2.41] \\ [0.05\ 0.06] & [0.03\ 0.04] \end{bmatrix}, \boldsymbol{C}^{I\cdot 2} = \begin{bmatrix} [0.87\ 1.01] & [0.03\ 0.10] \\ [2.07\ 2.28] & [0.68\ 0.80] \end{bmatrix},$$

$$\boldsymbol{D}^{I\cdot 1} = \begin{bmatrix} [0.09\ 0.11] & [3.14\ 3.32] \\ [-0.05\ 0.13] & [0.43\ 0.54] \end{bmatrix}, \boldsymbol{D}^{I\cdot 2} = \begin{bmatrix} [0.11\ 0.35] & [-0.02\ 0.10] \\ [2.87\ 3.00] & [0.05\ 0.13] \end{bmatrix},$$

$$\boldsymbol{G}^{I\cdot 1} = \begin{bmatrix} [0.19\ 0.21] & [0.20\ 0.22] \\ [0.11\ 0.13] & [0.22\ 0.24] \end{bmatrix}, \boldsymbol{G}^{I\cdot 2} = \begin{bmatrix} [0.17\ 0.20] & [0.15\ 0.18] \\ [0.09\ 0.12] & [0.28\ 0.31] \end{bmatrix},$$

$$\boldsymbol{U}^3 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \boldsymbol{C}^3 = \begin{bmatrix} -0.2 & 1 \\ 1 & -0.1 \end{bmatrix}, \boldsymbol{D}^3 = \begin{bmatrix} -0.4 & 0.3 \\ 0.1 & 0.6 \end{bmatrix}, \boldsymbol{G}^3 = \begin{bmatrix} 0.2 & 0.3 \\ -0.2 & 0.4 \end{bmatrix},$$

$$\boldsymbol{U}^4 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \boldsymbol{C}^4 = \begin{bmatrix} 0.1 & -1.9 \\ 0.1 & -2.4 \end{bmatrix}, \boldsymbol{D}^4 = \begin{bmatrix} 0.3 & -1 \\ 0.3 & -0.4 \end{bmatrix}, \boldsymbol{G}^4 = \begin{bmatrix} 0.2 & 0.1 \\ 0.3 & -0.2 \end{bmatrix},$$

By simple calculate yields

$$\underline{\boldsymbol{U}}^1 = \begin{bmatrix} 3.99 & 0 \\ 0 & 2.99 \end{bmatrix}, \boldsymbol{C}^{1\cdot *} = \begin{bmatrix} 1.21 & 2.41 \\ 0.06 & 0.04 \end{bmatrix}, \boldsymbol{D}^{1\cdot *} = \begin{bmatrix} 0.11 & 3.32 \\ 0.13 & 0.54 \end{bmatrix}, \boldsymbol{G}^{1\cdot *} = \begin{bmatrix} 0.21 & 0.22 \\ 0.13 & 0.24 \end{bmatrix},$$

$$\underline{\boldsymbol{U}}^2 = \begin{bmatrix} 2.81 & 0 \\ 0 & 3.60 \end{bmatrix}, \boldsymbol{C}^{2\cdot *} = \begin{bmatrix} 1.01 & 0.10 \\ 2.28 & 0.80 \end{bmatrix}, \boldsymbol{D}^{2\cdot *} = \begin{bmatrix} 0.35 & 0.10 \\ 3.00 & 0.13 \end{bmatrix}, \boldsymbol{G}^{1\cdot *} = \begin{bmatrix} 0.20 & 0.18 \\ 0.12 & 0.31 \end{bmatrix}.$$

And $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ are M-matrices, whereas $\boldsymbol{Q}_3$ and $\boldsymbol{Q}_4$ are not classified as M-matrices. Thus, Π_1 and Π_2 belong to $\mathcal{M}_1$, while Π_3 and Π_4 belong to $\mathcal{M}_2$. Let $\lambda = 0.8$, which satisfies inequality (3.4). Additionally, based on the states of modes Π_3 and Π_4, the estimated value of $\bar{\lambda}$ can be set to 0.2. For numerical simulations, define $\boldsymbol{U}^i = \underline{\boldsymbol{U}}^{i\cdot *}$, $\boldsymbol{C}^i = \boldsymbol{C}^{i\cdot *}$, $\boldsymbol{D}^i = \boldsymbol{D}^{i\cdot *}$, and $\boldsymbol{G}^i = \boldsymbol{G}^{i\cdot *}$ for $i = 1, 2$, with the initial state given by $[\cos(2s) - 0.4, \sin(2s) + 0.4]$.

The state responses for the four modes are displayed in Figures 2–5. The simulation results confirm that modes Π_1 and Π_2 remain stable, while modes Π_3 and Π_4 exhibit instability. Figures 6 and 7 demonstrate that an incorrect switching signal can lead to instability of system (4.1).

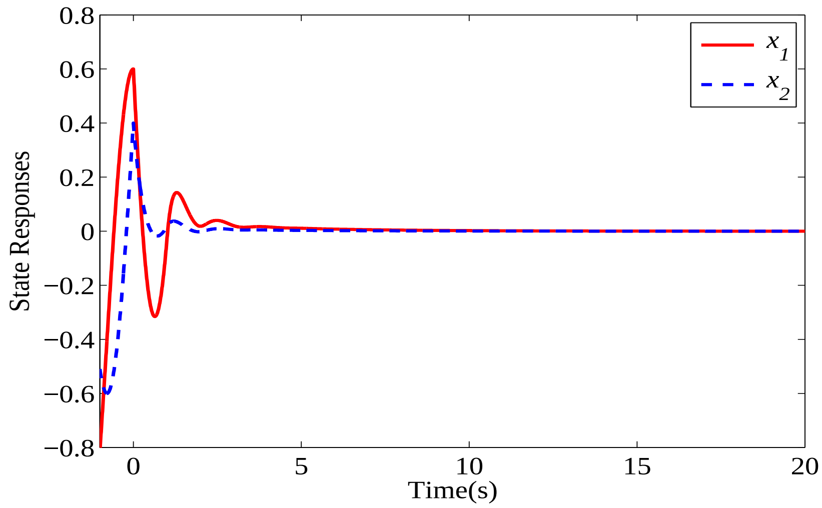

Figure 2. State responses of mode 1 of the considered switched system.

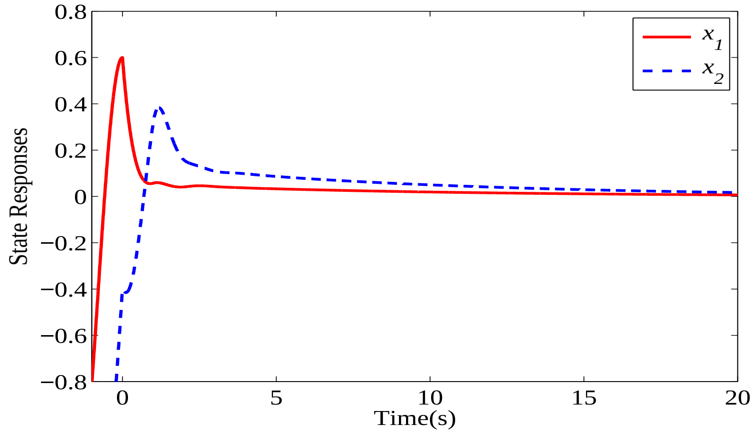

Figure 3. State responses of mode 2 of the considered switched system.

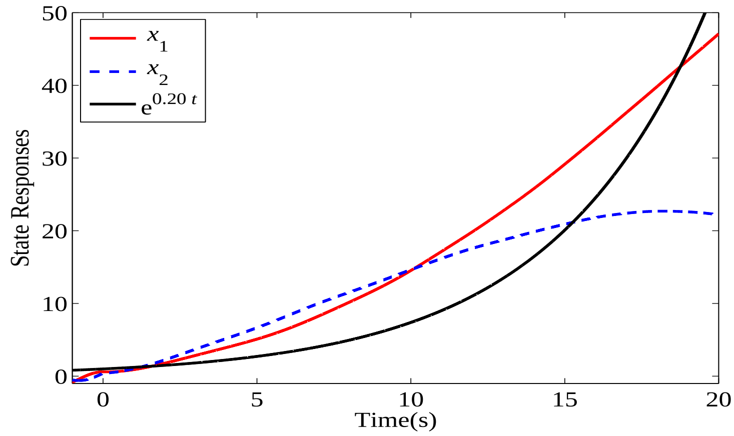

Figure 4. State responses of mode 3 of the considered switched system.

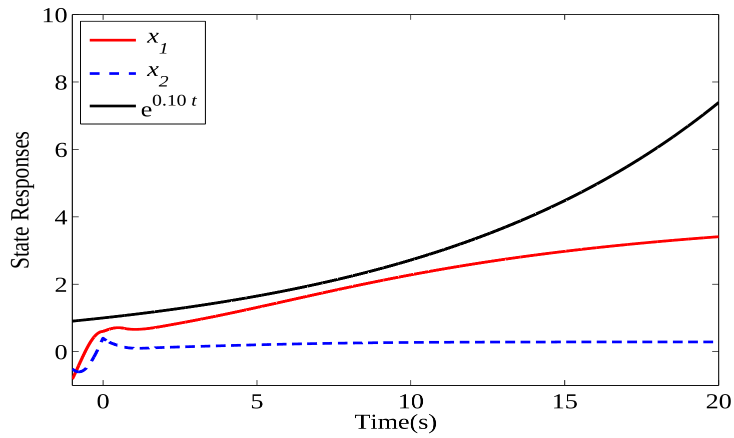

Figure 5. State responses of mode 4 of the considered switched system.

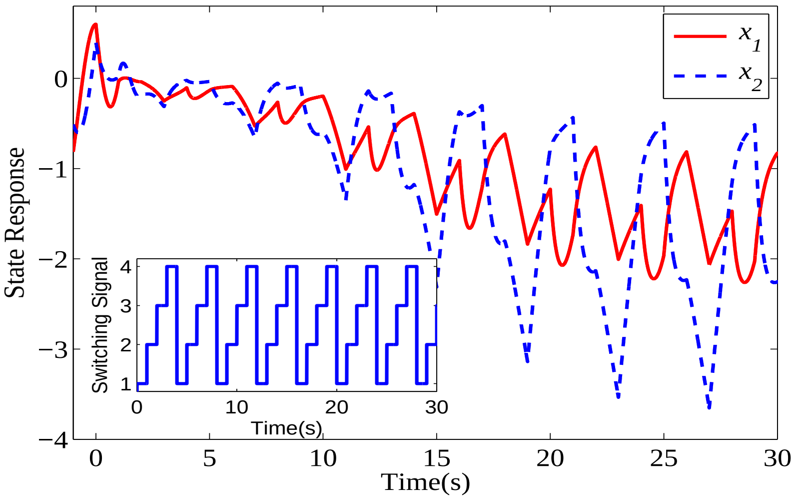

Figure 6. Unstable switching signal for system (4.1).

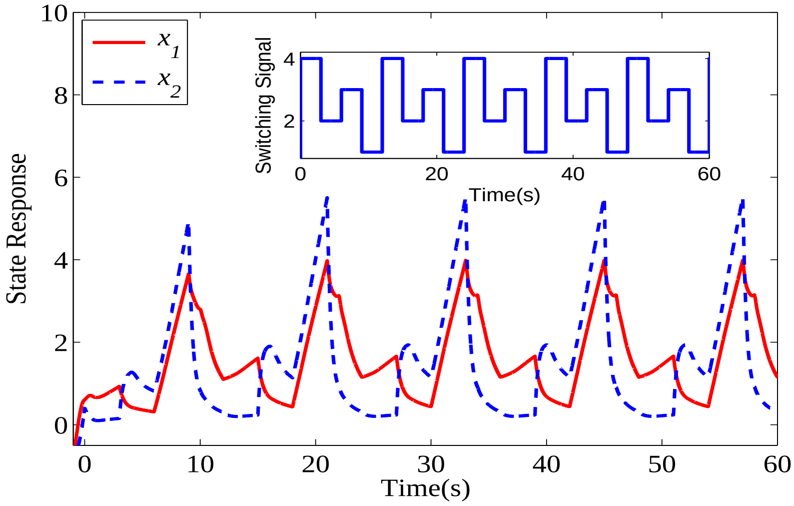

Figure 7. Unstable switching signal for system (4.1).

To stabilize the system defined by equation (4.1), define a nominal switching signal $\Sigma(t)$ as follows: for $t \in [10k, 2 + 10k)$, let $\Sigma(t) = 4$; for $t \in [2 + 10k, 4 + 10k)$, let $\Sigma(t) = 1$; for $t \in [4 + 10k, 6 + 10k)$, let $\Sigma(t) = 3$; and for $t \in [6 + 10k, 10 + 10k)$, let $\Sigma(t) = 2$, where $k = 0, 1, \ldots$. Also, set $\mathcal{R} = 0.5$. As illustrated in Figure 8, the system rapidly converges to the equilibrium point $\boldsymbol{e} = \mathbf{0}$ under this nominal switching signal $\Sigma(t)$.

We now examine whether the nominal switching signal $\Sigma(t)$ can still stabilize system (4.1) when uncertain disturbances are present, which lead to generalized mode-changing ratios $\{0.6, 0.4\}$ and a 0.5-second delay in the switching signal. According to condition (3.22) in Theorem 2, the system will remain exponentially stable if $\tau_1/\tau_2 > \bar{\lambda}c/\lambda(1 - c) = 0.375$. The delay of 0.5 seconds in the switching signal results in the composite signal $\overline{\sigma}(t)$ (or generalized nominal signal $\overline{\Sigma}(t)$) experiencing double the number of switching events. Additionally, we compute $\tau_1/\tau_2 = 1.17/1.5 = 0.778 > 0.375$. Therefore, we conclude that system (4.1) remains robustly exponentially stable. As shown in Figures 9 and 10, exponential stability is maintained under $\overline{\sigma}(t)$.

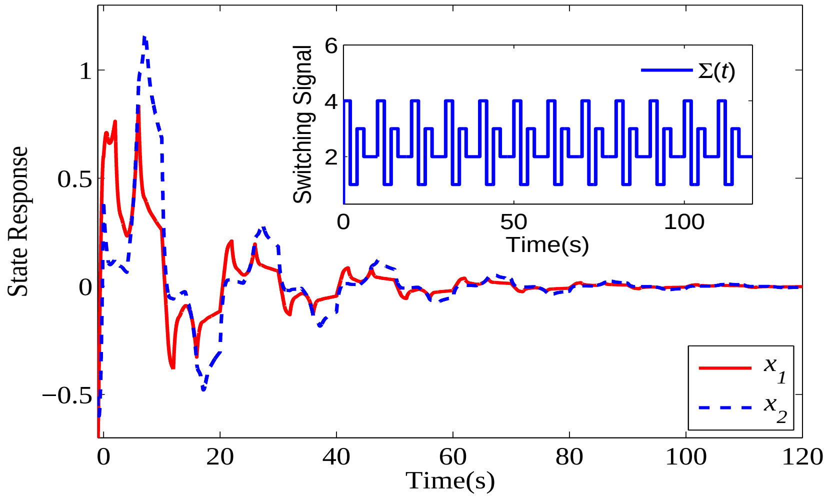

Figure 8. Stable switching signal for system (4.1).

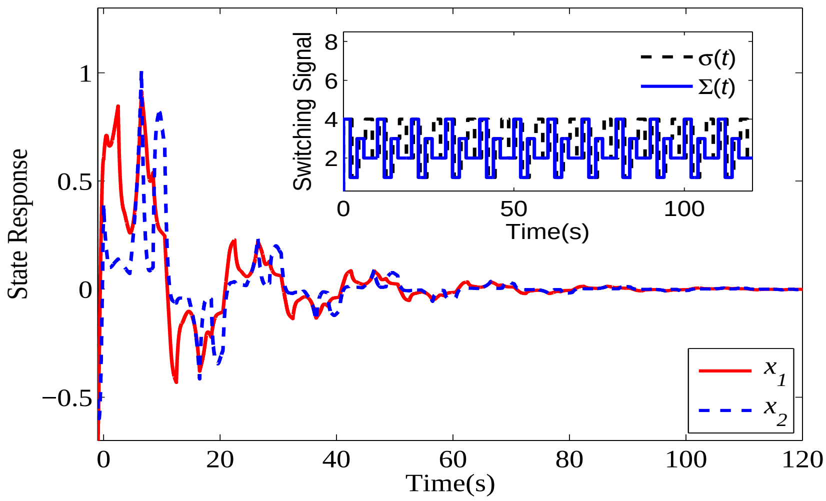

Figure 9. Stable switching signal for system (4.1).

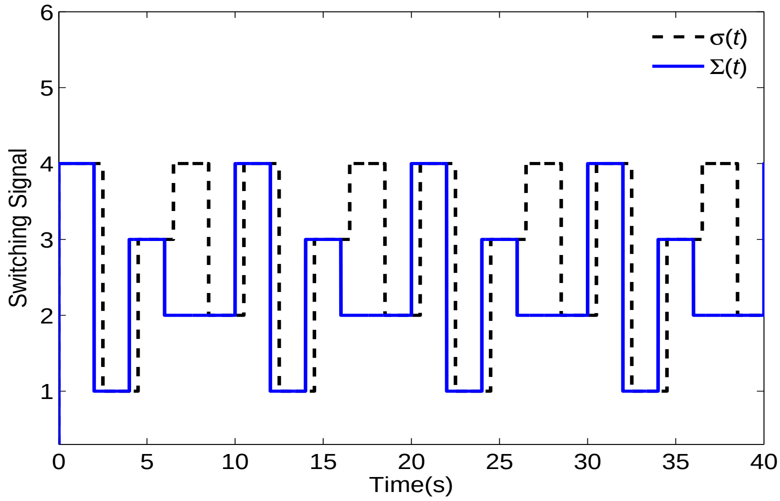

Figure 10. Partial enlarged view of the switching signal in Figure 9.

5. Conclusions

This paper establishes robust stability conditions for switched-delay interconnected systems with unstable modes, both time-varying and continuously distributed state delays, along with uncertainties in switching and system parameters. By introducing two novel concepts—the composite switching signal and the generalized nominal switching signal—along with a new index, the generalized mode-changing rate, this study effectively addresses the influence of switching uncertainties on system stability. Sufficient conditions for ensuring robust exponential stability are derived using the average dwell time approach and vector Lyapunov functions. In this paper, all modes of the considered switched system have been assumed to share a common state space. Consequently, the dimension of the state vector remains invariant, irrespective of which modes become activated during the evolution of the system. However, another important class of switched systems, characterized by subsystems with distinct state spaces, has attracted growing research interest. Future studies will therefore focus on addressing the robust exponential stability problem of interconnected switched systems whose subsystems possess heterogeneous state spaces.

Author contributions

Conceptualization, H.X. and X.Y.; methodology, H.X. and X.Y.; software, H.X.; investigation, H.X. and X.Y.; writing—original draft preparation, H.X. and X.Y.; writing—review and editing, H.X. and X.Y.; funding acquisition, H.X. and X.Y. All authors have read and agreed to the published version of the manuscript.

Acknowledgments

This work was supported in part by the National Natural Science Foundation (NSF) of China under Grant 12372009 and Grant 12271132, in part by the NSF of Guangdong under Grant

2024A1515010532, in part by the NSF of Hanshan Normal University under Grant PNB221103, and in part by the Education Science Planning Project of Guangdong under Grant 2023GXJK385.

Use of Generative-AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

References

1. D. Liberzon, A. S. Morse, Basic problems in stability and design of switched systems, *IEEE Control Syst. Mag.*, **19** (1999), 59–70. https://doi.org/10.1109/37.793443
2. H. Xue, J. Zhang, Robust exponential stability of interconnected switched systems with mixed delays and impulsive effect, *Nonlinear Dynam.*, **97** (2019), 679–696. https://doi.org/10.1007/s11071-019-05006-5
3. J. Daafouz, P. Riedinger, C. Iung, Stability analysis and control synthesis for switched systems: A switched Lyapunov function approach, *IEEE Trans. Autom. Control*, **47** (2002), 1883–1887. https://doi.org/10.1109/TAC.2002.804474
4. A. V. Il'in, Yu. M. Mosolova, V. V. Fomichev, A. S. Fursov, On the problem of switched system stabilization, *Moscow Univ. Comput. Math. Cybern.*, **48** (2024), 303–316. https://doi.org/10.3103/S0278641924700201
5. G. Zhai, B. Hu, K. Yasuda, A. N. Michel, Stability analysis of switched systems with stable and unstable subsystems: An average dwell time approach, *Int. J. Syst. Sci.*, **32** (2001), 1055–1061. https://doi.org/10.1080/00207720116692
6. H. Yang, B. Jiang, V. Cocquempot, A survey of results and perspectives on stabilization of switched nonlinear systems with unstable modes, *Nonlinear Anal. Hybrid Syst.*, **13** (2014), 45–60. https://doi.org/10.1016/j.nahs.2013.12.005
7. H. Yang, V. Cocquempot, B. Jiang, On stabilization of switched nonlinear systems with unstable modes, *Syst. Control Lett.*, **58** (2009), 703–708. https://doi.org/10.1016/j.sysconle.2009.06.007
8. M. Zhang, L. Gao, Input-to-state stability for impulsive switched nonlinear systems with unstable subsystems, *T. I. Meas. Control*, **40** (2018), 2167–2177. https://doi.org/10.1177/0142331217699057
9. J. Xing, B. Wu, Y. E. Wang, L. Liu, Exponential stability and non-weighted L_2-gain analysis for switched neutral systems with unstable subsystems, *Int. J. Syst. Sci.*, **56** (2025), 673–689. https://doi.org/10.1080/00207721.2024.2409847
10. H. Li, Z. Wei, Stability analysis of discrete-time switched systems with all unstable subsystems, *Discrete Cont. Dyn.-A*, **17** (2024), 2762–2777. https://doi.org/10.3934/dcdss.2024040

11. N. Zhang, Y. Sun, Stabilization of switched positive linear delay system with all subsystems unstable, *Int. J. Robust Nonlin.*, **34** (2024), 2441–2456. https://doi.org/10.1002/rnc.7091

12. L. X. Zhang, H. J. Gao, Asynchronously switched control of switched linear systems with average dwell time, *Automatica*, **46** (2010), 953–958. https://doi.org/10.1016/j.automatica.2010.02.021

13. G. Zong, R. Wang, W. X. Zheng, L. Hou, Finite-time stabilization for a class of switched time-delay systems under asynchronous switching, *Appl. Math. Comput.*, **219** (2013), 5757–5771. https://doi.org/10.1016/j.amc.2012.11.078

14. X. Wu, Y. Tang, J. Cao, W. Zhang, Distributed consensus of stochastic delayed multi-agent systems under asynchronous switching, *IEEE T. Cybernetics*, **46** (2015), 1817–1827. https://doi.org/10.1109/TCYB.2015.2453346

15. L. Liu, F. Deng, Stability and stabilization of nonlinear stochastic systems with synchronous and asynchronous switching parameters to the states, *IEEE T. Cybernetics*, **53** (2022), 4894–4907. https://doi.org/10.1109/TCYB.2022.3151976

16. S. Zhang, A. B. Nie, Exponential stability of fractional-order uncertain systems with asynchronous switching and impulses, *Int. J. Robust Nonlin.*, **34** (2024), 7285–7313. https://doi.org/10.1002/rnc.7345

17. L. Hetel, J. Daafouz, C. Iung, Stability analysis for discrete time switched systems with temporary uncertain switching signal, In: *Proceedings of 46th IEEE Conf. Decision Control*, New Orleans, LA, USA, 2007, 5623–5628. https://doi.org/10.1109/CDC.2007.4434794

18. Z. G. Li, Y. C. Soh, C. Y. Wen, Robust stability of quasi-periodic hybrid dynamic uncertain systems, *IEEE T. Automat. Contr.*, **46** (2001), 107–111. https://doi.org/10.1109/9.898700

19. L. Zhang, E. Boukas, Stability and stabilization of Markovian jump linear systems with partly unknown transition probabilities, *Automatica*, **45** (2009), 463–468. https://doi.org/10.1016/j.automatica.2008.08.010

20. H. Wei, Q. Li, S. Zhu, D. Fan, Y. Zheng, Event-triggered resilient asynchronous estimation of stochastic Markovian jumping CVNs with missing measurements: A co-design control strategy, *Inform. Sciences*, **712** (2025), 122167. https://doi.org/10.1016/j.ins.2025.122167

21. Q. Li, H. Wei, D. Hua, J. Wang, J. Yang, Stabilization of semi-Markovian jumping uncertain complex-valued networks with time-varying delay: A sliding-mode control approach, *Neural Processing Letters*, **56** (2024), 111. https://doi.org/10.1007/s11063-024-11585-1

22. H. Yang, B. Jiang, G. Tao, D. Zhou, Robust stability of switched nonlinear systems with switching uncertainties, *IEEE Trans. Autom. Control*, **61** (2016), 2531–2537. https://doi.org/10.1109/TAC.2015.2495619

23. Z. Sun, Robust switching of discrete-time switched linear systems, *Automatica*, **48** (2012), 239–242. https://doi.org/10.1016/j.automatica.2011.10.004

24. M. S. Mahmoud, F. M. AL-Sunni, Interconnected continuous-time switched systems: Robust stability and stabilization, *Nonlinear Anal. Hybri.*, **4** (2010), 531–542. https://doi.org/10.1016/j.nahs.2010.01.001

25. L. Long, Multiple Lyapunov functions-based small-gain theorems for switched interconnected nonlinear systems, *IEEE Trans. Autom. Control*, **62** (2017), 3943–3958. https://doi.org/10.1109/TAC.2017.2648740

26. Y. Hou, Y. J. Liu, S. Tong, Decentralized event-triggered fault-tolerant control for switched interconnected nonlinear systems with input saturation and time-varying full-state constraints, *IEEE Trans. Fuzzy Syst.*, **32** (2024), 3850–3860. https://doi.org/10.1109/TFUZZ.2024.3392632

27. G. Gurrala, I. Sen, Power system stabilizers design for interconnected power systems, *IEEE Trans. Power Syst.*, **25** (2010), 1042–1051. https://doi.org/10.1109/TPWRS.2009.2036778

28. Y. W. Zhao, H. Y. Zhang, Z. Y. Chen, H. Q. Wang, X. D. Zhao, Adaptive neural decentralised control for switched interconnected nonlinear systems with backlash-like hysteresis and output constraints, *Int. J. Syst. Sci.*, **53** (2022), 1545–1561. https://doi.org/10.1080/00207721.2021.2017063

29. D. Zhai, X. Liu, Y. J. Liu, Adaptive decentralized controller design for a class of switched interconnected nonlinear systems, *IEEE Trans. Cybern.*, **50** (2020), 1644–1654. https://doi.org/10.1109/TCYB.2018.2878578

30. J. Zhang, S. Li, C. K. Ahn, Z. R. Xiang, Adaptive fuzzy decentralized dynamic surface control for switched large-scale nonlinear systems with full-state constraints, *IEEE Trans. Cybern.*, **52** (2022), 10761–10772. https://doi.org/10.1109/TCYB.2021.3069461

31. Y. Chen, Fuzzy interactions compensation in adaptive control of switched interconnected systems, *IEEE Trans. Autom. Sci. Eng.*, **21** (2024), 48–55. https://doi.org/10.1109/TASE.2022.3205902

32. Y. Song, Y. Liu, W. Zhao, Approximately bi-similar symbolic model for discrete-time interconnected switched system, *IEEE/CAA J. Autom. Sinica*, **11** (2024), 2185–2187. https://doi.org/10.1109/JAS.2023.123927

33. U. T. Jőnsson, C. Y. Kao, H. Fujioka, Low dimensional stability criteria for large-scale interconnected systems, In: Proceedings of the 9th European Control Conference, Kos, Greece, 2007, 2741–2747. https://doi.org/10.23919/ECC.2007.7068616

34. M. S. Andersen, S. K. Pakazad, A. Hansson, A. Rantzer, Robust stability analysis of sparsely interconnected uncertain systems, *IEEE Trans. Autom. Control*, **59** (2014), 2151–2156. https://doi.org/10.1109/TAC.2014.2305934

35. N. T. Thanh, V. N. Phat, Decentralized stability for switched nonlinear large-scale systems with interval time-varying delays in interconnections, *Nonlinear Anal. Hybrid Syst.*, **11** (2014), 22–36. https://doi.org/10.1016/j.nahs.2013.04.002

36. H. Ren, G. Zong, L. Hou, Y. Yi, Finite-time control of interconnected impulsive switched systems with time-varying delay, *Appl. Math. Comput.*, **276** (2016), 143–157. https://doi.org/10.1016/j.amc.2015.12.012

37. I. Malloci, Z. Lin, G. Yan, Stability of interconnected impulsive switched systems subject to state dimension variation, *Nonlinear Anal. Hybrid Syst.*, **6** (2012), 960–971. https://doi.org/10.1016/j.nahs.2012.07.001

38. G. Yang, D. Liberzon, A Lyapunov-based small-gain theorem for interconnected switched systems, *Syst. Control Lett.*, **78** (2015), 47–54. https://doi.org/10.1016/j.sysconle.2015.02.001

39. J. Cao, D. Huang, Y. Qu, Global robust stability of delayed recurrent neural networks, *Chaos Solitons Fractals*, **23** (2005), 221–229. https://doi.org/10.1016/j.chaos.2004.04.002

40. T. Ensari, S. Arik, New results for robust stability of dynamical neural networks with discrete time delays, *Expert Syst. Appl.*, **37** (2010), 5925–5930. https://doi.org/10.1016/j.eswa.2010.02.013

41. O. Faydasicok, S. Arik, A new upper bound for the norm of interval matrices with application to robust stability analysis of delayed neural networks, *Neural Netw.*, **44** (2013), 64–71. https://doi.org/10.1016/j.neunet.2013.03.014

42. D. D. Siljak, *Large Scale Dynamic Systems: Stability and Structure*, North Holland, Amsterdam, The Netherlands, 1978.

43. X. Li, J. Cao, M. Perc, Switching laws design for stability of finite and infinite delayed switched systems with stable and unstable modes, *IEEE Access*, **6** (2018), 6677–6691. https://doi.org/10.1109/ACCESS.2017.2789165

44. Y. E. Wang, H. R. Karimi, D. Wu, Conditions for the stability of switched systems containing unstable subsystems, *IEEE Trans. Circuits Syst. II: Express Briefs*, **66** (2019), 617–621. https://doi.org/10.1109/TCSII.2018.2852766

45. H. Xue, J. Zhang, H. Wang, B. Jiang, Robust stability of switched interconnected systems with time-varying delays, *J. Comput. Nonlinear Dyn.*, **13** (2018), 021004. https://doi.org/10.1115/1.4038203

46. J. Qi, C. Li, T. Huang, W. Zhang, Exponential stability of switched time-varying delayed neural networks with all modes being unstable, *Neural Process. Lett.*, **43** (2016), 553–565. https://doi.org/10.1007/s11063-015-9428-3