Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(8): 18232–18251.
DOI: 10.3934/math.2025814
Received: 24 June 2025
Revised: 06 August 2025
Accepted: 08 August 2025
Published: 13 August 2025

Research article

Geometric aspects of weakly symmetric and almost pseudo symmetric K-contact manifolds admitting a non-symmetric non-metric connection

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Abstract: In this paper, we explore the conditions for several geometric structures-specifically, weakly symmetric, weakly Ricci symmetric, almost pseudo symmetric, and almost pseudo Ricci symmetric– on a K-contact manifold equipped with a non-symmetric non-metric connection. We present key theoretical results that characterize these structures in the context of such a connection. To illustrate the applicability of our findings, we construct an explicit example of a 3-dimensional K-contact manifold with a non-symmetric non-metric connection. This example not only confirms the validity of the derived conditions but also offers a concrete model for further investigation in the field. Our results extend the understanding of geometric structures on K-contact manifolds beyond the classical framework of the Levi-Civita connection, paving the way for new directions in differential geometry and mathematical physics.

Keywords: non-symmetric non-metric connection; Levi-Civita connection; curvature tensors; K-contact manifold; weakly symmetric
Mathematics Subject Classification:53C05, 53C15, 53C25, 53D10

1. Introduction

The development of the infinitesimal viewpoint of connections in Riemannian geometry began, to some extent, with Christoffel, who introduced the formalism for differentiating vector fields. Building on this foundation, Levi-Civita recognized that connections also enable the concept of parallel transport and emphasized their role as differential operators. In the 20th century, Cartan advanced the theory by expressing connections in terms of differential forms, thereby enriching the geometric framework. Later, in 1950, Koszul provided an algebraic formulation of connections as differential operators on vector bundles. The two primary geometric invariants associated with such connections are torsion and curvature [13, 14]. The torsion tensor is a bilinear map $T : \chi(N) \times \chi(N) \rightarrow \chi(N)$ defined by $T(X, Y) = D_X Y - D_Y X - [X, Y]$, for arbitrary vector fields $X, Y \in \chi(N)$, where $\chi(N)$ denotes the collection of all smooth vector fields of N. A connection is termed *torsion-free* or *symmetric* if $T = 0$; otherwise, it is called *non-symmetric*. Moreover, a connection D is said to be *metric* if it preserves a Riemannian metric g, that is, if $Dg = 0$. The Levi-Civita connection is the unique connection that is both symmetric and metric. In contrast, semi-symmetric and quarter-symmetric connections are notable examples of non-symmetric connections. The foundational study of semi-symmetric connections in Riemannian geometry was initiated by Pak [18] and Yano [27]. Subsequently, Agashe and Chafle [1] introduced the notion of a semi-symmetric non-metric connection in 1992.

In more recent developments, Chaubey [5] proposed a distinct class of non-symmetric, non-metric connections in 2007. He later explored the behavior of this connection in various contact manifolds [4, 8, 9], prompting further investigations by many geometers. Parallel to these developments, Sasakian manifolds, originally introduced by Shigeo Sasaki in 1960 [3, 11, 22], have attracted significant interest. Their geometric structure has been extensively analyzed and expanded upon by numerous researchers [12, 15, 19, 21].

This research article is organized as follows: After the introduction in Section 2, we highlights the necessary identities of K-contact manifolds along with certain definitions. Section 3 explores certain curvature tensors like the Riemannian curvature tensor $\bar{R}$, the Ricci tensor $\bar{S}$, and the scalar curvature tensor $\bar{r}$ with respect to a non-symmetric non-metric connection $NSNMC\ \bar{D}$ in a K-contact manifold, as well as considering that there cannot be Ricci flat in regard to $NSNMC\ \bar{D}$ under a K-contact manifold. Section 4 reveals that there exists no weakly symmetric K-contact manifold (N, g) $(2n + 1 > 3)$ admitting a non-symmetric non-metric connection $NSNMC\ \bar{D}$ denoted by $[(WRS)_n, \bar{D}]$, unless $G + I + J$ vanishes everywhere. Moreover, we had discussed that there exists no weakly Ricci symmetric K-contact manifold admitting $NSNMC\ \bar{D}$ denoted by $[(WRS)_n, \bar{D}]$, unless $\alpha + \beta + \gamma$ vanishes everywhere. Section 5 talks about locally symmetric and locally ϕ-symmetric K-contact manifolds with respect to $NSNMC\ \bar{D}$. Both the conditions satisfy an η-Einstein manifold under transformations. Section 6 discusses almost pseudo-symmetric K-contact manifolds admitting $NSNMC\ \bar{D}$, symbolized as $[(APS)_n, \bar{D}]$. This condition showcases no $[(APS)_n, \bar{D}]$, unless $3A + B = 0$, where A and B are the two 1-forms. Later, in Section 7, we again test the existence of an almost pseudo Ricci symmetric K-contact manifold with respect to $NSNMC\ \bar{D}$, represented by $[(APRS)_n, \bar{D}]$, which still holds no $[(APRS)_n, \bar{D}]$, unless $3A + B$ vanishes everywhere. Finally, an example of a 3-dimensional K-contact manifold equipped with $NSNMC\ \bar{D}$ is provided for the validation of the results obtained.

2. Preliminaries

Let (N, g) be a $(2n+1)$-dimensional almost contact metric manifold [3] with contact metric structure (ϕ, ζ, η, g), where ϕ is a (1,1) tensor field, ζ is a vector field, η is a 1-form, and g is a Riemannian metric on (N, g) such that

$$\phi^2 X_1^\psi = -X_1^\psi + \eta(X_1^\psi)\zeta, \quad \phi\zeta = 0, \quad \eta(\phi X_1^\psi) = 0, \quad \eta(\zeta) = 1, \tag{2.1}$$

$$g(\phi X_1^\psi, \phi X_2^\psi) = g(X_1^\psi, X_2^\psi) - \eta(X_1^\psi)\eta(X_2^\psi), \tag{2.2}$$

$$\eta(X_1^\psi) = g(X_1^\psi, \zeta), \quad g(X_1^\psi, \phi X_2^\psi) = -g(\phi X_1^\psi, X_2^\psi), \tag{2.3}$$

for all vector fields X_1^ψ, X_2^ψ on (N, g).

If, moreover, $\zeta \in \chi(N)$ is a Killing vector field that satisfies $\mathcal{L}_\zeta g = 0$, where $\mathcal{L}_\zeta$ is the Lie derivative along the vector field ζ, then the manifold (N, g) with the contact metric structure (ϕ, ζ, η, g) is called a K-contact manifold [3, 23].

In a $(2n + 1)$-dimensional K-contact manifolds, the following relations hold [17]:

$$D_{X_1^\psi}\zeta = -\phi X_1^\psi, \quad D_\zeta\zeta = 0, \tag{2.4}$$

$$(D_{X_1^\psi}\eta)(X_2^\psi) = g(X_1^\psi, \phi X_2^\psi) = -g(\phi X_1^\psi, X_2^\psi), \tag{2.5}$$

$$\eta(R(X_1^\psi, X_2^\psi)X_3^\psi) = g(X_2^\psi, X_3^\psi)\eta(X_1^\psi) - g(X_1^\psi, X_3^\psi)\eta(X_2^\psi), \tag{2.6}$$

$$R(X_1^\psi, X_2^\psi)\zeta = \eta(X_2^\psi)X_1^\psi - \eta(X_1^\psi)X_2^\psi, \tag{2.7}$$

$$(D_{X_1^\psi}\phi)(X_2^\psi) = R(\zeta, X_1^\psi)X_2^\psi = g(X_1^\psi, X_2^\psi)\zeta - \eta(X_2^\psi)X_1^\psi = -R(X_1^\psi, \zeta)X_2^\psi, \tag{2.8}$$

$$S(X_1^\psi, \zeta) = 2n\eta(X_1^\psi), \quad S(\zeta, \zeta) = 2n, \tag{2.9}$$

$$QX^\psi = 2nX^\psi, \quad Q\zeta = 2n\zeta, \tag{2.10}$$

$$S(\phi X_1^\psi, \phi X_2^\psi) = S(X_1^\psi, X_2^\psi) - 2n\eta(X_1^\psi)\eta(X_2^\psi), \tag{2.11}$$

where S is the Ricci tensor, R signifies the Riemannian curvature tensor, and Q denotes the Ricci operator defined by $g(QX^\psi, X_2^\psi) = S(X_1^\psi, X_2^\psi)$ with respect to the Levi–Civita connection.

A linear connection $\bar{D}$ on K-contact manifold is defined by

$$\bar{D}_{X_1^\psi}X_2^\psi = D_{X_1^\psi}X_2^\psi + g(\phi X_1^\psi, X_2^\psi)\zeta, \tag{2.12}$$

where D is the Levi-Civita connection on N. Using (2.12), the torsion tensor $\bar{T}$ of N with respect to $\bar{D}$ gives

$$\bar{T}(X_1^\psi, X_2^\psi) = \bar{D}_{X_1^\psi}X_2^\psi - \bar{D}_{X_2^\psi}X_1^\psi - [X_1^\psi, X_2^\psi] = 2g(\phi X_1^\psi, X_2^\psi)\zeta. \tag{2.13}$$

Further, using (2.13), we get

$$(\bar{D}_{X_1^\psi}g)(X_2^\psi, X_3^\psi) = -\eta(X_3^\psi)g(\phi X_1^\psi, X_2^\psi) - \eta(X_2^\psi)g(\phi X_1{}^\psi, X_3^\psi) \neq 0, \tag{2.14}$$

for arbitrary vector field $X_1^\psi, X_2^\psi, X_3^\psi \in \chi(N)$. A linear connection $\bar{D}$ defined in (2.12) satisfying (2.13) and (2.14) is called a non-symmetric non-metric connection [5, 9] (briefly, $NSNMC$). The non-symmetric non-metric connection on Sasakian manifolds has also been studied by a few others [4, 20]. Moreover, the non-symmetric, non-metric connection on a different type of manifolds has been studied by [2, 16, 24].

Definition 2.1. *A* $(2n+1)$*-dimensional K-contact manifold* (N, g) *is said to be an* η*-Einstein manifold if its Ricci tensor* S *is of the form*

$$S(X_1^\psi, X_2^\psi) = p_1 g(X_1^\psi, X_2^\psi) + p_2 \eta(X_1^\psi)\eta(X_2^\psi),$$

where p_1 *and* p_2 *are smooth functions on* (N^n, g)*. If* $p_2 = 0$*, then* (N, g) *becomes an Einstein manifold.*

Definition 2.2. *[25] A non-flat Riemannian manifold* (N, g) $(2n+1 > 2)$ *is called weakly symmetric if there exist 1-forms* G, H, I, J *and a vector field* F *that satisfy the condition*

$$\begin{aligned}(D_{X_1^\psi}R)(X_2^\psi, X_3^\psi)V^\psi &= G(X_1^\psi)R(X_2^\psi, X_3^\psi)V^\psi + H(X_2^\psi)R(X_1^\psi, X_3^\psi)V^\psi \\ &+ I(X_3^\psi)R(X_2^\psi, X_1^\psi)V^\psi + J(V^\psi)R(X_2^\psi, X_3^\psi)X \\ &+ g(R(X_2^\psi, X_3^\psi)V^\psi, X_1^\psi)F,\end{aligned} \tag{2.15}$$

for all vector fields $X_1^\psi, X_2^\psi, X_3^\psi, V^\psi \in \chi(N)$*,* G, H, I, J *are not simultaneously zero and are defined by* $G(X_1^\psi) = g(X_1^\psi, \delta_1)$*,* $H(X_1^\psi) = g(X_1^\psi, \delta_2)$*,* $I(X_1^\psi) = g(X_1^\psi, \delta_3)$ *and* $J(X_1^\psi) = g(X_1^\psi, \delta_4)$*, where* $\delta_1, \delta_2, \delta_3, \delta_4$ *are the arbitrary vector fields corresponding to the 1-forms* G, H, I, J*, respectively, and such a manifold is denoted by* $[(WS)_n, D]$*.*

Definition 2.3. *[26] A Riemannian manifold* (N, g) $(2n+1 > 2)$ *is called weakly Ricci symmetric if there exists a 1-forms* α, β, γ*, and the Ricci tensor* S *is non-zero, which satisfies the condition*

$$(D_{X_1^\psi}S)(X_2^\psi, X_3^\psi) = \alpha(X_1^\psi)S(X_2^\psi, X_3^\psi) + \beta(X_2^\psi)S(X_1^\psi, X_3^\psi) + \gamma(X_3^\psi)S(X_2^\psi, X_1^\psi), \tag{2.16}$$

for all vector fields $X_1^\psi, X_2^\psi, X_3^\psi \in \chi(N)$*, where the 1-forms* α, β*, and* γ *are not simultaneously zero. Such a manifold is denoted by* $[(WRS)_n, D]$*.*

Definition 2.4. *A non-flat Riemannian manifold* (N, g) $(2n+1 > 3)$ *is said to be locally symmetric if the following relation holds*

$$(D_{V^\psi}R)(X_1^\psi, X_2^\psi)X_3^\psi = 0, \tag{2.17}$$

for all vector fields $X_1^\psi, X_2^\psi, X_3^\psi, V^\psi \in \chi(N)$*.*

Definition 2.5. *A non-flat Riemannian manifold* (N, g) $(2n+1 > 3)$ *is said to be locally* ϕ*-symmetric if its curvature tensor* R *satisfies the following relation:*

$$\phi^2((D_{V^\psi}R)(X_1^\psi, X_2^\psi)X_3^\psi) = 0, \tag{2.18}$$

for all vector fields $X_1^\psi, X_2^\psi, X_3^\psi, V^\psi$ *orthogonal to* ζ*.*

Definition 2.6. *A non-flat Riemannian manifold* (N, g) $(2n+1 > 3)$ *is said to be* ϕ*-symmetric if the following relation is satisfied*

$$\phi^2((D_{V^\psi}R)(X_1^\psi, X_2^\psi)X_3^\psi) = 0, \tag{2.19}$$

for all vector fields $X_1^\psi, X_2^\psi, X_3^\psi, V^\psi \in \chi(N)$*.*

Definition 2.7. *[10] A non-flat Riemannian manifold* (N, g) $(2n + 1 \geq 2)$ *is said to be almost pseudo symmetric, denoted by* $(APS)_n$*, if the following relation holds*

$$(D_{X_1^\psi} R)(X_2^\psi, X_3^\psi)V^\psi = [A(X_1^\psi) + B(X_1^\psi)]R(X_2^\psi, X_3^\psi)V^\psi + A(X_2^\psi)R(X_1^\psi, X_3^\psi)V^\psi + A(X_3^\psi)R(X_2^\psi, X_1^\psi)V^\psi + A(V^\psi)R(X_2^\psi, X_3^\psi)X_1^\psi + g(R(X_2^\psi, X_3^\psi)V^\psi, X_1^\psi)P, \tag{2.20}$$

where A and B are two non zero 1-forms defined by $A(X_1^\psi) = g(X_1^\psi, P)$ *and* $B(X_1^\psi) = g(X_1^\psi, L)$*, and P and L are the arbitrary vector fields corresponding to the 1-forms A and B, respectively. If* $A = B$ *in* (2.20)*, then the manifold reduces to a pseudo-symmetric manifold introduced by [6].*

Definition 2.8. *[7] A Riemannian manifold* (N, g) *is called an almost pseudo Ricci symmetric manifold, represented by* $(APRS)_n$*, if its Ricci tensor S of type (0,2) is not identically zero and there exists two non zero 1-forms defined by* $A(X_1^\psi) = g(X_1^\psi, P)$ *and* $B(X_1^\psi) = g(X_1^\psi, L)$ *such that*

$$(D_{X_1^\psi} S)(X_2^\psi, X_3^\psi) = [A(X_1^\psi) + B(X_1^\psi)]S(X_2^\psi, X_3^\psi) + A(X_2^\psi)S(X_1^\psi, X_3^\psi) + A(X_3^\psi)S(X_1^\psi, X_2^\psi), \tag{2.21}$$

for all $X_1^\psi, X_2^\psi, X_3^\psi$ *on* (N, g)*, where P and L are the arbitrary vector fields corresponding to the 1-forms A and B, respectively.*

3. Curvature tensors on K-contact manifold with respect to $NSNMC\ \bar{D}$

Let (N, g) be a $(2n + 1)$-dimensional K-contact manifold admitting an $NSNMC\ \bar{D}$. If the curvature tensor $\bar{R}$ of $\bar{D}$ is defined by

$$\bar{R}(X_1^\psi, X_2^\psi)X_3^\psi = \bar{D}_{X_1^\psi}\bar{D}_{X_2^\psi}X_3^\psi - \bar{D}_{X_2^\psi}\bar{D}_{X_1^\psi}X_3^\psi - \bar{D}_{[X_1^\psi, X_2^\psi]}X_3^\psi,$$

for all X_1^ψ, X_2^ψ, and X_3^ψ on (N, g), then the Riemannian curvature tensor R of the Levi-Civita connection D is defined by

$$R(X_1^\psi, X_2^\psi)X_3^\psi = D_{X_1^\psi}D_{X_2^\psi}X_3^\psi - D_{X_2^\psi}D_{X_1^\psi}X_3^\psi - D_{[X_1^\psi, X_2^\psi]}X_3^\psi,$$

for all X_1^ψ, X_2^ψ, and X_3^ψ on (N, g) and connected by the relation

$$\bar{R}(X_1^\psi, X_2^\psi)X_3^\psi = R(X_1^\psi, X_2^\psi)X_3^\psi + g(X_2^\psi, X_3^\psi)\eta(X_1^\psi)\zeta - g(X_1^\psi, X_3^\psi)\eta(X_2^\psi)\zeta + g(\phi X_1^\psi, X_3^\psi)\phi X_2^\psi - g(\phi X_2^\psi, X_3^\psi)\phi X_1^\psi. \tag{3.1}$$

Taking the inner product of (3.1) with V^ψ, we obtain

$$g(\bar{R}(X_1^\psi, X_2^\psi)X_3^\psi, V^\psi) = g(R(X_1^\psi, X_2^\psi)X_3^\psi, V^\psi) + g(X_2^\psi, X_3^\psi)\eta(X_1^\psi)\eta(V^\psi) - g(X_1^\psi, X_3^\psi)\eta(X_2^\psi)\eta(V^\psi) + g(\phi X_1^\psi, X_3^\psi)g(\phi X_2^\psi, V^\psi) - g(\phi X_2^\psi, X_3^\psi)g(\phi X_1^\psi, V^\psi). \tag{3.2}$$

Contracting equation (3.2) over X_1^ψ and V^ψ, we obtain

$$\bar{S}(X_2^\psi, X_3^\psi) = S(X_2^\psi, X_3^\psi) + g(X_2^\psi, X_3^\psi) - \eta(X_2^\psi)\eta(X_3^\psi). \tag{3.3}$$

Again, contracting equation (3.3) over X_2^ψ and X_3^ψ, we obtain

$$\bar{r} = r + 2n. \tag{3.4}$$

From (3.1), it follows that

$$\bar{R}(X_1^\psi, X_2^\psi)\zeta = \eta(X_2^\psi)X_1^\psi - \eta(X_1^\psi)X_2^\psi, \tag{3.5}$$

$$\bar{R}(\zeta, X_2^\psi)X_3^\psi = 2g(X_2^\psi, X_3^\psi)\zeta - [X_2^\psi + \eta(X_2^\psi)\zeta]\eta(X_3^\psi) = -\bar{R}(X_2^\psi, \zeta)X_3^\psi. \tag{3.6}$$

Also, from (3.3), we are able to obtain

$$\bar{S}(X_2^\psi, \zeta) = 2n\eta(X_2^\psi), \tag{3.7}$$

$$\bar{Q}X_2^\psi = (2n+1)X_2^\psi - \eta(X_2^\psi)\zeta, \tag{3.8}$$

$$\bar{Q}\zeta = 2n\zeta. \tag{3.9}$$

Theorem 3.1. *Let* (N, g) *be a* $(2n+1)$*-dimensional K-contact manifold equipped with* $NSNMC\ \bar{D}$*, then the curvature tensor* $\bar{R}$ *is given by* (3.1)*, the Ricci tensor* $\bar{S}$ *is given by* (3.3)*, the scalar curvature tensor* $\bar{r}$ *is given by* (3.4)*, and* $\bar{S}$ *is symmetric.*

Now, let us consider that a $(2n+1)$-dimensional K-contact manifold (N, g) with respect to $NSNMC\ \bar{D}$ is Ricci flat, that is, $\bar{S} = 0$. Then from (3.3), we have

$$S(X_2^\psi, X_3^\psi) = -g(X_2^\psi, X_3^\psi) + \eta(X_2^\psi)\eta(X_3^\psi).$$

If $X_3^\psi = \zeta$ in the above equation, then

$$S(X_2^\psi, \zeta) = -\eta(X_2^\psi) + \eta(X_2^\psi) = 0,$$

which contradicts Eq (2.9). This means that the manifold cannot be Ricci flat with respect to the $NSNMC\ \bar{D}$. Thus, we can conclude the following:

Theorem 3.2. *A* $(2n+1)$*-dimensional K-contact manifold* (N, g) *cannot be Ricci flat with respect to* $NSNMC\ \bar{D}$*.*

4. Weakly symmetric and weakly Ricci symmetric K-contact manifold admitting $NSNMC\ \bar{D}$

On the basis of Definition 2.2, we define a weakly symmetric K-contact manifold (N, g) of $(2n+1)$-dimension with respect to $NSNMC\ \bar{D}$, whose curvature tensor satisfies the following condition:

$$\begin{aligned}(\bar{D}_{X_1^\psi}\bar{R})(X_2^\psi, X_3^\psi)V^\psi = {} & G(X_1^\psi)\bar{R}(X_2^\psi, X_3^\psi)V^\psi + H(X_2^\psi)\bar{R}(X_1^\psi, X_3^\psi)V^\psi \\ & + I(X_3^\psi)\bar{R}(X_2^\psi, X_1^\psi)V^\psi + J(V^\psi)\bar{R}(X_2^\psi, X_3^\psi)X_1^\psi \\ & + g(\bar{R}(X_2^\psi, X_3^\psi)V^\psi, X_1^\psi)F,\end{aligned} \tag{4.1}$$

for all vector fields $X_1^\psi, X_2^\psi, X_3^\psi, V^\psi \in \chi(N)$. Such a manifold is denoted by $[(WS)_n, \bar{D}]$.

Taking a frame field of (4.1) with respect to X_2^ψ, we obtain

$$\begin{aligned}(\bar{D}_{X_1^\psi}\bar{S})(X_3^\psi, V^\psi) = {} & G(X_1^\psi)\bar{S}(X_3^\psi, V^\psi) + H(\bar{R}(X_1^\psi, X_3^\psi)V^\psi) + I(X_3^\psi)\bar{S}(X_1^\psi, V^\psi) \\ & + J(V^\psi)\bar{S}(X_1^\psi, X_3^\psi) + E(\bar{R}(X_1^\psi, V^\psi)X_3^\psi),\end{aligned} \tag{4.2}$$

where $E(X_1^\psi) = g(X_1^\psi, F)$.

Plugging $V^\psi = \zeta$ and using (3.3)–(3.7) in (4.2), we obtain

$$\begin{aligned} S(\phi X_1^\psi, X_3^\psi) &= (4n-1)g(\phi X_1^\psi, X_3^\psi) + 4n\eta(D_{X_1^\psi} X_3^\psi) + G(X_1^\psi)2n\eta(X_3^\psi) \\ &+ H[\eta(X_3^\psi)X_1^\psi - \eta(X_1^\psi)X_3^\psi] + I(X_3^\psi)2n\eta(X_1^\psi) \\ &+ J(\zeta)[S(X_1^\psi, X_3^\psi) + g(X_1^\psi, X_3^\psi) - \eta(X_1^\psi)\eta(X_3^\psi)] \\ &- 2E(\zeta)g(X_1^\psi, X_3^\psi) + E(X_1^\psi)\eta(X_3^\psi) + E(\zeta)\eta(X_1^\psi)\eta(X_3^\psi). \end{aligned} \tag{4.3}$$

Again, setting $X_1^\psi = X_3^\psi = \zeta$ and using (2.1), (2.3), (2.4), and (2.9) in (4.3), we have

$$2n[G(\zeta) + I(\zeta) + J(\zeta)] = 0, \tag{4.4}$$

since $2n \neq 0$. Therefore, Eq (4.4) becomes

$$G(\zeta) + I(\zeta) + J(\zeta) = 0.$$

Now, replacing X_3^ψ by ζ in (4.2), we obtain

$$\begin{aligned} (\bar{D}_{X_1^\psi}\bar{S})(\zeta, V^\psi) &= G(X_1^\psi)\bar{S}(\zeta, V^\psi) + H(\bar{R}(X_1^\psi, \zeta)V^\psi) + I(\zeta)\bar{S}(X_1^\psi, V^\psi) \\ &+ J(V^\psi)\bar{S}(X_1^\psi, \zeta) + E(\bar{R}(X_1^\psi, V^\psi)\zeta). \end{aligned} \tag{4.5}$$

Also,

$$(\bar{D}_{X_1^\psi}\bar{S})(\zeta, V^\psi) = (1-4n)g(\phi X_1^\psi, V^\psi) + S(\phi X_1^\psi, V^\psi), \tag{4.6}$$

$$G(X_1^\psi)\bar{S}(\zeta, V^\psi) = G(X_1^\psi)2n\eta(V^\psi), \tag{4.7}$$

$$H(\bar{R}(X_1^\psi, \zeta)V^\psi) = H(\zeta)[\eta(X_1^\psi)\eta(V^\psi) - 2g(X_1^\psi, V^\psi)] + H(X_1^\psi)\eta(V^\psi), \tag{4.8}$$

$$I(\zeta)\bar{S}(X_1^\psi, V^\psi) = I(\zeta)[S(X_1^\psi, V^\psi) + g(X_1^\psi, V^\psi) - \eta(X_1^\psi)\eta(V^\psi)], \tag{4.9}$$

$$J(V^\psi)\bar{S}(X_1^\psi, \zeta) = J(V^\psi)2n\eta(X_1^\psi), \tag{4.10}$$

$$E(\bar{R}(X_1^\psi, V^\psi)\zeta) = E(X_1^\psi)\eta(V^\psi) - E(V^\psi)\eta(X_1^\psi). \tag{4.11}$$

Applying (4.6) to (4.11) in (4.5), we obtain

$$\begin{aligned} S(\phi X_1^\psi, V^\psi) &= (4n-1)g(\phi X_1^\psi, V^\psi) + G(X_1^\psi)2n\eta(V^\psi) \\ &+ H(\zeta)[\eta(X_1^\psi)\eta(V^\psi) - 2g(X_1^\psi, V^\psi)] \\ &+ H(X_1^\psi)H(V^\psi) + J(V^\psi)2n\eta(X_1^\psi) \\ &+ I(\zeta)[S(X_1^\psi, V^\psi) + g(X_1^\psi, V^\psi) - \eta(X_1^\psi)\eta(V^\psi)] \\ &+ E(X_1^\psi)\eta(V^\psi) - E(V^\psi)\eta(X_1^\psi). \end{aligned} \tag{4.12}$$

Setting $V^\psi = \zeta$ and with the help of (2.1), (2.3), and (2.9) in (4.12), we obtain

$$0 = 2nG(X_1^\psi) + H(X_1^\psi) + [2n\{I(\zeta) + J(\zeta)\} - H(\zeta)]\eta(X_1^\psi) + E[X_1^\psi - \eta(X_1^\psi)\zeta]. \tag{4.13}$$

Similarly, putting $X_1^\psi = \zeta$ and using (2.1), (2.3), and (2.9) in (4.12), we have

$$0 = 2nJ(V^\psi) + 2n[G(\zeta) + I(\zeta)]\eta(V^\psi) - E[V^\psi - \eta(V^\psi)\zeta]. \quad (4.14)$$

Replacing V^ψ by X_1^ψ in (4.14), we are able to obtain

$$0 = 2nJ(X_1^\psi) + 2n[G(\zeta) + I(\zeta)]\eta(X_1^\psi) - E[X_1^\psi - \eta(X_1^\psi)\zeta]. \quad (4.15)$$

Adding (4.13) and (4.15), it results in

$$0 = 2n[G(X_1^\psi) + J(X_1^\psi)] + H(X_1^\psi) + [2n\{G(\zeta) + 2I(\zeta) + J(\zeta)\} - H(\zeta)]\eta(X_1^\psi). \quad (4.16)$$

Again, plugging $X_1^\psi = \zeta$ in (4.3), we obtain

$$0 = 2nG(\zeta)\eta(X_3^\psi) + H[\eta(X_3^\psi)\zeta - X_3^\psi] + 2nI(X_3^\psi) + 2nJ(\zeta)\eta(X_3^\psi). \quad (4.17)$$

Changing X_3^ψ by X_1^ψ in (4.17), we have

$$0 = 2nG(\zeta)\eta(X_1^\psi) + H[\eta(X_1^\psi)\zeta - X_1^\psi] + 2nI(X_1^\psi) + 2nJ(\zeta)\eta(X_1^\psi). \quad (4.18)$$

Summing (4.16) and (4.18), we obtain

$$2n[G(X_1^\psi) + I(X_1^\psi) + J(X_1^\psi)] = 0, \quad (4.19)$$

since $2n \neq 0$. Therefore, (4.19) implies

$$G(X_1^\psi) + I(X_1^\psi) + J(X_1^\psi) = 0. \quad (4.20)$$

Thus, this results in the statement:

Theorem 4.1. *There is no weakly symmetric K-contact manifold* (N, g) $(2n + 1 > 3)$ *admitting a non-symmetric non-metric connection NSNMC* $\bar{D}$ *denoted by* $[(WRS)_n, \bar{D}]$*, unless* $G + I + J$ *vanishes everywhere.*

Following from Definition 2.3, we define a $(2n+1)$-dimensional weakly Ricci symmetric K-contact manifold (N, g) with respect to a $NSNMC$ $\bar{D}$ if there exist 1-forms α, β, γ, and the Ricci tensor $\bar{S}$ is non-zero, which satisfies the condition

$$(\bar{D}_{X_1^\psi}\bar{S})(X_2^\psi, X_3^\psi) = \alpha(X_1^\psi)\bar{S}(X_2^\psi, X_3^\psi) + \beta(X_2^\psi)\bar{S}(X_1^\psi, X_3^\psi) + \gamma(X_3^\psi)\bar{S}(X_2^\psi, X_1^\psi), \quad (4.21)$$

for all vector fields $X_1^\psi, X_2^\psi, X_3^\psi \in \chi(N)$, where the 1-forms α, β, and γ are not simultaneously zero. Such a manifold is denoted by $[(WRS)_n, \bar{D}]$.

Setting $X_3^\psi = \zeta$ in (4.21), we have

$$(\bar{D}_{X_1^\psi}\bar{S})(X_2^\psi, \zeta) = \alpha(X_1^\psi)\bar{S}(X_2^\psi, \zeta) + \beta(X_2^\psi)\bar{S}(X_1^\psi, \zeta) + \gamma(\zeta)\bar{S}(X_2^\psi, X_1^\psi), \quad (4.22)$$

where

$$(\bar{D}_{X_1^\psi}\bar{S})(X_2^\psi, \zeta) = \bar{D}_{X_1^\psi}\bar{S}(X_2^\psi, \zeta) - \bar{S}(\bar{D}_{X_1^\psi}Y, \zeta) - \bar{S}(X_2^\psi, \bar{D}_{X_1^\psi}\zeta). \quad (4.23)$$

Using (2.12), (3.3), (3.7), and (4.23), we obtain

$$S(\phi X_1^\psi, X_2^\psi) = (4n-1)g(\phi X_1^\psi, X_2^\psi) + 2n[\alpha(X_1^\psi)\eta(X_2^\psi) + \beta(X_2^\psi)\eta(X_1^\psi)] \\ + \gamma(\zeta)[S(X_1^\psi, X_2^\psi) + g(X_1^\psi, X_2^\psi) - \eta(X_1^\psi)\eta(X_2^\psi)]. \tag{4.24}$$

Setting $X_1^\psi = X_2^\psi = \zeta$ and using (2.1), (2.3), and (2.9) in (4.24), we obtain

$$2n[\alpha(\zeta) + \beta(\zeta) + \gamma(\zeta)] = 0, \tag{4.25}$$

since $2n \neq 0$. Therefore, (4.25) becomes

$$\alpha(\zeta) + \beta(\zeta) + \gamma(\zeta) = 0. \tag{4.26}$$

Setting $X_1^\psi = \zeta$ in (4.24), we obtain

$$\alpha(\zeta)\eta(X_2^\psi) + \beta(X_2^\psi) + \gamma(\zeta)\eta(X_2^\psi) = 0. \tag{4.27}$$

Changing X_2^ψ by X_1^ψ in (4.27), we obtain

$$\alpha(\zeta)\eta(X_1^\psi) + \beta(X_1^\psi) + \gamma(\zeta)\eta(X_1^\psi) = 0. \tag{4.28}$$

From (4.26) and (4.28), we have

$$\beta(X_1^\psi) = \beta(\zeta)\eta(X_1^\psi). \tag{4.29}$$

Again, setting $X_2^\psi = \zeta$ and using (2.1), (2.3), and (2.9) in (4.24), we obtain

$$\alpha(X_1^\psi) + \beta(\zeta)\eta(X_1^\psi) + \gamma(\zeta)\eta(X_1^\psi) = 0. \tag{4.30}$$

From (4.26) and (4.30), we obtain

$$\alpha(X_1^\psi) = \alpha(\zeta)\eta(X_1^\psi). \tag{4.31}$$

Moreover, putting $X_1^\psi = X_2^\psi = \zeta$ in (4.21), we obtain

$$\alpha(\zeta)\eta(X_3^\psi) + \beta(\zeta)\eta(X_3^\psi) + \gamma(Z) = 0. \tag{4.32}$$

Replacing X_3^ψ by X_1^ψ in (4.32), we obtain

$$\alpha(\zeta)\eta(X_1^\psi) + \beta(\zeta)\eta(X_1^\psi) + \gamma(X_1^\psi) = 0. \tag{4.33}$$

Again, from (4.26) and (4.33), we obtain

$$\gamma(X_1^\psi) = \gamma(\zeta)\eta(X_1^\psi). \tag{4.34}$$

Finally, adding (4.29), (4.31), and (4.34), and with the help of (4.26), we have

$$\alpha(X_1^\psi) + \beta(X_1^\psi) + \gamma(X_1^\psi) = 0. \tag{4.35}$$

Therefore, from (4.35), we can state the following:

Theorem 4.2. *There exists no weakly Ricci-symmetric K-contact manifold* (N, g) $(2n+1 > 3)$ *admitting a non-symmetric non-metric connection NSNMC* $\bar{D}$ *denoted by* $[(WRS)_n, \bar{D}]$*, unless* $\alpha+\beta+\gamma$ *vanishes everywhere.*

5. Locally symmetric and ϕ-symmetric K-contact manifold with respect to $NSNMC\ \bar{D}$

In this section, based on Definition 2.4, we define the following:

Definition 5.1. *A $(2n+1)$-dimensional K-contact manifold (N,g) with respect to $NSNMC\ \bar{D}$ is said to be locally symmetric if the curvature tensor $\bar{R}$ satisfies*

$$(\bar{D}_{V^\psi}\bar{R})(X_1^\psi, X_2^\psi)X_3^\psi = 0, \tag{5.1}$$

for all vector fields $X_1^\psi, X_2^\psi, X_3^\psi, V^\psi \in \chi(N)$.

Contracting (5.1) over X_1^ψ, we obtain

$$(\bar{D}_{V^\psi}\bar{S})(X_2^\psi, X_3^\psi) = \bar{D}_{V^\psi}\bar{S}(X_2^\psi, X_3^\psi) - \bar{S}(\bar{D}_{V^\psi}X_2^\psi, X_3^\psi) - \bar{S}(X_2^\psi, \bar{D}_{V^\psi}X_3^\psi) = 0. \tag{5.2}$$

Setting $X_3^\psi = \zeta$ in (5.2), we obtain

$$\bar{D}_{V^\psi}\bar{S}(X_2^\psi, \zeta) - \bar{S}(\bar{D}_{V^\psi}X_2^\psi, \zeta) - \bar{S}(X_2^\psi, \bar{D}_{V^\psi}\zeta) = 0. \tag{5.3}$$

In view of (3.1), (3.3), (3.7), and (5.3), we obtain

$$S(X_2^\psi, \phi V^\psi) = (4n-1)g(X_2^\psi, \phi V^\psi). \tag{5.4}$$

Replacing V^ψ by ϕV^ψ and using (2.1) and (2.9) in (5.4), we have

$$S(X_2^\psi, V^\psi) = (4n-1)g(X_2^\psi, V^\psi) + (1-2n)\eta(X_2^\psi)\eta(V^\psi). \tag{5.5}$$

Therefore, (5.5) satisfies Definition 2.1. Thus, the theorem states:

Theorem 5.2. *Let (N,g) be a $(2n+1)$-dimensional locally symmetric K-contact manifold with respect to $NSNMC\ \bar{D}$. Then the manifold (N,g) is an η-Einstein manifold.*

Next, we will discuss for locally ϕ-symmetric K-contact manifold admitting $NSNMC\ \bar{D}$. Therefore, we state the following:

Definition 5.3. *A $(2n+1)$-dimensional K-contact manifold (N,g) with respect to $NSNMC\ \bar{D}$ is said to be locally ϕ-symmetric if its curvature tensor $\bar{R}$ satisfies the following relation*

$$\phi^2((\bar{D}_{V^\psi}\bar{R})(X_1^\psi, X_2^\psi)X_3^\psi) = 0, \tag{5.6}$$

for all vector fields $X_1^\psi, X_2^\psi, X_3^\psi, V^\psi$ orthogonal to ζ.

From (2.12) and (3.1), we have

$$(\bar{D}_{V^\psi}\bar{R})(X_1^\psi, X_2^\psi)X_3^\psi = (D_{V^\psi}\bar{R})(X_1^\psi, X_2^\psi)X_3^\psi + g(\phi V^\psi, \bar{R}(X_1^\psi, X_2^\psi)X_3^\psi)\zeta. \tag{5.7}$$

Now, differentiating (3.1) covariantly with respect to V^ψ, we have

$$\begin{aligned}(D_{V^\psi}\bar{R})(X_1^\psi, X_2^\psi)X_3^\psi &= (D_{V^\psi}R)(X_1^\psi, X_2^\psi)X_3^\psi + (D_{V^\psi}g)(X_2^\psi, X_3^\psi)\eta(X_1^\psi)\zeta \\ &+ g(X_2^\psi, X_3^\psi)g(\phi X_1^\psi, V^\psi)\zeta - g(X_2^\psi, X_3^\psi)\eta(X_1^\psi)\phi V^\psi \\ &- (D_{V^\psi}g)(X_1^\psi, X_3^\psi)\eta(X_2^\psi)\zeta - g(X_1^\psi, X_3^\psi)g(\phi X_2^\psi, V^\psi)\zeta \\ &+ g(X_1^\psi, X_3^\psi)\eta(X_2^\psi)\phi V^\psi + (D_{V^\psi}g)(\phi X_1^\psi, X_3^\psi)\phi X_2^\psi \\ &+ \eta(X_3^\psi)g(X_1^\psi, V^\psi)\phi X_2^\psi - \eta(X_1^\psi)g(X_3^\psi, V^\psi)\phi X_2^\psi \\ &+ g(X_2^\psi, V^\psi)g(\phi X_1^\psi, X_3^\psi)\zeta - \eta(X_2^\psi)g(\phi X_1^\psi, X_3^\psi)V^\psi \\ &+ g(\phi X_1^\psi, X_3^\psi)\phi(D_{V^\psi}X_2^\psi) - (D_{V^\psi}g)(\phi X_2^\psi, X_3^\psi)\phi X_1^\psi \\ &- \eta(X_3^\psi)g(X_2^\psi, V^\psi)\phi X_1^\psi + \eta(X_2^\psi)g(X_3^\psi, V^\psi)\phi X_1^\psi \\ &- g(X_1^\psi, V^\psi)g(\phi X_2^\psi, X_3^\psi)\zeta + \eta(X_1^\psi)g(\phi X_2^\psi, X_3^\psi)V^\psi.\end{aligned} \tag{5.8}$$

Inserting (2.1), (2.2), (3.1), and (5.8) in (5.7), we obtain

$$\begin{aligned}(\bar{D}_{V^\psi}\bar{R})(X_1^\psi, X_2^\psi)X_3^\psi &= (D_{V^\psi}R)(X_1^\psi, X_2^\psi)X_3^\psi + (D_{V^\psi}g)(X_2^\psi, X_3^\psi)\eta(X_1^\psi)\zeta \\ &+ g(X_2^\psi, X_3^\psi)g(\phi X_1^\psi, V^\psi)\zeta - g(X_2^\psi, X_3^\psi)\eta(X_1^\psi)\phi V^\psi \\ &- (D_{V^\psi}g)(X_1^\psi, X_3^\psi)\eta(X_2^\psi)\zeta - g(X_1^\psi, X_3^\psi)g(\phi X_2^\psi, V^\psi)\zeta \\ &+ g(X_1^\psi, X_3^\psi)\eta(X_2^\psi)\phi V^\psi + (D_{V^\psi}g)(\phi X_1^\psi, X_3^\psi)\phi X_2^\psi \\ &+ \eta(X_3^\psi)g(X_1^\psi, V^\psi)\phi X_2^\psi - \eta(X_1^\psi)g(X_3^\psi, V^\psi)\phi X_2^\psi \\ &+ g(X_2^\psi, V^\psi)g(\phi X_1^\psi, X_3^\psi)\zeta - \eta(X_2^\psi)g(\phi X_1^\psi, X_3^\psi)V^\psi \\ &+ g(\phi X_1^\psi, X_3^\psi)\phi(D_{V^\psi}X_2^\psi) - (D_{V^\psi}g)(\phi X_2^\psi, X_3^\psi)\phi X_1^\psi \\ &- \eta(X_3^\psi)g(X_2^\psi, V^\psi)\phi X_1^\psi + \eta(X_2^\psi)g(X_3^\psi, V^\psi)\phi X_1^\psi \\ &- g(X_1^\psi, V^\psi)g(\phi X_2^\psi, X_3^\psi)\zeta + \eta(X_1^\psi)g(\phi X_2^\psi, X_3^\psi)V^\psi \\ &+ g(X_2^\psi, X_3^\psi)g(X_1^\psi, \phi V^\psi)\zeta - g(X_1^\psi, X_3^\psi)g(X_2^\psi, \phi V^\psi)\zeta \\ &+ g(X_2^\psi, V^\psi)g(\phi X_1^\psi, X_3^\psi)\zeta - g(\phi X_1^\psi, X_3^\psi)\eta(X_2^\psi)\eta(V^\psi)\zeta \\ &- g(X_1^\psi, V^\psi)g(\phi X_2^\psi, X_3^\psi)\zeta + g(\phi X_2^\psi, X_3^\psi)\eta(X_1^\psi)\eta(V^\psi)\zeta.\end{aligned} \tag{5.9}$$

Applying ϕ^2 on both sides of (5.9), we obtain

$$\begin{aligned}\phi^2((\bar{D}_{V^\psi}\bar{R})(X_1^\psi, X_2^\psi)X_3^\psi) &= \phi^2((D_{V^\psi}R)(X_1^\psi, X_2^\psi)X_3^\psi) - g(X_2^\psi, X_3^\psi)\eta(X_1^\psi)\phi^2(\phi V^\psi) \\ &+ g(X_1^\psi, X_3^\psi)\eta(X_2^\psi)\phi^2(\phi V^\psi) + (D_{V^\psi}g)(\phi X_1^\psi, X_3^\psi)\phi^2(\phi X_2^\psi) \\ &+ \eta(X_3^\psi)g(X_1^\psi, V^\psi)\phi^2(\phi X_2^\psi) - \eta(X_1^\psi)g(X_3^\psi, V^\psi)\phi^2(\phi X_2^\psi) \\ &- \eta(X_2^\psi)g(\phi X_1^\psi, X_3^\psi)\phi^2 V^\psi + g(\phi X_1^\psi, X_3^\psi)\phi^2(\phi(D_{V^\psi}X_2^\psi)) \\ &- (D_{V^\psi}g)(\phi X_2^\psi, X_3^\psi)\phi^2(\phi X_1^\psi) - \eta(X_3^\psi)g(X_2^\psi, V^\psi)\phi^2(\phi X_1^\psi) \\ &+ \eta(X_2^\psi)g(X_3^\psi, V^\psi)\phi^2(\phi X_1^\psi) + \eta(X_1^\psi)g(\phi X_2^\psi, X_3^\psi)\phi^2 V^\psi.\end{aligned} \tag{5.10}$$

Taking $X_1^\psi, X_2^\psi, X_3^\psi$, and V^ψ as orthogonal to ζ, then (5.10) yields

$$\phi^2((\bar{D}_{V^\psi}\bar{R})(X_1^\psi, X_2^\psi)X_3^\psi) = \phi^2((D_{V^\psi}R)(X_1^\psi, X_2^\psi)X_3^\psi) + \mu, \tag{5.11}$$

where
$$\mu = (D_{V^\psi} g)(\phi X_2^\psi, X_3^\psi)\phi X_1^\psi - (D_{V^\psi} g)(\phi X_1^\psi, X_3^\psi)\phi X_2^\psi - g(\phi X_1^\psi, X_3^\psi)\phi(D_{V^\psi} X_2^\psi).$$
From (5.11), we thus arrive at the following theorem:

Theorem 5.4. *A $(2n+1)$-dimensional K-contact manifold (N, g) admitting $NSNMC$ $\bar{D}$ is not locally ϕ-symmetric, unless μ vanishes everywhere.*

Further, let us consider a ϕ-symmetric K-contact manifold equipped with $NSNMC$ $\bar{D}$. Then the following definition states:

Definition 5.5. *A $(2n+1)$-dimensional K-contact manifold (N, g) with respect to $NSNMC$ $\bar{D}$ is said to be ϕ-symmetric if the curvature tensor $\bar{R}$ of $\bar{D}$ takes the form*

$$\phi^2((\bar{D}_{V^\psi}\bar{R})(X_1^\psi, X_2^\psi)X_3^\psi) = 0, \tag{5.12}$$

for all vector fields $X_1^\psi, X_2^\psi, X_3^\psi, V^\psi \in \chi(N)$.

By virtue of (2.1) and (5.12), we have

$$-((\bar{D}_{V^\psi}\bar{R})(X_1^\psi, X_2^\psi)X_3^\psi) + \eta((\bar{D}_{V^\psi}\bar{R})(X_1^\psi, X_2^\psi)X_3^\psi)\zeta = 0. \tag{5.13}$$

The inner product of (5.13) with V^ψ gives

$$-g((\bar{D}_{V^\psi}\bar{R})(X_1^\psi, X_2^\psi)X_3^\psi, V^\psi) + \eta((\bar{D}_{V^\psi}\bar{R})(X_1^\psi, X_2^\psi)X_3^\psi)g(\zeta, V^\psi) = 0. \tag{5.14}$$

Let $\{e_i\}$, $(i = 1, 2, ..., 2n+1)$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X_1^\psi = V^\psi = e_i$ in (5.14) and taking summation over i, $1 \le i \le 2n+1$, we have

$$-(\bar{D}_{V^\psi}\bar{S})(X_2^\psi, X_3^\psi) + \sum_{i=1}^{2n+1} \eta((\bar{D}_{V^\psi}\bar{R})(e_i, X_2^\psi)X_3^\psi)g(\zeta, e_i) = 0. \tag{5.15}$$

Replacing X_3^ψ by ζ in (5.15), we obtain

$$-(\bar{D}_{V^\psi}\bar{S})(X_2^\psi, \zeta) + \sum_{i=1}^{2n+1} \eta((\bar{D}_{V^\psi}\bar{R})(e_i, X_2^\psi)\zeta)\eta(e_i) = 0. \tag{5.16}$$

The second term of (5.16) takes the form

$$\begin{aligned}\eta((\bar{D}_{V^\psi}\bar{R})(e_i, X_2^\psi)\zeta)\eta(e_i) = {} & [g(\bar{D}_{V^\psi}\bar{R}(e_i, X_2^\psi)\zeta, \zeta) - g(\bar{R}(\bar{D}_{V^\psi}e_i, X_2^\psi)\zeta, \zeta) \\ & - g(\bar{R}(e_i, \bar{D}_{V^\psi}X_2^\psi)\zeta, \zeta) - g(\bar{R}(e_i, X_2^\psi)\bar{D}_{V^\psi}\zeta, \zeta)]\eta(e_i).\end{aligned} \tag{5.17}$$

Since $\{e_i\}$ is an orthonormal basis, therefore, $\bar{D}_{V^\psi}e_i = 0$ at the point of the manifold.

Also, from (2.7), we obtain

$$g(\bar{R}(e_i, \bar{D}_{V^\psi}X_2^\psi)\zeta, \zeta) = 0. \tag{5.18}$$

Using (5.18) in (5.17), we have

$$\eta((\bar{D}_{V^\psi}\bar{R})(e_i, X_2^\psi)\zeta) = g(\bar{D}_{V^\psi}\bar{R}(e_i, X_2^\psi)\zeta, \zeta) - g(\bar{R}(e_i, X_2^\psi)\bar{D}_{V^\psi}\zeta, \zeta). \tag{5.19}$$

Since

$$g(\bar{R}(e_i, X_2^\psi)\zeta, \zeta) = -g(\bar{R}(\zeta, \zeta)X_2^\psi, e_i) = 0.$$

Therefore, using the above equation and (5.19), Eq (5.17) becomes

$$g(\bar{D}_{V^\psi}\bar{R}(e_i, X_2^\psi)\zeta, \zeta) + g(\bar{R}(e_i, X_2^\psi)\zeta, \bar{D}_{V^\psi}\zeta) = 0. \tag{5.20}$$

Inserting (5.20) in (5.19), we obtain

$$g((\bar{D}_{V^\psi}\bar{R})(e_i, X_2^\psi)\zeta, \zeta) = -g(\bar{R}(e_i, X_2^\psi)\zeta, \bar{D}_{V^\psi}\zeta) - g(\bar{R}(e_i, X_2^\psi)\bar{D}_{V^\psi}\zeta, \zeta). \tag{5.21}$$

From (2.12), we are able to obtain

$$\bar{D}_{V^\psi}\zeta = -\phi V^\psi. \tag{5.22}$$

Again, with the insertion of (5.22) in (5.21), we have

$$g((\bar{D}_{V^\psi}\bar{R})(e_i, X_2^\psi)\zeta, \zeta) = 0. \tag{5.23}$$

Thus, (5.16) and (5.22) produce $(\bar{D}_{V^\psi}\bar{S})(X_2^\psi, \zeta) = 0$, which yields

$$\bar{D}_{V^\psi}\bar{S}(X_2^\psi, \zeta) - \bar{S}(\bar{D}_{V^\psi}X_2^\psi, \zeta) - \bar{S}(X_2^\psi, \bar{D}_{V^\psi}\zeta) = 0. \tag{5.24}$$

Applying (3.3), (3.7), and (5.22) in (5.24), we obtain

$$S(X_2^\psi, \phi V^\psi) = (4n-1)g(X_2^\psi, \phi V^\psi). \tag{5.25}$$

Substituting X_2^ψ by ϕX_2^ψ and using (2.2) and (2.16) in (5.25), we obtain

$$S(X_2^\psi, V^\psi) = (4n-1)g(X_2^\psi, V^\psi) + (1-2n)\eta(X_2^\psi)\eta(V^\psi).$$

Consequently, the statement can be written as:

Theorem 5.6. *A $(2n+1)$-dimensional ϕ-symmetric K-contact manifold (N, g) with respect to a $NSNMC$ $\bar{D}$ is an η-Einstein manifold.*

6. Almost pseudo symmetric K-contact manifold with respect to $NSNMC$ $\bar{D}$

The following definition states:

Definition 6.1. *A K-contact manifold (N, g) $(2n+1 \geq 2)$ is said to be almost pseudo symmetric with respect to $NSNMC$ $\bar{D}$ denoted by $[(APS)_n, \bar{D}]$, if there exists a 1-forms A and B and vector fields P and L such that*

$$\begin{aligned}(\bar{D}_{X_1^\psi}\bar{R})(X_2^\psi, X_3^\psi)V^\psi &= [A(X_1^\psi) + B(X_1^\psi)]\bar{R}(X_2^\psi, X_3^\psi)V^\psi \\ &+ A(X_2^\psi)\bar{R}(X_1^\psi, X_3^\psi)V^\psi \\ &+ A(X_3^\psi)\bar{R}(X_2^\psi, X_1^\psi)V^\psi \\ &+ A(V^\psi)\bar{R}(X_2^\psi, X_3^\psi)X_1^\psi \\ &+ g(\bar{R}(X_2^\psi, X_3^\psi)V^\psi, X_1^\psi)P,\end{aligned} \tag{6.1}$$

where $A(X_1^\psi) = g(X_1^\psi, P)$ and $B(X_1^\psi) = g(X_1^\psi, L)$, respectively.

Taking the inner product of (6.1) with U^{ψ}, we obtain

$$\begin{aligned} g((\bar{D}_{X_1^{\psi}}\bar{R})(X_2^{\psi}, X_3^{\psi}), U^{\psi}) = {} & [A(X_1^{\psi}) + B(X_1^{\psi})]g(\bar{R}(X_2^{\psi}, X_3^{\psi})V^{\psi}, U^{\psi}) \\ & + A(X_2^{\psi})g(\bar{R}(X_1^{\psi}, X_3^{\psi})V^{\psi}, U^{\psi}) \\ & + A(X_3^{\psi})g(\bar{R}(X_2^{\psi}, X_1^{\psi})V^{\psi}, U^{\psi}) \\ & + A(V^{\psi})g(\bar{R}(X_2^{\psi}, X_3^{\psi})X_1^{\psi}, U^{\psi}) \\ & + g(\bar{R}(X_2^{\psi}, X_3^{\psi})V^{\psi}, X_1^{\psi})g(P, U^{\psi}). \end{aligned}$$

Contracting the above equation over X_2^{ψ} and U^{ψ}, we obtain

$$\begin{aligned} (\bar{D}_{X_1^{\psi}}\bar{S})(X_3^{\psi}, V^{\psi}) = {} & [A(X_1^{\psi}) + B(X_1^{\psi})]\bar{S}(X_3^{\psi}, V^{\psi}) + A(\bar{R}(X_1^{\psi}, X_3^{\psi})V^{\psi}) \\ & + A(X_3^{\psi})\bar{S}(X_1^{\psi}, V^{\psi}) + A(V^{\psi})\bar{S}(X_3^{\psi}, X_1^{\psi}) \\ & + A(\bar{R}(X_2^{\psi}, V^{\psi})X_3^{\psi}). \end{aligned} \tag{6.2}$$

Arranging $V^{\psi} = \zeta$ and using (3.3) and (3.5)–(3.7) in (6.2), we obtain

$$\begin{aligned} (\bar{D}_{X_1^{\psi}}\bar{S})(X_3^{\psi}, \zeta) = {} & 2[(n+1)A(X_1^{\psi}) + nB(X_1^{\psi})]\eta(X_3^{\psi}) + (2n-1)A(X_3^{\psi})\eta(X_1^{\psi}) \\ & + A(\zeta)[S(X_1^{\psi}, X_3^{\psi}) - g(X_1^{\psi}, X_3^{\psi})]. \end{aligned} \tag{6.3}$$

It is known that

$$(\bar{D}_{X_1^{\psi}}\bar{S})(X_3^{\psi}, X_2^{\psi}) = \bar{D}_{X_1^{\psi}}\bar{S}(X_3^{\psi}, X_2^{\psi}) - \bar{S}(\bar{D}_{X_1^{\psi}}X_3^{\psi}, X_2^{\psi}) - \bar{S}(X_3^{\psi}, \bar{D}_{X_1^{\psi}}X_2^{\psi}). \tag{6.4}$$

Replacing X_2^{ψ} by ζ and using (3.3), (3.7), and (5.22) in (6.4), we have

$$(\bar{D}_{X_1^{\psi}}\bar{S})(X_3^{\psi}, \zeta) = (1-4n)g(\phi X_1^{\psi}, X_3^{\psi}) + S(\phi X_1^{\psi}, X_3^{\psi}). \tag{6.5}$$

In view of (6.3) and (6.5), we obtain

$$\begin{aligned} S(\phi X_1^{\psi}, X_3^{\psi}) = {} & (4n-1)g(\phi X_1^{\psi}, X_3^{\psi}) + 2[(n+1)A(X_1^{\psi}) + nB(X_1^{\psi})]\eta(X_3^{\psi}) \\ & + (2n-1)A(X_3^{\psi})\eta(X_1^{\psi}) + A(\zeta)[S(X_1^{\psi}, X_3^{\psi}) - g(X_1^{\psi}, X_3^{\psi})]. \end{aligned} \tag{6.6}$$

Substituting $X_1^{\psi} = X_3^{\psi} = \zeta$ and using (2.1) and (2.10) in (6.6), we obtain

$$2n[3A(\zeta) + B(\zeta)] = 0. \tag{6.7}$$

Similarly, setting $X_3^{\psi} = \zeta$ and with the help of (6.5) in (6.2), we have

$$\begin{aligned} S(\phi X_1^{\psi}, V^{\psi}) = {} & (4n-1)g(\phi X_1^{\psi}, V^{\psi}) + 2[(n+1)A(X_1^{\psi}) + nB(X_1^{\psi})]\eta(V^{\psi}) \\ & + A(\zeta)S(X_1^{\psi}, V^{\psi}) + (2n-1)A(V^{\psi})\eta(X_1^{\psi}) - A(\zeta)g(X_1^{\psi}, V^{\psi}). \end{aligned} \tag{6.8}$$

Taking $X_1^{\psi} = \zeta$ and inserting (2.1) and (2.9) in (6.8), we obtain

$$[(4n+1)A(\zeta) + 2nB(\zeta)]\eta(V^{\psi}) + (2n-1)A(V^{\psi}) = 0. \tag{6.9}$$

Again, putting $V^{\psi} = \zeta$ and using (2.1) and (2.9) in (6.8), we obtain

$$2[(2n-1)A(\zeta)\eta(X_1^{\psi}) + (n+1)A(X_1^{\psi}) + nB(X_1^{\psi})] = 0. \tag{6.10}$$

The addition of (6.9) and (6.10) becomes

$$\begin{aligned}&[(4n+1)A(\zeta) + 2nB(\zeta)]\eta(V^{\psi}) + (2n-1)A(V^{\psi})\\ &+ 2(2n-1)A(\zeta)\eta(X_1^{\psi}) + 2(n+1)A(X_1^{\psi}) + 2nB(X_1^{\psi}) = 0.\end{aligned} \tag{6.11}$$

Changing V^{ψ} by X_1^{ψ} in (6.11), we obtain

$$[(8n-1)A(\zeta) + 2nB(\zeta)]\eta(X_1^{\psi}) + (4n+1)A(X_1^{\psi}) + 2nB(X_1^{\psi}) = 0. \tag{6.12}$$

Furthermore, replacing V^{ψ} by X_1^{ψ} in (6.9), we have

$$[(4n+1)A(\zeta) + 2nB(\zeta)]\eta(X_1^{\psi}) + (2n-1)A(X_1^{\psi}) = 0. \tag{6.13}$$

Again, adding (6.12) and (6.13), we obtain

$$4n[3A(\zeta) + B(\zeta)]\eta(X_1^{\psi}) + 2n[3A(X_1^{\psi}) + B(X_1^{\psi})] = 0. \tag{6.14}$$

Now, with the help of (6.7), Eq (6.14) becomes

$$2n[3A(X_1^{\psi}) + B(X_1^{\psi})] = 0,$$

since $2n \neq 0$. Therefore, the above equation yields

$$3A(X_1^{\psi}) + B(X_1^{\psi}) = 0.$$

As a result, we conclude that:

Theorem 6.2. *There exists no almost pseudo symmetric K-contact manifold* (N, g) *of* $(2n+1)$*-dimension admitting* $NSNMC$ $\bar{D}$*, unless* $3A + B$ *vanishes everywhere.*

7. Almost pseudo Ricci symmetric K-contact manifold with respect to $NSNMC$ $\bar{D}$

We define the following definition as follows:

Definition 7.1. *A* $(2n+1)$*-dimensional K-contact manifold* (N, g) $(2n+1 \geq 2)$ *is said to be an almost pseudo Ricci symmetric with respect to* $NSNMC$ $\bar{D}$*, denoted by* $[(APRS)_n, \bar{D}]$*, if its Ricci tensor* $\bar{S}$ *of* $\bar{D}$ *is not identically zero and there exists two non-zero 1-forms defined by* $A(X_1^{\psi}) = g(X_1^{\psi}, P)$ *and* $B(X_1^{\psi}) = g(X_1^{\psi}, L)$ *such that*

$$\begin{aligned}(\bar{D}_{X_1^{\psi}}\bar{S})(X_2^{\psi}, X_3^{\psi}) = &[A(X_1^{\psi}) + B(X_1^{\psi})]\bar{S}(X_2^{\psi}, X_3^{\psi}) + A(X_2^{\psi})\bar{S}(X_1^{\psi}, X_3^{\psi})\\ &+ A(X_3^{\psi})\bar{S}(X_1^{\psi}, X_2^{\psi}),\end{aligned} \tag{7.1}$$

for all $X_1^{\psi}, X_2^{\psi}, X_3^{\psi}$ *on* (N, g)*.*

Replacing X_3^ψ by ζ in (7.1), we obtain

$$\begin{aligned}(\bar{D}_{X_1^\psi}\bar{S})(X_2^\psi,\zeta) = [A(X_1^\psi) + B(X_1^\psi)]\bar{S}(X_2^\psi,\zeta) + A(X_2^\psi)\bar{S}(X_1^\psi,\zeta)\\ + A(\zeta)\bar{S}(X_1^\psi,X_2^\psi).\end{aligned} \tag{7.2}$$

With the help of (3.3), (3.7), (6.5), and (7.2), we obtain

$$\begin{aligned}S(\phi X_1^\psi,X_2^\psi) = (4n-1)g(\phi X_1^\psi,X_2^\psi) + 2n[A(X_1^\psi) + B(X_1^\psi)]\eta(X_2^\psi)\\ + A(\zeta)[S(X_1^\psi,X_2^\psi) + g(X_1^\psi,X_2^\psi) - \eta(X_1^\psi)\eta(X_2^\psi)]\\ + 2nA(X_2^\psi)\eta(X_1^\psi).\end{aligned} \tag{7.3}$$

Substituting X_1^ψ and X_2^ψ by ζ and using (2.1), (2.3), and (2.9) in (7.3), we obtain

$$3A(\zeta) + B(\zeta) = 0. \tag{7.4}$$

Again, plugging $X_1^\psi = \zeta$ and using (2.1), (2.3), and (2.9) in (7.3), we obtain

$$2n[2A(\zeta)\eta(X_2^\psi) + B(\zeta)\eta(X_2^\psi) + A(X_2^\psi)] = 0. \tag{7.5}$$

Similarly, setting $X_2^\psi = \zeta$ and applying (2.1), (2.3), and (2.9) in (7.3), we have

$$2n[A(X_1^\psi) + B(X_1^\psi) + 2A(\zeta)\eta(X_1^\psi)] = 0. \tag{7.6}$$

Changing X_2^ψ by X_1^ψ in (7.5) and adding with (7.6), we obtain

$$2n[3A(\zeta) + B(\zeta)]\eta(X_1^\psi) + 2n[A(\zeta)\eta(X_1^\psi) + 2A(X_1^\psi) + B(X_1^\psi)] = 0. \tag{7.7}$$

In view of (7.4) and (7.7), we obtain

$$2n[A(\zeta)\eta(X_1^\psi) + 2A(X_1^\psi) + B(X_1^\psi)] = 0. \tag{7.8}$$

Replacing X_2^ψ by X_1^ψ in (7.5) and adding with (7.8), the expression becomes

$$[3A(\zeta) + B(\zeta)]\eta(X_1^\psi) + 3A(X_1^\psi) + B(X_1^\psi) = 0. \tag{7.9}$$

Finally, from (7.4) and (7.9), the result yields

$$3A(X_1^\psi) + B(X_1^\psi) = 0.$$

Accordingly, we now present the following theorem:

Theorem 7.2. *There exists no almost pseudo Ricci symmetric K-contact manifold* (N, g) *of* $(2n + 1)$*-dimension admitting* $NSNMC$ $\bar{D}$*, unless* $3A + B$ *vanishes everywhere.*

8. Example

Let $N^3 = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$ be a 3-dimensional K-contact manifold, where (x, y, z) are regarded as the standard coordinates in $\mathbb{R}^3$.

Let s_1, s_2, and s_3 be vector fields defined on N^3 that are linearly independent at each point of N^3 and are expressed as follows:

$$s_1 = x(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}) - 2y\frac{\partial}{\partial z}, \quad s_2 = \frac{\partial}{\partial y}, \quad s_3 = \frac{\partial}{\partial z} = \zeta.$$

A Riemannian metric g on N^3 can be defined as

$$g(s_i, s_j) = 1 \quad \text{for} \quad i = j, \quad g(s_i, s_j) = 0 \quad \text{for} \quad i \neq j,$$

where $i, j = 1, 2, 3$.

Let $\zeta = s_3$ and the 1-form $\eta(X_1^\psi) = g(X_1^\psi, s_3)$ be defined on N^3. Also, the (1,1)-type tensor ϕ on N^3 is defined by

$$\phi s_1 = -s_2, \quad \phi s_2 = s_1, \quad \phi s_3 = 0.$$

The Lie brackets for the vector fields s_1, s_2, and s_3 are given by

$$[s_1, s_2] = 2s_3, \quad [s_1, s_3] = 0, \quad [s_2, s_3] = 0.$$

Now, the Koszul's formula is given by

$$\begin{aligned} 2g(D_{X_1^\psi} X_2^\psi, X_3^\psi) = {} & X_1^\psi g(X_2^\psi, X_3^\psi) + X_2^\psi g(X_3^\psi, X_1^\psi) - X_3^\psi g(X_1^\psi, X_2^\psi) \\ & - g(X_1^\psi, [X_2^\psi, X_3^\psi]) - g(X_2^\psi, [X_1^\psi, X_3^\psi]) \\ & + g(X_3^\psi, [X_1^\psi, X_2^\psi]). \end{aligned}$$

Using the above formula, we find

$$\begin{aligned} D_{s_1} s_1 &= 0, & D_{s_1} s_2 &= s_3, & D_{s_1} s_3 &= -s_2, \\ D_{s_2} s_1 &= -s_3, & D_{s_2} s_2 &= 0, & D_{s_2} s_3 &= s_1, \\ D_{s_3} s_1 &= -s_2, & D_{s_3} s_2 &= s_1, & D_{s_3} s_3 &= 0, \end{aligned}$$

where D is the Levi-Civita connection. Also, by using the above results and equation (2.12), the non-zero components of $\bar{D}_{s_i} s_j$ with respect to $NSNMC$ $\bar{D}$ are obtained by

$$\bar{D}_{s_1} s_3 = -s_2, \quad \bar{D}_{s_3} s_1 = -s_2, \quad \bar{D}_{s_3} s_2 = s_1.$$

The non-vanishing components of R and S are given below

$$\begin{aligned} R(s_1, s_2)s_2 &= -3s_1 & R(s_1, s_3)s_3 &= s_1 & R(s_2, s_3)s_3 &= s_2 \\ R(s_3, s_2)s_2 &= s_3 & R(s_2, s_1)s_1 &= -3s_2 & R(s_3, s_1)s_1 &= s_3 \\ S(s_1, s_2) &= -2 & S(s_2, s_2) &= -2 & S(s_3, s_3) &= 2. \end{aligned}$$

Again, the non-vanishing components of $\bar{R}$ and $\bar{S}$ of $NSNMC$ $\bar{D}$ are given below, respectively

$$\begin{array}{lll} \bar{R}(s_1, s_2)s_2 = -4s_1 & \bar{R}(s_1, s_3)s_3 = s_1 & \bar{R}(s_2, s_3)s_3 = s_2 \\ \bar{R}(s_3, s_2)s_2 = 2s_3 & \bar{R}(s_2, s_1)s_1 = -4s_2 & \bar{R}(s_3, s_1)s_1 = 2s_3 \\ \bar{S}(s_1, s_2) = -1 & \bar{S}(s_2, s_2) = -1 & \bar{S}(s_3, s_3) = 2. \end{array}$$

Using the above curvature values $\bar{R}$ of $NSNMC$ $\bar{D}$, (4.1) and (6.1), it is obvious that

$$G(s_i) + I(s_i) + J(s_i) = 0, \quad \text{for all } i = 1, 2, 3$$

and

$$3A(s_i) + B(s_i) = 0, \quad \text{for all } i = 1, 2, 3.$$

Thus, these last two equations verify Theorem 4.1 and Theorem 6.2, respectively.

From (4.21), (7.1) and using the values of $\bar{S}$ of $NSNMC$ $\bar{D}$, it is clear that

$$\alpha(s_i) + \beta(s_i) + \gamma(s_i) = 0, \quad \text{for all } i = 1, 2, 3$$

and

$$3A(s_i) + B(s_i) = 0, \quad \text{for all } i = 1, 2, 3.$$

These final two expressions completely satisfy Theorem 4.2 and Theorem 7.2, respectively.

9. Conclusions

In this paper, we examined weakly symmetric and almost pseudo-symmetric K-contact manifolds equipped with a non-symmetric non-metric connection. We derived several fundamental geometric properties and curvature conditions that characterize such manifolds, highlighting the impact of this special connection on their underlying structure-particularly with regard to symmetry. We established essential conditions under which a K-contact manifold with this type of connection exhibits weak symmetry or almost pseudo-symmetry. Furthermore, we analyzed the interaction between the connection and the metric structure, offering new insights into the resulting curvature behavior. These findings deepen our understanding of the geometric and topological consequences of introducing non-symmetric non-metric connection in contact geometry. Future research could extend this work to broader classes of manifolds or investigate potential applications in theoretical physics, where such connections naturally arise.

Author contributions

Rajesh Kumar: Methodology, Validation, Formal analysis, Investigation, Writing-original draft, Writing-review & editing; Laltluangkima Chawngthu: Validation, Formal analysis, Investigation, Writing-original draft, Writing-review & editing; Oguzhan Bahadir: Validation, Formal analysis, Investigation, Writing-original draft, Supervision, Writing-review & editing; Md Aquib: Conceptualization, Methodology, Validation, Formal analysis, Investigation, Visualization, Writing-original draft, Supervision, Project administration, Funding acquisition, Writing–review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported and funded by the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) (grant number IMSIU-DDRSP2502).

Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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