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Research article

The dynamical behavior of a rational difference equation through KAM theory

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Abstract: In this study, we employ the Kolmogorov-Arnold-Moser (KAM) theory to investigate the stability of solutions for the following difference equation:

$$\zeta_{n+1} = \frac{1}{\zeta_{n-1} - \beta\zeta_n\zeta_{n-1}}, n = 0, 1, ...$$

In this context, the parameter β and the initial conditions ζ_{-1}, ζ_0 are positive. We construct a novel logarithmic transformation that satisfies the area-preserving mapping to demonstrate the existence of a positive elliptic equilibrium and examine the stability. Then, the Birkhoff normal form locally simplifies the nonlinear system into a higher-order standard form near the equilibrium or periodic orbits. Consequently, we apply the KAM theory to prove that there exist numerous invariant closed curves in the smooth invariant region of any non-degenerate elliptic fixed point. Additionally, we conduct multiple numerical simulations to support our research findings using the Matlab software.

Keywords: KAM theory; Birkhoff normal form; difference equation; area preserving map
Mathematics Subject Classification: 39A30

1. Introduction

Nonlinear difference equations [1–3] serve as fundamental mathematical models for complex phenomena across physics [4–6], biology, and economics. These systems exhibit rich behaviors including bifurcations, chaos, and quasi-periodicity that defy linear approximation techniques. The analysis of such equations remains challenging due to: a sensitivity to the initial conditions and parameters, the emergence of singularities and bifurcations, and a lack of general analytical solutions. For conservative systems that preserve the phase-space structure, the Kolmogorov-Arnold-Moser (KAM) theory provides a powerful analytical framework to address these challenges. Specifically, this cornerstone theory in Hamiltonian dynamical systems offers a profound framework to understand the persistence of quasi-periodic motion under small perturbations. Originating from the seminal works of Kolmogorov (1954), Arnold (1963), and Moser (1962), this theory addresses the stability of integrable systems when subjected to weak nonlinear perturbations. At its core, the KAM theory establishes that, under non-degeneracy and non-resonance conditions, a majority of invariant tori—associated with quasi-periodic trajectories—survive in perturbed systems, thus ensuring long-term stability in weakly chaotic regimes.

Traditionally applied to celestial mechanics and plasma physics [7–9], the KAM theory has recently found novel applications in diverse fields, including ecology [10, 11], economics [12], and population dynamics [13], where complex nonlinear interactions govern the system behavior. In ecological contexts, the theory offers a rigorous mathematical tool to analyze the stability of species coexistence, predict bifurcations, and characterize transitions from ordered to chaotic dynamics in discrete-time models. Specifically, we investigate the following singular recurrence relation:

$$\zeta_{n+1} = \frac{1}{\zeta_{n-1} - \beta\zeta_n\zeta_{n-1}}, \quad n = 0, 1, \ldots, \tag{1.1}$$

which models density-dependent population interactions with environmental stress. This non-integrable system exhibits a complex singularity structure that challenges conventional stability analyses, thereby providing a mathematical testbed for KAM-based methods.

In [14],

$$\chi_{n+1} = \frac{A\chi_n^k + B}{a\chi_{n-1}}, n = 0, 1, ..., \tag{1.2}$$

and

$$\chi_{n+1} = \frac{A\chi_n^2 + F}{e\chi_{n-1}}, n = 0, 1, ... \tag{1.3}$$

The equations under consideration in this study were analyzed using two methods: the first is a combination of algebraic and geometric techniques, which is typically grounded in the identification of invariants, as discussed in previous literature [15]; and the second approach is the KAM theory. However, for Eq (1.3) examined in this study, no invariant capable of generating an area preserving map could be identified. Consequently, the KAM theory emerged as the sole viable technique for analysis. A similar situation was observed for Eq (1.1), where no invariant could be found. It is worth noting that a previous work [16] proposed effective methods for determining the presence of continuous invariants in difference equations.

In this paper, we demonstrate that Eq (1.1) exhibits three distinct qualitative scenarios: no positive equilibrium points, a single non-hyperbolic parabolic equilibrium point, and two positive fixed points, non-hyperbolic elliptic and saddle, respectively. Symmetries play a crucial role in our analysis, particularly in the context of area-preserving maps, as they give rise to unique dynamic behaviors.

Therefore, the study of KAM Theory, periodicity, Normal Forms, and symmetries for certain rational difference equations represents a significant area of research, with far-reaching implications to understand the complex dynamics of these systems. Further exploration and analyses in these areas are crucial to advance our knowledge and to unlock new insights into the behavior of rational difference equations.

The fixed points of (1.1) are the non-negative solutions of the following equation:

$$\beta\zeta^3 - \zeta^2 + 1 = 0. \tag{1.4}$$

Set $g(\zeta) = \beta\zeta^3 - \zeta^2 + 1$, where g has a local maximum $g_{max} = 1$ when $\zeta = 0$ and has a local minimum when $\zeta = \frac{2}{3\beta}$. Equation (1.4) can be expressed as follows:

a) The equation has no positive roots if $g(\frac{2}{3\beta}) > 0$, which is equivalent to $\beta > \frac{2}{3\sqrt{3}}$;

b) The equation merely possesses a singular positive root if $g(\frac{2}{3\beta}) = 0$, which is equivalent to $\beta = \frac{2}{3\sqrt{3}}$;

c) The equation has two positive roots if $g(\frac{2}{3\beta}) < 0$, which is equivalent to $\frac{2}{3\sqrt{3}} > \beta > 0$.

The following has the same meaning for Eq (1.4):

a_1) has no positive equilibrium points for $\beta > \frac{2}{3\sqrt{3}}$;

b_1) has only one positive equilibrium point $P = P_1 = P_2 = \sqrt{3}$;

c_1) has two positive equilibrium points P_1 and P_2 ($1 < P_1 < \frac{2}{3\beta}$ and $P_2 > \frac{2}{3\beta} > \sqrt{3}$).

In this study, it will be demonstrated that $P = P_1$ for $0 < \beta < \frac{2}{3\sqrt{3}}$ acts as an elliptic point, given that the eigenvalues of the Jacobian matrix $J_\Psi(P_1)$ manifest as a complex conjugate pair $\lambda, \bar{\lambda}$ with pure imaginary numbers. This characteristic implies that Eq (1.1) exhibits distinct dynamics, which warrants the utilization of the KAM theory to analyze its behavior. Consequently, it will be shown that the map Ψ associated with Eq (1.1) is, in fact, an area-preserving map.

Definition 1. *[15] Let $\Psi : R^2 \to R^2$ be an area-preserving map, that has a Jacobian matrix J_Ψ, which meets the following condition:*

$$det J_\Psi(X) = 1, \quad for\ all\ points\ X\ within\ the\ domain\ of\ \Psi.$$

Example 1. *In the context of the linear transformation defined by $x \mapsto Ax$ and the associated system, the criterion for area preservation can be expressed as follows:*

$$\det A = 1.$$

Given that the determinant of a square matrix is equivalent to the product of its eigenvalues λ_1 and λ_2, this condition can be reformulated as follows:

$$\lambda_1\lambda_2 = 1.$$

Consequently, this requirement gives rise to three qualitatively distinct cases:

(i) Saddle: Both λ_1 and λ_2 are real and of the same sign, with $|\lambda_1| < 1 < |\lambda_2|$;

(ii) Parabolic: $\lambda_1 = \lambda_2 = 1$ or $\lambda_1 = \lambda_2 = -1$;

(iii) Elliptic: λ_1 and λ_2 are complex conjugates, with $|\lambda_1| = |\lambda_2| = 1$.

The saddle point case (i) is characterized by hyperbolicity, which allows for a conclusive determination of instability based on its linearization. Furthermore, this linearization provides insights into the local stable and unstable manifolds associated with the system. In contrast, the remaining two cases are classified as nonhyperbolic, thus rendering the linearization inconclusive in these instances. Notably, the elliptic case frequently arises in conservative mechanical systems, which exhibit a degree of stability alongside complex dynamics. The stability of such equilibrium points can be established by simplifying the nonlinear terms through suitable coordinate transformations, thereby reformulating them into the so-called normal forms, where the stability (or instability) can be readily assessed.

Theorem 1 (Birkhoff normal form)**.** *Given the map $\Psi : R^2 \to R^2$, which is area-preserving and n-times continuously differentiable, it has an equilibrium point at the origin $(0,0)$. Furthermore, the eigenvalues λ and $\bar{\lambda}$ are complex conjugates situated within the unit disk. Assume*

$$4 \leq d \leq n+1$$

under certain conditions on an integer d and given constraints on eigenvalues meeting specific criteria

$$\lambda^k \neq 1 \; for \; k = 3, 4, ..., d.$$

Consider $r = [\frac{d}{2}]$, and a function $h(m, \bar{m})$ exists, and is differentiable, with zero values when it derivatives up to order $r-1$ at $m = 0$. This is accompanied by a real polynomial Υ

$$\Upsilon(\tau) = \Upsilon_1 \tau + \Upsilon_2 \tau^2 + \ldots + \Upsilon_r \tau^r.$$

In such a manner, the mapping Ψ can be converted to the standard form by means of appropriate transformations.

$$m \to \Psi(m, \bar{m}) = \lambda m e^{i\Upsilon(m\bar{m})} + h(m, \bar{m}).$$

Essentially, the difference equations system

$$\zeta_{n+1} = \Psi(\zeta_n)$$

is simplified to the following form:

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix} + \begin{bmatrix} O_d \\ O_d \end{bmatrix}, \tag{1.5}$$

where

$$\tau = \sum_{k=0}^{M} \delta_k (u_n^2 + v_n^2)^k, M = [\frac{d}{2}] - 1. \tag{1.6}$$

Here, O_d is a convergent series power in u_n and v_n with $d \geq m$, which vanishes at zero and $[\zeta] \leq \zeta$.

The twist coefficients, $\delta_1, ..., \delta_k$, are important parameters. Using Theorem 1, an elliptic fixed point stability can be explained, and referred to as the KAM Theorem, which is described in [11–13].

Theorem 2 (KAM theorem)**.** *Let map $\Psi : R^2 \to R^2$ be area-preserving and has an elliptic fixed point at the origin; then, the conditions from Theorem 1 are satisfied. Assuming $\Upsilon(|m|^2)$ is not precisely zero, the origin $(0,0)$ is a stable equilibrium, that is, supposing there exists $\delta_k \neq 0$ in (1.6) for $k \in 1, ..., M$, the origin $(0,0)$ is recognized as a stable equilibrium point.*

Remark 1. *[15] Discuss an invariant annulus ($\hbar_\epsilon = m : \epsilon < |m| < 2\epsilon$), where ϵ is a sufficiently small positive number near an elliptic fixed point, where the linear part of a normal form approximation preserves circular shapes and induces specific rotational angles on each circle. Suppose there exists at least one non-zero twist coefficient; then, differential rotation and twisting occur along radial directions. According to the KAM theory, introducing a residual term ensures that most of these unchanging circles remain closed curves under the initial mapping, thus forming a stable configuration. This conclusion is referenced in sources [17–19].*

Theorem 3. *If $\Upsilon(m\bar{m})$ significantly departs from zero and ϵ is adequately small, then the map Ψ demonstrates invariant closed curves with positive Lebesgue measure in the vicinity of initial invariant circles. As ϵ approaches 0, the set of the surviving invariant curves's relative measure is approaches a full measure. These lasting closed curves have dense irrational orbits.*

The conclusion is drawn based on the principle of Moser's distortion mapping theorem, which is mentioned in the references [20].

Theorem 4. *An area-preserving diffeomorphism $\Psi : R^2 \to R^2$ is considered, where $(\bar{\zeta}, \bar{\iota})$ is a nondegenerate elliptic fixed point. It is shown that in the vicinity of $(\bar{\zeta}, \bar{\iota})$, there exist arbitrarily large periodic points.*

The findings from Theorem 3 suggest multiple intervals between consecutive invariant curves that contain periodic points near the fixed point. Moreover, these intervals may contain orbits of hyperbolic and elliptic periodic points. These observations cannot be inferred from visual representations generated by computer simulations. The linearized component of Eq (1.5) induces a rotation by an angle τ, such that if τ is a rational multiple of π, then all solutions will exhibit periodic behavior characterized by a uniform period. Conversely, the potential for chaotic solutions emerges when τ takes the form of an irrational multiple of π.

2. KAM theory utilized in Eq (1.1)

In the subsequent section, the KAM theorem will be utilized to illustrate the stability of the equilibrium point $P = P_1$ for $0 < \beta < \frac{2}{3\sqrt{3}}$, and the presence of a boundless number of periodic solutions will be confirmed.

Theorem 5. *Assume Eq (1.1) has the following properties depending on β:*

i) If β is chosen as $\frac{2}{3\sqrt{3}}$, then (1.1) will exhibit a sole positive equilibrium point at $P = \sqrt{3}$, thus indicating a non-hyperbolic behavior with parabolic features; and

ii) If β lies within the interval $\left(0, \frac{2}{3\sqrt{3}}\right)$, then (1.1) will have two positive equilibrium points:

  a) $P = P_1$, where $1 < P < \frac{2}{3\beta}$, which is stable and non-hyperbolic of the elliptic type; and

  b) $P_2 > \frac{2}{3\beta} > \sqrt{3}$, which is a saddle point.

Proof. Let $\zeta_n = Pe^{\zeta_n}$

$$\begin{cases} \zeta_{n+1} &= -ln(1 - \beta Pe^{\zeta_n}) - \iota_n - 2lnP, \\ \iota_{n+1} &= \zeta_n. \end{cases} \tag{2.1}$$

Subsequently, the equilibrium point P is mapped to the origin $(0, 0)$.

The corresponding map Ψ for system (2.1) takes the following form:

$$\Psi\begin{pmatrix} \zeta \\ \iota \end{pmatrix} = \begin{pmatrix} -\iota_n - ln(1 - \beta Pe^{\zeta_n}) - 2lnP) \\ \zeta \end{pmatrix}. \tag{2.2}$$

The map's Jacobian matrix evaluates at the point (ζ, ι) is characterized by the following expression:

$$J_\Psi\begin{pmatrix} \zeta \\ \iota \end{pmatrix} = \begin{pmatrix} \frac{\beta Pe^{\zeta_n}}{1-\beta Pe^{\zeta_n}} & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.3}$$

It is easy to see that

$$detJ_{\Psi}(\zeta, \iota) = 1.$$

Namely, the mapping Ψ is a measure-preserving mapping. This indicates that the system (2.1) is amenable to the application of the KAM theory.

Observe that

$$J_0 = J_{\Psi}(0,0) = \begin{pmatrix} \frac{\beta P}{1-\beta P} & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.4}$$

At $(0,0)$, the characteristic equation is

$$f(\lambda) = \lambda^2 - \frac{\beta P}{1-\beta P}\lambda + 1,$$

and the characteristic roots are

$$\lambda = \frac{\frac{\beta P}{1-\beta P} + \sqrt{(\frac{\beta P}{1-\beta P})^2 - 4}}{2}, \quad \bar{\lambda} = \frac{\frac{\beta P}{1-\beta P} - \sqrt{(\frac{\beta P}{1-\beta P})^2 - 4}}{2}.$$

1) For the equilibrium $P = P_1 = P_2 = \sqrt{3}$, when $\beta = \frac{2}{3\sqrt{3}}$, we have that $\lambda_1 = \lambda_2 = 1$ (P is non-hyperbolic equilibrium of parabolic type), and the KAM theory can not be applied in this situation.

2) If $0 < \beta < \frac{2}{3\sqrt{3}}$, then, we have that $P_1 < \frac{2}{3\beta}$ for the equilibrium $P = P_1 < P_2$, which implies $(\frac{\beta P}{1-\beta P})^2 - 4 < 0$ and

$$\lambda(P_2) = \frac{\frac{\beta P}{1-\beta P} + i\sqrt{4 - (\frac{\beta P}{1-\beta P})^2}}{2}, \quad \bar{\lambda}(P_2) = \frac{\frac{\beta P}{1-\beta P} - i\sqrt{4 - (\frac{\beta P}{1-\beta P})^2}}{2}.$$

It signifies that P is a non-hyperbolic elliptic fixed point, thus enabling the application of the KAM theory in this case. Similarly, we have that $P_3 > \frac{2}{3\beta}$ for the equilibrium P_3, which implies that $(\frac{\beta P}{1-\beta P})^2 - 4 < 0$ and the equilibrium P_3 is a saddle point. The KAM theory is applied to analyze the equilibrium $P = P_2,\ P_2 < \frac{2}{3\beta},\ \beta \in (0, \frac{2}{3\sqrt{3}})$.

Clearly, $|\lambda(P)| = 1, \lambda^3 \neq 1, \lambda^4 \neq 1$ for $\beta \in (0, \frac{2}{3\sqrt{3}})$. Indeed,

$$\lambda^2(P_2) = \frac{(\frac{\beta P}{1-\beta P})^2}{2} - 1 + 2i\frac{\beta P}{1-\beta P}\sqrt{4 - (\frac{\beta P}{1-\beta P})^2},$$

$$\lambda^3(P_2) = \frac{\frac{\beta P}{1-\beta P}}{8}(7(\frac{\beta P}{1-\beta P})^2 - 24) + \frac{i\sqrt{4 - (\frac{\beta P}{1-\beta P})^2}}{8}(7(\frac{\beta P}{1-\beta P})^2 - 4),$$

$$\lambda^4(P_2) = \frac{1}{4}(17(\frac{\beta P}{1-\beta P})^4 - 64(\frac{\beta P}{1-\beta P})^2 - 4\frac{\beta P}{1-\beta P} + 4) + 4i\frac{\beta P}{1-\beta P}\sqrt{4 - (\frac{\beta P}{1-\beta P})^2}(\frac{(\frac{\beta P}{1-\beta P})^2}{2} - 1).$$

If $7(\frac{\beta P}{1-\beta P})^2 - 4 = 0, \quad \lambda^3 \neq 1$, if $\frac{(\frac{\beta P}{1-\beta P})^2}{2} - 1 = 0, \quad \lambda^4 \neq 1$.

Therefore, the conditions outlined in Theorem 1 are satisfied when $d = 4$, and we shall determine the Birkhoff normal form of Eq (2.1) through the series of transformations.

First transformation: The matrix that represents the linearized system at the origin is presented as follows:

$$J_0 = \begin{pmatrix} \frac{\beta P}{1-\beta P} & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.5}$$

and the eigenvector matrix corresponding to the eigenvalues λ and $\bar{\lambda}$ of J_0 is

$$R = \begin{pmatrix} 1 & 1 \\ \bar{\lambda} & \lambda \end{pmatrix}. \tag{2.6}$$

The equations in system (2.1) at the right hand sides will be expanded at $P(0,0)$ up to the order $d-1 = 3$ to obtain the Birkhoff normal form as follows:

$$\begin{cases} \zeta_{n+1} = \frac{\beta P}{P^2}\zeta_n + \frac{\beta P^2(P+\beta)}{2P^4}(\zeta_n^2 + \frac{P(P+2\beta)}{3P^2}\zeta_n^3) - \iota_n + O_4, \\ \iota_{n+1} = \zeta_n. \end{cases} \tag{2.7}$$

The change of variables

$$\begin{bmatrix} \zeta_n \\ \iota_n \end{bmatrix} = R \begin{bmatrix} s_n \\ t_n \end{bmatrix} = \begin{bmatrix} s_n + t_n \\ \bar{\lambda}s_n + \lambda t_n \end{bmatrix} \tag{2.8}$$

transforms system (2.7) into

$$\begin{aligned} s_{n+1} &= \lambda s_n + \sigma((s_n + t_n)^2 + \frac{P(P+2\beta)}{3P^2}(s_n + t_n)^3) + O_4, \\ t_{n+1} &= \bar{\lambda} t_n + \bar{\sigma}((s_n + t_n)^2 + \frac{P(P+2\beta)}{3P^2}(s_n + t_n)^3) + O_4, \end{aligned} \tag{2.9}$$

where

$$\sigma = \frac{\lambda}{\lambda - \bar{\lambda}} \frac{\beta P^2(P+\beta)}{2P^4} = \frac{\beta(P+\beta)}{4P^2\sqrt{(\frac{\beta P}{1-\beta P})^2 - 4}} \left(\sqrt{(\frac{\beta P}{1-\beta P})^2 - 4} - \frac{i\beta P}{1-\beta P} \right).$$

Second transformation: The second transformation is intended to capture the non-linear terms up to the $d-1$ order in the normal form. A change of variables will be introduced in the following manner:

$$\begin{cases} s_n = \xi_n + \sum_{k=0}^{2}(a_{2k}\xi_n^{2-k}\eta_n^k) + \sum_{k=0}^{3}(a_{3k}\xi_n^{3-k}\eta_n^k), \\ t_n = \eta_n + \sum_{k=0}^{2}(\bar{a}_{2k}\xi_n^k\eta_n^{2-k}) + \sum_{k=0}^{3}(\bar{a}_{3k}\xi_n^k\eta_n^{3-k}), \end{cases} \tag{2.10}$$

where

$$a_{20} = \frac{\sigma}{\lambda(\lambda-1)}, \quad a_{21} = \frac{2\sigma}{1-\lambda}, \quad a_{22} = \frac{\sigma}{\bar{\lambda}^2 - \lambda}.$$

Therefore,

$$a_{20} = \frac{\beta(P+\beta)(\sqrt{4 - (\frac{\beta P}{1-\beta P})^2} + i(\frac{\beta P}{1-\beta P} - 2))}{4P^2\sqrt{4 - (\frac{\beta P}{1-\beta P})^2}(\frac{\beta P}{1-\beta P} - 2)},$$

$$a_{21} = \frac{\beta(P+\beta)(-\sqrt{4 - (\frac{\beta P}{1-\beta P})^2} + i(\frac{\beta P}{1-\beta P} - 2))}{2P^2\sqrt{4 - (\frac{\beta P}{1-\beta P})^2}(\frac{\beta P}{1-\beta P} - 2)},$$

$$a_{22} = \frac{\beta(P+\beta)((\frac{\beta P}{1-\beta P}-1)\sqrt{4-(\frac{\beta P}{1-\beta P})^2} - i(\frac{\beta P}{1-\beta P}-2)(\frac{\beta P}{1-\beta P}+1))}{8P^2\sqrt{4-(\frac{\beta P}{1-\beta P})^2}(2-\frac{\beta P}{1-\beta P})(\frac{\beta P}{1-\beta P}+1)},$$

which simplifies system (2.9) to the form

$$\begin{aligned} \xi_{n+1} &= (\lambda\xi_n + \beta_2\xi_n^2\eta_n) + O_4, \\ \eta_{n+1} &= (\bar{\lambda}\eta_n + \bar{\beta_2}\xi\eta_n^2) + O_4, \end{aligned} \tag{2.11}$$

where

$$\Upsilon_2 = 2(a_{21} + \bar{a_{21}})\sigma + 2(a_{20} + \bar{a_{22}})\sigma.$$

Thus,

$$Re(\Upsilon_2) = \frac{\beta^4(P+\beta)^4(3-(\frac{\beta P}{1-\beta P})^2)}{16P^8(2+\frac{\beta P}{1-\beta P})(\frac{\beta P}{1-\beta P}-2)^2(\frac{\beta P}{1-\beta P}+1)}.$$

Third transformation: The transformation of variables

$$\begin{aligned} \xi_n &= u_n + iv_n, \\ \eta_n &= u_n - iv_n, \end{aligned}$$

changes system (2.11) into

$$\begin{aligned} u_{n+1} &= \mu_1 u_n - \mu_2 v_n + O_4, \\ v_{n+1} &= \mu_2 u_n - \mu_1 v_n + O_4, \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} \mu_1 &= Re(\lambda) + Re(\Upsilon_2)(u_n^2 + v_n^2) + O_4, \\ \mu_2 &= Im(\lambda) + Im(\Upsilon_2)(u_n^2 + v_n^2) + O_4, \end{aligned}$$

System (2.12) can be written in the following form:

$$\begin{aligned} u_{n+1} &= cos\tau u_n - sin\tau v_n + O_4, \\ v_{n+1} &= sin\tau u_n + cos\tau v_n + O_4, \end{aligned} \tag{2.13}$$

where

$$\tau = \delta_0 + \delta_1(u_n^2 + v_n^2).$$

Subsequently, for the twist coefficients δ_0 and δ_1, and from the equation $\beta = \frac{P^2-1}{P^3}$, the following is deduced:

$$cos\delta_0 = Re(\lambda) = \lambda(P_2) = \frac{\frac{\beta P}{1-\beta P}}{2} = \frac{(P+1)(P-1)}{2} \in (0,1) \; and \; \delta_1 = -\frac{Re(\Upsilon_2)}{sin\delta_0}.$$

For system (2.7), system (2.13) is the Birkhoff normal form because $1 < P < \frac{2}{3\beta}$ implies the following:

$$\beta^4(P+\beta)^4(3-(\frac{\beta P}{1-\beta P})^2) > 0.$$

Since $\beta \in (0, \frac{2}{3\sqrt{3}})$, this implies $\delta_1 \neq 0$.

Thus, the following result has been proven. □

Theorem 6. *The positive fixed point solution P of* (1.1) *is stable for* $\beta \in (0, \frac{2}{3\sqrt{3}})$.

Figures 1–6 exhibit some orbits of Eq (1.1) and Ψ, Figure 7 illustrates that multiple bifurcation points appear in the diagram as the control parameter β gradually increases, and Figure 8 illustrates the minimal possible period near (0,0) for map Ψ. These points mark transitions in the system stability, particularly when it approaches a critical value, thus leading to significant changes in the system behavior, such as shifts from periodic orbits to chaotic states.

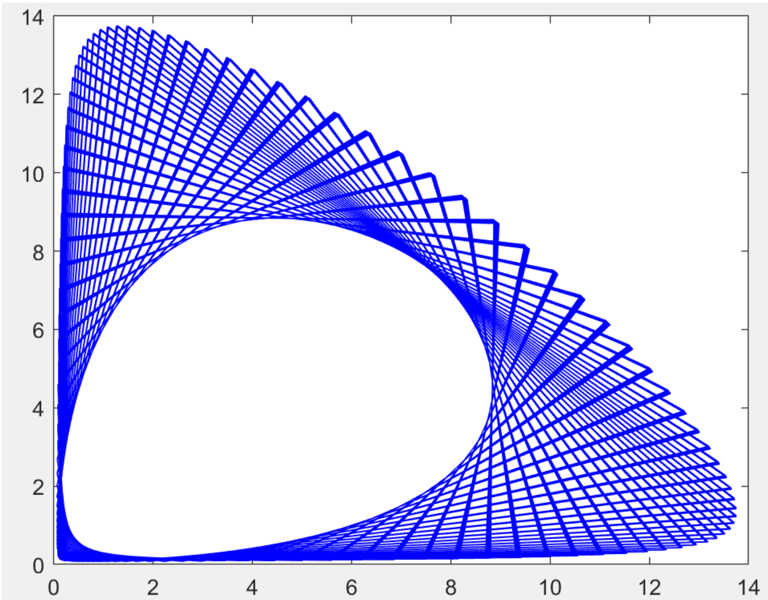

Figure 1. The trajectories of the map T associated with Eq (1.1) at $\beta = \frac{1}{30}$.

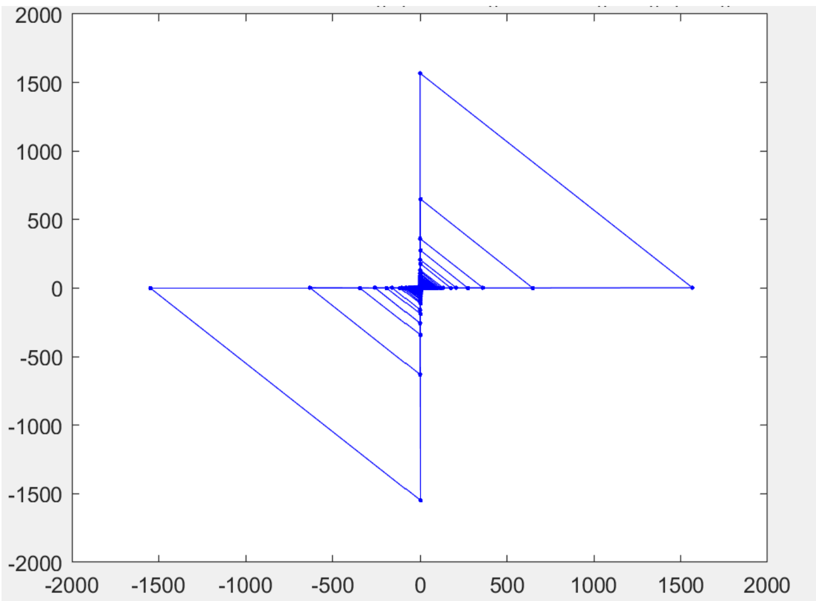

Figure 2. The trajectories of the map T associated with Eq (1.1) at $\beta = \frac{1}{3}$.

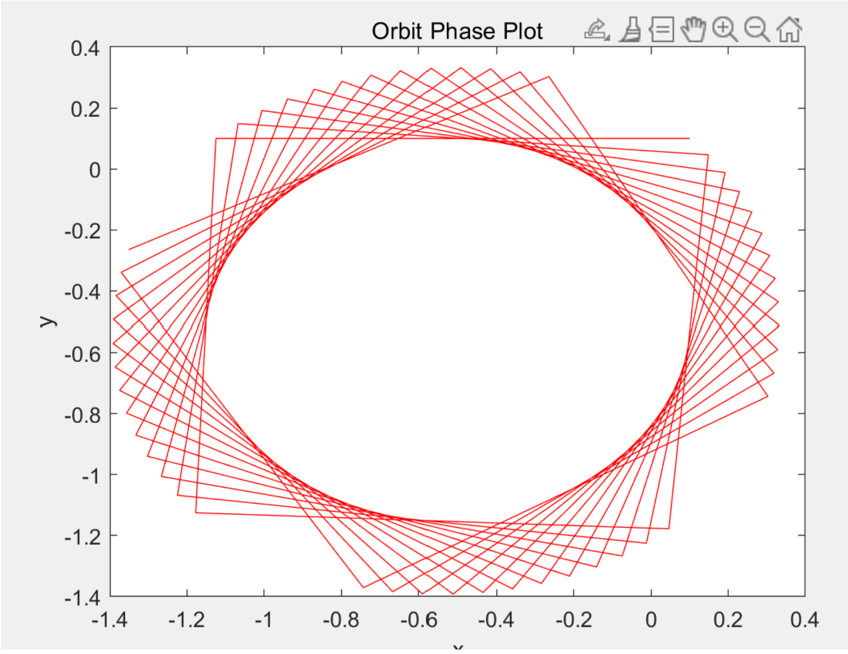

Figure 3. Some orbits of the map Ψ at $\beta = \frac{1}{30}$ for 50 iterations.

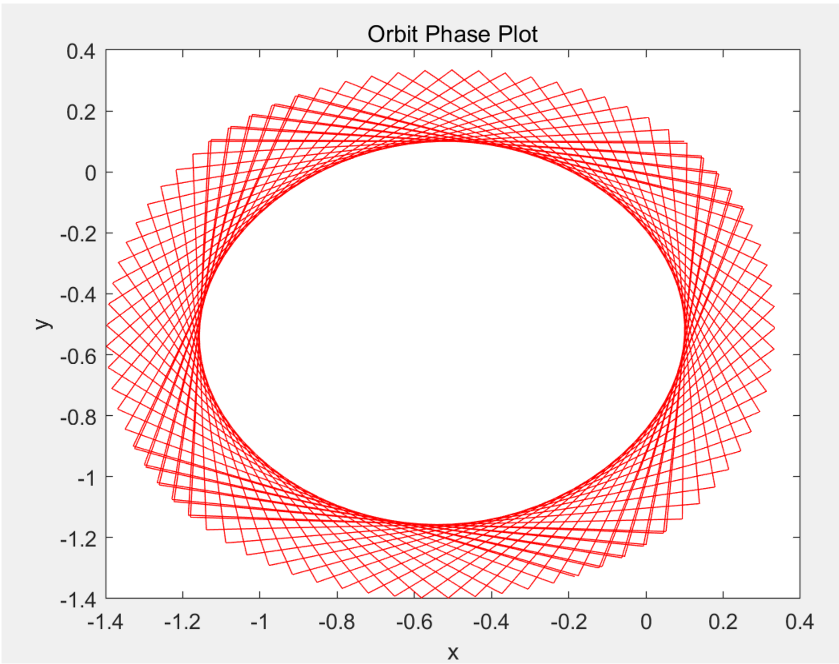

Figure 4. Some orbits of the map Ψ at $\beta = \frac{1}{30}$ for 100 iterations.

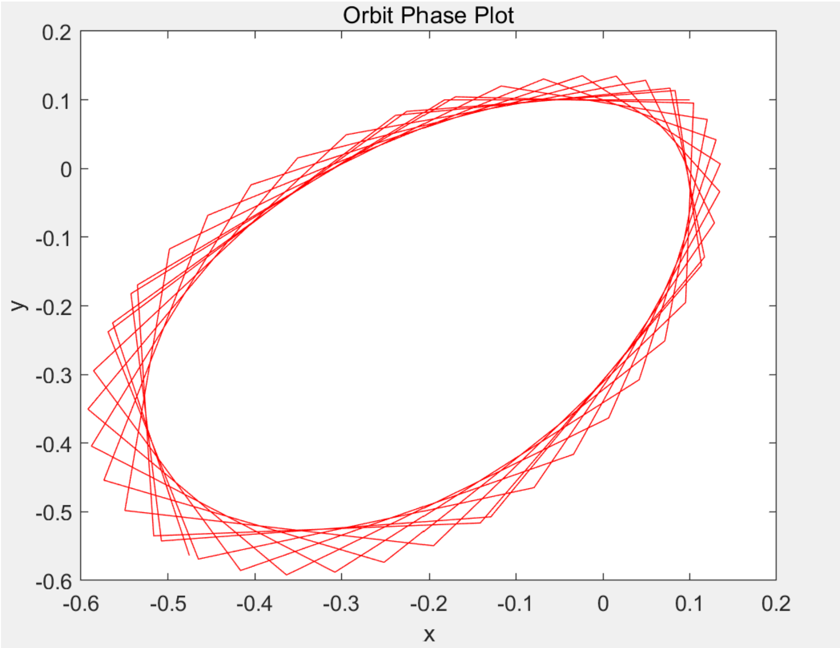

Figure 5. Some orbits of the map Ψ at $\beta = \frac{1}{3}$ for 50 iterations.

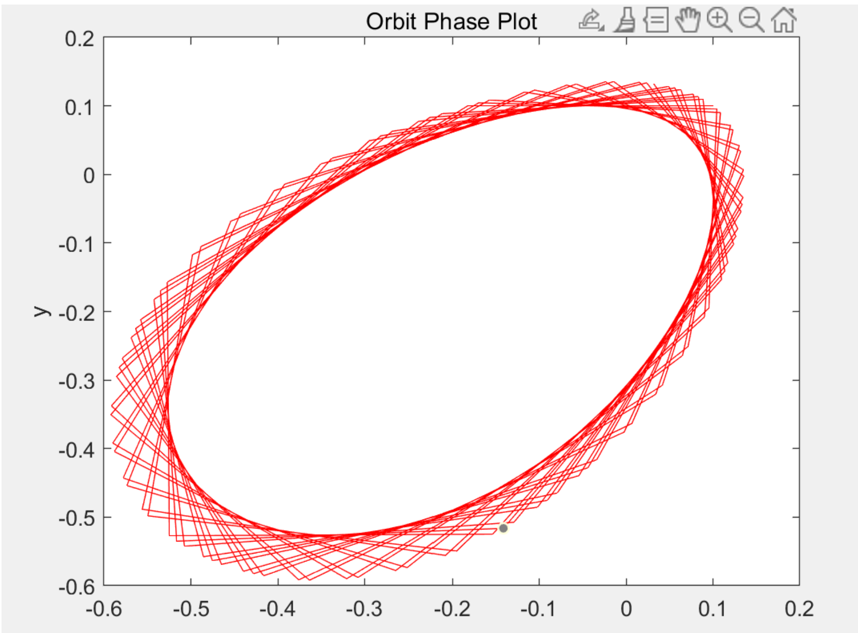

Figure 6. Some orbits of the map Ψ at $\beta = \frac{1}{3}$ for 100 iterations.

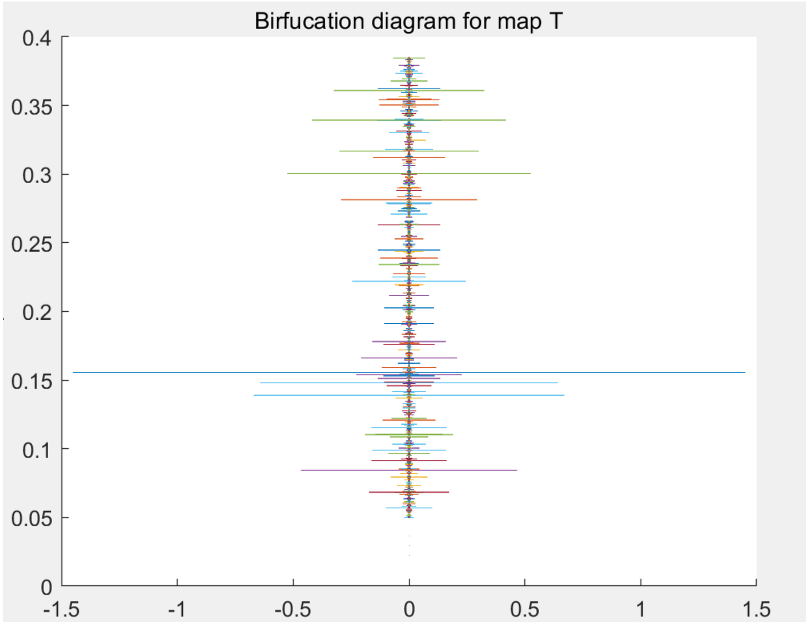

Figure 7. Birfucation diagram for map T associated with Eq (1.1).

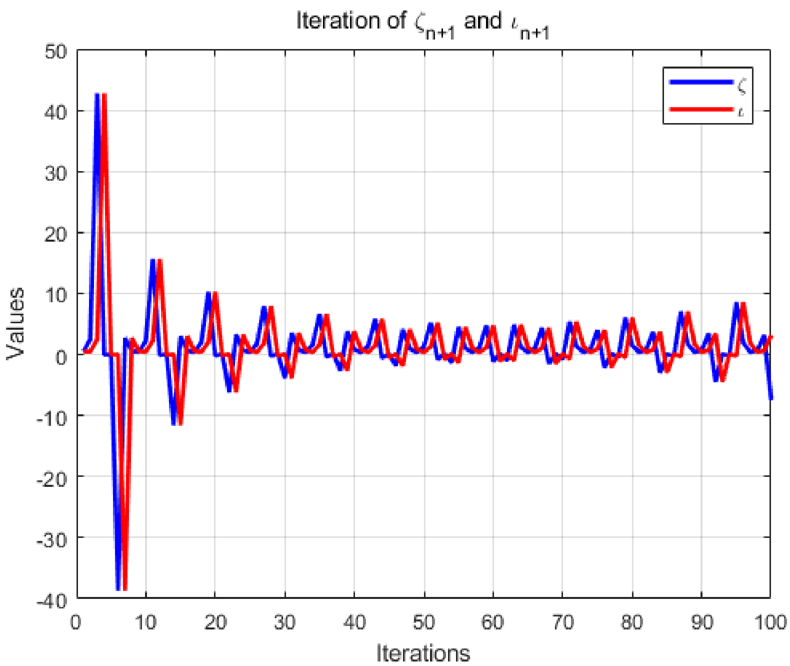

Figure 8. The minimal possible period near (0,0) for map Ψ.

Remark 2. *The subsequent findings are founded on Moser's twist map theorem, which establishes the presence of periodic points with extended periods in the vicinity of* $P = P_1$*. Previous investigations [17–19] demonstrate that the elliptic equilibrium point* P *can be expressed as* $\lambda = e^{\pm i\theta}$ *with* λ *and* $\bar{\lambda}$*, where* $\frac{\pi}{2} > \theta > 0$*. Additionally, the motion period around point* P *has to satisfy* $q > \frac{2\pi}{\theta} > 4$.

therefore, the Eq (1.1) *in a neighborhood of the elliptic fixed point has orbits more than or equal to* 4. *For example, if* $\beta = 0.3$*, then* $\frac{2\pi}{\theta} \approx 4.96271722$*. Consequently, the smallest possible duration for a periodic orbit in the vicinity of the elliptic fixed point is established to be* 5.

3. Symmetries

Symmetries can be applied to the map Ψ, which conjugates to its inverse through an involution. Additionally, based on symmetries, the method of time reversal symmetry will be utilized to accurately determine specific feasible periods and corresponding trajectories of the mapping Ψ.

As referred to the previous studies [17–19], symmetries play a crucial role in the area-preserving map analysis, as they lead to distinctive dynamic behaviors. If the equation $W^{-1} \circ \Psi \circ W = \Psi^{-1}$ is equivalent to $W \circ \Psi \circ W = \Psi^{-1}$, then the transformation W of the plane can be interpreted as a symmetry reversing time for Ψ. This enables us to express Ψ as a combination of two convolutions: $\Psi = I_1 \circ I_0$, where $I_0 = W$, and $I_1 = \Psi \circ W$. Importantly, if $I_0 = W$ functions as an inverter, then so does $I_1 = \Psi \circ W$. In addition, the j involution (defined as $I_j = \Psi_j \circ W$) acts as a reversal.

The collections $S_j = \{(\zeta, \iota) | I_j(\zeta, \iota) = (\zeta, \iota), j = 1, 2$ are characterized by one-dimensional features and regarded as invariant sets under the involution mappings. These sets are referred to as the symmetry lines of the map Ψ. The significance of the following conclusion lies in its relevance to the exploration of periodic orbits, as discussed in [13].

Theorem 7. *If* $(\zeta, \iota) \in S_{0,1}$*, then* $\Psi^n(\zeta, \iota) = (\zeta, \iota)$ *if and only if*

$$\begin{cases} \Psi^{\frac{n}{2}}(\zeta, \iota) \in S_{0,1}, & n \text{ is even}, \\ \Psi^{\frac{n\pm1}{2}}(\zeta, \iota) \in S_{1,0}, & n \text{ is odd}. \end{cases}$$

Periodic trajectories of various orders exist where the symmetry lines, $S_j, j = 1, 2, ...,$ intersect, with $S_{2j-i} = I_j(S_j), S_{2j+i} = \Psi^j(S_i)$, for every i, j. If (ζ, ι) belongs to the intersection of S_j and S_k, then $\Psi^{j-k}(\zeta, \iota) = (\zeta, \iota)$.

Regarding $\beta \in (0, \dfrac{2}{3\sqrt{3}})$,

$$\zeta_{n+1} = \frac{1}{\iota_n(1 - \beta\zeta_n)}, \quad \iota_{n+1} = \zeta_n. \tag{3.1}$$

Accordingly, the map Ψ for system (3.1) is determined by the following:

$$\Psi(\zeta, \iota) = (\frac{1}{\iota(1 - \beta\zeta)}, \zeta).$$

The map inverse of (3.1) applies to the positive quadrant Q in R^2 as follows:

$$\Psi^{-1}(\zeta, \iota) = (\iota, \frac{1}{\zeta(1 - \beta\iota)}).$$

Note that the involution $W(\zeta, \iota) = (\iota, \zeta)$ is a reversor for (3.1):

$$(W \circ \Psi \circ W)(\zeta, \iota) = (W \circ \Psi)(\iota, \zeta) = W(\frac{1}{\zeta(1 - \beta\iota)}, \iota) = (\iota, \frac{1}{\zeta(1 - \beta\iota)}) = \Psi^{-1}(\zeta, \iota).$$

Therefore, it can be deduced that $\Psi = I_1 \circ I_0$, where $I_0(\zeta, \iota) = W(\zeta, \iota) = (\iota, \zeta)$ and

$$I_1(\zeta, \iota) = (\Psi \circ W)(\zeta, \iota) = (\frac{1}{\zeta(1 - \beta\iota)}, \iota).$$

Associated with I_0 and I_1, the symmetry lines are as follows:

$$S_0 = \{(\zeta, \iota) : \iota = \zeta\}, S_1 = \{(\zeta, \iota) : \zeta = \sqrt{\frac{1}{1 - \beta\iota}}\}.$$

Searching for even-periodic orbits on the symmetry line S_0 begins with points $(\zeta_0, \zeta_0) \in S_0$ and imposes that $(\zeta_{\frac{n}{2}}, \iota_{\frac{n}{2}}) \in S_0$, where

$$(\zeta_{\frac{n}{2}}, \iota_{\frac{n}{2}}) = \Psi^{\frac{n}{2}}(\zeta_0, \zeta_0).$$

It can find the root of an equation $\zeta_{\frac{n}{2}} = \iota_{\frac{n}{2}}$ in one dimension for an unknown variable ζ_0. Odd-period orbits on S_0 can be found through tackling an equation $\zeta^2_{\frac{n+1}{2}} = (\frac{1}{\iota_{\frac{n+1}{2}}(1-\beta\zeta_n)})^2$ for ζ_0, where

$$(\zeta_{\frac{n+1}{2}}, \iota_{\frac{n+1}{2}}) = \Psi^{\frac{n+1}{2}}(\zeta_0, \zeta_0).$$

4. Conclusions

Under specific constraints on the parameter β, the stability of the zero equilibrium of (1.1) was established through the application of the KAM theory and the utilization of Birkhoff normal forms. Furthermore, through the employment of symmetries, we demonstrated the existence of periodic solutions characterized by specific periods.

Author contributions

Ruiqi Yan: Writing-original draft, Revising and Editing; Qianhong Zhang: Writing-review, Funding acquisition and Guiding the revision of the paper.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors assert the absence of any competing interests.

References

1. J. Wang, X. Jiang, H. Zhang, X. Yang, A new fourth-order nonlinear difference scheme for the nonlinear fourth-order generalized Burgers-type equation, *J. Appl. Math. Comput.*, in press. https://doi.org/10.1007/s12190-025-02467-3
2. J. Zhang, X. Yang, S. Wang, The ADI difference and extrapolation scheme for high-dimensional variable coefficient evolution equations, *Electron. Res. Arch.*, **33** (2025), 3305–3327. https://doi.org/10.3934/era.2025146
3. J. Wang, X. Jiang, H. Zhang, A BDF3 and new nonlinear fourth-order difference scheme for the generalized viscous Burgers' equation, *Appl. Math. Lett.*, **151** (2024), 109002. https://doi.org/10.1016/j.aml.2024.109002
4. X. Yang, Z. Zhang, Analysis of a new NFV scheme preserving DMP for two-dimensional sub-diffusion equation on distorted meshes, *J. Sci. Comput.*, **99** (2024), 80. https://doi.org/10.1007/s10915-024-02511-7
5. X. Yang, Z. Zhang, Superconvergence analysis of a robust orthogonal Gauss collocation method for 2D fourth-order subdiffusion equations, *J. Sci. Comput.*, **100** (2024), 62. https://doi.org/10.1007/s10915-024-02616-z
6. Y. Shi, X. Yang, The pointwise error estimate of a new energy-preserving nonlinear difference method for supergeneralized viscous Burgers' equation, *Comp. Appl. Math.*, **44** (2025), 257. https://doi.org/10.1007/s40314-025-03222-x
7. E. Martínez, J. Vidarte, J. Zapata, Periodic orbits and KAM tori of a particle around a homogeneous elongated body, *Dynam. Syst.*, **40** (2025), 91–101. https://doi.org/10.1080/14689367.2024.2426585
8. X. Niu, K. Wang, Y. Li, Quasi-periodic swing via weak KAM theory, *Physica D*, **474** (2025), 134559. https://doi.org/10.1016/j.physd.2025.134559
9. A. Celletti, I. De Blasi, S. Di Ruzza, Perturbative methods and synchronous resonances in Celestial mechanics, *Appl. Math. Model.*, **143** (2025), 116040. https://doi.org/10.1016/j.apm.2025.116040
10. K. Senada, E. Pilav, Stability of May's host-parasitoid model with variable stocking upon parasitoids, *Int. J. Biomath.*, **15** (2022), 2150072. https://doi.org/10.1142/S1793524521500728
11. W. Jamieson, On the global behaviour of May's host-parasitoid model, *J. Differ. Equ. Appl.*, **25** (2019), 583–596. https://doi.org/10.1080/10236198.2019.1613387
12. C. Lautenschlager, N. Tzempelikos, Towards an integration of corporate foresight in key account management, *Ind. Market. Manag.*, **120** (2024), 90–99. https://doi.org/10.1016/j.indmarman.2024.05.009
13. M. Gidea, J. Meiss, I. Ugarcovici, H. Weiss, Applications of KAM theory to population dynamics, *J. Biol. Dynam.*, **5** (2011), 44–63. https://doi.org/10.1080/17513758.2010.488301

14. M. Garić-Demirović, M. Nurkanović, Z. Nurkanović, Stability, periodicity and symmetries of certain second-order fractional difference equation with quadratic terms via KAM theory, *Math. Method. Appl. Sci.*, **40** (2017), 306–318. https://doi.org/10.1002/mma.4000

15. M. Kulenović, Invariants and related Liapunov functions for difference equations, *Appl. Math. Lett.*, **13** (2000), 1–8. https://doi.org/10.1016/S0893-9659(00)00068-9

16. A. Cima, A. Gasull, V. Mañosa, Non-integrability of measure preserving maps via Lie symmetries, *J. Differ. Equations*, **259** (2015), 5115–5136. https://doi.org/10.1016/j.jde.2015.06.019

17. T. Ibrahim, Z. Nurkanović, Kolmogorov-Arnold-Moser theory and symmetries for a polynomial quadratic second order difference equation, *Mathematics*, **7** (2019), 790. https://doi.org/10.3390/math7090790

18. E. Dennete, M. Kulenovic, E. Pilav, Birkhoff normal forms, KAM theory and time reversal symmetry for certain rational map, *Mathematics*, **4** (2016), 20. https://doi.org/10.3390/math4010020

19. M. Kulenović, Z. Nurkanović, E. Pilav, Birkhoff normal forms and KAM theory for Gumowski-Mira equation, *Sci. World J.*, **2014** (2014), 819290. https://doi.org/10.1155/2014/819290

20. M. Kulenovic, O. Merino, *Discrete dynamical systems and difference equations with mathematica*, New York: Chapman and Hall/CRC, 2002. https://doi.org/10.1201/9781420035353