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Research article

Explicit travelling wave solutions to the modified Korteweg-de Vries-Zakharov-Kuznetsov and the time-regularized long-wave equations using two efficient integration techniques

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Abstract: Numerous scientific disciplines such as fluid dynamics, non linear optics and laser physics are the sources of non-linear evolution equations. The modified Korteweg-de Vries-Zakharov-Kuznetsov equation can be applied to analyze the evolution of ion-acoustic perturbations in magnetized plasma made up of two negative ion ingredients that have distinct temperatures. The analysis of shallow water waves gives rise to the time-regularized long-wave equation. In this work, traveling wave solutions to the non-linear evolution equations are obtained by using the modified auxiliary equation method and the simple mapping method. This study demonstrates the presence of traveling wave solutions for the modified Korteweg-de Vries-Zakharov-Kuznetsov equation and the time-regularized long-wave equation. An ordinary differential equation is transformed into a non linear form by applying the traveling wave solutions, which arise from Lie symmetry with infinite dimensions. The results show how rich the analyzed models are in explicit solutions. This leads to the discovery of exact traveling wave solutions for the proposed study including periodic and quasi-periodic solitons, bright and dark solitons, kink, anti-peakon, bell shape, anti-bell shape, W-shape, and M-shape soliton solutions by using the proposed methods. Adding the solutions into the original equation allows for an analysis of their accuracy. It shows how the free parameters affect the amplitudes and wave characteristics. The study provides thorough two-dimensional (2D) and three-dimensional (3D) graphical illustrations of the outcomes that improve comprehension of their physical attributes and show how effectively the recommended approaches work to solve challenging non-linear equations. It is crucial to remember that the suggested techniques are capable, reliable, and engaging analytical instruments for resolving non linear partial differential equations.

Keywords: Modified Korteweg-de Vries-Zakharov-Kuznetsov equation; time-regularized long-wave

equation; modified auxiliary equation method; simple mapping method; solitons; traveling wave
Mathematics Subject Classification: 39A12, 9B62, 33B10, 26A48, 26A51

1. Introduction

A wide range of complex physical processes are defined by non linear evolution equations (NEEs), which are widely employed. Currently, the NEEs are being investigated in other non linear domains, such as physics, chemical kinematics, meteorology, applied mathematics, chemistry, propagation of shallow water waves, protein chemistry, and chemically reactive materials. Work on soliton wave solutions of NEEs is becoming increasingly important. In the process of studying these physical processes, finding exact solutions to NEEs assumes greater significance, and importance. Within the last few decades, the literature has produced influential techniques that allow one to create precise solutions to non linear equations using traveling waves. Non linear partial differential equations (NPDEs) [1] are used to explain complex phenomena in a variety of fields, notably physics, engineering, mathematics, and many other scientific fields. In the fields of hydrodynamics, nuclear science, material science, plasma physics, and related disciplines, NPDEs are utilized to investigate physical resonance phenomena with varied uses, and dynamical techniques. There are several effective methods for solving NPDEs, but no single method is suitable for all NPDE types. A variety of NPDEs are used in fluid dynamics to describe shallow-water waves. These equations include the Bateman-Burgers equation, the Boomeron equation, the Boltzmann equation, the Buckmaster equation, the Chafee-Infante equation, and the Davey-Stewartson equation. Determining the traveling-wave solutions of NPDEs is the main physical issue. Several effective techniques have been presented for solving NPDEs. These include the unified transform method [2], the tanh-coth function method [3], Hirota's bilinear transformation [4], the recursive prediction error method [5], the two-grid discretization techniques [6], the rational-expansion technique [7], the P-expansion method [8], and the generalized exponential rational function method [9], as well as the five factor inventory method [10], the two-variable-expansion method [11], and variational methods [12]. A unique kind of solitary wave known as a soliton is one that, even after colliding with another solitary wave, maintains its amplitude, form, and velocity. Optical fibers, high-energy mechanics, solid-state mechanics, elastic media, chemical kinematics, biophysics, plasma physics, nuclear mechanics, geochemistry, and meteorology are among the fields in which soliton phenomena are seen. Thus, the closed form solitary wave solutions of NEEs have been examined by numerous researchers interested in non linear phenomena [13].

Other studies that have been documented in the literature, and embrace the non linearity of real-world problems are noteworthy. Ion-acoustic waves in a magnetized plasma that are weakly non linear are described by the Korteweg–de Vries (KdV) equation. Solitons move at different speeds, and Calogero, and Degasperis modified the KdV, and Schrödinger equations to account for them. They found a correlation between the polarization effects, and the speed of the solitons. This led to the emergence of two distinct forms of soliton: one is defined as an accelerated soliton that, in the distant past, boomeranged back with the same speed, and the other as being imprisoned, and oscillating in a space domain while continually changing direction around a fixed point. Consider the coupled

Boomeron equation of the following form:

$$\begin{cases} r_t(s,t) = \overrightarrow{\beta}.\overrightarrow{w}_s(s,t), \\ \overrightarrow{w}_t(s,t) = r_s(s,t)\overrightarrow{\beta} + \overrightarrow{\alpha} \wedge \overrightarrow{w}(s,t) + 2\int_s^{\infty} ds'\overrightarrow{w}_s(s',t) \wedge [\overrightarrow{w}_s(s',t) \wedge \overrightarrow{\beta}], \end{cases} \tag{1.1}$$

where $\overrightarrow{w}(s,t)$ is a vector field, and $r(s,t)$ is a scalar field, and $\overrightarrow{\alpha}$, and $\overrightarrow{\beta}$ are two vector quantities in 3D space.

The Korteweg–de Vries-Zakharov-Kuznetsov (KdV-ZK) equation is another common model. However, the soliton's character may change from compressive to rarefactive or vice versa at critical densities, and temperatures in more complicated plasma compositions, so the modified Korteweg–de Vries–Zakharov–Kuznetsov (mKdV-ZK) equation should be used instead. When ion-acoustic disturbances evolve in a magnetized plasma with two different temperature negative ion components, the mKdV-ZK develops. The mKdV-ZK equation [14], which governs the oblique propagation of non linear electrostatic oscillations of the type is in this article. Methods that have previously employed on mKdV-ZK equation are the subsidiary ordinary differential equation method, the Sardar method, the $(\frac{G'}{G})$-expansion method, and the improved generalized Riccati equation mapping (IGREM) method.

$$r_t + \alpha r^2 r_s + r_{sss} + r_{syy} + r_{szz} = 0. \tag{1.2}$$

The KdV equation in its most basic form is provided by the following:

$$r_t + \alpha r r_s + r_{sss} = 0. \tag{1.3}$$

There are two alternatives to KDV equation. The first alternate form is the equation,

$$r_s + r_t + \alpha r r_s - r_{stt} = 0. \tag{1.4}$$

This equation is known as the Benjamin-Bona-Mahony equation or the regularized long wave (RLW) equation [15, 16]. Waqar, et al. [17] obtained diverse wave solutions for the (2+1)-dimensional Zoomeron equation using the modified extended direct algebraic approach. Alqhtani, et al. [18] obtained approximate similarity solutions, and error control of the functional Boiti-Leon-Mana-Pempinelli equation. Dereli asserts that Peregrine first created the equation to describe how it formed, and changed, and Benjamin, Bona, and Mahony then looked into it as a way to improve the KdV. The second alternative of the KdV equation is

$$r_s + r_t + \alpha r r_s + r_{stt} = 0, \tag{1.5}$$

given by Jeffrey, Egri, and Joseph. This equation is named the time-regularized long-wave (TRLW) equation. Here α is a non-zero parameter in the three equations above. Peregrine first proposed this equation to explain the behaviour of the undular bore, and Benjamin et al. later adopted it. One important model for examining shallow water wave phenomena is the TRLW equation. By including dispersion, and regularization features that are essential in transmission line analogues, and electrical pulse propagation, the TRLW equation provides a more accurate approximation than classical models like the KdV or Boussinesq equations, which mainly describe unidirectional wave propagation, and assume small amplitudes, and long wavelengths. The assumptions of the KdV equation are violated for wider wave profiles or when higher-order nonlinear effects are not trivial, even though it successfully simulates solitary wave propagation in a one-dimensional (1D) shallow sea setting. The TRLW equation, on the other hand, takes these higher-order dispersive effects into consideration, which makes it more suitable for situations in which energy dispersion, and bidirectional wave interactions are significant. The Joseph-Egri TRLW equation is essential to the study of nonlinear waves, and can be used to represent many significant scientific phenomena, including ion-acoustic plasma waves and shallow water waves [16,19–21]. The soliton, and additional solitary wave solutions used in this investigation of mKdV-ZK equation, and the TRLW equation are found using the simple mapping method (SMM) [22]. The SMM was used for the stochastic mKdV equation. The SMM was used by Rehman et al. to derive the optical solitons of the Biswas-Arshed model [23, 24]. The (G'/G)-expansion method, based on expressing solutions as polynomials in (G'/G) with G satisfying a second-order linear ordinary differential equation (ODE), is widely used for obtaining exact traveling wave solutions of nonlinear equations [25]. For assessing the stability, and robustness of the generated solutions under realistic physical conditions, one possible approach is to include experimental or simulation-based studies. Numerical simulations [26], in particular, may supplement analytical findings by providing light on the effects of perturbations or starting conditions on soliton's behaviour.

This paper's goal is to use the modified auxiliary equation method (MAEM), and SMM to obtain the exact solutions followed by the solitary wave solutions for the TRLW equation, and the mKdV-ZK equation. Examining the impact of the free variables on the resulting traveling wave solutions is our secondary objective.

This is how the paper is structured. In Section 2, the description of analytical methods are performed. The prescribed analytical techniques are utilized in Section 3. Implementation of the methods is discussed in Section 4. In Section 5, graphs and physical discussions are employed. Section 7 concludes the study with a few closing remarks.

2. Description of analytical methods

2.1. Modified auxiliary equation method

In this section, the MAEM is covered. Let us consider a NPDE with the following form:

$$H(r, r_s, r_t, r_{ss}, r_{tt}, ...) = 0, \tag{2.1}$$

where H is a polynomial in $r(s,t)$, and its partial derivatives in which the highest-order derivatives, and non linear terms are involved. In the following, we give the steps of this method.

Step 1. Using the traveling wave transformation

$$r(s,t) = r(\phi), \tag{2.2}$$

where $\phi = s + y + ct$. Using this transformation in Eq (2.1), the NPDEs is converted into an ODE with the following form:

$$H(r, r', r'', ...) = 0. \tag{2.3}$$

Step 2. Assume that the solutions of the ODE in Eq (2.3) have the following form:

$$r(\phi) = \alpha_0 + \sum_{j=1}^{m} \alpha_j \chi^{jh(\phi)} + \sum_{j=1}^{m} \beta_j \chi^{-jh(\phi)}, \tag{2.4}$$

where $\chi > 0$; $\chi \neq 1$; and α_0 , α_j, and β_j are arbitrary constants that will be determined later.

Here, $h(\phi)$ satisfies the following ODE:

$$h'(\phi) = \frac{1}{\ln(a)}(\gamma\chi^{-h(\phi)} + \delta + \eta\chi^{h(\phi)}), \tag{2.5}$$

where δ affects the damping or growth rate and γ controls the potential or restoring force; together, they influence the solution type, depending on the discriminant.

Step 3. Find out the positive integer m in Eq (2.4) by balancing the highest-order derivatives, and the non linear terms.

Step 4. Insert Eq (2.4) in conjunction with Eq (2.5) into Eq (2.3) combining all values of $\chi^{jh(\phi)}$ where $j = -m, ..., m$, and equating them to zero to obtain the values of $\alpha_j, \beta_j, \gamma, \delta$, and η. The function $\chi^{jh(\phi)}$ assumes the following solutions,

Case 1. If $\delta^2 - 4\gamma\eta > 0$ and $\eta \neq 0$, then

$$\chi^{h(\phi)} = -\frac{1}{2\eta}(\delta + \sqrt{\delta^2 - 4\gamma\eta})(\tanh(\frac{\sqrt{\delta^2 - 4\gamma\eta}\phi}{2})), \tag{2.6}$$

$$\chi^{h(\phi)} = -\frac{1}{2\eta}(\delta + \sqrt{\delta^2 - 4\gamma\eta})(\coth(\frac{\sqrt{\delta^2 - 4\gamma\eta}\phi}{2})). \tag{2.7}$$

Case 2. If $\delta^2 - 4\gamma\eta < 0$ and $\eta \neq 0$, then

$$\chi^{h(\phi)} = \frac{1}{2\eta}(-\delta + \sqrt{4\gamma\eta - \delta^2})(\tan(\frac{\sqrt{4\gamma\eta - \delta^2}\phi}{2})), \tag{2.8}$$

$$\chi^{h(\phi)} = -\frac{1}{2\eta}(\delta + \sqrt{4\gamma\eta - \delta^2})(\cot(\frac{\sqrt{4\gamma\eta - \delta^2}\phi}{2})). \tag{2.9}$$

Case 3. If $\delta^2 - 4\gamma\eta = 0$ and $\eta \neq 0$, then

$$\chi^{h(\phi)} = (-\frac{2 + \delta\phi}{2\eta\phi}). \tag{2.10}$$

The exact solution of Eq (2.1) is obtained by replacing the all values of the constants $\alpha_j, \beta_j, \gamma, \delta$, and η and substituting the corresponding values of $\chi^{h(\phi)}$ from Equation (2.6)–(2.10) into Equation (2.4).

2.2. Simple mapping method

This section describes the SMM for obtaining traveling wave solutions of NEEs. Consider the following NPDEs, which have two independent time, and space variables.

$$G(\phi, \phi_s, \phi_t, \phi_{ss}, \phi_{tt}, \phi_{st}, ...) = 0, \tag{2.11}$$

where $G(s, t) = G(\phi)$. $G(s, t)$, and its partial derivatives comprise the polynomial G, which include highest-order derivatives, and non linear terms.

Step 1. Equation transformation into ODE can be accomplished using the wave transformation relation. The wave transformation is as follows:

$$G(s, t) = G(\phi), \quad \phi = \alpha s - \beta t, \tag{2.12}$$

where α, and β are the wave number, and velocity, respectively. Putting Eq (2.12) in Eq (2.11) yields an ODE of $G(\phi)$,

$$G(w, w', w'', w''', ...) = 0, \tag{2.13}$$

where G is a polynomial in $G(\phi)$ with derivatives of $G(\phi)$, i.e., $w'(\phi) = \frac{dG}{d\phi}$, $w''(\phi) = \frac{d^2G}{d\phi^2}$, and so forth. If Eq (2.11) is integrable, then take each integral constant to be zero, and integrate it as often as required.

Step 2. Suppose that the general solution of Eq (2.11) has the follwoing:

$$G(\phi) = \sum_{u=0}^{l} \nu_u \kappa^u(\phi), \tag{2.14}$$

where l is a positive integer that needs to be calculated, and ν_u represents the real constants to be determined such that $\nu_u \neq 0$ is to be identified, while κ satisfies the first kind of elliptic equation,

$$\begin{cases} \kappa'' = a\kappa + b\kappa^3, \\ \kappa'^2 = a\kappa^2 + \frac{1}{2}b\kappa^4 + c. \end{cases} \tag{2.15}$$

Here, the prime indicates the derivative with respect to ϕ.

Step 3. The non linear term in the Eq (2.11), and the power of non linearity in Eq (2.12) are used to balance the linear term of the highest-order derivative in order to establish the value of l.

Step 4. After replacing Eq (2.14) with Eq (2.15), the coefficients ν_u, β, a, b, and c may be obtained into the ODE in Eq (2.13). It will be considered arbitrary for the solution of Equation (2.11) if one of them is left undetermined. As a result, Equation (2.13) establishes an algebraic mapping relation between the solutions of Eqs (2.11), and (2.14).

Step 5. The following solutions comes from the Eq (2.15):

$$\begin{cases} \kappa(\phi) = \tanh(\phi), \\ \kappa(\phi) = \sec h(\phi), \\ \kappa(\phi) = sn(\phi), \text{ or } \kappa(\phi) = cd(\phi), \\ \kappa(\phi) = cn(\phi), \\ \kappa(\phi) = dn(\phi), \\ \kappa(\phi) = ns(\phi), \text{ or } \kappa(\phi) = dc(\phi), \end{cases} \tag{2.16}$$

where the Jacobi elliptic functions are denoted by $sn(\phi)$, $cn(\phi)$, $dn(\phi)$, $ns(\phi)$, $dc(\phi)$, and $cd(\phi)$, respectively. Since the tanh-function yields shock waves, and the sec h-function yields solitary waves, Eq (2.15) is taken into consideration. When the parameters a, b, and c are set appropriately; however, the periodic waves are derived in terms of Jacobi elliptic functions. The Jacobi elliptic functions $sn = sn(\phi \mid t)$, $cn = cn(\phi \mid t)$, $dn = dn(\phi \mid t)$, $ns = ns(\phi \mid t)$, $cd = cd(\phi \mid t)$, and $dc = dc(\phi \mid t)$, where $(0 < t < 1)$ is the modulus of the elliptic function, are double-periodic, and possess the properties of triangular functions, namely,

$$\begin{cases} sn^2(\phi) + cn^2(\phi) = 1, \\ dn^2(\phi) + t^2 sn^2(\phi) = 1, \\ sn(\phi)' = cn(\phi)dn(\phi), \\ cn(\phi)' = -sn(\phi)dn(\phi), \\ dn(\phi)' = -t^2 sn(\phi)cn(\phi). \end{cases} \tag{2.17}$$

When $t \to 0$, the Jacobi elliptic function degenerates to the triangular functions, i.e.,

$$\begin{cases} sn(\phi) \to \sin(\phi), \\ cn(\phi) \to \cos(\phi), \\ dn(\phi) \to 1. \end{cases} \tag{2.18}$$

When $t \to 1$, the Jacobi elliptic function degenerates to the hyperbolic functions, i.e.,

$$\begin{cases} sn(\phi) \to \tanh(\phi), \\ cn(\phi) \to \sec h(\phi), \\ dn(\phi) \to \sec h(\phi). \end{cases} \tag{2.19}$$

To find the exact solutions of physical importance for certain NPDEs, it should be noted that any of the other nine Jacobi elliptic functions can be expressed using $sn(\phi)$, $cn(\phi)$, and $dn(\phi)$.

3. Mathematical computation

3.1. The modified Korteweg-de Vries-Zakharov-Kuznetsov equation

The PDE named as the mKdV-ZK equation is given below:

$$R_t + \xi R^2 R_s + R_{sss} + R_{syy} + R_{szz} = 0, \tag{3.1}$$

where ξ is a non-zero parameter.

The traveling wave transformation is,

$$R(s, y, z, t) = R(\phi), \quad \phi = s + y + z - \lambda t. \tag{3.2}$$

After differentiating equation above and by replacing in Eq (3.1) it becomes

$$-\lambda R' + \xi R^2 R' + 3R''' = 0. \tag{3.3}$$

Integrating the equation above, we get

$$3R'' - \lambda R + \frac{\xi}{3} R^3 + c_1 = 0. \tag{3.4}$$

3.2. Time regularized long-wave equation

The PDE named the TRLW equation is given as follows

$$R_t + R_s + \xi R R_s + R_{stt} = 0, \tag{3.5}$$

where ξ is a non-zero parameter.

The traveling wave transformation is,

$$R(s, t) = R(\phi), \quad \phi = s - \lambda t. \tag{3.6}$$

After differentiating the equation above and by putting it into Eq (3.5) it becomes

$$(1 - \lambda) R' + \xi R R' + \lambda^2 R''' = 0. \tag{3.7}$$

Integrating equation above, we get

$$\lambda^2 R'' + (1 - \lambda) R + \frac{\xi}{2} R^2 + c_2 = 0. \tag{3.8}$$

4. Application of analytical methods

4.1. Modified auxiliary equation method

4.1.1. The modified Korteweg-de Vries-Zakharov-Kuznetsov equation

After balancing the non linear term R^3, and the highest-order derivative term R'' from the Eq (3.4) , $3j = j + 2$, which results in $j = 1$. Consequently, the equation's solution has the following form:

$$r(\phi) = \alpha_0 + \alpha_1\chi^{h(\phi)} + \beta_1\chi^{-h(\phi)}, \tag{4.1}$$

where α_0, α_1, and β_1 are the unknown coefficients to be determined. In the auxiliary equation, the function is stated as $h(\phi)$

$$\ln(a) * h'(\phi) = \delta + \gamma\chi^{-h(\phi)} + \eta\chi^{h(\phi)}, \tag{4.2}$$

Putting Eq (4.2) into Eq (3.4), we get

$$\begin{cases} -\lambda\alpha_0 + \frac{\xi}{3}\alpha_0^3 + 3\delta\gamma\alpha_1 + 3\delta^2\alpha_1\chi^h + 6\gamma\eta\chi^h + 9\delta\eta\alpha_1\chi^{2h} + 6\eta^2\alpha_1\chi^{3h} - \lambda\alpha_1\chi^h + \xi\alpha_0^2\alpha_1\chi^h + \\ \xi\alpha_0\alpha_1^2\chi^{2h} + \frac{\xi}{3}\chi^{3h}\alpha_1^3 + 3\delta\eta\beta_1 + 6\gamma^3\beta_1\chi^{-3h} + 9\delta\eta\beta_1\chi^{-2h} + 3\delta^2\beta_1\chi^{-h} + 6\gamma\eta\beta_1\chi^{-h} - \delta\beta_1\chi^{-h} + \\ \xi\alpha_0^2\beta_1\chi^{-h} + 2\xi\alpha_0\alpha_1\beta_1 + \xi\alpha_1^2\beta_1\chi^h + \xi\alpha_0\beta_1^2\chi^{-2h} + \xi\alpha_1\beta_1^2\chi^{-h} + \frac{\xi}{3}\beta_1^3\chi^{-3h} = 0. \end{cases} \tag{4.3}$$

Since $(p = -3, -2, -1, 0, 1, 2, 3)$, put each coefficient of χ^{ph} is equal to zero, we have

$$\begin{cases} 6\gamma^3\beta_1 + \frac{\xi}{3}\beta_1^3 = 0, \\ 9\delta\eta\beta_1 + \xi\alpha_0\beta_1^2 = 0, \\ 3\delta^2\beta_1 + 6\gamma\eta\beta_1 - \delta\beta_1 + \xi\alpha_0^2\beta_1 + \xi\alpha_1\beta_1^2 = 0, \\ -\lambda\alpha_0 + \frac{\xi}{3}\alpha_0^3 + 3\delta\gamma\alpha_1 + 2\xi\alpha_0\alpha_1\beta_1 + 3\delta\eta\beta_1 = 0, \\ 3\delta^2\alpha_1 + 6\gamma\eta\alpha_1 - \lambda\alpha_1 + \xi\alpha_0^2\alpha_1 + \xi\alpha_1^2\beta_1 = 0, \\ 9\delta\eta\alpha_1 + \xi\alpha_0\alpha_1^2 = 0, \\ 6\eta^2\alpha_1 + \frac{\xi}{3}\alpha_1^3 = 0. \end{cases} \tag{4.4}$$

In order for a mathematical statement to be classified as an algebraic equation, it must always equal zero. By solving Eq (4.4), we get the following families:

Family 1.

Let $\alpha_0 = \sqrt{\frac{-3\delta^2-6\gamma\eta+\lambda}{\xi}}$, $\alpha_1 = 3\sqrt{\frac{2}{\xi}}\eta\iota$, and $\beta_1 = 0$.

Substituting the values of α_0, α_1, and β_1 into (4.2) along with the solutions of $\chi^{h(\phi)}$ from Eqs (2.6)–(2.10) we obtain the following.

Case 1. If $\delta^2 - 4\gamma\eta > 0$, and $\eta \neq 0$, then

$$r_{1,1}(\phi) = \sqrt{\frac{-3\delta^2 - 6\gamma\eta + \lambda}{\xi}} + 3\sqrt{\frac{2}{\xi}}\eta\iota(-\frac{1}{2\eta}(\delta + \sqrt{\delta^2 - 4\gamma\eta})(\tanh(\frac{\sqrt{\delta^2 - 4\gamma\eta}\phi}{2})). \tag{4.5}$$

$$r_{1,2}(\phi) = \sqrt{\frac{-3\delta^2 - 6\gamma\eta + \lambda}{\xi}} + 3\sqrt{\frac{2}{\xi}}\eta\iota(-\frac{1}{2\eta}(\delta + \sqrt{\delta^2 - 4\gamma\eta})(\coth(\frac{\sqrt{\delta^2 - 4\gamma\eta}\phi}{2})). \tag{4.6}$$

Case 2. If $\delta^2 - 4\gamma\eta < 0$, and $\eta \neq 0$, then

$$r_{1,3}(\phi) = \sqrt{\frac{-3\delta^2 - 6\gamma\eta + \lambda}{\xi}} + 3\sqrt{\frac{2}{\xi}}\eta\iota(\frac{1}{2\eta}(-\delta + \sqrt{4\gamma\eta - \delta^2})(\tan(\frac{\sqrt{4\gamma\eta - \delta^2}\phi}{2})). \tag{4.7}$$

$$r_{1,4}(\phi) = \sqrt{\frac{-3\delta^2 - 6\gamma\eta + \lambda}{\xi}} + 3\sqrt{\frac{2}{\xi}}\eta\iota(-\frac{1}{2\eta}(\delta + \sqrt{4\gamma\eta - \delta^2})(\cot(\frac{\sqrt{4\gamma\eta - \delta^2}\phi}{2})). \tag{4.8}$$

Case 3. If $\delta^2 - 4\gamma\eta = 0$, and $\eta \neq 0$, then

$$r_{1,5}(\phi) = \sqrt{\frac{-3\delta^2 - 6\gamma\eta + \lambda}{\xi}} + 3\sqrt{\frac{2}{\xi}}\eta\iota(-\frac{2 + \delta\phi}{2\eta\phi}). \tag{4.9}$$

Family 2.

Let $\alpha_0 = \frac{3\delta\iota}{\sqrt{2\xi}}$, $\alpha_1 = 0$, and $\beta_1 = 3\sqrt{\frac{2}{\xi}}\gamma\iota$.

The cases for Eq (4.2) are as follows:

Case 1. If $\delta^2 - 4\gamma\eta > 0$, and $\eta \neq 0$, then

$$r_{2,1}(\phi) = \frac{3\delta\iota}{\sqrt{2\xi}} + 3\sqrt{\frac{2}{\xi}}\gamma\iota(-\frac{1}{2\eta}(\delta + \sqrt{\delta^2 - 4\gamma\eta}\tanh(\frac{\sqrt{\delta^2 - 4\gamma\eta}\phi}{2})))^{-1}. \tag{4.10}$$

$$r_{2,2}(\phi) = \frac{3\delta\iota}{\sqrt{2\xi}} + 3\sqrt{\frac{2}{\xi}}\gamma\iota(-\frac{1}{2\eta}(\delta + \sqrt{\delta^2 - 4\gamma\eta}\coth(\frac{\sqrt{\delta^2 - 4\gamma\eta}\phi}{2})))^{-1}. \tag{4.11}$$

Case 2. If $\delta^2 - 4\gamma\eta < 0$, and $\eta \neq 0$, then

$$r_{2,3}(\phi) = \frac{3\delta\iota}{\sqrt{2\xi}} + 3\sqrt{\frac{2}{\xi}}\gamma\iota(\frac{1}{2\eta}(-\delta + \sqrt{4\gamma\eta - \delta^2}\tan(\frac{\sqrt{4\gamma\eta - \delta^2}\phi}{2})))^{-1}. \tag{4.12}$$

$$r_{2,4}(\phi) = \frac{3\delta\iota}{\sqrt{2\xi}} + 3\sqrt{\frac{2}{\xi}}\gamma\iota(-\frac{1}{2\eta}(\delta + \sqrt{4\gamma\eta - \delta^2}\coth(\frac{\sqrt{4\gamma\eta - \delta^2}\phi}{2})))^{-1}. \tag{4.13}$$

Case 3. If $\delta^2 - 4\gamma\eta = 0$, and $\eta \neq 0$ then,

$$r_{2,5}(\phi) = \frac{3\delta\iota}{\sqrt{2\xi}} + 3\sqrt{\frac{2}{\xi}}\gamma\iota(-\frac{2 + \delta\phi}{2\eta\phi})^{-1}. \tag{4.14}$$

Family 3.

Let $\alpha_0 = \frac{3\delta\iota}{\sqrt{2\xi}}$, $\alpha_1 = \frac{(-3\sqrt{2}\delta^2 + 12\sqrt{2}\gamma\eta - 2\sqrt{2}\lambda)\iota}{12\gamma\sqrt{\xi}}$, and $\beta_1 = 3\sqrt{\frac{2}{\xi}}\gamma\iota$.

The cases for Eq (4.2) are as follows.

Case 1. If $\delta^2 - 4\gamma\eta > 0$, and $\eta \neq 0$, then

$$r_{3,1}(\phi) = \sqrt{\frac{-3\delta^2 - 6\gamma\eta + \lambda}{\xi}} + (\frac{(-3\sqrt{2}\delta^2 + 12\sqrt{2}\gamma\eta - 2\sqrt{2}\lambda)\iota}{12\gamma\sqrt{\xi}})(-\frac{1}{2\eta}(\delta + \sqrt{\delta^2 - 4\gamma\eta} \tanh(\frac{\sqrt{\delta^2 - 4\gamma\eta}\phi}{2}))) + (3\sqrt{\frac{2}{\xi}}\eta\iota)(-\frac{1}{2\eta}(\delta + \sqrt{\delta^2 - 4\gamma\eta}\tanh(\frac{\sqrt{\delta^2 - 4\gamma\eta}\phi}{2})))^{-1}. \tag{4.15}$$

$$r_{3,2}(\phi) = \sqrt{\frac{-3\delta^2 - 6\gamma\eta + \lambda}{\xi}} + (\frac{(-3\sqrt{2}\delta^2 + 12\sqrt{2}\gamma\eta - 2\sqrt{2}\lambda)\iota}{12\gamma\sqrt{\xi}})(-\frac{1}{2\eta}(\delta + \sqrt{\delta^2 - 4\gamma\eta} \coth(\frac{\sqrt{\delta^2 - 4\gamma\eta}\phi}{2}))) + (3\sqrt{\frac{2}{\xi}}\eta\iota)(-\frac{1}{2\eta}(\delta + \sqrt{\delta^2 - 4\gamma\eta}\coth(\frac{\sqrt{\delta^2 - 4\gamma\eta}\phi}{2})))^{-1}. \tag{4.16}$$

Case 2. If $\delta^2 - 4\gamma\eta < 0$, and $\eta \neq 0$, then

$$r_{3,3}(\phi) = \sqrt{\frac{-3\delta^2 - 6\gamma\eta + \lambda}{\xi}} + (\frac{(-3\sqrt{2}\delta^2 + 12\sqrt{2}\gamma\eta - 2\sqrt{2}\lambda)\iota}{12\gamma\sqrt{\xi}})(\frac{1}{2\eta}(-\delta + \sqrt{4\gamma\eta - \delta^2} \tan(\frac{\sqrt{\delta^2 - 4\gamma\eta}\phi}{2}))) + (3\sqrt{\frac{2}{\xi}}\eta\iota)(\frac{1}{2\eta}(-\delta + \sqrt{4\gamma\eta - \delta^2}\tan(\frac{\sqrt{4\gamma\eta - \delta^2}\phi}{2})))^{-1}. \tag{4.17}$$

$$r_{3,4}(\phi) = \sqrt{\frac{-3\delta^2 - 6\gamma\eta + \lambda}{\xi}} + (\frac{(-3\sqrt{2}\delta^2 + 12\sqrt{2}\gamma\eta - 2\sqrt{2}\lambda)\iota}{12\gamma\sqrt{\xi}})(-\frac{1}{2\eta}(\delta + \sqrt{4\gamma\eta - \delta^2} \cot(\frac{\sqrt{4\gamma\eta - \delta^2}\phi}{2}))) + (3\sqrt{\frac{2}{\xi}}\eta\iota)(-\frac{1}{2\eta}(\delta + \sqrt{4\gamma\eta - \delta^2}\cot(\frac{\sqrt{4\gamma\eta - \delta^2}\phi}{2})))^{-1}. \tag{4.18}$$

Case 3. If $\delta^2 - 4\gamma\eta = 0$, and $\eta \neq 0$, then

$$r_{3,5}(\phi) = \sqrt{\frac{-3\delta^2 - 6\gamma\eta + \lambda}{\xi}} + (\frac{(-3\sqrt{2}\delta^2 + 12\sqrt{2}\gamma\eta - 2\sqrt{2}\lambda)\iota}{12\gamma\sqrt{\xi}})(-\frac{2 + \delta\phi}{2\eta\phi}) + (3\sqrt{\frac{2}{\xi}}\eta\iota)(-\frac{2 + \delta\phi}{2\eta\phi})^{-1}. \tag{4.19}$$

4.1.2. Time-regularized long-wave equation

After balancing the non linear term R^2, and the highest-order derivative term R'' from Eq (3.8), $2j = j + 2$, which results in $j = 2$. Consequently, the equation's solution has the following form:

$$r(\phi) = \alpha_0 + \alpha_1\chi^{h(\phi)} + \beta_1\chi^{-h(\phi)} + \alpha_2\chi^{2h(\phi)} + \beta_2\chi^{-2h(\phi)}, \tag{4.20}$$

where α_0 , α_1 , β_1 , α_2, and β_2 are the unknown coefficients to be determined. In the auxiliary equation, the function is stated as $r(\phi)$;

$$\ln(a) * h'(\phi) = \delta + \gamma\chi^{-h(\phi)} + \eta\chi^{h(\phi)}, \tag{4.21}$$

for arbitrary constant values of δ, γ, and η, such that $\rho > 0$, and $\rho \neq 1$.

Putting Eq (4.21) into Eq (3.8) we get,

$$\alpha_0-\lambda\alpha_0+\frac{\xi\alpha_0^2}{2}+\alpha_1\chi^h-\lambda\alpha_1\chi^h+\delta\gamma\lambda^2\alpha_1+\delta^2\lambda^2\alpha_1\chi^h+2\gamma\eta\lambda^2\alpha_1\chi^h+3\delta\eta\lambda^2\alpha_1\chi^{2h}+2\eta^2\lambda^2\alpha_1\chi^{3h}+\xi\alpha_0\alpha_1\chi^h+$$
$$\frac{1}{2}\xi\alpha_1^2\chi^{2h}+\alpha_2\chi^{2h}-\lambda\alpha_2\chi^{2h}+2\gamma^2\lambda^2\alpha_2+6\delta\gamma\lambda^2\alpha_2\chi^h+4\delta^2\lambda^2\alpha_2\chi^{2h}+8\gamma\eta\lambda^2\alpha_2\chi^{2h}+10\delta\eta\lambda^2\alpha_2\chi^{3h}+$$
$$6\eta^2\lambda^2\alpha_2\chi^{4h}+\xi\alpha_0\alpha_2\chi^{2h}+\xi\alpha_1\alpha_2\chi^{3h}+\frac{1}{2}\xi\alpha_2^2\chi^{4h}+\beta_1\chi^{-h}-\lambda\beta_1\chi^{-h}+\delta\eta\beta_1\lambda^2+2\gamma^2\lambda^2\beta_1\chi^{-3h}+3\delta\gamma\lambda^2\beta_1\chi^{-2h}+$$
$$\delta^2\lambda^2\beta_1\chi^{-h}+2\gamma\eta\lambda^2\beta_1\chi^{-h}+\xi\alpha_0\beta_1\chi^{-h}+\xi\alpha_1\beta_1+\xi\alpha_2\beta_1\chi^h+\frac{1}{2}\xi\beta_1^2\chi^{-2h}+\beta_2\chi^{-2h}-\lambda\beta_2\chi^{-2h}+$$
$$2\eta^2\lambda^2\beta_2+6\gamma^2\lambda^2\beta_2\chi^{-4h}+10\delta\gamma\lambda^2\beta_2\chi^{-3h}+4\delta^2\lambda^2\beta_2\chi^{-2h}+8\gamma\eta\lambda^2\beta_2\chi^{-2h}+6\delta\gamma\lambda^2\beta_2\chi^{-h}+\xi\alpha_0\beta_2\chi^{-2h}+$$
$$\xi\alpha_1\beta_2\chi^{-h}+\xi\alpha_2\beta_2+\xi\beta_1\beta_2\chi^{-3h}+\frac{1}{2}\xi\beta_2^2\chi^{-4h}=0. \quad (4.22)$$

Since ($p=-4,-3,-2,-1,0,1,2,3,4$), and each coefficient of χ^{hp} is equal to zero, we have

$$\begin{cases}
\frac{1}{2}\xi\beta_2^2+6\gamma^2\lambda^2\beta_2=0,\\
2\gamma^2\lambda^2\beta_1+10\delta\gamma\lambda^2\beta_2+\xi\beta_1\beta_2=0,\\
3\delta\gamma\lambda^2\beta_1+\frac{1}{2}\xi\beta_1^2+\beta_2-\lambda\beta_2+4\delta^2\lambda^2\beta_2+8\gamma\eta\lambda^2\beta_2+\xi\alpha_0\beta_2=0,\\
\beta_1-\lambda\beta_1\delta^2\lambda^2\beta_1+2\gamma\eta\lambda^2\beta_1+\xi\alpha_0\beta_1+6\delta\gamma\lambda^2\beta_2+\xi\alpha_1\beta_2=0,\\
\alpha_0-\lambda\alpha_0+\frac{\xi\alpha_0^2}{2}+\delta\gamma\lambda^2\alpha_1+2\gamma^2\lambda^2\alpha_2+\delta\eta\beta_1\lambda^2+\xi\alpha_1\beta_1+2\eta^2\lambda^2\beta_2+\xi\alpha_2\beta_2=0,\\
\alpha_1-\lambda\alpha_1+\delta^2\lambda^2\alpha_1+2\gamma\eta\lambda^2\alpha_1+\xi\alpha_0\alpha_1+6\delta\gamma\lambda^2\alpha_2+\xi\alpha_2\beta_1=0,\\
3\delta\eta\lambda^2\alpha_1+\frac{1}{2}\xi\alpha_1^2+\alpha_2-\lambda\alpha_2+4\delta^2\lambda^2\alpha_2+8\gamma\eta\lambda^2\alpha_2+\xi\alpha_0\alpha_2=0,\\
2\eta^2\lambda^2\alpha_1+10\delta\eta\lambda^2\alpha_2+\xi\alpha_1\alpha_2=0,\\
6\eta^2\lambda^2\alpha_2+\frac{1}{2}\xi\alpha_2^2=0.
\end{cases} \quad (4.23)$$

In order for a mathematical statement to be classified as an algebraic equation, it must always equal zero. By solving Eq (4.23) we get the following families.

Family 1.

Let $\alpha_0=\frac{-1+\lambda-\delta^2\lambda^2-8\gamma\eta\lambda^2}{\xi}$, $\alpha_1=\frac{12\delta\eta\lambda^2}{\xi}$, $\alpha_2=\frac{12\eta^2\lambda^2}{\xi}$, and $\beta_1=\frac{-12\delta\gamma\lambda^2}{\xi}$, and $\beta_2=\frac{-12\gamma^2\lambda^2}{\xi}$.

And the cases for Eq (4.21) are as follows:

Case 1. If $\delta^2-4\gamma\eta>0$, and $\eta\neq 0$, then

$$r_{1,1}(\phi)=\frac{-1+\lambda-\delta^2\lambda^2-8\gamma\eta\lambda^2}{\xi}+\frac{12\delta\eta\lambda^2}{\xi}(-\frac{1}{2\eta}(\delta+\sqrt{\delta^2-4\gamma\eta}$$
$$\tanh(\frac{\sqrt{\delta^2-4\gamma\eta}}{2})))+\frac{12\eta^2\lambda^2}{\xi}(-\frac{1}{2\eta}(\delta+\sqrt{\delta^2-4\gamma\eta}\tanh(\frac{\sqrt{\delta^2-4\gamma\eta}\phi}{2})))^2-$$
$$\frac{12\delta\gamma\lambda^2}{\xi}(-\frac{1}{2\eta}(\delta+\sqrt{\delta^2-4\gamma\eta}\tanh(\frac{\sqrt{\delta^2-4\gamma\eta}\phi}{2})))^{-1}-\frac{12\gamma^2\lambda^2}{\xi}(-\frac{1}{2\eta}(\delta+\sqrt{\delta^2-4\gamma\eta}\tanh(\frac{\sqrt{\delta^2-4\gamma\eta}\phi}{2})))^{-2}. \quad (4.24)$$

$$r_{1,2}(\phi) = \frac{-1+\lambda-\delta^2\lambda^2-8\gamma\eta\lambda^2}{\xi} + \frac{12\delta\eta\lambda^2}{\xi}(-\frac{1}{2\eta}(\delta + \sqrt{\delta^2-4\gamma\eta} \coth(\frac{\sqrt{\delta^2-4\gamma\eta}}{2}))) + \frac{12\eta^2\lambda^2}{\xi}(-\frac{1}{2\eta}(\delta + \sqrt{\delta^2-4\gamma\eta}\coth(\frac{\sqrt{\delta^2-4\gamma\eta}\phi}{2})))^2 - \frac{12\delta\gamma\lambda^2}{\xi}(-\frac{1}{2\eta}(\delta+\sqrt{\delta^2-4\gamma\eta}\coth(\frac{\sqrt{\delta^2-4\gamma\eta}\phi}{2})))^{-1} - \frac{12\gamma^2\lambda^2}{\xi}(-\frac{1}{2\eta}(\delta+\sqrt{\delta^2-4\gamma\eta}\coth(\frac{\sqrt{\delta^2-4\gamma\eta}\phi}{2})))^{-2}. \tag{4.25}$$

Case 2. If $\delta^2 - 4\gamma\eta < 0$, and $\eta \neq 0$, then

$$r_{1,3}(\phi) = \frac{-1+\lambda-\delta^2\lambda^2-8\gamma\eta\lambda^2}{\xi} + \frac{12\delta\eta\lambda^2}{\xi}(\frac{1}{2\eta}(-\delta + \sqrt{\delta^2-4\gamma\eta} \tan(\frac{\sqrt{4\gamma\eta-\delta^2}}{2}))) + \frac{12\eta^2\lambda^2}{\xi}(\frac{1}{2\eta}(-\delta + \sqrt{\delta^2-4\gamma\eta}\tan(\frac{\sqrt{4\gamma\eta-\delta^2}\phi}{2})))^2 - \frac{12\delta\gamma\lambda^2}{\xi}(\frac{1}{2\eta}(-\delta+\sqrt{\delta^2-4\gamma\eta}\tan(\frac{\sqrt{4\gamma\eta-\delta^2}\phi}{2})))^{-1} - \frac{12\gamma^2\lambda^2}{\xi}(-\frac{1}{2\eta}(\delta+\sqrt{\delta^2-4\gamma\eta}\tan(\frac{\sqrt{4\gamma\eta-\delta^2}\phi}{2})))^{-2}. \tag{4.26}$$

$$r_{1,4}(\phi) = \frac{-1+\lambda-\delta^2\lambda^2-8\gamma\eta\lambda^2}{\xi} + \frac{12\delta\eta\lambda^2}{\xi}(-\frac{1}{2\eta}(\delta + \sqrt{\delta^2-4\gamma\eta} \cot(\frac{\sqrt{4\gamma\eta-\delta^2}}{2}))) + \frac{12\eta^2\lambda^2}{\xi}(-\frac{1}{2\eta}(\delta + \sqrt{\delta^2-4\gamma\eta}\cot(\frac{\sqrt{4\gamma\eta-\delta^2}\phi}{2})))^2 - \frac{12\delta\gamma\lambda^2}{\xi}(-\frac{1}{2\eta}(\delta+\sqrt{\delta^2-4\gamma\eta}\cot(\frac{\sqrt{4\gamma\eta-\delta^2}\phi}{2})))^{-1} - \frac{12\gamma^2\lambda^2}{\xi}(-\frac{1}{2\eta}(\delta+\sqrt{\delta^2-4\gamma\eta}\cot(\frac{\sqrt{4\gamma\eta-\delta^2}\phi}{2})))^{-2}. \tag{4.27}$$

Case 3. If $\delta^2 - 4\gamma\eta = 0$, and $\eta \neq 0$, then

$$r_{1,5}(\phi) = \frac{-1+\lambda-\delta^2\lambda^2-8\gamma\eta\lambda^2}{\xi} + \frac{12\delta\eta\lambda^2}{\xi}(-\frac{2+\delta\phi}{2\eta\phi}) + \frac{12\eta^2\lambda^2}{\xi}(-\frac{2+\delta\phi}{2\eta\phi})^2 - \frac{12\delta\gamma\lambda^2}{\xi}(-\frac{2+\delta\phi}{2\eta\phi})^{-1} - \frac{12\gamma^2\lambda^2}{\xi}(-\frac{2+\delta\phi}{2\eta\phi})^{-2}. \tag{4.28}$$

Family 2.

Suppose $\alpha_0 = \frac{-1+\lambda-\delta^2\lambda^2-8\gamma\eta\lambda^2}{\xi}$, $\alpha_1 = \alpha_2 = 0$, $\beta_1 = \frac{-12\delta\gamma\lambda^2}{\xi}$, and $\beta_2 = \frac{-12\gamma^2\lambda^2}{\xi}$.

And the cases for Eq (4.21) are as follows.

Case 1. If $\delta^2 - 4\gamma\eta > 0$, and $\eta \neq 0$, then

$$r_{2,1}(\phi) = \frac{-1+\lambda-\delta^2\lambda^2-8\gamma\eta\lambda^2}{\xi} + \frac{-12\delta\gamma\lambda^2}{\xi}(-\frac{1}{2\eta}(\delta + \sqrt{\delta^2-4\gamma\eta}\tanh(\frac{\sqrt{\delta^2-4\gamma\eta}\phi}{2})))^{-1} + \frac{-12\gamma^2\lambda^2}{\xi}(-\frac{1}{2\eta}(\delta + \sqrt{\delta^2-4\gamma\eta}\tanh(\frac{\sqrt{\delta^2-4\gamma\eta}\phi}{2})))^{-2}. \tag{4.29}$$

$$r_{2,2}(\phi) = \frac{-1+\lambda-\delta^2\lambda^2-8\gamma\eta\lambda^2}{\xi} + \frac{-12\delta\gamma\lambda^2}{\xi}(-\frac{1}{2\eta}(\delta + \sqrt{\delta^2-4\gamma\eta}\coth(\frac{\sqrt{\delta^2-4\gamma\eta}\phi}{2})))^{-1} + \frac{-12\gamma^2\lambda^2}{\xi}(-\frac{1}{2\eta}(\delta + \sqrt{\delta^2-4\gamma\eta}\coth(\frac{\sqrt{\delta^2-4\gamma\eta}\phi}{2})))^{-2}. \quad (4.30)$$

Case 2. If $\delta^2 - 4\gamma\eta < 0$, and $\eta \neq 0$, then

$$r_{2,3}(\phi) = \frac{-1+\lambda-\delta^2\lambda^2-8\gamma\eta\lambda^2}{\xi} + \frac{-12\delta\gamma\lambda^2}{\xi}(\frac{1}{2\eta}(-\delta + \sqrt{4\gamma\eta-\delta^2}\tan(\frac{\sqrt{4\gamma\eta-\delta^2}\phi}{2})))^{-1} + \frac{-12\gamma^2\lambda^2}{\xi}(\frac{1}{2\eta}(-\delta + \sqrt{4\gamma\eta-\delta^2}\tan(\frac{\sqrt{4\gamma\eta-\delta^2}\phi}{2})))^{-2}. \quad (4.31)$$

$$r_{2,4}(\phi) = \frac{-1+\lambda-\delta^2\lambda^2-8\gamma\eta\lambda^2}{\xi} + \frac{-12\delta\gamma\lambda^2}{\xi}(-\frac{1}{2\eta}(\delta + \sqrt{4\gamma\eta-\delta^2}\cot(\frac{\sqrt{4\gamma\eta-\delta^2}\phi}{2})))^{-1} + \frac{-12\gamma^2\lambda^2}{\xi}(-\frac{1}{2\eta}(\delta + \sqrt{4\gamma\eta-\delta^2}\cot(\frac{\sqrt{4\gamma\eta-\delta^2}\phi}{2})))^{-2}. \quad (4.32)$$

Case 3. If $\delta^2 - 4\gamma\eta = 0$, and $\eta \neq 0$, then

$$r_{2,5}(\phi) = \frac{-1+\lambda-\delta^2\lambda^2-8\gamma\eta\lambda^2}{\xi} - \frac{12\delta\gamma\lambda^2}{\xi}(-\frac{2+\delta\phi}{2\eta\phi})^{-1} - \frac{12\gamma^2\lambda^2}{\xi}(-\frac{2+\delta\phi}{2\eta\phi})^{-2}. \quad (4.33)$$

Family 3.

Let $\alpha_0 = \frac{-1+\lambda-\delta^2\lambda^2-8\gamma\eta\lambda^2}{\xi}$, $\alpha_1 = \frac{-12\delta\eta\lambda^2}{\xi}$, $\alpha_2 = \frac{-12\eta^2\lambda^2}{\xi}$, and $\beta_1 = \beta_2 = 0$.

The cases for Eq (4.21) are as follows.

Case 1. If $\delta^2 - 4\gamma\eta > 0$, and $\eta \neq 0$, then.

$$r_{3,1}(\phi) = \frac{-1+\lambda-\delta^2\lambda^2-8\gamma\eta\lambda^2}{\xi} + \frac{-12\delta\eta\lambda^2}{\xi}(-\frac{1}{2\eta}(\delta + \sqrt{\delta^2-4\gamma\eta}\tanh(\frac{\sqrt{\delta^2-4\gamma\eta}\phi}{2}))) + \frac{-12\eta^2\lambda^2}{\xi}(-\frac{1}{2\eta}(\delta + \sqrt{\delta^2-4\gamma\eta}\tanh(\frac{\sqrt{\delta^2-4\gamma\eta}\phi}{2})))^2. \quad (4.34)$$

$$r_{3,2}(\phi) = \frac{-1+\lambda-\delta^2\lambda^2-8\gamma\eta\lambda^2}{\xi} + \frac{-12\delta\eta\lambda^2}{\xi}(-\frac{1}{2\eta}(\delta + \sqrt{\delta^2-4\gamma\eta}\coth(\frac{\sqrt{\delta^2-4\gamma\eta}\phi}{2}))) + \frac{-12\eta^2\lambda^2}{\xi}(-\frac{1}{2\eta}(\delta + \sqrt{\delta^2-4\gamma\eta}\coth(\frac{\sqrt{\delta^2-4\gamma\eta}\phi}{2})))^2. \quad (4.35)$$

Case 2. If $\delta^2 - 4\gamma\eta < 0$, and $\eta \neq 0$, then.

$$r_{3,3}(\phi) = \frac{-1+\lambda-\delta^2\lambda^2-8\gamma\eta\lambda^2}{\xi} + \frac{-12\delta\eta\lambda^2}{\xi}(\frac{1}{2\eta}(-\delta+\sqrt{4\gamma\eta-\delta^2}\tan(\frac{\sqrt{4\gamma\eta-\delta^2}\phi}{2}))) + \frac{-12\eta^2\lambda^2}{\xi}(\frac{1}{2\eta}(-\delta+\sqrt{4\gamma\eta-\delta^2}\tan(\frac{\sqrt{4\gamma\eta-\delta^2}\phi}{2})))^2. \quad (4.36)$$

$$r_{3,4}(\phi) = \frac{-1+\lambda-\delta^2\lambda^2-8\gamma\eta\lambda^2}{\xi} + \frac{-12\delta\eta\lambda^2}{\xi}(-\frac{1}{2\eta}(\delta+\sqrt{4\gamma\eta-\delta^2}\cot(\frac{\sqrt{4\gamma\eta-\delta^2}\phi}{2}))) + \frac{-12\eta^2\lambda^2}{\xi}(-\frac{1}{2\eta}(\delta+\sqrt{4\gamma\eta-\delta^2}\cot(\frac{\sqrt{4\gamma\eta-\delta^2}\phi}{2})))^2. \quad (4.37)$$

Case 3. If $\delta^2 - 4\gamma\eta = 0$, and $\eta \neq 0$, then

$$r_{3,5}(\phi) = \frac{-1+\lambda-\delta^2\lambda^2-8\gamma\eta\lambda^2}{\xi} - \frac{12\delta\eta\lambda^2}{\xi}(-\frac{2+\delta\phi}{2\eta\phi}) - \frac{12\eta^2\lambda^2}{\xi}(-\frac{2+\delta\phi}{2\eta\phi})^2. \quad (4.38)$$

4.2. Simple mapping method

4.2.1. The modified Korteweg-de Vries-Zakharov-Kuznetsov equation

After balancing the non linear term R^3, and the highest-order derivative term R'' from Eq (3.4) , $3u = u + 2$, which results in $u = 1$. Consequently, the equation's solution has the following form:

$$G(\phi) = \nu_0 + \nu_1\kappa(\phi), \quad (4.39)$$

where ν_0, and ν_1 are the unknown coefficients to be determined.

Putting Eq (2.15) into Eq (3.4)

$$-\lambda\nu_0 + \frac{\xi}{3}\nu_0^3 + 3a\nu_1\kappa(\phi) - \lambda\nu_1\kappa(\phi) + 3b\nu_1(\kappa(\phi))^3 + \xi\nu_0^2\nu_1\kappa(\phi) + \xi\nu_0\nu_1^2(\kappa(\phi))^2 + \frac{\xi}{3}\nu_1^3(\kappa(\phi))^3 = 0. \quad (4.40)$$

Since ($p = 0, 1, 2, 3$), and each coefficient of $\kappa(\phi)^p$ is equal to zero, we have

$$\begin{aligned} -\lambda\nu_0 + \frac{\xi}{3}\nu_0^3 &= 0, \\ 3a\nu_1 - \lambda\nu_1 + \xi\nu_0^2\nu_1 &= 0, \\ \xi\nu_0\nu_1^2 &= 0, \\ 3b\nu_1 + \frac{\xi}{3}\nu_1^3 &= 0. \end{aligned} \quad (4.41)$$

By solving Eq (4.41), we get the following families that are.

Family 1.

$$\nu_0 = -3\sqrt{\frac{a}{2\xi}}\iota, \nu_1 = 3\sqrt{\frac{b}{\xi}}\iota \ , and \ \lambda = \frac{-3a}{2}. \quad (4.42)$$

Thus, the solution of Eq (4.39) is

$$G(\phi) = -3\sqrt{\frac{a}{2\xi}}\iota + 3\sqrt{\frac{b}{\xi}}\iota\kappa(\phi). \quad (4.43)$$

Case 1.

Here, $\kappa(\phi) = \sec h(\phi)$. Then the solution becomes

$$G_{1,1}(\phi) = -3\sqrt{\frac{a}{2\xi}}\iota + 3\sqrt{\frac{b}{\xi}}\iota \sec h(\phi). \tag{4.44}$$

Case 2.

Here, $\kappa(\phi) = \tanh(\phi)$. Then the solution becomes

$$G_{1,2}(\phi) = -3\sqrt{\frac{a}{2\xi}}\iota + 3\sqrt{\frac{b}{\xi}}\iota \tanh(\phi). \tag{4.45}$$

Case 3.

Here, $\kappa(\phi) = sn(\phi)$. Then the solution becomes

$$G_{1,3}(\phi) = -\frac{3\sqrt{a}\iota}{\sqrt{2\xi}} + 3\sqrt{\frac{b}{\xi}}\iota sn(\phi). \tag{4.46}$$

Case 4.

Here, $\kappa(\phi) = cn(\phi)$. Then the solution becomes

$$G_{1,4}(\phi) = -3\sqrt{\frac{a}{2\xi}}\iota + 3\sqrt{\frac{b}{\xi}}\iota cn(\phi). \tag{4.47}$$

Case 5.

Here, $\kappa(\phi) = dn(\phi)$. Then the solution becomes

$$G_{1,5}(\phi) = -3\sqrt{\frac{a}{2\xi}}\iota + 3\sqrt{\frac{b}{\xi}}\iota dn(\phi). \tag{4.48}$$

Case 6.

Here, $\kappa(\phi) = ns(\phi)$. Then the solution becomes

$$G_{1,6}(\phi) = -3\sqrt{\frac{a}{2\xi}}\iota + 3\sqrt{\frac{b}{\xi}}\iota ns(\phi). \tag{4.49}$$

Family 2.

$$\nu_0 = 3\sqrt{\frac{a}{2\xi}}\iota, \nu_1 = 3\sqrt{\frac{b}{\xi}}\iota \ , and \ \lambda = \frac{-3a}{2}. \tag{4.50}$$

Thus, the solution of Eq (4.39) is

$$G(\phi) = 3\sqrt{\frac{a}{2\xi}}\iota + 3\sqrt{\frac{b}{\xi}}\iota\kappa(\phi). \tag{4.51}$$

Case 1.

Here, $\kappa(\phi) = \sec h(\phi)$. Then the solution becomes

$$G_{2,1}(\phi) = 3\sqrt{\frac{a}{2\xi}}\iota + 3\sqrt{\frac{b}{\xi}}\iota \sec h(\phi). \tag{4.52}$$

Case 2.

Here, $\kappa(\phi) = \tanh(\phi)$. Then the solution becomes

$$G_{2,2}(\phi) = 3\sqrt{\frac{a}{2\xi}}\iota + 3\sqrt{\frac{b}{\xi}}\iota \tanh(\phi). \tag{4.53}$$

Case 3.

Here, $\kappa(\phi) = sn(\phi)$. Then the solution becomes

$$G_{2,3}(\phi) = 3\sqrt{\frac{a}{2\xi}}\iota + 3\sqrt{\frac{b}{\xi}}\iota sn(\phi). \tag{4.54}$$

Case 4.

Here, $\kappa(\phi) = cn(\phi)$. Then the solution becomes

$$G_{2,4}(\phi) = 3\sqrt{\frac{a}{2\xi}}\iota + 3\sqrt{\frac{b}{\xi}}\iota cn(\phi). \tag{4.55}$$

Case 5.

Here, $\kappa(\phi) = dn(\phi)$. Then the solution becomes

$$G_{2,5}(\phi) = 3\sqrt{\frac{a}{2\xi}}\iota + 3\sqrt{\frac{b}{\xi}}\iota dn(\phi). \tag{4.56}$$

Case 6.

Here, $\kappa(\phi) = ns(\phi)$. Then the solution becomes

$$G_{2,6}(\phi) = 3\sqrt{\frac{a}{2\xi}}\iota + 3\sqrt{\frac{b}{\xi}}\iota ns(\phi). \tag{4.57}$$

4.2.2. Time-regularized long-wave equation

After balancing the non linear term R^2, and the highest-order derivative term R'' from the Eq (3.8), $2u = u + 2$, which results in $u = 2$. Consequently, the equation's solution has the following form:

$$G(\phi) = \nu_0 + \nu_1\kappa(\phi) + \nu_2(\kappa(\phi))^2, \tag{4.58}$$

where ν_0, ν_1, and ν_2 are the unknown coefficients to be determined.

Putting Eq (2.15) into Eq (3.8), we have

$$\nu_0 - \lambda\nu_0 + \frac{\xi}{2}\nu_0^2 + \nu_1\kappa(\phi) - \lambda\nu_1\kappa(\phi) + a\lambda^2\nu_1\kappa(\phi) + b\lambda^2\nu_1(\kappa(\phi))^3 + \xi\nu_0\nu_1\kappa(\phi) + \frac{\xi}{2}\nu_1^2(\kappa(\phi))^2 + \nu_2(\kappa(\phi))^2$$
$$+ 2c\lambda^2\nu_2 - \lambda\nu_2(\kappa(\phi))^2 + 4a\lambda^2\nu_2(\kappa(\phi))^2 + 3b\lambda^2\nu_2(\kappa(\phi))^4 + \xi\nu_0\xi_2(\kappa(\phi))^2 + \xi\nu_1\nu_2(\kappa(\phi))^3 + \frac{\xi}{2}\nu_2^2(\kappa(\phi))^4 = 0. \quad (4.59)$$

Since ($p = 0, 1, 2, 3, 4$), and put each coefficient of $\kappa(\phi)^p$ is equal to zero, we have

$$\begin{aligned}
\nu_0 - \lambda\nu_0 + \frac{\xi}{2}\nu_0^2 + 2c\lambda^2\nu_2 &= 0,\\
\nu_1 - \lambda\nu_1 + a\lambda^2\nu_1 + \xi\nu_0\nu_1 &= 0,\\
\frac{\xi}{2}\nu_1^2 + \nu_2 - \lambda\nu_2 + 4a\lambda^2\nu_2 + \xi\nu_0\nu_2 &= 0,\\
b\lambda^2\nu_1 + \xi\nu_1\nu_2 &= 0,\\
3b\lambda^2\nu_2 + \frac{\xi}{2}\nu_2^2 &= 0.
\end{aligned} \quad (4.60)$$

By solving Eq (4.60), we get the following family **Family 1.**

$$\nu_0 = -\frac{-2\xi + 2\xi\lambda + \sqrt{96bc\xi^2\lambda^4 + (-2\xi + 2\xi\lambda)^2}}{2\xi^2}, \nu_1 = 0 \ , and \ \nu_2 = \frac{-6b\lambda^2}{\xi}. \quad (4.61)$$

Thus, the solution of Eq (4.58) is

$$G(\phi) = -\frac{-2\xi + 2\xi\lambda + \sqrt{96bc\xi^2\lambda^4 + (-2\xi + 2\xi\lambda)^2}}{2\xi^2} + \frac{-6b\lambda^2}{\xi}(\kappa(\phi))^2. \quad (4.62)$$

Case 1.

Here, $\kappa(\phi) = \sec h(\phi)$. Then the solution becomes

$$G_{1,1}(\phi) = -\frac{-2\xi + 2\xi\lambda + \sqrt{96bc\xi^2\lambda^4 + (-2\xi + 2\xi\lambda)^2}}{2\xi^2} + \frac{-6b\lambda^2}{\xi}(\sec h(\phi))^2. \quad (4.63)$$

Case 2.

Here, $\kappa(\phi) = \tanh(\phi)$. Then the solution becomes

$$G_{1,2}(\phi) = -\frac{-2\xi + 2\xi\lambda + \sqrt{96bc\xi^2\lambda^4 + (-2\xi + 2\xi\lambda)^2}}{2\xi^2} + \frac{-6b\lambda^2}{\xi}(\tan h(\phi))^2. \quad (4.64)$$

Case 3.

Here, $\kappa(\phi) = sn(\phi)$. Then the solution becomes

$$G_{1,3}(\phi) = -\frac{-2\xi + 2\xi\lambda + \sqrt{96bc\xi^2\lambda^4 + (-2\xi + 2\xi\lambda)^2}}{2\xi^2} + \frac{-6b\lambda^2}{\xi}(sn(\phi))^2. \quad (4.65)$$

Case 4.

Here, $\kappa(\phi) = cn(\phi)$. Then the solution becomes

$$G_{1,4}(\phi) = -\frac{-2\xi + 2\xi\lambda + \sqrt{96bc\xi^2\lambda^4 + (-2\xi + 2\xi\lambda)^2}}{2\xi^2} + \frac{-6b\lambda^2}{\xi}(cn(\phi))^2. \quad (4.66)$$

Case 5.

Here, $\kappa(\phi) = dn(\phi)$. Then the solution becomes

$$G_{1,5}(\phi) = -\frac{-2\xi + 2\xi\lambda + \sqrt{96bc\xi^2\lambda^4 + (-2\xi + 2\xi\lambda)^2}}{2\xi^2} + \frac{-6b\lambda^2}{\xi}(dn(\phi))^2. \tag{4.67}$$

Case 6.

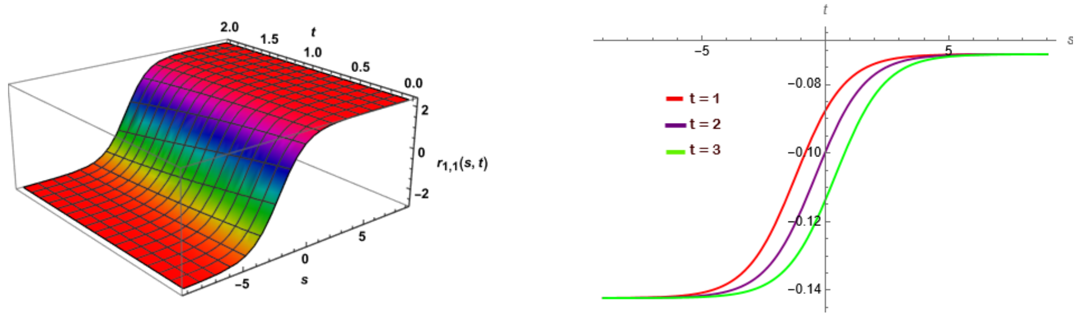
Here, $\kappa(\phi) = ns(\phi)$. Then the solution becomes

$$G_{1,6}(\phi) = -\frac{-2\xi + 2\xi\lambda + \sqrt{96bc\xi^2\lambda^4 + (-2\xi + 2\xi\lambda)^2}}{2\xi^2} + \frac{-6b\lambda^2}{\xi}(ns(\phi))^2. \tag{4.68}$$

5. Results, and discussion

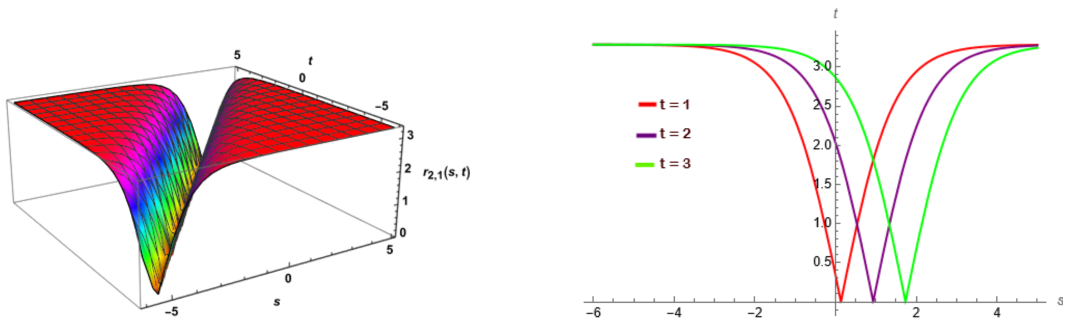
In this article, we have solved the mKdV-ZK equation and the TRLW equation. The MAEM and SMM were used to arrive at these solutions. Various soliton solutions derived in this study, including bright, and dark solitons, anti-peakons,, and W-shaped or M-shaped structures, have important applications in different physical settings. When examining negative energy pulses in magnetized plasmas, anti-peakons, which are distinguished by their acute troughs, can be utilized to mimic localized wave energy depressions. In nonlinear fiber optics, bright, and dark solitons, which represent localized bursts or voids of light intensity, respectively, are especially significant. The complex wave patterns seen in plasma environments, where there are several conflicting nonlinear, and dispersive processes, can be described by W-shaped, and M-shaped solitons. These links highlight how useful the solutions are in practice, and offer a starting point for investigating real-world wave phenomena in several fields. With the trigonometric, and hyperbolic function solutions of the mKdV-ZK equation in $r_{1,1}(s,t)$, we determine the kink soliton solution for the 2D, and 3D graphs by considering the values $\delta = 3$, $\gamma = 1$, $\eta = 2$, $\xi = 0.7$, and $\lambda = 0.8$, as shown in Figure 1, using the MAEM. For the trigonometric and hyperbolic function solutions in 2D, and 3D graphs of the mKdV-ZK equation in $r_{2,1}(s,t)$, we calculate the dark soliton solution for $\delta = 2$, $\gamma = 0.2$, $\eta = 2$, $\xi = 0.75$, and $\lambda = 0.8$, as shown in Figure 2, by using the MAEM. Figure 3 represents trignometric and hyperbolic solutions of the mKdV-ZK equation in $r_{3,1}(s,t)$, and gives the anti-peakon solution for $\delta = 3$, $\gamma = 1$, $\eta =1$, $\xi = 0.8$ and $\lambda = 0.7$ by using the MAEM. For the trigonometric and hyperbolic function solutions of TRLW equation in $r_{1,1}(s,t)$, we calculate the bright soliton solution for $\delta = 4$, $\gamma = 1$, $\eta =1$, $\xi = 1$, and $\lambda = 1$ in Figure 4 by using the MAEM. Figure 5 represents the trignometric and hyperbolic solutions of the TRLW equation in $r_{2,1}(s,t)$, we get the W-shaped soliton solution for $\delta = 1$, $\gamma = 0.1$, $\eta =0.4$, $\xi = 0.7$, and $\lambda = 0.8$ by using the MAEM. For the trigonometric function solution of the TRLW equation in $r_{3,1}(s,t)$, we calculate the bell-shaped soliton solution for $\delta = 3$, $\gamma = 1$, $\eta =1$, $\xi = 3$, and $\lambda = 2$ by using the MAEM, as shown in Figure 6. Figure 7 represents the trigonometric and hyperbolic solutions of the mKdV-ZK equation in $G_{1,1}(s,t)$, and we obtain the anti-bell soliton solution for b = -2, c = 0, a =1, $\xi = 0.01$, and $y = z = 1$ by using the SMM. For the Jacobi solution of the mKdV-ZK equation in $G_{2,3}(s,t)$, we calculate the periodic soliton solution by taking into account the values b = 0.5, c = 1, a = -1.25, $\xi = 0.1$ and $y = z = 1$, using SMM, in Figure 8. Figure 9 represents Jacobi solution of the mKdV-ZK equation in $G_{2,4}(s,t)$, and we get the M-shaped soliton solution for b = -0.1, c = 0.91, a = -0.82, $\xi = 1.5$, and $y = z = 1$ by using the SMM. For the Jacobi solution of the TRLW equation in

$G_{1,3}(s,t)$, we get the quasi-periodic solution for a = -1.25, b = 0.5, c =1, ξ = 0.1, and λ = 0.5 by using the SMM, as shown in Figure 10. Figure 11 represents the Jacobi solution of the TRLW equation in $G_{1,4}(s,t)$, being we get the periodic soliton solution for a = -0.82, b = -0.18, c =0.91, ξ = 0.1, and λ = 0.5 by using the SMM. Figure 12 represents the Jacobi solution of the TRLW equation in $G_{1,5}(s,t)$, being we get the W-periodic soliton solution for a = 1.64, b = -2, c = -0.64, ξ = 1.5, and λ = 0.75 by using the SMM. The comparison between the suggested methods and Hirota's bilinear method, and the (G'/G)-expansion method is given in Table 1.



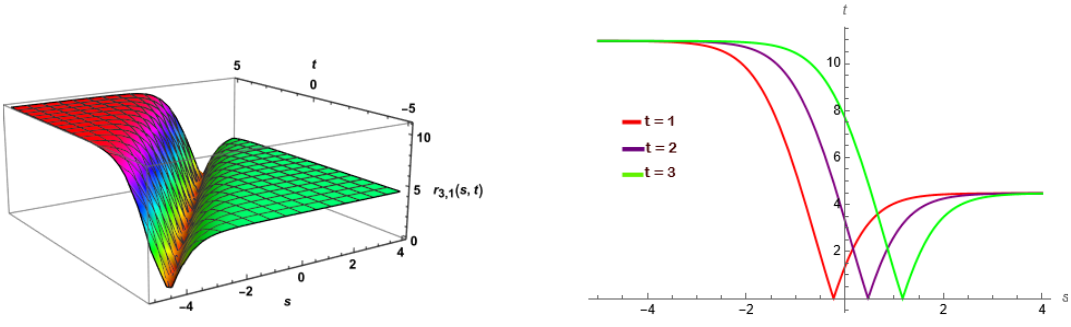
(a) 3D plot (b) 2D plot

Figure 1. The mKdV-ZK equation's kink profile obtained by using the MAEM is associated with the solution $r_{1,1}(s,t)$ in Eq (4.5) when δ = 3, γ = 1, η = 2, ξ = 0.7, and λ = 0.8. (a) 3D plot; (b) 2D plot.



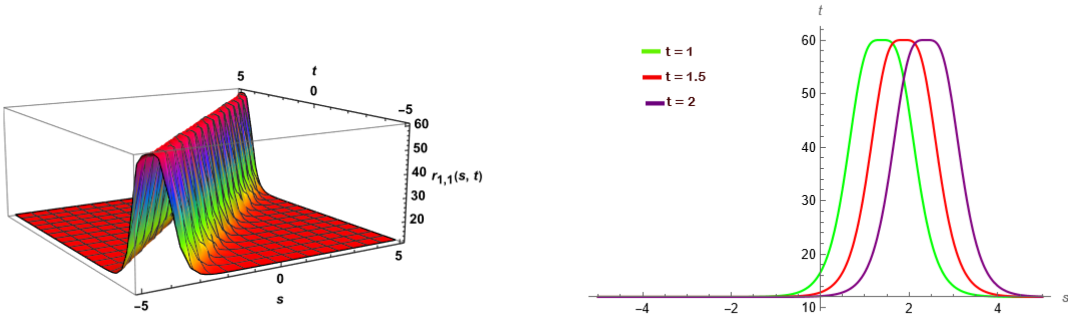
(a) 3D plot (b) 2D plot

Figure 2. The mKdV-ZK equation's dark soliton profile obtained by using the MAEM is associated with the solution $r_{2,1}(s,t)$ in Eq (4.8) when δ = 2, γ = 0.2, η = 2, ξ = 0.75, and λ = 0.8. (a) 3D plot; (b) 2D plot.



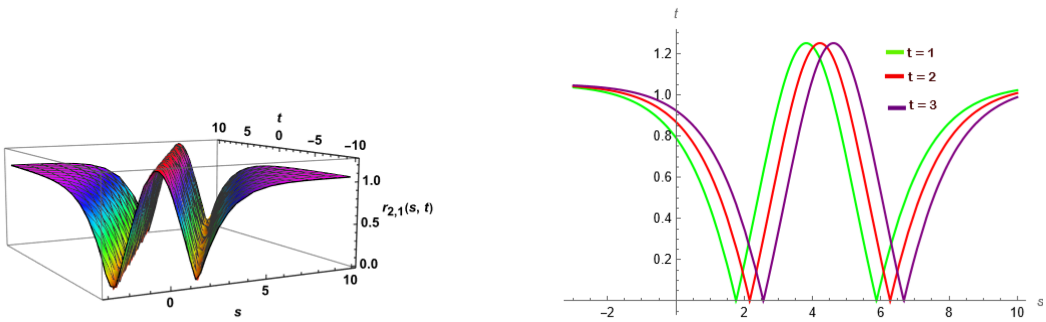
(a) 3D plot (b) 2D plot

Figure 3. The mKdV-ZK equation's anti-peakon profile obtained by using the MAEM is associated with the solution $r_{3,1}(s,t)$ in Eq (4.11) when δ = 3, γ = 1, η =1, ξ = 0.8, and λ = 0.7. (a) 3D plot; (b) 2D plot.



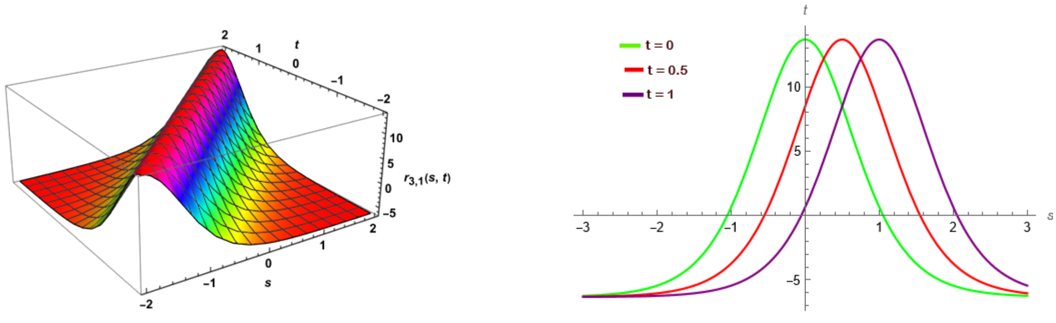
(a) 3D plot (b) 2D plot

Figure 4. The TRLW equation's bright soliton profile obtained by using the MAEM is associated with the solution $r_{1,1}(s,t)$ in Eq (4.18) when δ = 4, γ = 1, η =1, ξ = 1, and λ = 1. (a) 3D plot; (b) 2D plot.



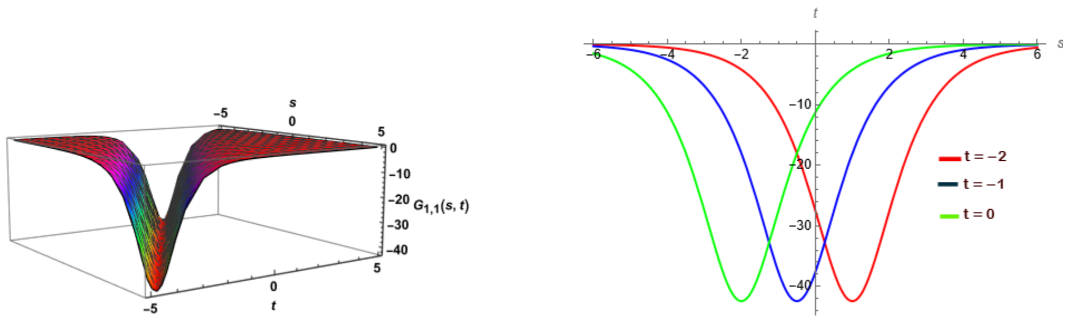
(a) 3D plot (b) 2D plot

Figure 5. The TRLW equation's W-shaped soliton profile obtained by using the MAEM is associated with the solution $r_{2,1}(s,t)$ in Eq (4.21) when δ = 1, γ = 0.1, η =0.4, ξ = 0.7, and λ = 0.8. (a) 3D plot; (b) 2D plot.



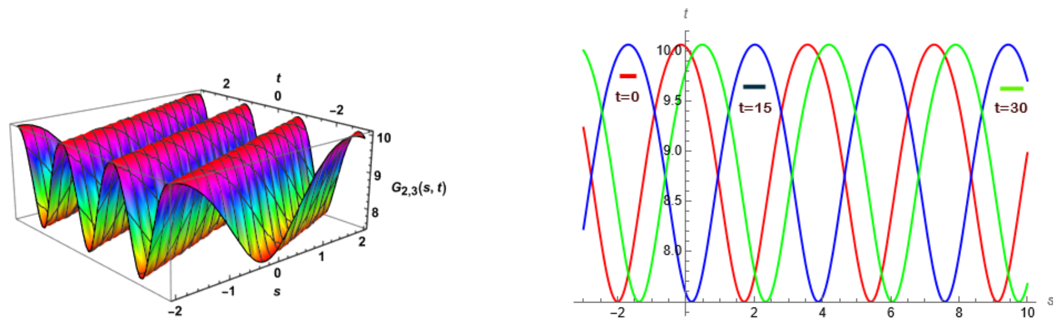
(a) 3D plot (b) 2D plot

Figure 6. The TRLW equation's bell-shaped soliton profile obtained by using the MAEM is associated with the solution $r_{3,1}(s, t)$ in Eq (4.24) when δ = 3, γ = 1, η =1, ξ = 3, and λ = 2. (a) 3D plot; (b) 2D plot.



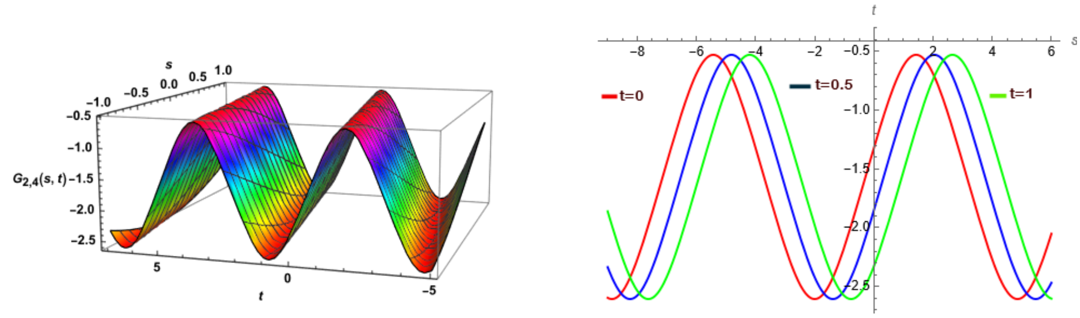
(a) 3D plot (b) 2D plot

Figure 7. The mKdV-ZK equation's anti-bell shaped soliton profile obtained by using SMM is associated with the solution $G_{1,1}(s, t)$ in Eq (4.32) when b = -2, c = 0, a =1, ξ = 0.01, and $y = z$ = 1. (a) 3D plot; (b) 2D plot.



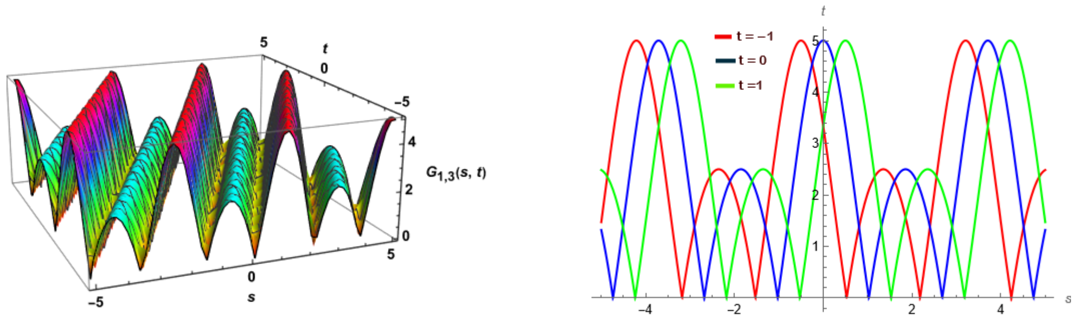
(a) 3D plot (b) 2D plot

Figure 8. The mKdV-ZK equation's periodic profile obtained by using the SMM is associated with the solution $G_{2,3}(s, t)$ in Eq (4.42) when b = 0.5, c = 1, a = -1.25, ξ = 0.1, and $y = z$ = 1. (a) 3D plot; (b) 2D plot.



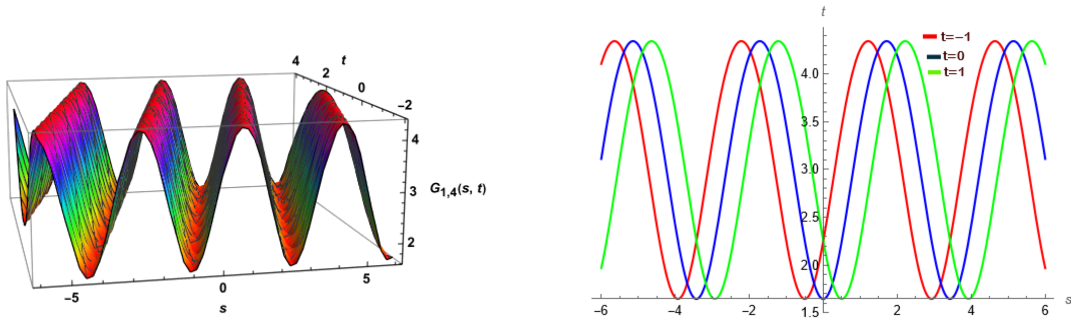
(a) 3D plot (b) 2D plot

Figure 9. The mKdV-ZK equation's M-shaped profile obtained by using the SMM is associated with the solution $G_{2,4}(s,t)$ in Eq (4.43) when b = -0.1, c = 0.91, a = -0.82, ξ = 1.5, and $y = z = 1$. (a) 3D plot; (b) 2D plot.



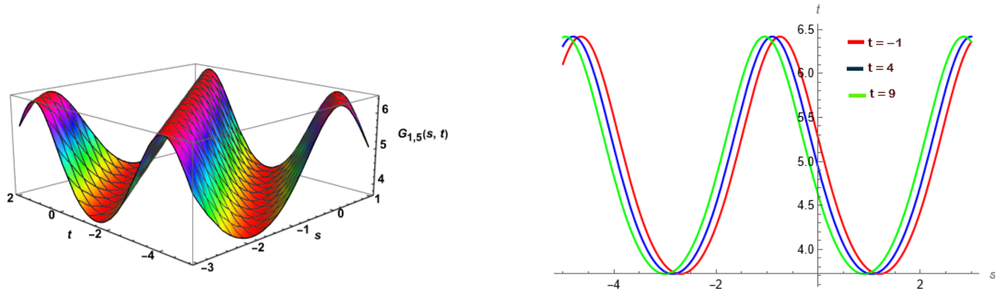
(a) 3D plot (b) 2D plot

Figure 10. The TRLW equation's quasi periodic profile obtained by using the SMM is associated with the solution $G_{1,3}(s,t)$ in Equation (4.53) when a = -1.25, b = 0.5, c =1, ξ = 0.1, and λ = 0.5. (a) 3D plot; (b) 2D plot.



(a) 3D plot (b) 2D plot

Figure 11. The TRLW equation's periodic profile obtained by using the SMM is associated with the solution $G_{1,4}(s,t)$ in Eq (4.54) when a = -0.82, b = -0.18, c =0.91, ξ = 0.1, and λ = 0.5. (a) 3D plot; (b) 2D plot.



(a) 3D plot (b) 2D plot

Figure 12. The TRLW equation's W-periodic profile obtained by using the SMM is associated with the solution $G_{1,5}(s, t)$ in Eq (4.55) when a = 1.64, b = -2, c = -0.64, ξ = 1.5, and λ = 0.75. (a) 3D plot; (b) 2D plot.

Table 1. Comparison of the MAEM, and SMM with traditional methods.

Feature	**MAEM**	**SMM**	**Hirota's bilinear method**	**(G'/G)-expansion method**
Complexity of computation	Moderate	Low	High (especially for multi-soliton)	Moderate
Type of solutions	Broad (includes bell, anti-bell, W-M-shaped, anti-peakons)	Broad (includes bell, anti-bell, W-M-shaped, anti-peakons)	Mostly bright/dark solitons	Mainly basic solitary, and periodic waves
Flexibility	High (easily adapted to various PDEs)	High	Requires bilinear transformation	Limited by the form of the auxiliary function
Applicability to higher-dimensional PDEs	Yes	Yes	Often complex or impractical	Less flexible
Symbolic implementation feasibility	High (suitable for computer software like Mathematica/Maple)	Very high	Often difficult to implement without special packages	Moderate
Novel structures generated	Anti-peakons, M/W-shaped solitons, kink-type waves	Anti-peakons, M/W-shaped solitons, kink-type waves	Mostly standard soliton profiles	Fewer complex structures

6. Stability analysis of the obtained soliton solutions

The analytical solutions obtained by the MAEM, and the SMM were verified via direct substitution into the TRLW, and mKdV-ZK equations. However, such substitution-based validation does not ensure the dynamical stability of the soliton solutions. To assess their physical viability, we consider a linear stability analysis by introducing small perturbations around the exact solution.

Let $R(s,t)$ be a known exact soliton solution. We introduce a perturbed solution of the form:

$R(s,t) = R_0(s,t) + \epsilon \tilde{R}(s,t)$, where $R_0(s,t)$ is the exact solution, $\tilde{R}(s,t)$ is a small perturbation function,, and $\epsilon \ll 1$ is a small parameter.

Substituting this into the governing nonlinear equations, and linearizing by retaining only the terms up to the first order in ϵ, we obtain a linearized perturbation equation of the follwoing form:

$$\mathcal{L}[\tilde{R}(s,t)] = 0,$$

where $\mathcal{L}$ is a linear differential operator depending on the background solution $R_0(x,t)$.

We analyze the stability by assuming a normal mode expansion

$$\tilde{R}(s,t) = e^{i(ks-\omega t)},$$

and substituting it into the linearized equation to derive a dispersion relation between ω, and k. If the imaginary part of ω is non-positive, the perturbation decays or remains bounded,, and the solution is considered to be linearly stable. Conversely, if imaginary part of ω is positive, the solution is unstable.

7. Conclusions

In this study, the exact solutions to the TRLW, and mKdV-ZK equations have been discovered using the MAEM, and SMM. For every equation under consideration that has some unknown parameters, we have created solitary wave solutions. The equation is frequently applied to a homogeneous magnetized electron-positron plasma in order to regulate weakly non linear ion-acoustic waves. The outcomes of previous reports are compared to build and validate new analytical solutions. A wide range of significant physical phenomena, including shallow water waves, and ion-acoustic plasma waves, can be represented using the TRLW equation, which is crucial to the study of non linear waves. The slanted propagation of nonlinear electrostatic modes is controlled by the mKdV-ZK equation. Utilizing these methods for the above mentioned equations, one can obtain periodic, and quasi-periodic waves, anti-peakons, bright and dark solitons, sharp, and smooth W-shaped, M-shaped soliton solutions, bell-shaped, anti-bell-shaped soliton solutions, and kink type soliton solutions. The solutions discovered here will help with research on problems related to engineering, mechanical theory, tsunamis, and tidal waves. Finally, Figures (1)–(12) are displayed. Furthermore, these techniques may become more applicable in practice if they are extended to more complicated nonlinear models, such as higher-dimensional or variable-coefficient variants of the TRLW, and mKdV-ZK equations. These developments could lead to new applications in multidisciplinary domains.

Authors contributions

Manal Alqhtani: Software, investigation, writing-original draft; Rabia Gul: Methodology, writing-original draft, writing-review & editing; Muhammad Abbas: Supervision, writing-original draft, writing-review & editing; Alina Alb Lupas: Visualization, investigation, writing-review & editing; Khaled M. Saad: Software, investigation, writing-review & editing. All authors have read, and agreed to publish the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest to report regarding the present study.

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