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Research article

Optimality and scalarization for approximate Benson proper efficient solutions in vector equilibrium problems

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Abstract: This paper examines the optimality and scalarization theorems for approximate quasi-Benson proper efficient solutions in constrained vector equilibrium problems. The optimality conditions are established based on the generalized convexity and convex separation theorems. Additionally, two scalarization theorems are developed by combining the cone scalarization function with the properties of generating cones. The proposed definitions and conclusions are supported by specific numerical examples.

Keywords: vector equilibrium; generalized convexity; optimality conditions; generating cones; scalarization theorems
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1. Introduction

The vector equilibrium problem, which was initially proposed by Blum and Oettli in 1994 [1], establishes a unified theoretical framework that unifies key mathematical paradigms such as vector optimization, variational inequalities, complementarity problems, and fixed point problems (see [2–4] for foundational extensions). Beyond its theoretical significance, this framework has found broad applicability in economics, engineering, and decision-making systems where the simultaneous optimization of interdependent criteria is critical [5–7]. Therefore, many scholars have conducted in-depth research on its related theories, including the definition of solutions, optimality conditions, scalarization, and the duality theory; for more detailed content, please refer to references [8–10]. In applications, deriving the exact solutions to mathematical models is often infeasible due to inherent complexities, thus necessitating computational techniques to approximate solutions. Compared to the exact solutions, these approximations exhibit relaxed requirements for existence, as demonstrated in [11, 12]. Consequently, investigating the approximate solutions to vector equilibrium problems is of critical importance in both theoretical and applied contexts.

The optimality conditions and scalarization of solutions are two important research topics in the study of vector equilibrium problems. Additionally, convexity and its generalization of functions serve as powerful tools in establishing the optimality conditions of vector equilibrium problems. In 1998, Chen and Rong [13] introduced the concept of generalized cone-subconvexlikeness for vector-valued mappings in locally convex Hausdorff topological vector spaces. Additionally, they conducted in-depth research on the scalarization, optimality conditions, and saddle point characterization of efficient solutions to vector optimization problems using generalized cone-subconvexlikeness. Shortly afterward, Yang and Li et al. [14] extended this convexity, thereby introducing the near generalized cone subclass convexity of set valued mappings, and obtained the optimality theory for vector optimization problems under their assumptions. In recent years, based on the convexity of near generalized cone classes, Qiu [15], Long and Huang et al. [16], and Qiu [17] established the optimality conditions for global, weak, Henig efficient, and super efficient solutions of vector equilibrium problems, respectively. Moreover, Long and Huang et al. [16] pointed out that, compared to other convexities, near generalized subclass convexity requires a looser assumption of function convexity, which makes near generalized subclass convexity more flexible and applicable in dealing with optimization problems. Therefore, this paper aims to construct a class of near generalized conic class convexities that are weaker than the above convexity and suitable for vector equilibrium problems, and, combined with the convex set separation theorem, study the optimality conditions for a class of constrained vector equilibrium problems (VEP) about approximate Benson proper efficient solutions.

The scalarization method serves as a cornerstone in the analysis of VEP (see [18–20]). This technique reformulates solutions of VEP as solutions to associated scalar equilibrium problems, thereby enabling the transfer of established scalar-level theoretical and computational tools to address the vector case. Notably, both linear and nonlinear scalarization frameworks to approximate solutions were rigorously developed in [21–23]. For instance, Qiu and Yang [24], using the Tammer nonlinear scalarization function, formulated scalarization theorems for both approximate weak effective solutions and approximate Henig effective solutions of vector equilibrium problems. By leveraging the Baire theorem, Gong [18] provided a scalarization theorem for super-efficient solutions of VEP. As we know, there is currently relatively little research on scalars that approximate true effective solutions. This paper develops a scalarization theorem for the approximate Benson proper efficient solutions of VEP, thereby leveraging the structural properties of cone functions and augmented cones.

The remainder of this article is organized as follows: in Section 2, we present some symbols, concepts, and lemmas used in subsequent sections; in Section 3, the optimality conditions for approximate Benson proper efficient solutions of VEP are derived; and Section 4 uses the cone functions and augmented cones to construct a scalarizations theorem of approximate Benson proper efficient solutions for VEP.

2. Preliminaries

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space, $\mathbb{B}(\bar{x}, r)$ represent the open ball of radius $r > 0$ centered at $\bar{x} \in \mathbb{R}^n$, and $\|\cdot\|$ denote the norm in $\mathbb{R}^n$. The inner product between vectors x and y is denoted by $\langle x, y\rangle$ for any $x, y \in \mathbb{R}^n$. For a set $K \subset \mathbb{R}^n$, we denote its closure, cone hull, and interior by $\mathrm{cl}\, K$, $\mathrm{cone}\, K$, and $\mathrm{int}\, K$, respectively. Let $\mathbb{R}^n_+$ denote the set of nonnegative real numbers, and define

$$\mathbb{R}^n_+ := \{(y_1, \ldots, y_n) \in \mathbb{R}^n : y_i \geq 0,\ i = 1, \ldots, n\}.$$

A convex cone $C \subset \mathbb{R}^m$ is termed a pointed cone if $C \cap (-C) = \{0\}$, with its dual cone C^* and quasi-interior $C^\sharp$ defined as follows:

$$C^* = \{y^* \in \mathbb{R}^m : \langle y^*, y\rangle \geq 0,\ \forall y \in C\},$$
$$C^\sharp = \{y^* \in \mathbb{R}^m : \langle y^*, y\rangle > 0,\ \forall y \in C\backslash\{0\}\}.$$

To further analyze the properties of such cones, we present the following lemma, which provides key insights into the behavior of functionals in the dual and strict dual cones.

Lemma 1. *(see [25]) Let $C \subset \mathbb{R}^m$ be a closed, convex pointed cone. Then,*
(i) if $\lambda \in C^*\backslash\{0\}$ and $y \in \mathrm{int}\, C$, then $\langle \lambda, y\rangle > 0$, and
(ii) if $\lambda \in \mathrm{int}\, C^*$ and $y \in C\backslash\{0\}$, then $\langle \lambda, y\rangle > 0$.

Based on the concept of near generalized subconvexlike mappings introduced in reference [13], we define the notion of approximate generalized subconvexlike mappings below. Under the assumptions of this definition, we establish the optimality conditions. To this end, we first present the definitions of near C-subconvexlike mappings and approximate vector-valued maps.

Definition 1. *(see [13]) Let $K \subset \mathbb{R}^n$ be a nonempty closed set, $C \subset \mathbb{R}^m$ be convex cone, and $f : K \to \mathbb{R}^m$ be vector-valued map. The function f is called near C-subconvexlike if* $\mathrm{cl}(\mathrm{cone}(f(K) + C))$ *is convex.*

Definition 2. *Let $K \subset \mathbb{R}^n$ be a nonempty closed set, $C \subset \mathbb{R}^m$ be a convex cone, and $\varepsilon \geq 0$, $\bar{x} \in K$, $e \in C \setminus \{0\}$, $f : K \to \mathbb{R}^m$ be a vector-valued map. The map $f_{\varepsilon e}$ is defined as follows:*

$$f_{\varepsilon e}(x) = f(x) + \varepsilon\|x - \bar{x}\|e, \qquad \forall x \in K.$$

Building on the previously defined concepts of near C-subconvexlikeness and $f_{\varepsilon e}$, we introduce an approximate near C-subconvexlikeness property and derive the corresponding optimality conditions under convexity assumptions in the subsequent analysis.

Definition 3. *Let $K \subset \mathbb{R}^n$ be a nonempty closed set, $C \subset \mathbb{R}^m$ be a convex cone, and $f : K \to \mathbb{R}^m$ be a vector-valued map. Suppose $\varepsilon \geq 0$, $\bar{x} \in K$, and $e \in C \setminus \{0\}$. If* $\mathrm{cl}(\mathrm{cone}\, f_{\varepsilon e}(K) + C)$ *is convex, then $f_{\varepsilon e}$ is called an εe-near C-subconvexlike mapping.*

Remark 1. *If $\varepsilon = 0$, then the εe-near C-subconvexlikeness defined by Definition 1 degenerates to the near C-subconvexlikeness.*

Now, let $K \subset \mathbb{R}^n$ be a non-empty closed set, and $C \subset \mathbb{R}^m$, and $D \subset \mathbb{R}^p$ be a closed convex, pointed cone. Let $F : K \times K \to \mathbb{R}^m$, $G : K \to \mathbb{R}^p$ be vector-valued mappings. Consider the following constrained VEP:

$$\text{(VEP)} \quad \text{find } \ \bar{x} \in K \ \text{ such that } \ F(\bar{x}, x) \notin -\mathrm{int}C, \quad \forall x \in K.$$

The feasible set of the VEP is defined as follows:

$$\Omega := \{x \in K \mid G(x) \in -D\},$$

and we assume $\Omega \neq \emptyset$ throughout this work. We adopt the following standing assumptions: $F(\bar{x}, \bar{x}) = 0$ holds for every reference point $\bar{x} \in K$, and the bifunction satisfies $F_{\bar{x}}(x) := F(\bar{x}, x)$ for all $x \in K$. Notably, if $F_{\bar{x}}(x) = f(x) - f(\bar{x})$, where $f : K \to Y$ is a vector-valued mapping, then the VEP reduces to a vector optimization problem (see [26]). Furthermore, in the case where

$$F_{\bar{x}}(x) = (f_1(x), \ldots, f_m(x)) - (f_1(\bar{x}), \ldots, f_m(\bar{x})),$$

with $f_i : K \to \mathbb{R}$ $(i = 1, \ldots, m)$, the VEP simplifies to a multi-objective optimization problem [27]. This generality underscores the theoretical significance of studying VEPs, as they provide a unifying framework to analyze diverse classes of optimization problems.

In this paper, we study the approximate Benson proper efficient solutions of VEPs as follows.

Definition 4. *Let $\varepsilon \geq 0$, $e \in intC$, and $\bar{x} \in \Omega$. Then, $\bar{x}$ is said to be an approximate quasi-Benson proper efficient solution of VEPs, if*

$$\mathrm{cl}(\mathrm{cone}(F_{\bar{x}}(x) + C + \varepsilon\|x - \bar{x}\|e)) \cap (-C) = \{0\}, \quad \forall\, x \in \Omega.$$

Below, we provide an example to demonstrate the validity of the above definition.

Example 1. *In the VEP, let $K = C = D = \mathbb{R}^2_+$. Define the vector-valued mappings $F : K \times K \to \mathbb{R}^2$ and $G : K \to \mathbb{R}^2$ as follows:*

$$F_{\bar{x}}(x) = \begin{pmatrix} ln(x_1 + x_2 + 1) + \bar{x}_1 \sin \bar{x}_2 \\ x_1^2 + x_2^2 + \bar{x}_1 \cos \bar{x}_2 \end{pmatrix}, \quad G(x) = \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix}.$$

The feasible set Ω is given by the following:

$$\Omega = \{x \in \mathbb{R}^2_+ : G(x) \in -D\} = \{x \in \mathbb{R}^2_+ : -x_1 \leq 0, -x_2 \leq 0\} = \mathbb{R}^2_+.$$

Clearly, Ω is nonempty.

Let $\varepsilon = 0.1$, $e = (1, 1) \in int\, C$, and consider $\bar{x} = (0, 0) \in \Omega$. For any $x \in \Omega$, we have the following:

$$\begin{aligned} F_{\bar{x}}(x) - F_{\bar{x}}(\bar{x}) + C + \varepsilon\|x - \bar{x}\|e &= \begin{pmatrix} \ln(x_1 + x_2 + 1) \\ x_1^2 + x_2^2 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \mathbb{R}^2_+ + 0.1 \cdot \|x\| \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \\ &= \begin{pmatrix} \ln(x_1 + x_2 + 1) + \|x\| \\ x_1^2 + x_2^2 + \|x\| \end{pmatrix} + \mathbb{R}^2_+. \end{aligned}$$

The closure of the cone generated by this set is as follows:

$$\mathrm{cl}(\mathrm{cone}\,(F_{\bar{x}}(x) - F_{\bar{x}}(\bar{x}) + C + \varepsilon\|x - \bar{x}\|e)) = \mathbb{R}^2_+.$$

Therefore, we have the following:

$$\mathrm{cl}(\mathrm{cone}\,(F_{\bar{x}}(x) - F_{\bar{x}}(\bar{x}) + C + \varepsilon\|x - \bar{x}\|e)) \cap (-C) = \mathbb{R}^2_+ \cap (-\mathbb{R}^2_+) = \{0\}, \ \forall x \in \Omega.$$

Therefore, $\bar{x} = (0, 0)$ is an approximate quasi-Benson proper efficient solution of the VEP.

3. Optimality conditions

In this section, we derive the optimality conditions for VEPs with respect to the approximate Benson proper efficiency by integrating the convex set separation theorem and the concept of approximate generalized subconvexlikeness. Specifically, we present a characterization of these conditions for approximate Benson proper efficient solutions in the framework of VEPs.

Theorem 1. *In the VEPs, let $\bar{x} \in \Omega$, $\varepsilon \geq 0$, and $e \in \operatorname{int} C$. Suppose that $F_{\bar{x}}$ is εe-near C-subconvexlike with respect to $\bar{x}$ on Ω, G is near D-subconvexlike on Ω, and there exists some $x' \in \Omega$ such that $G(x') \in -\operatorname{int} D$. Then, $\bar{x}$ is an approximate quasi-Benson efficient solution of the VEPs if and only if there exist $\lambda^* \in C^* \setminus \{0\}$ and $\mu^* \in D^*$ such that*

$$\lambda^*(F_{\bar{x}}(x)) + \varepsilon\|x - \bar{x}\|\lambda^*(e) + \mu^*(G(x)) \geq 0, \quad \forall x \in \Omega, \tag{3.1}$$

and

$$\mu^*(G(\bar{x})) = 0, \tag{3.2}$$

where $\mu^*(G(\bar{x})) = \langle \mu^*, G(\bar{x}) \rangle$.

Proof. If $\bar{x}$ is an approximate quasi-Benson efficient solution of the VEP, then

$$\operatorname{clcone}(F_{\bar{x}}(x) + C + \varepsilon\|x - \bar{x}\|e) \cap (-C) = \{0\}, \quad \forall\, x \in \Omega.$$

Since C is a close convex pointed cone, we obtain the following:

$$(F_{\bar{x}}(x) + C + \varepsilon\|x - \bar{x}\|e) \cap (-\operatorname{int}C) = \emptyset, \quad \forall\, x \in \Omega.$$

Let $H(\Omega) = \bigcup_{x\in\Omega}(F_{\bar{x}}(x) + \varepsilon\|x - \bar{x}\|e, G(x))$. The following proof demonstrates that

$$(H(\Omega) + C \times D) \cap (-(\operatorname{int}C \times \operatorname{int}D)) = \emptyset. \tag{3.3}$$

For a contradiction, assume that there exists a point $(\bar{y}, \bar{z})$ such that

$$(\bar{y},\ \bar{z}) \in (H(\Omega) + C \times D) \cap (-(\operatorname{int}C \times \operatorname{int}D)).$$

Hence, we have $\bar{y} \in -\operatorname{int}C$, $\bar{z} \in -\operatorname{int}D$, and there is $(e_n, d_n) \in C \times D$, $x_n \in \Omega$, $\eta_n > 0$ such that

$$\bar{y} = \lim_{n\to\infty} \eta_n(F_{\bar{x}}(x_n) + \varepsilon\|x_n - \bar{x}\|e + e_n), \tag{3.4}$$

and

$$\bar{z} = \lim_{n\to\infty} \eta_n(G(x_n) + d_n).$$

Therefore, there is $\bar{z}_n \in -\operatorname{int}D$ and $\bar{z}_n \to \bar{z}(n \to \infty)$ such that

$$\bar{z}_n = \eta_n(G(x_n)), \tag{3.5}$$

that is,

$$\frac{1}{\eta_n}\bar{z}_n = (G(x_n)). \tag{3.6}$$

Because int$D$ is an open set, then D is a point, closed convex cone, and $\eta_n > 0$; hence, $\frac{1}{\eta_n}\bar{z}_n \in -\mathrm{int}D$, and we have the following:

$$G(x_n) \in -\mathrm{int}D - d_n.$$

Combined with $d_n \in D$ and $D + intD = intD$, we obtain the following:

$$G(x_n) \in -\mathrm{int}D.$$

Additionally, due to $x_n \in \Omega$, $\bar{y} \in -\mathrm{int}C$, and combined with (3.4), we obtain the following:

$$\bar{y} \in (F_{\bar{x}}(x_n) + C + \varepsilon\|x_n - \bar{x}\|e) \cap (-\mathrm{int}C).$$

This leads to a contradiction with (3.3); therefore, (3.4) is established. Since $F_{\bar{x}}$ is εe-near C-subconvexlike on Ω, and G is near D-subconvexlike on Ω , then $clcone(H(\Omega) + C \times D)$ is a closed convex set, and because $-(intC \times intD)$ is convex. Therefore, from the separation theorem of convex sets (see [28]), it is known that there exists $(\lambda^*,\ \mu^*) \in Y^* \times Z^* \setminus \{(0,\ 0)\}$ such that

$$(\lambda^*,\ \mu^*)\mathrm{cl\ cone}(H(\Omega) + C \times D) \geq 0 \geq (\lambda^*,\ \mu^*)(-(\mathrm{int}D \times \mathrm{int}C). \tag{3.7}$$

Since cl cone$(H(\Omega) + C \times D)$ is a closed convex cone and $(\lambda^*,\ \mu^*)$ has a lower bound, we have

$$(\lambda^*,\ \mu^*)\mathrm{cl\ cone}(H(\Omega) + C \times D) \geq 0.$$

Since $(0, 0) \in C \times D$, we have the following:

$$(\lambda^*,\ \mu^*)\mathrm{clcone}(H(\Omega)) \geq 0. \tag{3.8}$$

Hence,

$$\lambda^*(F_{\bar{x}}(x) + \varepsilon\|x - \bar{x}\|e) + \mu^*(G(x) \geq 0, \quad \forall x \in \Omega. \tag{3.9}$$

Therefore, for any $x \in \Omega,\ \beta_1,\ \beta_2 \geq 0,\ \alpha \in C_\epsilon(B)$, and $d \in D$, we obtain the following:

$$\lambda^*(F_{\bar{x}}(x) + \varepsilon\|x - \bar{x}\|e + \beta_1\alpha) + \mu^*(G(x) + \beta_2 d) \geq 0. \tag{3.10}$$

Next, we prove that $\mu^* \in D^*$, that is, for any $d \in D$ such that $\mu^*(d) \geq 0$, there is $d_0 \in D$, such that

$$\mu^*(d_0) < 0.$$

From (3.10), when $\beta_2 \to \infty$, there exists $\tilde{x} \in \Omega$, $\beta_1' > 0$, such that

$$\mu^*(\beta_2 d_0) = \beta_2\mu^*(d_0) < -\lambda^*(F_{\bar{x}}(\tilde{x}) + \varepsilon\|\tilde{x} - \bar{x}\|e + \beta_1'\alpha) - \mu^*(G(\tilde{x})).$$

This is contradictory; then, $\mu^* \in D^*$. Let $\lambda^* \in C^*\setminus\{0\}$. If $\lambda^* = 0$, according to $(\lambda^*,\ \mu^*) \in Y^*\times Z^*\setminus\{(0,\ 0)\}$, where we get $\mu^* \neq 0$, then

$$\mu^*(G(x)) \geq 0, \quad \forall x \in \Omega. \tag{3.11}$$

Because of $G(x) \in -D$, then there exists $\hat{x} \in \Omega$ such that $G(\hat{x}) \in -D \setminus \{0\}$; thus,

$$\mu^*(G(\hat{x}) < 0.$$

This contradicts condition (3.11). Therefore, $\lambda^* \neq 0$. Combining (3.7) and (3.8) , we obtain

$$\lambda^*(-\text{int}C) + \mu^*(-\text{int}D) \leq 0,$$

that is,

$$\mu^*(-\text{int}D) \leq \lambda^*(\text{int}C).$$

Since λ^* has a lower bound, then we have the following:

$$0 \leq \lambda^*(\text{int}C).$$

Because C is a convex cone, then

$$C \subset \text{cl}C = \text{cl int}C.$$

Therefore, for any $k \in C$, there exists $k_n \in \text{int}C$ such that

$$k = \lim_{n\to\infty} k_n.$$

Hence, we have

$$\lambda^*(k) = \lim_{n\to\infty} \lambda^*(k_n) \geq 0.$$

This means $\lambda^*(k) \geq 0$. Therefore, we get $\lambda^* \in C\setminus\{0\}$. In particular, $F(\bar{x}, \bar{x}) = 0$. If we take $x = \bar{x}$, then by (3.9) , we obtain the following:

$$\mu^*(G(\bar{x})) \geq 0. \tag{3.12}$$

Since $G(\bar{x}) \in -D,\ \mu^* \in D$, then

$$\mu^*(G(\bar{x})) \leq 0. \tag{3.13}$$

Combining (3.12) and (3.13), it can be seen that

$$\mu^*(G(\bar{x})) = 0.$$

Conversely, if $\bar{x}$ is not the approximate quasi-Benson efficient solution of the VEP, then there exists $\hat{x} \in \Omega$ such that

$$\text{clcone}(F_{\bar{x}}(\hat{x}) + C + \varepsilon\|\hat{x} - \bar{x}\|e) \cap (-C) \neq \{0\}. \tag{3.14}$$

By (3.14), we can deduce that there exist $t_n > 0$ and $c \in C$ such that

$$0 \neq t_n(F_{\bar{x}}(\hat{x}) + c + \varepsilon\|\hat{x} - \bar{x}\|e) \in -C.$$

Because $\lambda^* \in C^* \setminus \{0\}$, according to Lemma 1(i), we obtain

$$t_n\lambda^*(F_{\bar{x}}(\hat{x}) + c + \varepsilon\|\hat{x} - \bar{x}\|e) < 0,$$

that is,

$$t_n((\lambda^*(F_{\bar{x}}(\hat{x}) + \lambda^*(c) + \varepsilon\|\hat{x} - \bar{x}\|\lambda^*(e)) < 0.$$

Since $t_n > 0, \lambda^* \in C^* \setminus \{0\}$, and $c \in C$, we obtain the following:

$$\lambda^*(F_{\bar{x}}(\hat{x})) + \varepsilon\|\hat{x} - \bar{x}\|\lambda^*(e) < 0. \tag{3.15}$$

Combined with (3.15), $\mu \in D^*$ and $G(\hat{x}) \in -D$, we have the following:

$$\lambda^*(F_{\bar{x}}(\hat{x})) + \varepsilon\|\hat{x} - \bar{x}\|\lambda^*(e)) + \mu^*(G(\hat{x})) < 0. \tag{3.16}$$

Note that $F_{\bar{x}}(\bar{x}) = 0$. Combined with (3.1) and (3.2), it can be seen that

$$\lambda^*(F_{\bar{x}}(x)) + \varepsilon\|x - \bar{x}\|\lambda^*(e) + \mu^*(G(x)) \geq 0 = \lambda^*(F_{\bar{x}}(\bar{x})) + \varepsilon\|\bar{x} - \bar{x}\|\lambda^*(e) + \mu^*(G(\bar{x})).$$

This is contradictory with (3.16). Therefore, $\bar{x}$ is an approximate quasi-Benson proper efficient solution of the VEP. Thus, the proof is complete. □

Remark 2. *Our preceding conclusions establish the optimality conditions for the approximate quasi-Benson proper efficiency. The local variant is analogously defined to Definition 4 by restricting Ω to $\Omega \cap U$ for some neighborhood U of the candidate point. Crucially, when Ω is convex, the local and global efficiency coincide [29]—an equivalence that extends to both weak and strong variants. Thus, for convex Ω, all previous conclusions hold for the local variant.*

Below, we provide an example to demonstrate the validity of the above Theorem 1.

Example 2. *In the VEP, let $K = [0, 1] \subset \mathbb{R}$, $C = \mathbb{R}^2_+$, and $D = \mathbb{R}_+$. Define the vector-valued mappings $F : K \times K \to \mathbb{R}^2$ and $G : K \to \mathbb{R}$ as follows:*

$$F_{\bar{x}}(x) = \begin{pmatrix} x + \ln(\bar{x} + 1) \\ x^2 + \bar{x} \end{pmatrix}, \quad G(x) = -x.$$

The feasible set Ω is given by the following:

$$\Omega = \{x \in [0, 1] : G(x) \in -D\} = \{x \in [0, 1] : -x \leq 0\} = [0, 1].$$

Clearly, Ω is nonempty. Let $\varepsilon = 0.1$, $e = (1, 1) \in \operatorname{int} C$, and $\bar{x} = 0 \in \Omega$. Then,

$$\operatorname{cl}(\operatorname{cone}(F_{\bar{x}}(K) + C + \varepsilon\|x - \bar{x}\|e)) = \operatorname{cl}(\operatorname{cone}\left(\begin{pmatrix} x + 0.1x \\ x^2 + 0.1x \end{pmatrix} + \mathbb{R}^2_+\right)) = \mathbb{R}^2_+$$

is convex; this implies that the map $F_{\bar{x}}$ is an εe-near C-subconvexlike. Additionally, since

$$G(K) + D = [-1, 0] + \mathbb{R}_+ = [-1, \infty)$$

is convex, the map $G(x) = -x$ is near D-subconvexlike. Obviously, there exists $x' = 0.5 \in \Omega$ such that $G(x') = -0.5 \in -intD$.

Since $-C = -\mathbb{R}^2_+$, *we have*

$$\mathrm{cl}(\mathrm{cone}\,(F_{\bar{x}}(x) + C + \varepsilon\|x - \bar{x}\|e)) \cap (-C) = \mathbb{R}^2_+ \cap -\mathbb{R}^2_+ = \{0\},$$

which holds for all $x \in \Omega$. *Therefore,* $\bar{x} = 0$ *is indeed an approximate quasi-Benson proper efficient solution of the VEP. Taking* $\lambda^* = (1,1) \in C^* \setminus \{0\}$ *and* $\mu^* = 1 \in D^*$, *for any* $x \in \Omega$, *we have the following:*

$$\lambda^*(F_{\bar{x}}(x)) + \varepsilon\|x - \bar{x}\|\lambda^*(e) + \mu^*(G(x)) = (1,1) \cdot \begin{pmatrix} x \\ x^2 \end{pmatrix} + 0.1x \cdot (1,1) \cdot (1,1) + 1 \cdot (-x) = x^2 + 0.2x \geq 0.$$

Additionally, for $\bar{x} = 0$, *we obtain the following:*

$$\mu^*(G(\bar{x})) = 1 \cdot (-0) = 0.$$

Thus, $\bar{x} = 0$ *is an approximate quasi-Benson efficient solution of the VEP, and the Theorem 1 is validated by this example.*

4. Scalarization

In this section, we introduce a class of cone-monotonic functions and leverage their structural properties to construct a scalarization framework to characterize the approximate Benson proper efficient solutions in the VEP. To formalize this approach, we first revisit the foundational definitions and properties of cones, thus ensuring alignment with the proposed monotonicity axioms.

Recall that the dual cone C^* of $C \subset \mathbb{R}^m$ and its quasi-interior $C^\sharp$ are defined by

$$C^* = \{y^* \in \mathbb{R}^m : \langle y^*, y\rangle \geq 0,\ \forall y \in C\},$$

and

$$C^\sharp = \{y^* \in \mathbb{R}^m : \langle y^*, y\rangle > 0,\ \forall y \in C\backslash\{0\}\},$$

respectively. The following three cones, called augmented dual cones of C, were introduced in references [19, 30, 31]:

$$C^{\alpha *} = \{(y^*, \alpha) \in C^\sharp \times \mathbb{R}_+ : \langle y^*, y\rangle - \alpha\|y\| \geq 0,\ \forall y \in C\},$$
$$C^{\alpha\circ} = \{(y^*, \alpha) \in C^\sharp \times \mathbb{R}_+ : \langle y^*, y\rangle - \alpha\|y\| > 0,\ \forall y \in intC\},$$

and

$$C^{\alpha\sharp} = \{(y^*, \alpha) \in C^\sharp \times \mathbb{R}_+ : \langle y^*, y\rangle - \alpha\|y\| > 0,\ \forall y \in C \setminus \{0\}\}.$$

In the definition of $C^{\alpha\circ}$, the ordering cone C is assumed to have a nonempty interior.

Using the augmented dual cone $C^{\alpha\sharp}$, we analyze key properties of cone-monotonic functions (see [19, 30, 31]) and develop a scalarization theorem to the approximate Benson proper efficient solutions in the VEP. To this end, we define cone-monotonicity as follows, thus forming the basis for subsequent analysis.

Definition 5. *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a given function.*
(a) ψ is called monotonically increasing if, for all $y, z \in \mathbb{R}^n$ such that $y - z \in \mathbb{R}^m_+$ and

$$\psi(y) \geq \psi(z).$$

(b) ψ is called strictly monotonically increasing if, for all $y, z \in \mathbb{R}^m$ such that $y - z \in int(\mathbb{R}^m_+)$ and

$$\psi(y) > \psi(z).$$

(c) ψ is called strongly monotonically increasing if, for all $y, z \in \mathbb{R}^m$ such that $y - z \in \mathbb{R}^m_+ \setminus \{0\}$ and

$$\psi(y) > \psi(z).$$

The following conclusion provides the properties of cone-monotonic functions in generating cones. These properties play a crucial role in deriving the approximate Benson proper efficient solution scalarization theorem in the subsequent section.

Lemma 2. *(see [19, 30, 31]) Let $C \subset \mathbb{R}^m$ be a pointed closed convex cone, $w \in \mathbb{R}^m$, and $\alpha \in \mathbb{R}_+$. Define the function $g_{(w,\alpha)} : \mathbb{R}^m \to \mathbb{R}$ as follows:*

$$g_{(w,\alpha)}(y) = w^T y + \alpha \|y\|.$$

Then, $g_{(w,\alpha)}$ is monotonically increasing, strictly monotonically increasing, and strongly monotonically increasing on $\mathbb{R}^m$ if and only if $(w, \alpha) \in C^{\alpha *}$, $(w, \alpha) \in C^{\alpha \circ}$, and $(w, \alpha) \in C^{\alpha \sharp}$, respectively.

Next, we utilize the scalarization function defined above to formulate the following scalar optimization problem related to the VEP:

$$(\mathbf{P}_\varphi) \quad \min \{\varphi(F_{\bar{x}}(x))\}, \quad \text{s.t.} \quad x \in \Omega,$$

where $\bar{x} \in \Omega$. Let $\varepsilon \geq 0$ and $\bar{x} \in \Omega$. Then, $\bar{x}$ is called a quasi-optimal solution for $(\mathbf{P}_\varphi)$ if

$$\varphi(F_{\bar{x}}(x)) - \varphi(F_{\bar{x}}(\bar{x})) + \varepsilon \|x - \bar{x}\| \geq 0, \quad \forall x \in \Omega.$$

The following theorem bridges the quasi-optimal solutions of the scalarized problem $(\mathbf{P}_\varphi)$ to the approximate quasi-Benson proper efficient solutions of the VEPs, thus establishing a direct correspondence through a rigorous duality analysis.

Theorem 2. *Let $C \subset \mathbb{R}^m$ be a pointed closed convex cone, $\varepsilon \geq 0$, $e = (e_1, \ldots, e_m) \in \text{int } C$, and $\bar{x} \in \Omega$. If $(w, \alpha) \in C^{\alpha \sharp}$ and $\bar{x}$ is a quasi-optimal solution for problem $(\mathbf{P}_\varphi)$, then $\bar{x}$ is also an approximate quasi-Benson proper efficient solution for the VEP.*

Proof. Assume that $\bar{x}$ is a quasi-optimal solution for problem $(\mathbf{P}_\varphi)$, then, we obtain the following:

$$\varphi(F_{\bar{x}}(x)) - \varphi(F_{\bar{x}}(\bar{x})) + \varepsilon \|x - \bar{x}\| \geq 0, \quad \forall x \in \Omega. \tag{4.1}$$

If $\bar{x}$ is not an approximate quasi-Benson proper efficient solution for the VEP, then there exists $\hat{x} \in \Omega$ such that

$$\text{cl}(\text{cone}(F_{\bar{x}}(\hat{x}) + C + \varepsilon \|\hat{x} - \bar{x}\| e)) \cap (-C) \neq \emptyset.$$

Hence, there exist $t_n > 0$ and $c \in C$ such that

$$t_n(F_{\bar{x}}(\hat{x}) + c + \varepsilon\|\hat{x} - \bar{x}\|e) \in -C.$$

Since $F_{\bar{x}}(\bar{x}) = 0$, we obtain the following:

$$t_n(F_{\bar{x}}(\hat{x}) - F_{\bar{x}}(\bar{x}) + c + \varepsilon\|\hat{x} - \bar{x}\|e) \in -C.$$

Since the function φ is strongly monotonically increasing on $\mathbb{R}^m$ for any $(y^*, \alpha) \in C^{\alpha\sharp}$, then we have the following:

$$\varphi(t_n(F_{\bar{x}}(\hat{x})) < \varphi(t_n(F_{\bar{x}}(\bar{x}) - c - \varepsilon\|\hat{x} - \bar{x}\|e)). \tag{4.2}$$

According to the definition of function φ, (4.2) can be equivalently rewritten as follows:

$$\begin{aligned}\varphi(F_{\bar{x}}(\hat{x})) &< \varphi(F_{\bar{x}}(\bar{x}) - c - \varepsilon\|\hat{x} - \bar{x}\|e)\\ &\le \varphi(F_{\bar{x}}(\bar{x})) + \varphi(-c - \varepsilon\|\hat{x} - \bar{x}\|e)\\ &\le \varphi(F_{\bar{x}}(\bar{x})) + \langle w, -c - \varepsilon\|\hat{x} - \bar{x}\|e\rangle + \alpha\| - c - \varepsilon\|\hat{x} - \bar{x}\|e\|\\ &\le \varphi(F_{\bar{x}}(\bar{x})) - \langle w, c\rangle + \alpha\|c\| - \langle w, e\rangle\varepsilon\|\hat{x} - \bar{x}\| + \alpha\varepsilon\|\hat{x} - \bar{x}\|\|e\|.\end{aligned} \tag{4.3}$$

From Eq (4.3), we can infer that

$$\varphi(F_{\bar{x}}(\hat{x})) < \varphi(F_{\bar{x}}(\bar{x}) - \langle w, c\rangle + \alpha\|c\| - \langle w, e\rangle\varepsilon\|\hat{x} - \bar{x}\| + \alpha\varepsilon\|\hat{x} - \bar{x}\|\|e\|. \tag{4.4}$$

Based on the definition of the function $C^{\alpha\sharp}$, we have the following:

$$-\langle w, c\rangle + \alpha\|c\| < 0 \quad \text{and} \quad -\langle w, e\rangle + \alpha\|e\| < 0.$$

Then, from Eq (4.4), we can deduce that

$$\varphi(F_{\bar{x}}(\hat{x})) < \varphi(F_{\bar{x}}(\bar{x}) + (-\langle w, e\rangle + \alpha\|e\|)\,\varepsilon\|\hat{x} - \bar{x}\|.$$

This implies the following:

$$\varphi(F_{\bar{x}}(\hat{x})) - \varphi(F_{\bar{x}}(\bar{x}) + (\langle w, e\rangle - \alpha\|e\|)\,\varepsilon\|\hat{x} - \bar{x}\| < 0. \tag{4.5}$$

Since $-\langle w, e\rangle + \alpha\|e\| < 0$ and $\varepsilon \ge 0$, we can set $\varepsilon = (\langle w, e\rangle - \alpha\|e\|)\varepsilon$ in (4.5) to ensure $\varepsilon \ge 0$. Hence, (4.5) can be transformed into an equivalent form as follows:

$$\varphi(F_{\bar{x}}(\hat{x})) - \varphi(F_{\bar{x}}(\bar{x})) + \varepsilon\|\hat{x} - \bar{x}\| < 0.$$

However, this contradicts Eq (4.1), which means that $\bar{x}$ is an approximate quasi-Benson proper efficient solution to the VEPs. □

To illustrate the validity of Theorem 2, we present a concrete example in the following discussion.

Example 3. *In the VEP, let* $K = \mathbb{R}^3_+$, $C = \mathbb{R}^3_+$, *and* $D = \mathbb{R}_+$. *Define the vector-valued mappings* $F : K \times K \to \mathbb{R}^3$ *and* $G : K \to \mathbb{R}$ *as follows:*

$$F_{\bar{x}}(x) = \begin{pmatrix} (x_1^2 + x_2^2 + x_3^2)(1 + \bar{x}_1\bar{x}_2 + \bar{x}_3 \ln(1 + x_3)) \\ x_1 + x_2 + x_3 + (\bar{x}_1\bar{x}_2 + \bar{x}_2)e^{\bar{x}_3} \\ x_1x_2 + x_2x_3 + x_3x_1 + \bar{x}_3 \sin(\bar{x}_1) \end{pmatrix},$$

and

$$G(x) = x_1^2 + x_2^2 + x_3^2 - 1.$$

The feasible set Ω *is given by the following:*

$$\Omega = \{x \in K : G(x) \in -D\} = \{x \in \mathbb{R}^3_+ : x_1^2 + x_2^2 + x_3^2 \leq 1\}.$$

Let $\bar{x} = (0, 0, 0) \in \Omega$. *Then,* $F_{\bar{x}}(\bar{x}) = (0, 0, 0)$ *and*

$$F_{\bar{x}}(x) = \begin{pmatrix} x_1^2 + x_2^2 + x_3^2 \\ x_1 + x_2 + x_3 \\ x_1x_2 + x_2x_3 + x_3x_1 \end{pmatrix}.$$

For $C = \mathbb{R}^3_+$, *we have the following:*

$$C^\sharp = \{y^* \in \mathbb{R}^3 : \langle y^*, y\rangle > 0,\ \forall y \in \mathbb{R}^3_+\backslash\{0\}\} = \mathbb{R}^3_+\backslash\{0\},$$

Hence, we obtain the following:

$$C^{\alpha\sharp} = \{(y^*, \alpha) \in \mathbb{R}^3_+\backslash\{0\} \times \mathbb{R}_+ : \langle y^*, y\rangle - \alpha\|y\| > 0,\ \forall y \in \mathbb{R}^3_+ \setminus \{0\}\}.$$

Take $w = (2, 2, 2)$ *and* $\alpha = 0.1$*; thus,* $(w, \alpha) \in C^{\alpha\sharp}$. *Let* $\varepsilon = 1$, $e = (1, 1, 1) \in int\, C$. *Now verify that* $\bar{x}$ *is a quasi-optimal solution for* $(\mathbf{P}_\varphi)$. *For all* $x \in \Omega$, *the following holds*

$$\begin{aligned} &\varphi(F_{\bar{x}}(x)) - \varphi(F_{\bar{x}}(\bar{x})) + \varepsilon\|x - \bar{x}\| = w^T F_{\bar{x}}(x) + \alpha\|F_{\bar{x}}(x)\| + \|x\| \\ =&2(x_1^2 + x_2^2 + x_3^2) + 2(x_1 + x_2 + x_3) + 2(x_1x_2 + x_2x_3 + x_3x_1) + 0.1 \left\| \begin{pmatrix} x_1^2 + x_2^2 + x_3^2 \\ x_1 + x_2 + x_3 \\ x_1x_2 + x_2x_3 + x_3x_1 \end{pmatrix} \right\| + 0.1\|x\| \\ \geq&0. \end{aligned}$$

The non-negativity holds because all terms are non-negative for $x \in \mathbb{R}^3_+$ *with* $\|x\| \geq 0$. *Thus,* $\bar{x}$ *is quasi-optimal for* $(\mathbf{P}_\varphi)$.

Next, verify $\bar{x}$ *is an approximate quasi-Benson proper efficient solution as follows:*

$$\mathrm{cl}\,(\,\mathrm{cone}(F_{\bar{x}}(K) + C + \varepsilon\|x - \bar{x}\|e)) = \mathrm{cl}\,(\,\mathrm{cone}\,(\begin{pmatrix} x_1^2 + x_2^2 + x_3^2 \\ x_1 + x_2 + x_3 \\ x_1x_2 + x_2x_3 + x_3x_1 \end{pmatrix} + \mathbb{R}^3_+ + \|x - \bar{x}\|e)) = \mathbb{R}^3_+.$$

Thus, $\bar{x}$ *is an approximate quasi-Benson proper efficient solution.*

The following theorem establishes a duality correspondence between the approximate quasi-Benson proper efficient solutions of the VEP and quasi-optimal solutions of the scalarized problem ($\mathbf{P}_\varphi$), thus leveraging the monotonicity properties of the augmented dual cone $C^{\alpha\sharp}$.

Theorem 3. *Let $C \subset \mathbb{R}^m$ be a pointed closed convex cone, $\varepsilon \geq 0$, $e = (e_1, \ldots, e_m) \in int\, C$, and $\bar{x} \in \Omega$. If $(w, \alpha) \in C^{\alpha *}$ and $\bar{x}$ is an approximate quasi-Benson proper efficient solution for the VEP, then $\bar{x}$ is also a quasi-optimal solution for problem ($\mathbf{P}_\varphi$).*

Proof. If $\bar{x}$ is an approximate quasi-Benson proper efficient solution for the VEP, then

$$\mathrm{cl}(\mathrm{cone}(F_{\bar{x}}(x) + C + \varepsilon\|x - \bar{x}\|e)) \cap (-C) = \{0\}, \quad \forall x \in \Omega.$$

Additionally, $F_{\bar{x}}(\bar{x}) = 0$, we obtain the following:

$$\mathrm{cl}(\mathrm{cone}(F_{\bar{x}}(x) - F_{\bar{x}}(\bar{x}) + C + \varepsilon\|x - \bar{x}\|e)) \cap (-C) = \{0\}, \quad \forall x \in \Omega.$$

Since the function φ is monotonically increasing on $\mathbb{R}^m$ for all $(w, \alpha) \in C^{\alpha *}$. Therefore, for any $c \in C$ and $x \in \Omega$, we have the following:

$$\begin{aligned}\varphi(F_{\bar{x}}(\bar{x})) &\leq \varphi(F_{\bar{x}}(x) + c + \varepsilon\|x - \bar{x}\|e)\\ &\leq \varphi(F_{\bar{x}}(x)) + \varphi(c + \varepsilon\|x - \bar{x}\|e)\\ &\leq \varphi(F_{\bar{x}}(x)) + \langle w, c + \varepsilon\|x - \bar{x}\|e\rangle + \alpha\|c + \varepsilon\|x - \bar{x}\|e\|\\ &\leq \varphi(F_{\bar{x}}(x)) + \langle w, c\rangle + \alpha\|c\| + \langle w, e\rangle\varepsilon\|x - \bar{x}\| + \alpha\varepsilon\|x - \bar{x}\|\|e\|.\end{aligned} \tag{4.6}$$

Since $0 \in C$, from Eq (4.6), we obtain the following:

$$\begin{aligned}\varphi(F_{\bar{x}}(\bar{x})) &\leq \varphi(F_{\bar{x}}(x)) + \langle w, e\rangle\varepsilon\|x - \bar{x}\| + \alpha\varepsilon\|x - \bar{x}\|\|e\|\\ &\leq \varphi(F_{\bar{x}}(x)) + (\|w\| + \|e\|)\varepsilon\|x - \bar{x}\| + \alpha\varepsilon\|x - \bar{x}\|\|e\|\\ &\leq \varphi(F_{\bar{x}}(x)) + (\|w\| + \|e\| + \alpha\|e\|)\varepsilon\|x - \bar{x}\|.\end{aligned} \tag{4.7}$$

Let $\varepsilon' = (\|w\| + \|e\| + \alpha\|e\|)\varepsilon$. Form (4.7), we have

$$\varphi(F_{\bar{x}}(\bar{x}) \leq \varphi(F_{\bar{x}}(x)) + \varepsilon'\|x - \bar{x}\|.$$

Hence, $\bar{x}$ is a quasi-optimal solution to problem ($\mathbf{P}_\varphi$). This completes the proof. □

5. Conclusions

This paper presents the optimality and scalarization theorems for the approximate quasi-Benson proper efficient solutions within constrained VEPs. The optimality conditions are derived using the generalized convexity and convex separation theorems. Furthermore, two scalarization theorems were obtained by applying the cone scalarization function with the characteristics of generating cones. The introduced definitions and conclusions were validated through detailed numerical examples. However, obtaining the optimality conditions for local efficient solutions of uncentered vector equilibrium problems also remains an important yet unexplored problem. Moreover, developing more general scalarization functions and leveraging non-polyhedral cones to obtain (locally) efficient solutions constitutes a promising research direction.

Author contributions

Shan Cai: Conceptualization, Methodology, Data curation, Writing-original draft; Shengxin Hua: Conceptualization, Methodology, Formal analysis, Validation, Writing-original draft; Xiaoping Li: Validation, Writing-review and editing. All authors contributed significantly to the research and manuscript, approving the final version.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper. No financial or personal relationships with other people or organizations have influenced the research, the writing of the manuscript, or the decision to submit the paper for publication.

References

1. E. Blum, From optimization and variational inequalities to equilibrium problems, *Math. Student*, **63** (1994), 123–145.
2. Q. Ansari, E. Köbis, J. Yao, *Vector variational inequalities and vector optimization: theory and applications*, Cham: Springer, 2018. https://doi.org/10.1007/978-3-319-63049-6
3. F. Giannessi, *Vector variational inequalities and vector equilibria: mathematical theories*, New York: Springer, 2000. https://doi.org/10.1007/978-1-4613-0299-5
4. G. Kassay, V. Rădulescu, *Equilibrium problems and applications*, London: Academic Press, 2019. https://doi.org/10.1016/C2015-0-06685-0
5. S. Dempe, V. Kalashnikov, *Optimization with multivalued mappings: theory, applications and algorithms*, New York: Springer, 2006. https://doi.org/10.1007/0-387-34221-4
6. I. Konnov, J. Yao, Existence of solutions for generalized vector equilibrium problems, *J. Math. Anal. Appl.*, **233** (1999), 328–335. https://doi.org/10.1006/jmaa.1999.6312
7. F. Flores-Bazan, F. Flores-Bazan, Vector equilibrium problems under asymptotic analysis, *J. Global Optim.*, **26** (2003), 141–166. https://doi.org/10.1023/A:1023048928834
8. Q. Ansari, I. Konnov, J. Yao, Existence of a solution and variational principles for vector equilibrium problems, *J. Optimiz. Theory Appl.*, **110** (2001), 481–492. https://doi.org/10.1023/A:1017581009670

9. Y. Feng, Q. Qiu, Optimality conditions for vector equilibrium problems with constraint in Banach spaces, *Optim. Lett.*, **8** (2014), 1931–1944. https://doi.org/10.1007/s11590-013-0695-5
10. X. Gong, Scalarization and optimality conditions for vector equilibrium problems, *Nonlinear Anal.-Theor.*, **73** (2010), 3598–3612. https://doi.org/10.1016/j.na.2010.07.041
11. Q. Ansari, Vectorial form of Ekeland-type variational principle with applications to vector equilibrium problems and fixed point theory, *J. Math. Anal. Appl.*, **334** (2007), 561–575. https://doi.org/10.1016/j.jmaa.2006.12.076
12. M. Bianchi, G. Kassay, R. Pini, Ekeland's principle for vector equilibrium problems, *Nonlinear Anal.-Theor.*, **66** (2007), 1454–1464. https://doi.org/10.1016/j.na.2006.02.003
13. G. Chen, W. Rong, Characterizations of the Benson proper efficiency for nonconvex vector optimization, *J. Optimiz. Theory Appl.*, **98** (1998), 365–384. https://doi.org/10.1023/A:1022689517921
14. X. Yang, D. Li, S. Wang, Near-subconvexlikeness in vector optimization with set-valued functions, *J. Optimiz. Theory Appl.*, **110** (2001), 413–427. https://doi.org/10.1023/A:1017535631418
15. Q. Qiu, Optimality conditions of globally efficient solution for vector equilibrium problems with generalized convexity, *J. Inequal. Appl.*, **2009** (2009), 898213. https://doi.org/10.1155/2009/898213
16. X. Long, Y. Huang, Z. Peng, Optimality conditions for the Henig efficient solution of vector equilibrium problems with constraints, *Optim. Lett.*, **5** (2011), 717–728. https://doi.org/10.1007/s11590-010-0241-7
17. Q. Qiu, Optimality conditions for vector equilibrium problems with constraints, *J. Ind. Manag. Optim.*, **5** (2009), 783–790. https://doi.org/10.3934/jimo.2009.5.783
18. X. Gong, A characterization of super efficiency in vector equilibrium problems, *Optim. Lett.*, **5** (2011), 683–690. https://doi.org/10.1007/s11590-010-0234-6
19. R. Gasimov, Characterization of the Benson proper efficiency and scalarization in nonconvex vector optimization, In: *Multiple criteria decision making in the new millennium*, Berlin: Springer, 2001, 189–198. https://doi.org/10.1007/978-3-642-56680-6_17
20. E. Bishop, R. Phelps, The support functionals of a convex set, In: *Selected papers of Errett Bishop*, Singapore: World Scientific, 1986, 293–301. https://doi.org/10.1142/9789814415514_0020
21. C. Chen, X. Zuo, F. Lu, S. Li, Vector equilibrium problems under improvement sets and linear scalarization with stability applications, *Optim. Method. Softw.*, **31** (2016), 1240–1257. https://doi.org/10.1080/10556788.2016.1200043
22. Y. Han, N. Huang, Some characterizations of the approximate solutions to generalized vector equilibrium problems, *J. Ind. Manag. Optim.*, **12** (2016), 1135–1151. https://doi.org/10.3934/jimo.2016.12.1135
23. J. Ródenas Pedregosa, Caracterización de soluciones de problemas de equilibrio vectoriales, Ph.D Thesis, Universidad Nacional de Educación a Distancia, 2018.
24. Q. Qiu, X. Yang, Scalarization of approximate solution for vector equilibrium problems, *J. Ind. Manag. Optim.*, **9** (2013), 143–151. https://doi.org/10.3934/jimo.2013.9.143

25. F. Clarke, *Optimization and nonsmooth analysis*, Philadelphia: Society for Industrial and Applied Mathematics, 1990. https://doi.org/10.1137/1.9781611971309

26. R. Tyrrell Rockafellar, R. Wets, *Variational analysis*, Berlin: Springer, 2009. https://doi.org/10.1007/978-3-642-02431-3

27. K. Miettinen, *Nonlinear multiobjective optimization*, New York: Springer, 1998. https://doi.org/10.1007/978-1-4615-5563-6

28. J. Jahn, *Mathematical vector optimization in partially order linear spaces*, Frankfurt: Verlag Peter Lang, 1986.

29. B. Soleimani, C. Tammer, Concepts for approximate solutions of vector optimization problems with variable order structures, *Vietnam J. Math.*, **42** (2014), 543–566. https://doi.org/10.1007/s10013-014-0103-1

30. R. Kasimbeyli, A conic scalarization method in multi-objective optimization, *J. Glob. Optim.*, **56** (2013), 279–297. https://doi.org/10.1007/s10898-011-9789-8

31. R. Kasimbeyli, A nonlinear cone separation theorem and scalarization in nonconvex vector optimization, *SIAM J. Optimiz.*, **20** (2010), 1591–1619. https://doi.org/10.1137/070694089