Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(8): 18321–18336.
DOI: 10.3934/math.2025818
Received: 05 July 2025
Revised: 01 August 2025
Accepted: 06 August 2025
Published: 13 August 2025

Research article

Modulation instability and soliton families of the complex Ginzburg-Landau equation having the parabolic with nonlocal law of self-phase modulation

Wael W. Mohammed[1], Neslihan Ozdemir[2], Aydın Secer[3,4], Muslum Ozisik[3,5], Mustafa Bayram[4], Taha Radwan[6,*] and Karim K. Ahmed[7]

[1] Department of Mathematics, College of Science, University of Ha'il, Ha'il 2440, Saudi Arabia
[2] Department of Mathematics, Yildiz Technical University, Istanbul-Turkey
[3] Department of Mathematical Engineering, Yildiz Technical University, Istanbul-Turkey
[4] Department of Computer Engineering, Biruni University, Istanbul-Turkey
[5] Department of Mathematics, Faculty of Engineering and Natural Sciences, Istinye University, Istanbul-Turkey
[6] Department of Management Information Systems, College of Business and Economics, Qassim University, Buraydah 51452, Saudi Arabia
[7] Department of Mathematics, Faculty of Engineering, German International University (GIU), New Administrative Capital, Cairo, Egypt

* **Correspondence:** Email: t.radwan@qu.edu.sa.

Abstract: This paper acquaints the adopted parabolic with the nonlocal law of self-phase modulation form of the complex Ginzburg–Landau model, which regularizes the evolution of specific amplitudes of instability pulses in diverse dissipative systems. We employ the new Kudryashov and Sinh-Gordon equation expansion schemes to obtain bright and dark soliton families under particular conditions on the parameters of the physical model. Furthermore, the effect of diverse model parameters such as the chromatic dispersion, the parabolic law, and the nonlocal nonlinearity terms on the behaviors of bright and dark soliton solutions is also explored. We also search for modulation instability analysis for the model. The primary contribution of this study is the examination of a different version of the complex Ginzburg–Landau model, which is not yet available in the literature, along with the first comprehensive analysis of its modulation instability. Thus, this study highlights the practical and prompt results received by the Sinh-Gordon equation expansion scheme. Thus, this study is expected to offer meaningful implications for ongoing and future research within the framework of this model.

Keywords: the complex Ginzburg–Landau equation; nonlocal nonlinearity; soliton dynamics; partial differential equations; modulation instability
Mathematics Subject Classification: 35Q51, 35C08, 35Q56

1. Introduction

NLEEs, namely the nonlinear evolution equations and their soliton solutions, have enlarged academic curiosity. That is why a wide variety of scientists have developed the varied structures of NLEEs that model many problems, especially those encountered in daily life in optical physics and communication. To interpret the underlying mechanisms and operations of their dynamic systems and support their development and maintenance for the developed models, researchers also examine soliton solutions, which are solutions with a wide variety of physical behaviors. They are governed by various nonlinear equations. As is widely acknowledged, in studies involving the concept of soliton, the observation made by John Scott Russell in the Union Canal is frequently cited as the historical starting point. This was followed by foundational work conducted by the German mathematician Diederik Korteweg and his student Gustav de Vries, leading to the formulation of what is now known as the KdV (Korteweg–de Vries) equation. This equation has laid the theoretical groundwork for plenty of subsequent investigations. Following the First World War, developments in quantum mechanics, particularly the introduction of the wave model by Erwin Schrödinger, later formalized as the Schrödinger equation, initiated a new wave of research focused on atom-based wave modeling. This stimulated a broad spectrum of studies, leading to the development of various models by many distinguished researchers. In 1948, the invention of the transistor—facilitated by advancements in semiconductor technology—paved the way for significant progress in computing. These innovations, combined with the discovery of the maser in the 1950s and the laser in the 1960s, led to a surge of interest in wave phenomena, producing a vast body of theoretical and experimental literature. Notably, in the 1960s, Zabusky and Kruskal introduced the term soliton, and in the 1970s, advances in fiber optics enabled the experimental verification of bright solitons within the framework of nonlinear optics. These developments significantly accelerated the momentum of research in the field. One of the major challenges in this domain remains the limited availability of analytic or exact solution methods for the nonlinear models under consideration—especially NLEEs and nonlinear partial differential equations (NLPDEs)—despite progress in numerical approaches. For some models, integrability has been demonstrated using tools such as the Painlevé test and the inverse scattering transform, thus opening new theoretical pathways. From the 1980s onward, advancements in electronics and computing have facilitated parallel progress in the software industry, particularly in symbolic computation. This, in turn, has enabled new possibilities for researchers engaged in the analytic treatment of NLPDEs. Although a range of analytical methods has since been developed, their applicability to nonlinear optical models remains of particular significance. Consequently, the analytic solutions of such equations carry considerable importance not only in physics and nonlinear optics but also in a broader engineering context. These solution methods are often referred to in the literature as soliton solutions. In the case of water waves and fluid dynamics, the term solitary wave is commonly utilized, whereas in nonlinear optics, the terminology optical soliton is more prevalent. From the perspective of engineering applications, the derivation and analysis of soliton solutions for these models constitute a critical and expansive research area, which has seen remarkable progress over the past quarter-century.

In the context of quantum mechanics and fiber-optic communication, one of the most significant concepts is that of superconductivity. A major milestone in this area was achieved by Dutch physicist Heike Kamerlingh Onnes, who, on July 10, 1908, successfully liquefied helium in his laboratory at Leiden University. This experiment marked the beginning of low-temperature physics and, serendipitously, laid the groundwork for the discovery of superconductivity a few years later [1]. Superconductivity is characterized by the near-complete disappearance of electrical resistance in certain materials when cooled to temperatures approaching absolute zero (approximately −237.59 °C). Over the ensuing decades, the phenomenon has been extensively studied. A significant theoretical development occurred in 1950, with the introduction of the Ginzburg–Landau model, which provides a macroscopic description of superconductivity and is regarded as a limiting case of the Bardeen–Cooper–Schrieffer (BCS) quantum theory [2]. Given the growing technological relevance of superconductors in modern science and engineering, the importance of the complex Ginzburg–Landau equation is now more apparent than ever [3]. Here are some common nonlinear equations and their associated optical soliton solutions: The nonlinear Schrödinger equation [4, 5], the Davey-Stewartson equation [6–8], the Lakshmanan–Porsezian–Daniel equation [9], the Biswas–Milovic equation [10, 11], the Chen-Lee-Liu equation [12, 13], the Manakov model [14], the Gerdjikov-Ivanov equation [15, 16], the (3 + 1)-dimensional B-type Kadomtsev–Petviashvili equation [17], the Schrödinger-Hirota equation [18], the Radhakrishnan–Kundu–Lakshmanan equation [19], the nonlinear Kodama equation [20], the extended (3 + 1)-dimensional Boiti-Leon-Manna-Pempinelli model [21], and the (3 + 1)-dimensional nonlinear Schrödinger equation in optical fiber [22].

Optical solitons have practical applications in fiber optics communication, where they help maintain the integrity of data pulses over long distances. Understanding and manipulating optical solitons has significant implications for optical communication, nonlinear optics, and other areas of photonics. The study of solitons involves a combination of theoretical modeling, numerical simulations, and experimental observations to explore their properties and potential applications. Analytical methods are often utilized to investigate the soliton solutions of these nonlinear equations and understand the dynamics of optical solitons in different physical scenarios.

The complex Ginzburg–Landau model (CGLM) is also one of the NLEEs of great importance in physics and mathematics. The CGLM is an extension of the Ginzburg–Landau theory [23], which was originally developed to describe the behavior of superconductors near their critical temperature. The CGLM also mostly describes the propagation of optical solitons across optical fibers over extended distances. Thus, examining the dynamic structure of the CGLM is critical. As a result, the divergent variants of the CGLM have been enhanced, and the soliton solutions of these models have been derived in the literature. Some of these are the fractional CGLM with a non-local nonlinearity term [24], the generalized CGLM [25], the quintic CGLM [26], the fractal CGLM form with the cubic-quintic nonlinearity [27], the conformable space–time fractional CGLM [28], the CGLM with beta-derivative [29], the CGLM with Kudryashov's law of refractive index [30], the perturbed CGLM in Kerr law and cubic–quintic–septic nonlinearity [31], the higher-order (3+1)- dimensional CGLM with cubic–quintic–septic [32], the CGLM with the CGLM with anti-cubic nonlinearity [33], the CGLM of eighth-order with multiplicative white noise in the Itô sense [34], and the CGLM with the law of four powers of nonlinearity [35].

In this study, the complex Ginzburg–Landau equation [36] having parabolic with nonlocal law of self-phase modulation (CGL-P) is given as:

$$iU_t + aU_{xx} + \left(c_1|U|^2 + c_2|U|^4 + c_3\left(|U|^2\right)_{xx}\right)U = \frac{b}{|U|^2U^*}\left\{2|U|^2\left(|U|^2\right)_{xx} - \left[\left(|U|^2\right)_x\right]^2\right\} + \gamma U, \tag{1.1}$$

in which a expresses the coefficient of chromatic dispersion, and c_1 and c_2 are defined as the coefficients of the parabolic law nonlinearity term. The parameter c_3 characterizes the strength of the nonlocal nonlinear interaction, which becomes significant in media where the response depends not only on the local field intensity but also on its spatial derivatives, as in thermal nonlinear media and photorefractive crystals. The parameter b governs the higher-order nonlocal term on the right-hand side of Eq (1.1), which is relevant in modeling systems with spatially extended nonlinearities and long-range coupling effects, such as in fiber optic systems with higher-order dispersion, nonparaxial propagation, or even certain superconducting and quantum fluid models exhibiting nonlocal coherence. $U = U(x,t)$ is the wave profile, which is complex, whereas U^* is assigned to the complex conjugate of U.

The paper is divided into the following sections: Section 2 presents obtaining the NODE (nonlinear ordinary differential equation) structure of Eq (1.1). Sections 3 and 4 offer the fundamental procedures of the new Kudryahov scheme and ShGEEM and their implementation. Section 5 covers the modulation instability. Section 6 consists of the graphical representations and their comments. Section 7 belongs to the concluding remarks.

2. Mathematical analysis

To attain the NODE form of Eq (1.1), let us define:

$$U(x,t) = U(\xi)\, e^{i(-kx+\omega t+\psi_0)}, \;\; \xi = \lambda(x - \nu t), \tag{2.1}$$

where $U(\xi)$ denotes the soliton pulse profile, ψ_0 expresses the phase constant, and $k, \omega,$ and ν are the wave number, frequency, and velocity, respectively. Adjoining Eq (2.1) into Eq (1.1), the resulting equation yields the following imaginary and real components, respectively:

$$\lambda\,(2ak + \nu)\,U' = 0, \tag{2.2}$$

$$c_2U^5 + c_1U^3 - \left(a\,k^2 + \omega + \gamma\right)U + 2c_3\,\lambda^2(U')^2U + \lambda^2\,(a - 4b)\,U'' + 2c_3\,\lambda^2U^2U'' = 0. \tag{2.3}$$

Here $U = U(\xi)$, $U' = \frac{dU}{d\xi}$, and $U'' = \frac{d^2U}{d\xi^2}$. Equation (2.2) gives the following formula:

$$\nu = -2ak. \tag{2.4}$$

From the terms $U''U^2$ and U^5 in Eq (2.3), the balance constant is attained as 1. Therefore, the derivation of the NODE of Eq (1.1) and the determination of the positive integer balancing constant, which are prerequisites for the application of any chosen analytical method, have been completed. As a result, there remain no obstacles to the implementation of the selected method. The introduction and application of the method are presented in the subsequent section.

3. Elucidation and implementation of the new Kudryashov scheme

Sections 3 and 4 include the description and implementation of the methods utilized in this study. However, we believe that it would be beneficial to include the following explanation regarding the rationale behind the method selection. As is well known, especially in the last quarter-century, dozens of analytical methods have been introduced into the literature, including the tanh expansion scheme and its many variations, the G'/G-expansion method and its versions, the generalized expansion rational function method, the trial equation expansion method, the $\exp(-\phi(\xi))$-expansion method and its variants (such as the extended, improved, and modified forms), the first integral method, the generalized elliptic equation rational expansion method, the hyperbolic function method and its extensions, the simple equation method and its variations, the projective Riccati method, the residual power series method, the unified solver method, the Sardar subequation method and its variants, the sine-Gordon and sinh-Gordon methods and their extensions, the Kudryashov method and its versions [37], the variable separated ODE method, the new auxiliary equation method, the mapping method, the new generalized ϕ^6-model expansion method, the Laplace–Adomian composition method, Lie symmetry analysis, the Hirota bilinear method, among others. It is an evident fact that there are even more methods beyond those readily recalled by researchers active in this field. Thousands of studies utilizing these methods have been published in the literature. Naturally, each method has its own advantages and disadvantages. These can be briefly listed as: Ease of implementation, requiring fewer algebraic manipulations, applicability to a wide range of nonlinear partial differential equations, effectiveness in handling higher-order models in nonlinear optics, and the generation of physically meaningful solutions (such as appropriate solitary wave or optical wave solutions). As is well known, singular solutions are generally not preferred in physical contexts, as they lack clear physical interpretations. Therefore, methods that produce repetitive solutions or primarily singular ones are typically not regarded positively. Additionally, there exists a degree of equivalence between some of these methods, especially when certain algebraic, trigonometric, or hyperbolic transformations are employed (e.g., many solutions obtained via the G'/G-expansion method can also be generated using the modified tanh-expansion method). Furthermore, the reliability of a method and its usage by respected researchers are also important selection criteria. One of the most fundamental factors in choosing a method is the suitability of the method to the problem under investigation and the researcher's judgment in this regard. These considerations have significantly influenced our choice of the methods employed in this article. The new Kudryashov method [38] is a recently developed, reliable, and easy-to-implement method that has been widely adopted by reputable researchers. It does not yield repetitive solutions and offers physically meaningful optical soliton forms such as bright and dark solitons, making it highly suitable for problems in nonlinear optics. Similarly, the ShGEEM method is an effective and reliable method that has been utilized extensively by researchers for many years.

One of the essential reasons we chose the new Kudryashov scheme (NKS) [37, 38] is that it is powerful, does not call for numerous mathematical processes, and can be operated for NLEEs. The truncated series is considered a solution to Eq (2.3):

$$U(\xi) = \sum_{j=0}^{n} \Lambda_j \Upsilon^j(\xi), \quad \Lambda_n \neq 0, \tag{3.1}$$

in which Λ_j are real numbers, and n stands for the balancing constant. $\Upsilon(\xi)$ satisfies the following form:

$$\frac{d\Upsilon(\xi)}{d\xi} = \sqrt{\alpha^2 \Upsilon(\xi)^2 (1 - \beta \Upsilon(\xi)^2)}, \tag{3.2}$$

where α, β are non-zero arbitrary real numbers. The solution of Eq (3.2) is given as:

$$\Upsilon(\xi) = \frac{4L}{4L^2 e^{\alpha\xi} + \beta e^{-\alpha\xi}}, \; L \in \mathbb{R}. \tag{3.3}$$

Let us remember that with the balancing constant $n = 1$, Eq (3.1) is transformed as:

$$U(\xi) = \Lambda_0 + \Lambda_1 \Upsilon(\xi), \quad \Lambda_1 \neq 0. \tag{3.4}$$

Placing Eq (3.4) into Eq (2.3), and making use of Eq (3.2), we collect the terms involving Υ^j and equate them to zero. This procedure yields the following algebraic system:

$$\begin{aligned}
&\Upsilon^0 : \left(-c_2\Lambda_0^4 + a\,k^2 - c_1\Lambda_0^2 + \gamma + \omega\right)\Lambda_0 = 0,\\
&\Upsilon^1 : \left(5c_2\Lambda_0^4 + \left(2\alpha^2\lambda^2 c_3 + 3c_1\right)\Lambda_0^2 + \alpha^2\,(a - 4b)\,\lambda^2 - a\,k^2 - \omega - \gamma\right)\Lambda_1 = 0,\\
&\Upsilon^2 : \left(\alpha^2\lambda^2 c_3 + \frac{5\Lambda_0^2 c_2}{3} + \frac{c_1}{2}\right)\Lambda_0\Lambda_1^2 = 0,\\
&\Upsilon^3 : \left(\left(-4\alpha^2\lambda^2 c_3 - 10\Lambda_0^2 c_2 - c_1\right)\Lambda_1^2 + 2\alpha^2\lambda^2\beta\left(2c_3\Lambda_0^2 + a - 4b\right)\right)\Lambda_1 = 0,\\
&\Upsilon^4 : \Lambda_1^2\left(10\alpha^2\lambda^2 c_3\beta - 5\Lambda_1^2 c_2\right)\Lambda_0 = 0,\\
&\Upsilon^5 : -6\alpha^2\beta c_3\,\lambda^2\Lambda_1^3 + c_2\Lambda_1^5 = 0.
\end{aligned}$$

Solution of the system yields:

Set-1.

$$\left\{\begin{matrix}\Lambda_0 = 0, \Lambda_1 = \dfrac{\sqrt{6c_2c_3\beta}\,\lambda\alpha}{c_2}, c_1 = \dfrac{-12\alpha^2{c_3}^2\lambda^2 + ac_2 - 4bc_2}{3c_3},\\ \omega = \alpha^2\lambda^2 a - 4\alpha^2\lambda^2 b - a\,k^2 - \gamma\end{matrix}\right\}. \tag{3.5}$$

Taking into account the parameters in Set-1 with Eqs (2.1), (3.3), and (3.4), we produce:

$$U_1(x,t) = \frac{4\sqrt{6c_2c_3\beta}\,\lambda\alpha L}{c_2\left(4L^2 \mathrm{e}^{\alpha\lambda(2akt+x)} + \beta\,\mathrm{e}^{-\alpha\lambda(x+2akt)}\right)}\mathrm{e}^{\mathrm{i}\left(-kx+\left(\alpha^2\lambda^2 a - 4\alpha^2\lambda^2 b - a\,k^2 - \gamma\right)t + \psi_0\right)}. \tag{3.6}$$

4. Explanation and application of ShGEEM

This section offers the fundamental traits and operation of the sinh-Gordon equation expansion method [39].

The solution of Eq (2.3) is presumed as:

$$U(\xi) = U(\vartheta(\xi)) = \varsigma_0 + \sum_{j=1}^{n} \cosh^{j-1}(\vartheta)[\varsigma_j \cosh(\vartheta) + \sigma_j \sinh(\vartheta)], \tag{4.1}$$

in which ς_j and σ_j's are real constants, and $\varsigma_j^2 + \sigma_j^2 \neq 0$. n is also the balance constant, and $\vartheta = \vartheta(\xi)$ satisfies the following relation:

$$\frac{d\vartheta(\xi)}{d\xi} = \sqrt{\sinh^2(\xi) + K}, \tag{4.2}$$

in which K is the integration constant. If we assume that $K = 0$, the Eq (4.2) can be written as:

$$\sinh(\vartheta(\xi)) = \operatorname{csch}(\xi), \quad \cosh(\vartheta(\xi)) = \coth(\xi), \tag{4.3}$$

$$\sinh(\vartheta(\xi)) = i\,\mathrm{sech}(\xi), \quad \cosh(\vartheta(\xi)) = \tanh(\xi). \tag{4.4}$$

Since $n = 1$, Eq (4.1) is formed as

$$U(\vartheta) = \varsigma_0 + \varsigma_1 \cosh(\vartheta) + \sigma_1 \sinh(\vartheta), \tag{4.5}$$

where ς_1 and σ_1 cannot be zero at the same time. Inserting Eq (4.5) into Eq (2.3), utilizing Eq (4.2), considering $K = 0$, and gathering the terms $\cosh(\vartheta)^i \sinh(\vartheta)^j$, equalizing to zero, then we have:

$$
\begin{aligned}
\cosh(\vartheta)^0 :\ & c_2\,\varsigma_0^5 - \left(10c_2\,\sigma_1^2 - c_1\right)\varsigma_0^3 + 5c_2\,\sigma_1^4\varsigma_0 - \left(-4c_3\,\lambda^2 + 3c_1\right)\sigma_1^2\varsigma_0 \\
& + \left(2c_3\,\lambda^2\varsigma_1^2 - a\,k^2 - \gamma - \omega\right)\varsigma_0 = 0, \\
\cosh(\vartheta)^0 \sinh(\vartheta)^1 :\ & -c_2\,\sigma_1^5 + \left(10\varsigma_0^2 c_2 - 2c_3\,\lambda^2 + c_1\right)\sigma_1^3 + \left(2c_3\,\varsigma_0^2 - 2c_3\,\varsigma_1^2 + a - 4b\right)\lambda^2\sigma_1 \\
& - \left(5c_2\,\varsigma_0^4 + a\,k^2 - 3c_1\,\varsigma_0^2 + \gamma + \omega\right)\sigma_1 = 0, \\
\cosh(\vartheta)^1 :\ & \left(-12c_3\,\sigma_1^2 + 4c_3\,\varsigma_0^2 - 2c_3\,\varsigma_1^2 + 2a - 8b\right)\lambda^2\varsigma_1 + \left(30c_2\,\sigma_1^2 - 3c_1\right)\varsigma_0^2\varsigma_1 \\
& + \left(-5c_2\,\varsigma_0^4 - 5c_2\,\sigma_1^4 + a\,k^2 + 3\sigma_1^2 c_1 + \gamma + \omega\right)\varsigma_1 = 0, \\
\cosh(\vartheta)^1 \sinh(\vartheta)^1 :\ & \sigma_1\varsigma_0\left(-10c_2\,\sigma_1^2 + 10\varsigma_0^2 c_2 - 8c_3\,\lambda^2 + 3c_1\right)\varsigma_1 = 0, \\
\cosh(\vartheta)^2 :\ & \varsigma_0\left(-10c_2\,\sigma_1^4 + \left(10\varsigma_0^2 c_2 - 30c_2\,\varsigma_1^2 - 42c_3\,\lambda^2 + 3c_1\right)\sigma_1^2\right) \\
& + \varsigma_1^2\left(10\varsigma_0^2 c_2 - 12c_3\,\lambda^2 + 3c_1\right)\varsigma_0 = 0, \\
\cosh(\vartheta)^2 \sinh(\vartheta)^1 :\ & \left(-2c_2\,\sigma_1^4 + \left(-8c_3\,\lambda^2 + 10\varsigma_0^2 c_2 - 10c_2\,\varsigma_1^2 + c_1\right)\sigma_1^2\right)\sigma_1 \\
& + \left(2\left(2c_3\,\varsigma_0^2 - 9c_3\,\varsigma_1^2 + a - 4b\right)\lambda^2 + 3\varsigma_1^2\left(10\varsigma_0^2 c_2 + c_1\right)\right)\sigma_1 = 0, \\
\cosh(\vartheta)^3 :\ & \left(\left(-10c_2\,\sigma_1^2 + 10\varsigma_0^2 c_2 - 8c_3\,\lambda^2 + c_1\right)\varsigma_1^2 - 10c_2\,\sigma_1^4\right)\varsigma_1 \\
& + \left(\left(30\varsigma_0^2 c_2 - 30c_3\,\lambda^2 + 3c_1\right)\sigma_1^2 + 2\lambda^2\left(2c_3\,\varsigma_0^2 + a - 4b\right)\right)\varsigma_1 = 0, \\
\cosh(\vartheta)^3 \sinh(\vartheta)^1 :\ & \varsigma_1\left(\left(\sigma_1^2 + \varsigma_1^2\right)c_2 + c_3\,\lambda^2\right)\varsigma_0\sigma_1 = 0, \\
\cosh(\vartheta)^4 :\ & \left(\left(\sigma_1^4 + 6\varsigma_1^2\sigma_1^2 + \varsigma_1^4\right)c_2 + 2c_3\,\lambda^2\left(\sigma_1^2 + \varsigma_1^2\right)\right)\varsigma_0 = 0, \\
\cosh(\vartheta)^4 \sinh(\vartheta)^1 :\ & \left(c_2\,\sigma_1^4 + 10c_2\,\varsigma_1^2\sigma_1^2 + 5c_2\,\varsigma_1^4 + 6\sigma_1^2 c_3\,\lambda^2 + 18c_3\,\lambda^2\varsigma_1^2\right)\sigma_1 = 0, \\
\cosh(\vartheta)^5 :\ & \varsigma_1\left(5c_2\,\sigma_1^4 + 10c_2\,\varsigma_1^2\sigma_1^2 + c_2\,\varsigma_1^4 + 18\sigma_1^2 c_3\,\lambda^2 + 6c_3\,\lambda^2\varsigma_1^2\right) = 0.
\end{aligned}
$$

We retrieve the following solution set from the above system:

Set-2.

$$\left\{\begin{aligned} & \varsigma_0 = 0, \varsigma_1 = \frac{\sqrt{-6c_2c_3}\,\lambda}{c_2}, \sigma_1 = 0, c_1 = \frac{24{c_3}^2\lambda^2 + ac_2 - 4bc_2}{3c_3}, \\ & \omega = -\frac{12{c_3}^2\lambda^4 + ac_2\,k^2 + 2c_2\,\lambda^2(a - 4b) + c_2\gamma}{c_2} \end{aligned}\right\}. \tag{4.6}$$

Taking into account the parameters in Set-2 with Eqs (2.1), (4.3)–(4.5), the following solutions are derived:

$$U_{2,1}(x,t) = \frac{\sqrt{-6c_2c_3}\,\lambda\coth(\lambda\,(x - \nu t))}{c_2}\mathrm{e}^{\mathrm{i}\left(-kx - \frac{\left(12{c_3}^2\lambda^4 + ac_2\,k^2 + 2ac_2\,\lambda^2 - 8bc_2\,\lambda^2 + c_2\gamma\right)t}{c_2} + \psi_0\right)}, \tag{4.7}$$

$$U_{2,2}(x,t) = \frac{\sqrt{-6c_2c_3}\,\lambda\tanh(\lambda\,(x-\nu t))}{c_2}\mathrm{e}^{\mathrm{i}\left(-kx-\frac{\left(12c_3{}^2\lambda^4+ac_2\,k^2+2ac_2\,\lambda^2-8bc_2\,\lambda^2+c_2\gamma\right)t}{c_2}+\psi_0\right)}, \tag{4.8}$$

where $c_2c_3 < 0$ and $\nu = -2ak$.

5. Modulation instability (MI)

To derive the linear stability analysis [40] of Eq (1.1), we consider the following solution:

$$U(x,t) = \left(\sqrt{\kappa} + \Psi(x,t)\right)e^{i\varrho\kappa t}, \tag{5.1}$$

where κ comes to mean normalized optical power. Inserting Eq (5.1) into Eq (1.1), the linearized equation, which consists of $\Psi(x,t)$ and $\Psi^*(x,t)$ is extracted as:

$$\begin{aligned}&\kappa^{3/2}\left((\kappa c_3 + a - 2b)\left(\frac{\partial^2}{\partial x^2}\Psi\right) + (\kappa c_3 - 2b)\left(\frac{\partial^2}{\partial x^2}\Psi^*\right) + i\left(\frac{\partial}{\partial t}\Psi\right)\right)\\ &+ \kappa^{3/2}\left(\left(4\kappa^4 c_2 + (3c_1 - 2\varrho)\kappa - 2\gamma\right)\Psi^* + \left(4\kappa^2 c_2 + 3\kappa c_1 - 2\gamma - 2\varrho\kappa\right)\Psi\right) = 0.\end{aligned} \tag{5.2}$$

$$\begin{aligned}\Psi(x,t) &= \beta_1 e^{i(\Omega x - \Delta t)} + \beta_2 e^{-i(\Omega x - \Delta t)},\\ \Psi^*(x,t) &= \beta_1 e^{-i(\Omega x - \Delta t)} + \beta_2 e^{i(\Omega x - \Delta t)},\end{aligned} \tag{5.3}$$

where β_1 and β_2 are real constants, and Δ and Ω specify the frequency and normalized wave number. By integrating Eq (5.3) with Eq (5.2) and noting the coefficients of $e^{-i(\Omega x-\Delta t)}$ and $e^{i(\Omega x-\Delta t)}$, the following system is acquired:

$$\begin{aligned}M_1\beta_1 + M_2\beta_2 &= 0,\\ M_3\beta_1 + M_4\beta_2 &= 0,\end{aligned} \tag{5.4}$$

in which

$$M_1 = \kappa^{3/2}\left(-(\kappa c_3 - 2b)\,\Omega^2 + 4\kappa^2 c_2 + 3\kappa c_1 - 2\gamma - 2\varrho\kappa\right), \tag{5.5}$$

$$M_2 = \kappa^{3/2}\left(-(\kappa c_3 + a - 2b)\,\Omega^2 - \Delta + 4\kappa^2 c_2 + 3\kappa c_1 - 2\gamma - 2\varrho\kappa\right), \tag{5.6}$$

$$M_3 = \kappa^{3/2}\left(-(\kappa c_3 + a - 2b)\,\Omega^2 + \Delta + 4\kappa^2 c_2 + 3\kappa c_1 - 2\gamma - 2\varrho\kappa\right), \tag{5.7}$$

$$M_4 = \kappa^{3/2}\left(-(\kappa c_3 - 2b)\,\Omega^2 + 4\kappa^2 c_2 + 3\kappa c_1 - 2\gamma - 2\varrho\kappa\right). \tag{5.8}$$

The above system can be written in the following structure:

$$\begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}\begin{bmatrix}\beta_1 \\ \beta_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}. \tag{5.9}$$

When computing the determinant of the matrix in Eq (5.9) regarding Eqs (5.5)–(5.8) for Δ, the following equation form is derived:

$$\Delta = \sqrt{\left(-8\kappa^2 c_2 + \left(2\Omega^2 c_3 - 6c_1 + 4\varrho\right)\kappa + (a - 4b)\,\Omega^2 + 4\gamma\right) a\,\Omega}. \tag{5.10}$$

Thus, the modulation instability of Eq (1.1) occurs when:

$$\left(-8\kappa^2 c_2 + \left(2\Omega^2 c_3 - 6c_1 + 4\varrho\right)\kappa + (a - 4b)\,\Omega^2 + 4\gamma\right) a < 0. \tag{5.11}$$

We also examine the modulation instability gain spectrum $G(\Omega)$ as shown in Figure 1, which is utilized from the maximum absolute value for the imaginary component of the frequency and assigned as:

$$G(\Omega) = 2Im\,(\Delta)\,. \tag{5.12}$$

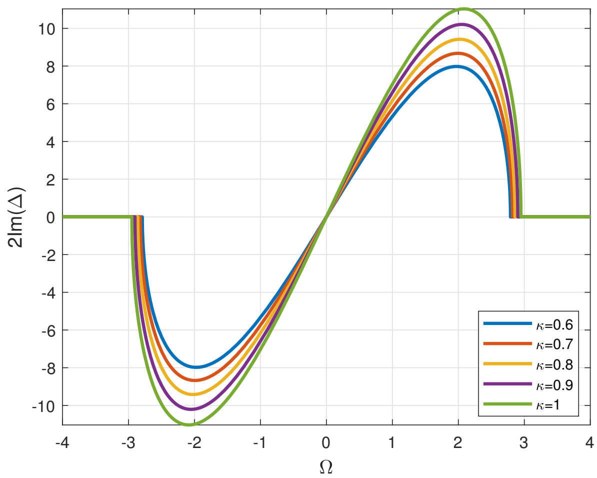


Figure 1. Gain spectrum of Eq (5.12) for $a = 0.7, b = 0.1, c_1 = 1, c_2 = 1, c_3 = 1, \gamma = -1, \varrho = -0.5$.

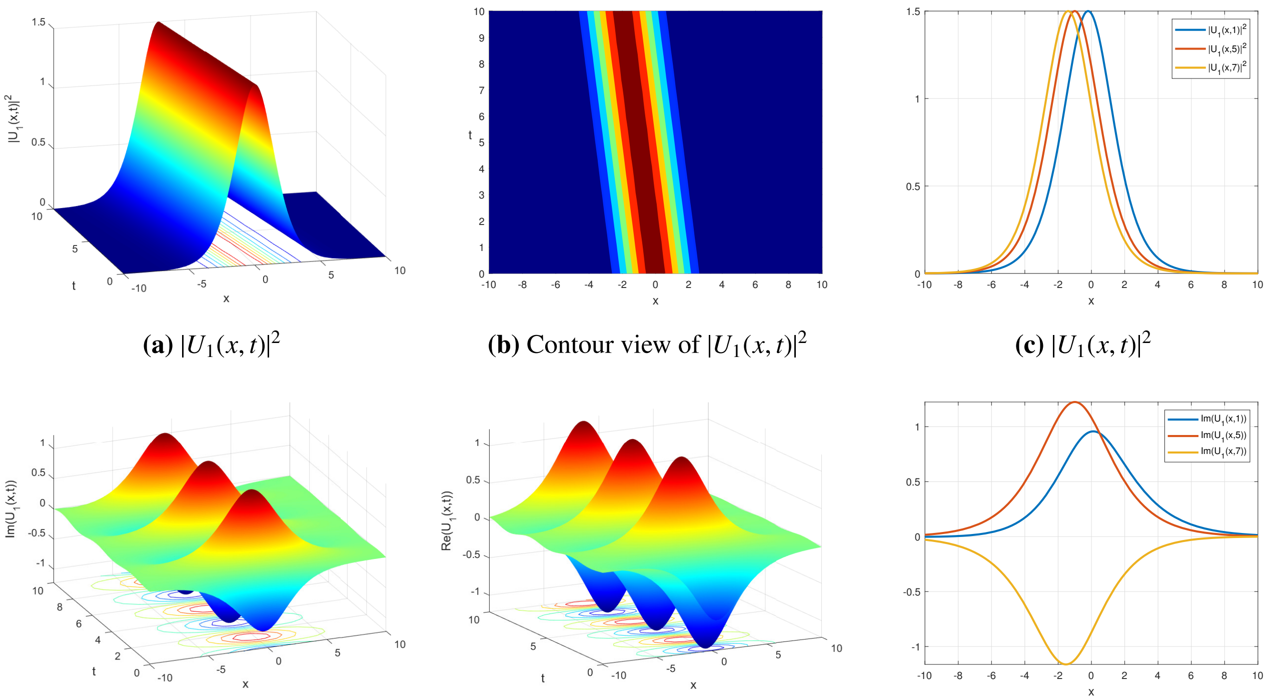
6. Results and discussion

This section is formed by the graphical projections of the $U_1(x,t)$ and $U_{2,2}(x,t)$ solutions resulting from the interpretations of these charts. All computations were conducted in Maple [41], while MATLAB [42] was employed for generating the graphical outputs. One of the diagrams introduced in this part is for the bright soliton indicated by Figure 2. All graphical representations in Figure 2 are drawings for the appropriate, particular, optimal, and selected parameter values created through an algorithm for Eq (3.6). These values are $a = 1, b = 1, c_2 = 1, c_3 = 1, \lambda = 0.5, k = 0.1, \gamma = 1, \psi_0 = 4, L = 0.1, \alpha = 1, \beta = 0.04$. Figures 2(a)–(c) belong to 3D, contour, and 2D representations of $|U_1(x,t)|^2$, sequentially. Figures 2(d)–(f) refer to 3D visualizations of $Im(U_1(x,t)), Re(U_1(x,t))$, and 2D depictions of $Im(U_1(x,t))$ for $t = 1, 5, 7$.

Figure 3 comprises drawings indicating the influences of the group velocity dispersion, the parabolic law, and the nonlocal nonlinearity term parameters on the structure of the bright soliton. According to the group velocity dispersion parameter, the soliton shifts to the left in Figure 3(a) as the parameter increases. Figure 3(b) is depicted to monitor the effect of the parabolic law c_2. The amplitude of the soliton declines concerning Figure 3(b) when c_2 increases. The inverse comment applies to Figure 3(c).

One of the charts presented in this part is for the dark soliton examination expressed by Figures 4 and 5. All projections are illustrations concerning the compatible, particular, optimal, and checked parameter values constructed via an algorithm for Eq (4.8). These values are $a = 0.5, b = 0.5, c_2 = 0.5, c_3 = -0.5, \gamma = 0.5, \lambda = 1, \psi_0 = 1, k = -0.05$. Figures 4(a)–(c) indicate 3D, contour, and 2D sketches of $|U_{2,2}(x,t)|^2$, while Figures 4(d)–(f) illustrate 3D and 2D graphs of the imaginary component for $t = 1, 5, 7$.

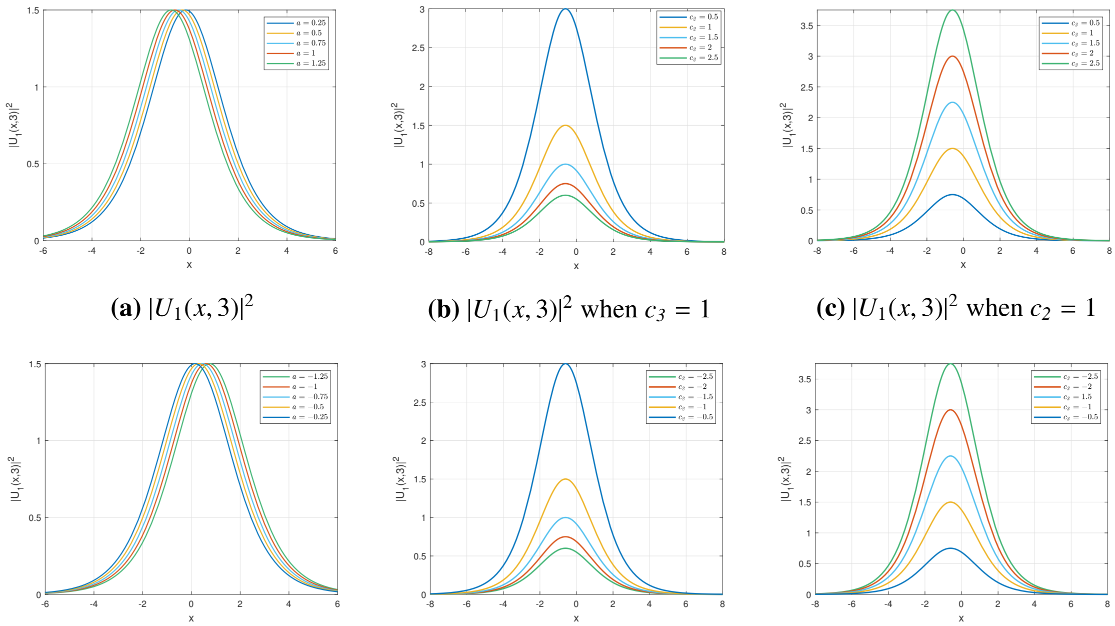
We analyze how some variety parameters in Eq (1.1) affect the dynamics of the soliton in Figure 5. When Figures 5(a),(d) are tested, it can be detected that as all values of a increase, the dark solitons progress to the right. When the parameter c_2 is probed in Figure 5(b), the vertical altitude of the soliton increases as positive values of c_2 decrease, while its horizontal amplitude decreases and it undergoes contraction. In Figure 5(e), the vertical altitude of the soliton grows as the negative values of c_2 decrease, while its horizontal amplitude decreases and it undergoes contraction. For Figure 5(c), the vertical altitude of the soliton becomes smaller when the negative values of c_3 increase, whereas its horizontal amplitude increases. Inversely, the positive values of c_3 decrease in Figure 5(f); a similar result is seen at negative values of c_3.



(a) $|U_1(x,t)|^2$ **(b)** Contour view of $|U_1(x,t)|^2$ **(c)** $|U_1(x,t)|^2$

(d) $Im(U_1(x,t))$ **(e)** $Re(U_1(x,t))$ **(f)** $Im(U_1(x,t))$

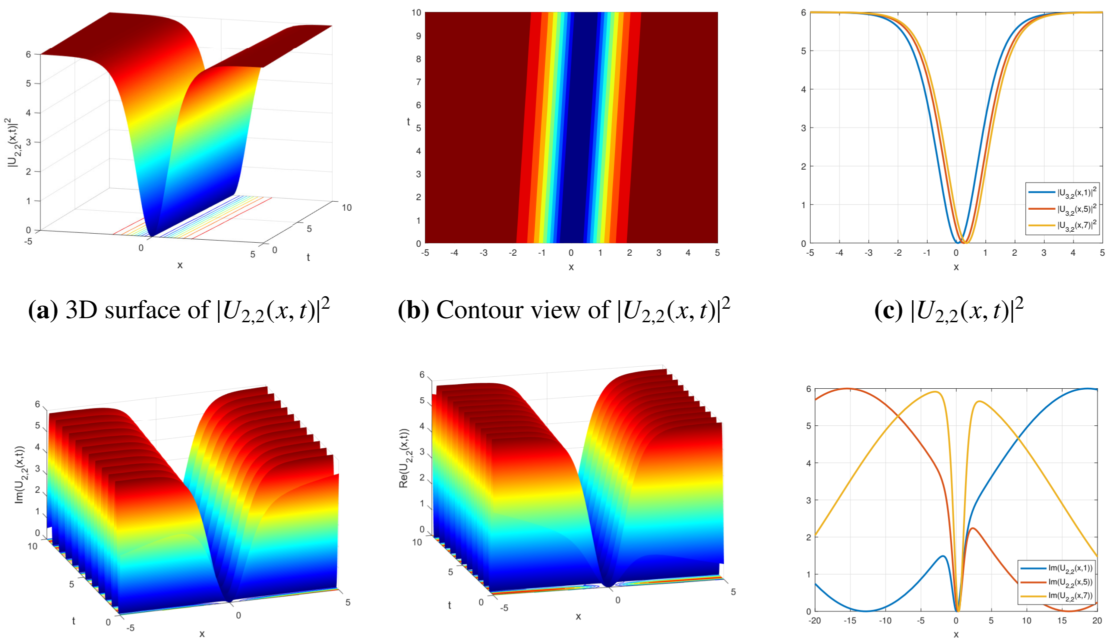
Figure 2. Some depictions of $U_1(x,t)$ for $a = 1, b = 1, c_2 = 1, c_3 = 1, \lambda = 0.5, k = 0.1, \gamma = 1, \psi_0 = 4, L = 0.1, \alpha = 1, \beta = 0.04$.



(a) $|U_1(x,3)|^2$ **(b)** $|U_1(x,3)|^2$ when $c_3 = 1$ **(c)** $|U_1(x,3)|^2$ when $c_2 = 1$

(d) $|U_1(x,3)|^2$ **(e)** $|U_1(x,3)|^2$ when $c_3 = -1$ **(f)** $|U_1(x,3)|^2$ when $c_2 = -1$

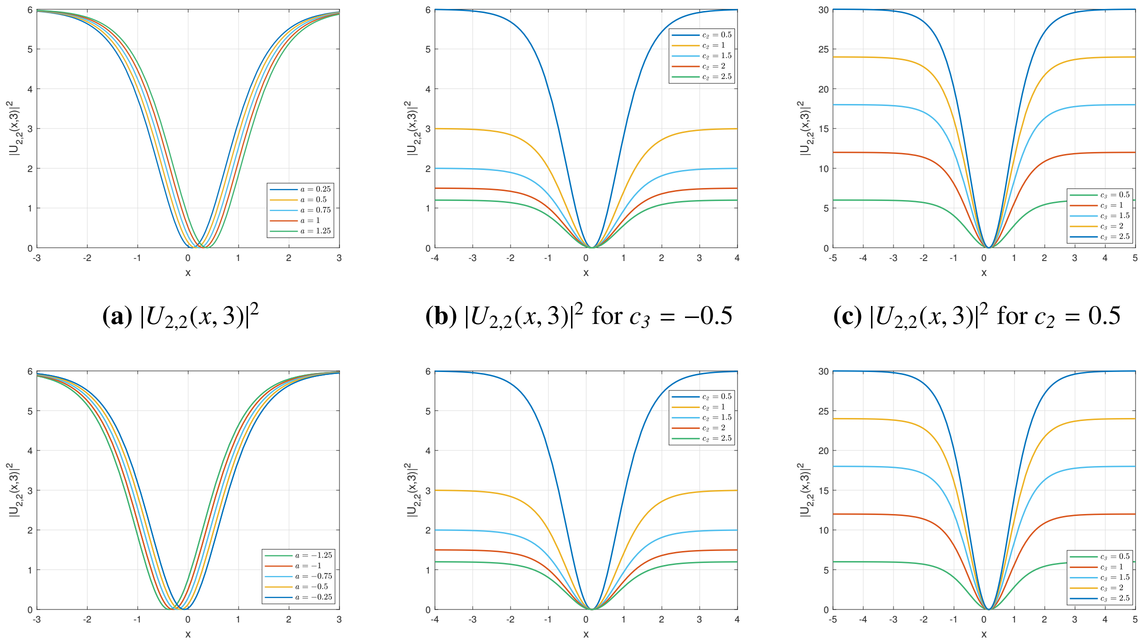
Figure 3. Various charts of $|U_1(x,t)|^2$ for $a > 0$ in (a), $c_2 > 0$ in (b), $c_3 > 0$ in (c), $a < 0$ in (d), $c_2 < 0$ in (e), $c_3 < 0$ in (f).



(a) 3D surface of $|U_{2,2}(x,t)|^2$ **(b)** Contour view of $|U_{2,2}(x,t)|^2$ **(c)** $|U_{2,2}(x,t)|^2$

(d) 3D plot of $Im(U_{2,2}(x,t))$ **(e)** Contour projection of $Im(U_{2,2}(x,t))$ **(f)** $Im(U_{2,2}(x,t))$

Figure 4. Some graphical records of $U_{2,2}(x,t)$ for $a = 0.5, b = 0.5, c_2 = 0.5, c_3 = -0.5, \gamma = 0.5, \lambda = 1, \psi_0 = 1, k = -0.05$.



(a) $|U_{2,2}(x,3)|^2$ **(b)** $|U_{2,2}(x,3)|^2$ for $c_3 = -0.5$ **(c)** $|U_{2,2}(x,3)|^2$ for $c_2 = 0.5$

(d) $|U_{2,2}(x,3)|^2$ **(e)** 2D of $|U_{2,2}(x,3)|^2$ for $c_3 = 0.5$ **(f)** $|U_{2,2}(x,3)|^2$ for $c_2 = -0.5$

Figure 5. Some graphs of $|U_{2,2}(x,3)|^2$ for $a > 0$ in (a), $c_2 > 0$ in (b), $c_3 > 0$ in (c), $a < 0$ in (d), $c_2 < 0$ in (e), $c_3 < 0$ in (f).

It is important to emphasize the following point: Figures 3 and 5 present visualizations corresponding to different values of certain parameters within the studied model. Specifically, Figure 3 illustrates the influence of selected parameters on the bright soliton form, whereas Figure 5 depicts their effect on the dark soliton form. In interpreting both figures, it should not be inferred that the observed variations in soliton amplitude or horizontal position represent changes in amplitude or velocity in the conventional physical sense. Rather, these differences arise from the influence of the model parameters a, c_2, and c_3, which are coefficients intrinsic to the main governing equation. Therefore, assigning different values to these parameters (provided that they comply with the mathematical constraints of the model and the method) does not alter the fundamental nature of the model represented by Eq (1.1), but instead produces different soliton solutions. In this context, Figures 3 and 5 should be interpreted as comparative illustrations of distinct bright and dark soliton profiles generated by the same underlying model under different parameter configurations. Any interpretation suggesting that these parameters directly alter the amplitude or velocity of the soliton would be inconsistent with the very definition of a soliton, which, by nature, is a wave-form that preserves both its shape and speed.

7. Conclusions

The parabolic law with weak nonlocal nonlinearity structure of the CGL-P model has been scrutinized, and a variety of soliton solutions have been procured utilizing the new Kudryashov and sinh-Gordon equation expansion schemes. The solution set encompasses both bright and dark solitons, each of which holds significant relevance in the study of nonlinear optical phenomena. The appropriate dimensional representations of the soliton solutions have been indicated in detail. Furthermore, the influence of some model parameters on the gained solitons was surveyed and presented in detail with 2D graphics. Finally, the modulation instability examination for the CGL-P was produced. Thus, this work represents an exhaustive analysis of some kinds of soliton solutions to the new structure of the CGL-P model and indicates the effectiveness of the ShGEEM. Moreover, the Kudryashov and the Sinh-Gordon expansion schemes contain a direct ansatz for the solution, usually expressed in terms of polynomial or hyperbolic/trigonometric functions. So, the Kudryashov and Sinh-Gordon expansion methods can be applied directly to a broader class of nonlinear partial differential equations, even those that are non-integrable. Kudryashov's approach is particularly suitable for higher-order nonlinear terms (e.g., polynomial nonlinearities of high degree) and complex forms of the nonlinear partial differential equations. In future work, investigations will consist of various fractional and stochastic structures of the CGL-P model. Applying the model to higher-dimensional systems could provide deeper insights into complex pattern formation and spatiotemporal chaos. Furthermore, a detailed parameter sensitivity analysis could yield a better understanding of the system's stability and bifurcation structures.

Author Contributions

Wael W. Mohammed: Formal analysis, software, methodology; Neslihan Ozdemir: Formal analysis, software, methodology; Aydin Secer: Validation, methodology; Muslum Ozisik: Investigation, writing-review and editing, supervision; Mustafa Bayram: Investigation, writing-review and editing, supervision; Taha Radwan: Data curation, resources, writing-review and editing; Karim K. Ahmed: Data curation, resources, writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence tools in the creation of this article.

Acknowledgment

The Researchers would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University for financial support (QU-APC-2025).

Conflict of interest

All authors declare no conflicts of interest in this paper.

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