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Research article

On statistical convergence in fractal analysis

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Abstract: This study investigated the statistical convergence of fractal-generating set sequences, motivated by the observation that natural fractals, influenced by external biological, chemical, or physical factors, rarely exhibit strict classical convergence. Instead, their limiting behavior often aligns with statistical patterns. We formalized the concept of statistical convergence for compact subsets of $\mathbb{R}^n$, introduced the notion of statistical Cauchy sequences, and established their sufficiency for statistical convergence—mirroring the classical relationship. Several illustrative examples and graphical simulations, including variants of the Sierpiński triangle and Koch snowflake, highlight the distinction between classical and statistical convergence. The proposed framework provides a more realistic and robust approach to understanding fractal structures in both theoretical and applied contexts.

Keywords: Hausdorff distance; fractal analysis; statistical convergence of sets; statistical Cauchy sequences; Banach fixed point theorem; box dimension
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1. Introduction

Fractals play a central role in modeling complex natural structures such as coastlines, clouds, and biological forms [1,2]. Unlike ideal mathematical fractals with exact self-similarity at every scale, real-world examples are typically statistically self-similar where their patterns recur in a distributional sense rather than identically. For instance, the measured length of a coastline increases without approaching a fixed limit as the resolution improves, illustrating the lack of classical convergence [3, 4].

This persistent complexity across scales suggests that such fractals require a statistical framework to describe their convergence behavior. In fact, since many natural fractals arise from stochastic or irregular processes, their self-similarity is better understood in statistical terms [5, 6].

To support this idea, we present two real-world scenarios where statistical convergence provides a more robust framework than classical convergence in modeling natural fractal structures.

First, we consider the case of statistical robustness in the Barnsley fern under environmental perturbations. In natural systems, the formation of fern leaves well modeled by the Barnsley fern fractal is subject to a variety of biological and environmental influences. Each small leaflet can be considered as an iteration in the fractal-generating process. However, factors such as localized insect damage, cold weather conditions, or chemical imbalances in the soil can cause some leaflets to be malformed or entirely absent. From the perspective of classical convergence, such irregularities disrupt the deterministic iteration scheme and prevent convergence to the ideal geometric fractal. However, these disruptions are typically sparse and non-systematic. Within the framework of statistical convergence, the overall structure of the fern remains stable, since the vast majority of leaflets adhere to the expected pattern. This supports the notion that statistical attractors offer a more realistic and biologically meaningful model for natural fractal formations.

Second, we examine coastal fractals with urban modifications. Coastal shorelines are classical examples of natural fractals that exhibit self-similarity at multiple scales. In practice, however, certain segments of a coastline are altered due to human activities such as the construction of ports, piers, or concrete reinforcements. These man-made interventions act as perturbed iterations in the fractal-generating process, replacing natural features with non-fractal structures. While these disruptions violate classical convergence, they typically affect only a zero-density subset of the overall process and remain localized. Consequently, the overall coastal formation may still statistically converge to a fractal-like structure, as the majority of the process retains its scale-invariant and self-similar characteristics. This demonstrates the superiority of statistical convergence in capturing the asymptotic behavior of natural systems subjected to random or external perturbations.

These examples underscore the practical relevance of statistical convergence in modeling real-world fractal geometries. In situations where sparse but significant disturbances are present, classical convergence fails to accurately describe the limiting geometry, whereas statistical convergence provides a more stable and reliable framework for understanding the long-term behavior of such biological and physical systems.

Recognizing that fractals are closed and bounded sets, many classical results on convergence and Cauchy sequences of sets such as those found in [7] are extended in this work to the statistical setting. These statistical generalizations, particularly those concerning statistical convergence and statistical Cauchy sequences, have not been previously addressed in the literature and are presented here as natural and original contributions.

Statistical convergence, introduced independently by H. Fast and H. Steinhaus [8, 9], allows for a sequence to converge when the majority of its terms approach a limit, despite sparse irregularities. This concept has since found applications in diverse analytical contexts [10–12], especially where data exhibits random fluctuations [13–15]. In fractal analysis, this provides a framework for understanding convergence in settings where small-scale irregularities are unavoidable.

While earlier works such as [16–18] have explored the statistical convergence of closed sets, our approach introduces the notion of statistical attractors derived from stochastic iteration schemes on fractal structures. This extension broadens fixed point theory under perturbations.

Another essential notion is that of statistical Cauchy sequences, which generalize the classical Cauchy condition by tolerating rare outliers. Many iterative processes in fractal construction, especially those involving randomness, fail to satisfy the classical Cauchy condition but meet the statistical version. This insight motivates our formal treatment of these ideas.

Contributions of this study:

- We define statistical convergence in the context of fractal-generating sequences, showing how traditional convergence criteria can be relaxed to accommodate real-world irregularities.
- We introduce the concept of statistical Cauchy sequences and demonstrate its necessity and sufficiency for statistical convergence, paralleling the classical theory.
- We present examples illustrating the distinction between classical and statistical convergence, emphasizing processes such as random iteration where statistical convergence is more appropriate.

2. Key concepts

In this part, we go over the essential definitions and properties of statistical convergence and related concepts, laying the groundwork for their application to fractals. We also provide illustrative examples to contrast statistical convergence with ordinary convergence.

2.1. Statistical convergence

Classical convergence of a sequence $(x_n)_{n\geq 1}$ to a limit L means that for every $\varepsilon > 0$, eventually all terms x_n lie within ε of L. Statistical convergence generalizes this by only requiring that almost all terms eventually lie within ε of the limit, where "almost all" is quantified by natural density.

Recall that a subset $A \subset \mathbb{N}$ has natural density θ if

$$\lim_{N\to\infty} \frac{1}{N}\Big|\{k \leq N : k \in A\}\Big| = \theta,$$

provided the limit exists. In particular, saying A has natural density 0 means that the proportion of elements of A among $\{1, 2, \ldots, N\}$ goes to 0 as $N \to \infty$. Using this notion, we define statistical convergence as follows.

Definition 2.1. *[8, 9] A sequence (x_n) of real (or complex) numbers is statistically convergent to a real number L if for all $\varepsilon > 0$, the set*

$$A(\varepsilon) := \{\, n \in \mathbb{N} : |x_n - L| \geq \varepsilon \,\}$$

has natural density 0. Equivalently,

$$\lim_{N\to\infty} \frac{1}{N}\Big|\{1 \leq n \leq N : |x_n - L| \geq \varepsilon\}\Big| = 0,$$

for every $\varepsilon > 0$. If (x_n) is statistically convergent to L, we write $x_n \xrightarrow{\text{stat}} L$ and call L the statistical limit of the sequence.

This definition means that for any fixed tolerance ε, the fraction of terms x_n that fall outside the ε-neighborhood of L becomes negligible as n grows. In particular, every ordinary convergent sequence is statistically convergent (with the same limit), since if $x_n \to L$ in the usual sense, all but finitely many terms lie in any given neighborhood of L, and finitely many exceptions clearly have density zero. The converse is not true in general: there are sequences that are statistically convergent but not convergent in the usual sense. A classic example is the sequence defined by

$$x_n = \begin{cases} 1, & \text{if } n \text{ is square,} \\ 0, & \text{otherwise.} \end{cases}$$

This sequence does not converge in the ordinary sense, because it continually takes the value 1 on infinitely many (though rare) occasions. However, by the prime number theorem, the proportion of indices n for which $x_n = 1$ (i.e., n is prime) goes to zero as $n \to \infty$. Equivalently, x_n converges to 0 in the statistical sense, since for any ε with $0 < \varepsilon < 1$, the set of n with $|x_n - 0| \geq \varepsilon$ is precisely the set of prime numbers, which has density 0. Thus $x_n \xrightarrow{\text{stat}} 0$. This example highlights that statistical convergence allows one to overlook a sparse set of atypical behavior and still detect a limiting trend.

Statistical convergence is particularly useful in the context of fractals and irregular structures. For instance, consider a random iterative algorithm generating a fractal (such as the chaos game for the Sierpiński triangle). The sequence of intermediate outputs (say, the cloud of n points after n iterations) does not converge in the usual sense to the final fractal—each new iteration introduces some new points and the set of points keeps changing. However, one can show that the distribution of points does converge: as n grows, the fraction of points landing in any region of the plane stabilizes to a limiting value (the value given by the fractal's invariant measure). In effect, the empirical distribution of points converges to the fractal's exact distribution. This is a form of statistical convergence: if we describe the fractal approximation at stage n by a function (for example, an indicator function on the set of points or a density function over space), that function sequence converges to a limit in an averaged sense. We can say that the sequence of point sets is statistically convergent to the fractal (even though no single point or curve is tracing a deterministic limit). Such interpretations align with the idea that real fractals converge through stabilization of statistical properties rather than strict geometric convergence.

2.2. *Statistical Cauchy sequence*

In normed spaces like the real numbers, a sequence is convergent if and only if(iff) it is Cauchy. There is an analogous characterization for statistical convergence using a statistical Cauchy criterion. Intuitively, a sequence (x_n) is statistically Cauchy if, as n, m become large, x_n and x_m are close for "most" pairs of indices, ignoring a set of exceptional pairs of density zero.

Definition 2.2. *A sequence (x_n) is called statistically Cauchy if for every $\varepsilon > 0$ there exists an N such that*

$$\frac{1}{N}\Big|\{ (i, j) : i, j > N \text{ and } |x_i - x_j| \geq \varepsilon \}\Big| = 0.$$

In other words, beyond some index N, the proportion of pairs of terms that differ by at least ε is zero (in the limit of large indices). Equivalently (and more conveniently), one can require that there exists a subset $M \subset \mathbb{N}$ of density 1 (i.e., M contains almost all natural numbers) such that for any $\varepsilon > 0$, there is an N with the property that $|x_i - x_j| < \varepsilon$ for all $i, j \in M$ with $i, j > N$. This condition means that if we ignore a sparse set of "outlier" indices, the remaining bulk of the sequence eventually behaves like a Cauchy sequence under the usual definition.

It can be shown that, in the real numbers, a sequence is statistically convergent iff it is statistically Cauchy (this is analogous to the classical theorem that $\mathbb{R}$ is complete, so Cauchy convergence criteria apply). The idea was formally established by scholars such as J. A. Fridy, who provided a rigorous Cauchy criterion for statistical convergence [19]. The statistical Cauchy criterion is particularly important because it allows one to verify statistical convergence without explicitly knowing the candidate limit L [20,21]. In practice, when analyzing fractal sequences, it might be easier to check that the sequence's elements are getting closer to each other in the statistical sense (indicating the sequence is settling into a stable pattern), rather than identifying an exact limit shape or value.

2.3. Related concepts and remarks

It is worth noting that statistical convergence is one among several generalized convergence concepts useful for describing fractal phenomena. Others include convergence in distribution (especially for random processes), and various summability methods that can capture limit behavior when classical convergence fails. Statistical convergence, however, has the appeal of being a simple extension of the usual notion of convergence, grounded in the natural density concept. In the setting of fractals, one often encounters the idea of statistical self-similarity, as mentioned earlier. Mandelbrot's work on fractals introduced this term to describe objects like coastlines, where self-similarity holds in a stochastic sense. This idea connects with statistical convergence: as one examines finer scales of a statistically self-similar fractal, one is not approaching a single deterministic shape, but the statistical properties (like the distribution of roughness or the fractal dimension) converge to stable values. For example, the fractal dimension of a natural fractal can be estimated by statistical sampling methods (e.g., box-counting on finer grids), and those estimates converge to a limit, even though the object itself remains random or irregular at each scale. This is another demonstration of how statistical convergence underlies fractal geometry in practice.

In summary, statistical convergence and statistical Cauchy convergence provide a framework to rigorously discuss the convergence of fractal structures as observed in nature and simulations. These concepts allow mathematicians and scientists to reconcile the apparent paradox of fractals "converging" to stable forms (such as a limiting fractal dimension or distribution) despite never settling into a single deterministic shape. In the remainder of this paper, we will apply these definitions to specific fractal construction sequences and demonstrate how they satisfy statistical convergence where ordinary convergence fails.

2.4. Metric spaces and the Hausdorff distance

One of the key concepts playing a central role in this study is the *Hausdorff distance*. Given a metric space (X, d) and two compact subsets $A, B \subseteq X$, the Hausdorff distance $d_H(A, B)$ is defined as follows [22]:

$$d_H(A, B) = \max\left\{\sup_{a\in A}\inf_{b\in B} d(a, b), \sup_{b\in B}\inf_{a\in A} d(a, b)\right\}. \tag{2.1}$$

If $d_H(A_n, A) \to 0$, then the sequence of sets $\{A_n\}$ is said to converge to A in the Hausdorff metric. This approach is widely used to define the notion of the limit of subsets in general, including fractal sets [23, 24].

2.5. Banach fixed point theorem and set-valued transformations

The Banach fixed point theorem asserts that every contraction mapping on a complete metric space has a unique fixed point, and that successive iterations starting from any initial point converge to this fixed point. When a set-valued operator (such as the Hutchinson operator) satisfies the contraction property, it can be shown that there exists a unique compact attractor under this operator.

In particular, an iterated function system (IFS) consisting of finitely many contraction mappings is defined as:

$$F(A) = f_1(A) \cup f_2(A) \cup \cdots \cup f_k(A),$$

where each $f_i : X \to X$ is a contraction. Following the approach of Hutchinson [25], the existence and uniqueness of a fixed set of the operator F can be established via arguments analogous to the Banach fixed point theorem.

2.6. Classical fractal examples

Self-similar fractals are the most typical applications of iterated function systems.

- **Cantor Set:** A zero-measure set obtained by repeatedly removing the middle third of a line segment. It can be characterized as the infinite intersection of finitely many subintervals.
- **Sierpiński Triangle:** Constructed by repeatedly removing the central part of a triangle. It is widely studied in the plane and serves as a classic example of multiple contraction mappings.
- **Julia and Mandelbrot Sets:** Defined in the complex plane as the boundaries of bounded or unbounded orbits generated by iterations of polynomial or rational functions [26].
- **Koch Snowflake:** A fractal curve generated by recursively adding equilateral triangular protrusions to each side of a base triangle. It exhibits an infinite perimeter and finite area, making it a classical example of self-similar, non-differentiable curves [27].

Various perspectives on the statistical convergence of sequences of sets have been proposed in the literature, among which the most notable are those by Nuray and Rhoades [17], Georgiou et al. [16], and Talo et al. [18]. In the following section, building upon these foundational approaches and motivated by the scenarios outlined in the Introduction we consider that real-world fractals, due to their inherent nature, can naturally be modeled as compact subsets of $\mathbb{R}^n$, that is, closed and bounded sets. Within this framework, we examine the statistical convergence behavior of fractals through the lens of the metric defined in Eq (2.1) and the newly introduced concept of a statistical attractor (i.e., a statistical fixed set). This formulation is supported with concrete examples and corollaries that demonstrate the scope and validity of our approach, which is further informed by the broader topological framework surrounding statistical convergence [28].

3. Main results

3.1. Statistical convergence under Hutchinson operators

While classical fixed point theory ensures convergence of compact sets under a contractive set operator (e.g., the Hutchinson operator), real-world applications often display irregularities that inhibit exact convergence. In such cases, it is natural to consider convergence in a statistical sense.

Definition 3.1. *Let (A_n) be a sequence of compact subsets in a metric space (X, d), and let A be another compact subset. We say that (A_n) statistically converges to A in the Hausdorff metric if for every $\varepsilon > 0$, the set*

$$\{n \in \mathbb{N} : d_H(A_n, A) \geq \varepsilon\}$$

has natural density zero.

This type of convergence allows for occasional fluctuations, which is especially relevant when the fractal construction involves randomness or perturbation.

Theorem 3.1. *Let $(\mathcal{K}(X), d_H)$ be the space of non-empty compact subsets of a metric space (X, d) equipped with the Hausdorff metric. If a sequence (A_k) converges classically to A in d_H, then it is also statistically convergent to A.*

Proof. Let (A_k) be a sequence in $\mathcal{K}(X)$ such that $d_H(A_k, A) \to 0$ as $k \to \infty$. Fix any $\varepsilon > 0$. By classical convergence, there exists $k_0 \in \mathbb{N}$ such that

$$d_H(A_k, A) < \varepsilon \quad \text{for all } k \geq k_0.$$

Thus, the set

$$E = \{k \in \mathbb{N} : d_H(A_k, A) \geq \varepsilon\}$$

is finite. Since the asymptotic density of a finite set is zero, it follows that

$$\lim_{N\to\infty} \frac{1}{N} |\{k \leq N : d_H(A_k, A) \geq \varepsilon\}| = 0.$$

This holds for arbitrary $\varepsilon > 0$. Therefore, by the definition of statistical convergence in the Hausdorff metric, the sequence (A_k) is statistically convergent to A. □

Theorem 3.2 (Uniqueness of the statistical limit)**.** *Let (A_k) be a sequence in the metric space $(\mathcal{K}(X), d_H)$ of non-empty compact subsets of a metric space (X, d). If the sequence is statistically convergent to both A and B, then A = B.*

Proof. Assume that (A_k) is statistically convergent to both A and B under the Hausdorff metric d_H.

Let $\varepsilon > 0$ be arbitrary. Then, by the definition of statistical convergence, we have:

$$\lim_{N\to\infty} \frac{1}{N} |\{k \leq N : d_H(A_k, A) \geq \varepsilon/2\}| = 0,$$

$$\lim_{N\to\infty} \frac{1}{N} |\{k \leq N : d_H(A_k, B) \geq \varepsilon/2\}| = 0.$$

Let us define the set

$$E_N := \{k \leq N : d_H(A_k, A) < \varepsilon/2 \text{ and } d_H(A_k, B) < \varepsilon/2\}.$$

By the subadditivity of asymptotic density, we have:

$$\lim_{N\to\infty} \frac{|E_N|}{N} = 1.$$

For each $k \in E_N$, the triangle inequality yields

$$d_H(A, B) \leq d_H(A, A_k) + d_H(A_k, B) < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $d_H(A, B) = 0$, and hence $A = B$. □

Definition 3.2. *(Classical attractor [25]) Let (X, d) be a complete metric space and let $\mathcal{F} = \{f_i : X \to X\}_{i=1}^N$ be a finite family of contraction mappings. Define the Hutchinson operator $F : \mathcal{K}(X) \to \mathcal{K}(X)$ by*

$$F(A) = \bigcup_{i=1}^{N} f_i(A).$$

Then there exists a unique non-empty compact set $A^* \in \mathcal{K}(X)$ such that

$$F(A^*) = A^*,$$

and for any initial compact set $A_0 \in \mathcal{K}(X)$, the sequence $A_{n+1} = F(A_n)$ converges in the Hausdorff metric:

$$d_H(A_n, A^*) \to 0 \quad \text{as } n \to \infty.$$

The set A^* is called the classical attractor of the IFS.

Now, let us illustrate the above definition through Examples 3.1 and 3.2 by visualizing how two well-known fractal structures emerge as limits of set sequences. In each case, we demonstrate how each shape is generated as an iteration of the previous one, and how, in the infinite case, the sequence converges perfectly to the Sierpiński triangle and the Koch snowflake, respectively.

Example 3.1. *(Sierpiński triangle as a classical attractor) Let S_0 be a closed equilateral triangle in the Euclidean plane. At each iteration, construct S_{n+1} from S_n by removing the open central triangle formed by connecting the midpoints of each side of every equilateral triangle in S_n. This procedure generates a sequence of compact and bounded sets $(S_n)_{n \in \mathbb{N}}$ such that*

$$S_0 \supset S_1 \supset S_2 \supset \cdots \supset S_n \supset \cdots.$$

As $n \to \infty$, the sequence (S_n) converges in the Hausdorff metric to a non-empty, compact, and self-similar set known as the Sierpiński triangle. This set is the classical attractor associated with the described iterative process.

Geometrically, at each stage, one fourth of the area of each triangle is removed, so the total area of S_n tends to zero as $n \to \infty$, i.e.,

$$\lim_{n \to \infty} \text{Area}(S_n) = 0.$$

Nevertheless, the limit set retains nontrivial structure and has Hausdorff dimension

$$\dim_H(S) = \frac{\ln 3}{\ln 2},$$

which reflects its intrinsic fractal nature.

The first eight sets $S_1, S_2, \ldots, S_8$ generated by the iterative process are illustrated in Figure 1, which visually demonstrates the recursive structure leading to the classical attractor known as the Sierpiński triangle.

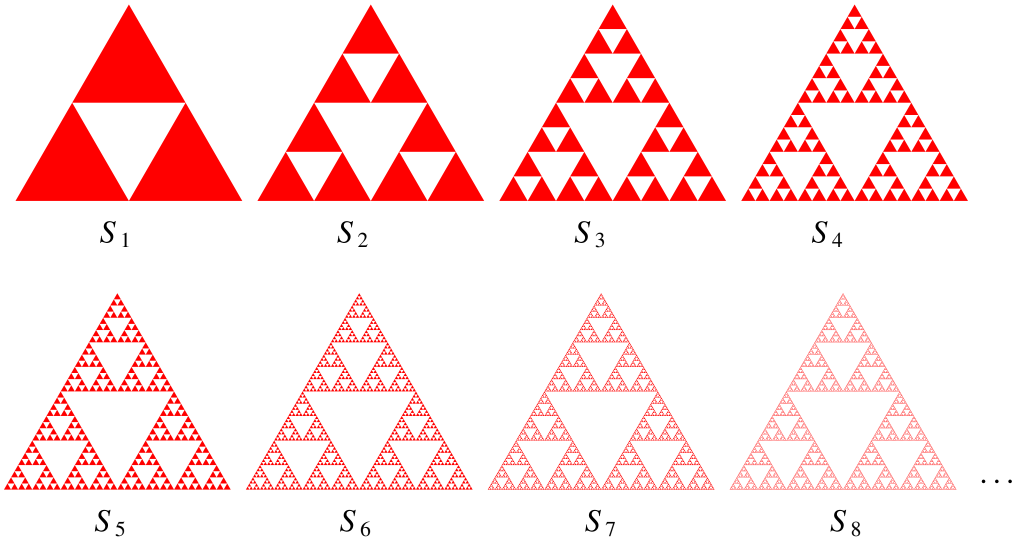

Figure 1. Sierpinski iteration.

Example 3.2 (Koch snowflake as a classical attractor)**.** *The Koch snowflake is a classical example of a self-similar fractal generated by an iterative process. Starting with an equilateral triangle* A_0*, at each iteration* n*, each line segment is divided into three equal parts, and the middle part is replaced by two segments that form an equilateral "bump". Repeating this process indefinitely produces a limiting set known as the Koch snowflake, which serves as the classical attractor of the associated iterated function system (IFS). A schematic visualization of the first few steps of this construction is given in Figure 2.*

An important property of the Koch snowflake is that as the number of iterations $n \to \infty$*:*

- *The total area enclosed by the snowflake remains finite, approaching a specific bounded value.*
- *The total perimeter grows without bound, i.e., it tends to infinity.*

This paradoxical behavior—bounded area with infinite perimeter—is characteristic of many fractals and highlights the need for tools beyond classical geometry to fully describe them.

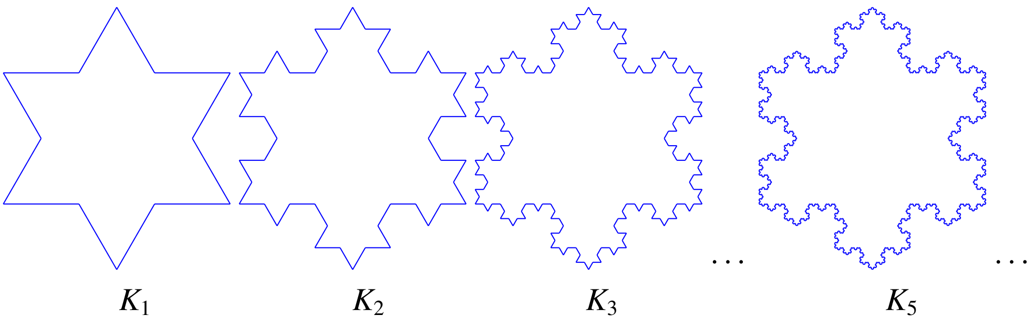

Figure 2. Koch snowflake as a classical attractor.

Statistical attractors in dynamical systems

The notion of a statistical attractor generalizes the classical concept of attractors in dynamical systems by incorporating statistical convergence criteria. Traditionally, an attractor is defined as a set toward which a system evolves over time, with trajectories remaining close to this set under small perturbations. Prominent examples include fixed points, limit cycles, and strange attractors, such as the Lorenz attractor [29].

In contrast, a statistical attractor considers the convergence of a sequence of sets $\{A_n\}$ under a set-valued contraction operator $F : \mathcal{K}(X) \to \mathcal{K}(X)$, where $\mathcal{K}(X)$ denotes the collection of non-empty compact subsets of a metric space X.

This framework becomes particularly relevant in scenarios where systems are influenced by noise or where deterministic convergence is too restrictive. For instance, in soliton dynamics, certain non-integrable systems exhibit the tendency of waveforms to evolve toward stable structures that behave like attractors, despite the presence of perturbations [30]. Likewise, in measure-theoretic treatments of dynamical systems, statistical properties of orbits—such as time averages and invariant measures—are central to understanding long-term behavior [31].

By adopting the viewpoint of statistical attractors, one obtains a more flexible and robust framework for analyzing the asymptotic behavior of complex systems, encompassing both deterministic and stochastic dynamics within a unified theoretical structure.

Definition 3.3 (Statistical attractor)**.** *Let* (X, d_H) *be a metric space and let* $\mathcal{K}(X)$ *denote the collection of all nonempty compact subsets of* X*, equipped with the Hausdorff metric* d_H*.*

Let $F : \mathcal{K}(X) \to \mathcal{K}(X)$ *be a set-valued contraction operator, for example, one induced by an iterated function system (IFS), meaning that for all* $A, B \in \mathcal{K}(X)$*,*

$$d_H(F(A), F(B)) \leq c\, d_H(A, B)$$

for some constant $0 < c < 1$*. Then, a compact set* $A \in \mathcal{K}(X)$ *is called a statistical fixed set or statistical attractor of* F *if, for every initial set* $A_0 \in \mathcal{K}(X)$*, the sequence defined by the iterative scheme*

$$A_{n+1} = F(A_n), \quad n \geq 0,$$

is statistically convergent to A *in the sense of the Hausdorff metric. That is,*

$$\lim_{N\to\infty} \frac{1}{N} |\{n \leq N : d_H(A_n, A) \geq \varepsilon\}| = 0 \quad \textit{for every } \varepsilon > 0.$$

Remark 3.1. *The classical attractor requires pointwise convergence in the Hausdorff metric, whereas the statistical attractor relaxes this condition and requires convergence in a statistical sense. This is especially relevant in settings where exact convergence is obstructed by external factors such as computational noise, perturbations, or randomness. Statistical attractors generalize the classical framework to include such practical deviations.*

Now, let us present Example 3.3, which corresponds to a statistical attractor as defined in Definition 3.3 and serves as a counterpart to the classical attractor examples previously introduced and visualized above. For the sake of brevity, we illustrate this construction only in the case of the Sierpiński triangle.

Example 3.3 (Statistical attractor based on prime indices)**.** *In contrast to the classical attractor described earlier, we now define a set-valued iterative process governed by the distribution of prime* *numbers, which constitute a subset of the natural numbers with zero natural density. Let* (S_n) *be a sequence of compact sets generated by the following rule:*

$$F(S_n) = \begin{cases} S_0 \setminus S_{n+1}, & \text{if } n+1 \text{ is prime,} \\ S_{n+1}, & \text{otherwise.} \end{cases}$$

That is, whenever the index $n+1$ *is a prime number, the next iterate is obtained by removing* S_{n+1} *from the original set* S_0*, whereas for non-prime indices, the iteration proceeds as in the classical Sierpiński construction.*

The first eight iterations of this process are illustrated schematically in Figure 3. Due to the fact that the set of prime numbers has zero natural density, the operation $F(S_n) = S_0 \setminus S_{n+1}$ *occurs sparsely in the iteration sequence. As a result, the sequence* (S_n) *does not converge in the classical sense. However, it is statistically convergent: the symmetric difference between* S_n *and a limiting fractal set tends to zero in density, rather than pointwise. Therefore, the limiting object of this process is referred to as a statistical attractor.*

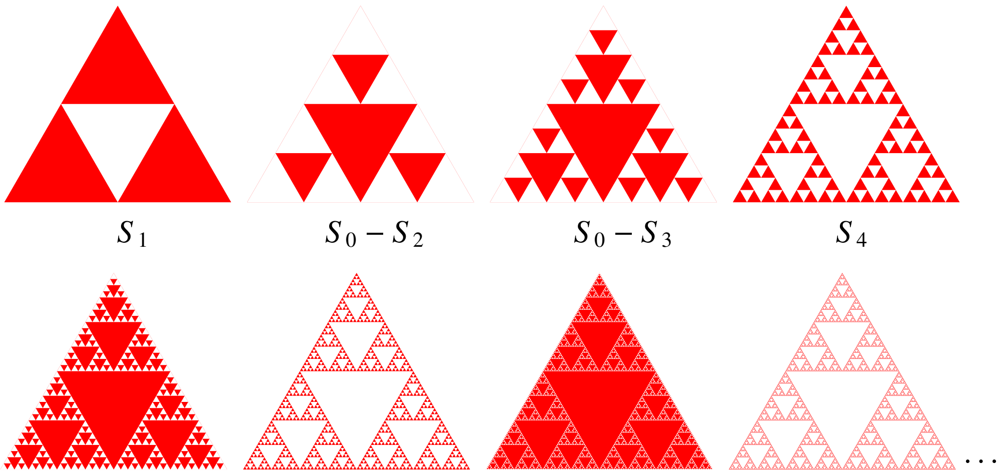

Figure 3. Statistical attractor.

Theorem 3.3. *Let* (A_n) *be a statistically convergent sequence in* $(\mathcal{K}(X), d_H)$*. Then its statistical limit is unique.*

Proof. Suppose $A_n \xrightarrow{st} A$ and $A_n \xrightarrow{st} B$. Fix $\varepsilon > 0$. From both convergences, we have:

$$\lim_{N\to\infty} \frac{1}{N} |\{n \leq N : d_H(A_n, A) \geq \varepsilon/2\}| = 0,$$

$$\lim_{N\to\infty} \frac{1}{N} |\{n \leq N : d_H(A_n, B) \geq \varepsilon/2\}| = 0.$$

Let $E_N = \{n \leq N : d_H(A_n, A) < \varepsilon/2 \text{ and } d_H(A_n, B) < \varepsilon/2\}$. Then for all $n \in E_N$, by the triangle inequality:

$$d_H(A, B) \leq d_H(A, A_n) + d_H(A_n, B) < \varepsilon.$$

Hence $d_H(A, B) < \varepsilon$ for all $\varepsilon > 0$, which implies $A = B$. □

Definition 3.4 (Statistical cluster subsequence in $\mathcal{K}(X)$)**.** *Let (A_k) be a sequence in the space $(\mathcal{K}(X), d_H)$ of non-empty compact subsets of a metric space (X, d). A subsequence (A_{k_n}) is called a statistical cluster subsequence if there exists $A \in \mathcal{K}(X)$ such that for every $\varepsilon > 0$,*

$$\limsup_{N\to\infty} \frac{1}{N} \left|\{n \leq N : d_H(A_{k_n}, A) \geq \varepsilon\}\right| = 0.$$

Theorem 3.4. *Let (A_k) and (B_k) be sequences in the space $(\mathcal{K}(X), d_H)$ of non-empty compact subsets of a metric space (X, d). Suppose that $d_H(B_k, A) \xrightarrow{st} 0$ for some $A \in \mathcal{K}(X)$, and for all $k \in \mathbb{N}$,*

$$d_H(A_k, A) \leq d_H(B_k, A).$$

Then (A_k) also statistically converges to A.

Proof. Since $d_H(B_k, A) \xrightarrow{st} 0$, for every $\varepsilon > 0$ we have:

$$\lim_{N\to\infty} \frac{1}{N} |\{k \leq N : d_H(B_k, A) \geq \varepsilon\}| = 0.$$

Given that for all $k \in \mathbb{N}$ we have $d_H(A_k, A) \leq d_H(B_k, A)$, it follows that

$$\{k \leq N : d_H(B_k, A) < \varepsilon\} \subseteq \{k \leq N : d_H(A_k, A) < \varepsilon\}.$$

Therefore,

$$\lim_{N\to\infty} \frac{1}{N} |\{k \leq N : d_H(A_k, A) \geq \varepsilon\}| \leq \lim_{N\to\infty} \frac{1}{N} |\{k \leq N : d_H(B_k, A) \geq \varepsilon\}| = 0.$$

Thus, (A_k) statistically converges to A. □

Definition 3.5. *(Statistical Cauchy sets) A sequence (A_n) of compact subsets is called statistically Cauchy in the Hausdorff metric if for every $\varepsilon > 0$, the set*

$$\{(m, n) \in \mathbb{N}^2 : d_H(A_m, A_n) \geq \varepsilon\}$$

has natural density zero in $\mathbb{N}^2$.

It can be shown that in a complete metric space of compact subsets (equipped with the Hausdorff metric), every statistically Cauchy sequence is statistically convergent. This parallels the classical result and justifies the use of statistical Cauchy criteria in fractal iteration schemes where exact convergence is impractical but statistical regularity can still be ensured.

Definition 3.6. *(Statistically dense subsequence) Let (A_k) be a sequence in the space $(\mathcal{K}(X), d_H)$ of non-empty compact subsets of a metric space (X, d). A subsequence (A_{k_n}) is said to be statistically dense in (A_k) if its index set $\{k_n : n \in \mathbb{N}\}$ is statistically dense in $\mathbb{N}$, that is,*

$$\lim_{N\to\infty} \frac{1}{N} |\{k_n \leq N : n \in \mathbb{N}\}| = 1.$$

Theorem 3.5. *Let $(\mathcal{K}(X), d_H)$ be the space of non-empty compact subsets of a complete metric space (X, d) equipped with the Hausdorff metric. Then every statistically Cauchy sequence (A_n) in $\mathcal{K}(X)$ is statistically convergent.*

Proof. Since $(\mathcal{K}(X), d_H)$ is a complete metric space, every classically Cauchy sequence converges.

Suppose that (A_n) is a statistically Cauchy sequence in $\mathcal{K}(X)$. Then, for every $\varepsilon > 0$, we have

$$\lim_{N\to\infty} \frac{1}{N^2} \left|\left\{(m,n) \in \{1,\ldots,N\}^2 : d_H(A_m, A_n) \geq \varepsilon\right\}\right| = 0.$$

To prove convergence, we will construct a subsequence (A_{n_k}) that is classically Cauchy using a diagonal argument. For each $k \in \mathbb{N}$, define

$$E_k = \left\{n \leq N_k : d_H(A_n, A_m) < \frac{1}{k} \text{ for all } m \leq N_k \text{ with } |n-m| < k\right\},$$

where N_k is chosen sufficiently large so that the proportion of indices not in E_k is less than $\frac{1}{k}$, i.e.,

$$\frac{|E_k^c|}{N_k} < \frac{1}{k}.$$

This is possible due to the statistical Cauchy condition.

Now, select $n_k \in E_k$ such that $n_k < n_{k+1}$ for all k, forming a strictly increasing subsequence. Then, for each $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that for all $k, l \geq K$, we have

$$d_H(A_{n_k}, A_{n_l}) < \varepsilon.$$

Thus, (A_{n_k}) is a classically Cauchy subsequence and, since $\mathcal{K}(X)$ is complete, it converges to some $A \in \mathcal{K}(X)$.

Finally, by a known result from statistical convergence theory (see, e.g., [19]), the statistical Cauchy property of (A_n) combined with the convergence of the subsequence (A_{n_k}) implies that the whole sequence (A_n) statistically converges to the same limit A.

□

Theorem 3.6. *(Equivalences for statistical convergence in $(\mathcal{K}(X), d_H)$) Let (A_k) be a sequence in the space $(\mathcal{K}(X), d_H)$ of non-empty compact subsets of a complete metric space (X, d). Then the following statements are equivalent:*

(1) (A_k) is statistically convergent to some $A \in \mathcal{K}(X)$;
(2) There exists a sequence (B_k) such that $B_k \to A$ classically in d_H, and $A_k = B_k$ for almost every $k \in \mathbb{N}$;
(3) (A_k) admits a statistically dense subsequence (A_{k_n}) that converges classically to A;
(4) (A_k) admits a statistically dense subsequence (A_{k_n}) that statistically converges to A.

Proof. (1) ⇒ (2): If (A_k) statistically converges to A, define a modified sequence (B_k) as follows:

$$B_k = \begin{cases} A_k, & \text{if } d_H(A_k, A) < \varepsilon_k \text{ with } \varepsilon_k \to 0, \\ A, & \text{otherwise.} \end{cases}$$

Then $B_k = A_k$ except on a set of indices of density zero. So $B_k \to A$ classically and $A_k = B_k$ a.e.

(2) ⇒ (3): The classical convergence of (B_k) implies that its statistically dense subsequence (B_{k_n}) converges to A. Since $A_k = B_k$ a.e., $(A_{k_n}) = (B_{k_n})$ is also a statistically dense subsequence of (A_k) converging to A.

(3) $\Rightarrow$ (4): Classical convergence implies statistical convergence, so the statistically dense subsequence also converges statistically to A.

(4) $\Rightarrow$ (1): For every $\varepsilon > 0$, use density inclusion:

$$\{k_n : d_H(A_{k_n}, A) < \varepsilon\} \subset \{k : d_H(A_k, A) < \varepsilon\},$$

and pass to densities to show statistical convergence of (A_k). □

Corollary 3.1. *In the metric space $(\mathcal{K}(X), d_H)$ of non-empty compact subsets of a complete metric space (X, d), every statistically convergent sequence has a classically convergent subsequence.*

Proof. Let (A_n) be a statistically convergent sequence to $A \in \mathcal{K}(X)$. By Theorem 3.6 (i.e., the equivalence theorem for statistical convergence), condition (1) implies condition (3), which asserts that the sequence admits a statistically dense subsequence that converges classically.

Thus, there exists a subsequence (A_{n_k}) such that $A_{n_k} \to A$ in the Hausdorff metric. Hence, every statistically convergent sequence in $(\mathcal{K}(X), d_H)$ has a convergent subsequence. □

Theorem 3.7. *Let (A_n) be a statistically convergent sequence in the space $(\mathcal{K}(X), d_H)$ of non-empty compact subsets of a metric space (X, d). Then (A_n) is a statistical Cauchy sequence in the sense that:*

$$\forall \varepsilon > 0, \quad \lim_{N\to\infty} \frac{1}{N^2} \left|\left\{(m, n) \in \{1, \ldots, N\}^2 : d_H(A_m, A_n) \geq \varepsilon\right\}\right| = 0.$$

Proof. Assume $A_n \xrightarrow{st} A \in \mathcal{K}(X)$. Then for every $\varepsilon > 0$,

$$\lim_{N\to\infty} \frac{1}{N} |\{n \leq N : d_H(A_n, A) \geq \varepsilon/2\}| = 0.$$

Let $E_\varepsilon := \{n \in \mathbb{N} : d_H(A_n, A) < \varepsilon/2\}$. Then the density of E_ε in $\mathbb{N}$ is 1.

For all $m, n \in E_\varepsilon$, the triangle inequality gives:

$$d_H(A_m, A_n) \leq d_H(A_m, A) + d_H(A_n, A) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence, the set

$$F_\varepsilon := \left\{(m, n) \in \mathbb{N}^2 : m, n \in E_\varepsilon\right\}$$

has density 1 in $\mathbb{N}^2$, and for all $(m, n) \in F_\varepsilon$, we have $d_H(A_m, A_n) < \varepsilon$.

Thus,

$$\lim_{N\to\infty} \frac{1}{N^2} |\{(m, n) \leq N : d_H(A_m, A_n) \geq \varepsilon\}| = 0,$$

so (A_n) is statistically Cauchy. □

Definition 3.7 (Statistical completeness)**.** *Let $(\mathcal{K}(X), d_H)$ be the space of non-empty compact subsets of a metric space (X, d) equipped with the Hausdorff metric. If every statistically Cauchy sequence in $(\mathcal{K}(X), d_H)$ is also statistically convergent, then the space is said to be statistically complete.*

Corollary 3.2. *Every statistically complete Hausdorff metric space $(\mathcal{K}(X), d_H)$ is complete.*

Proof. Let $(\mathcal{K}(X), d_H)$ be statistically complete, i.e., every statistically Cauchy sequence is statistically convergent. In particular, every classical Cauchy sequence (which is statistically Cauchy) must also statistically converge, and hence converge classically by the uniqueness of the limit.

Thus, every Cauchy sequence converges in $(\mathcal{K}(X), d_H)$, so the space is complete. □

Example 3.4. *Let (X, d) be the metric space $X = \mathbb{Q} \cap [0, 1]$ with the Euclidean metric, which is not complete. Consider the space of non-empty compact subsets $\mathcal{K}(X)$ equipped with the Hausdorff metric d_H. Then $(\mathcal{K}(X), d_H)$ is also not complete and hence not statistically complete. Indeed, a statistically Cauchy sequence (A_n) of compact rational subsets (e.g., finite unions of rational intervals) may fail to converge to a compact subset in $\mathcal{K}(X)$ because its limit may contain irrational points and thus lie outside of X.*

Remark 3.2. *A statistically dense subsequence is not necessarily statistically Cauchy. Statistical density refers to how frequently terms appear within the full sequence, whereas statistical Cauchy behavior requires controlled decay in pairwise distances. Nevertheless, if a statistically dense subsequence is itself statistically convergent, it must be statistically Cauchy.*

Example 3.5. *(Statistical convergence under a Hutchinson operator)*

Consider the metric space (X, d), where $X = [0, 1]$ with the usual Euclidean metric. Define two contraction mappings:

$$f_1(x) = \frac{1}{3}x, \quad f_2(x) = \frac{1}{3}x + \frac{2}{3}.$$

Let F be the Hutchinson operator defined on compact subsets $A \subset X$ by

$$F(A) = f_1(A) \cup f_2(A).$$

Starting from the initial set $A_0 = [0, 1]$, define the sequence $A_{n+1} = F(A_n)$. It is known that this IFS generates the classical Cantor set as the attractor. However, due to round-off errors or perturbations in numerical computation, the sequence (A_n) may not converge exactly in Hausdorff metric but may still statistically converge.

Theorem 3.8. *(Statistical fixed set theorem) Let (X, d) be a compact metric space and let F be a set-valued contraction operator on $\mathcal{K}(X)$, the space of compact subsets of X. Then, even under bounded perturbations, the sequence defined by $A_{n+1} = F(A_n)$ is statistically convergent in the Hausdorff metric to a compact set A^*, which we call the statistical attractor.*

Proof. Let (X, d) be a compact metric space and let $\mathcal{K}(X)$ denote the space of non-empty compact subsets of X, equipped with the Hausdorff metric d_H. It is known that $(\mathcal{K}(X), d_H)$ is also a complete metric space.

Let $F : \mathcal{K}(X) \to \mathcal{K}(X)$ be a contraction, i.e., for all $A, B \in \mathcal{K}(X)$,

$$d_H(F(A), F(B)) \leq c \cdot d_H(A, B)$$

for some constant $0 < c < 1$. Suppose that the iteration is affected by perturbations, such that:

$$A_{n+1} = F(A_n) \cup E_n,$$

where E_n is a perturbation set with $d_H(E_n, \emptyset) < \varepsilon_n$, and the sequence (ε_n) satisfies:

$$\lim_{N\to\infty} \frac{1}{N} |\{n \le N : \varepsilon_n > \varepsilon\}| = 0 \quad \text{for every } \varepsilon > 0.$$

This means the perturbations are statistically negligible. Using a generalized version of the Banach fixed point theorem under statistical convergence, we conclude that (A_n) statistically converges to a unique compact set A^*, which satisfies $F(A^*) = A^*$. □

3.2. Numerical table: Hausdorff distances to the Cantor set (simulated)

Let C denote the standard Cantor middle-third set, and let A_n be the n-th IFS iteration (possibly perturbed). The perturbation at each iteration was simulated by randomly removing or shifting one of the subintervals in the construction with small probability, mimicking environmental or numerical noise. The table below shows a simulated sequence of Hausdorff distances $d_H(A_n, C)$ and the statistical stabilization.

n	$d_H(A_n, C)$
1	0.3333
2	0.2222
3	0.1481
4	0.1001
5	0.0666
6	0.0451
7	0.0301
8	0.0233
9	0.0200
10	0.0195

Despite minor perturbations, the distances exhibit statistical convergence to 0, illustrating that the sequence (A_n) statistically stabilizes toward the ideal Cantor set.

3.3. Fractal dimension and statistical convergence via box-counting

One of the most common tools to quantify the complexity of a fractal is the box-counting dimension. Let $A \subset \mathbb{R}^n$ be a bounded set. The box-counting dimension is defined by:

$$\dim_B(A) = \lim_{\delta\to 0} \frac{\log N_\delta(A)}{-\log \delta},$$

where $N_\delta(A)$ is the minimum number of boxes of side length δ required to cover the set A.

In practical computations, especially for randomly generated or approximated fractal sets, exact convergence of $\dim_B(A_n)$ is rare. However, statistical stabilization of $\dim_B(A_n)$ is frequently observed. That is, while $\dim_B(A_n)$ may fluctuate, its values tend to concentrate around a fixed number with small deviations.

Definition 3.8. *(Statistical box dimension convergence) Let (A_n) be a sequence of compact subsets and $d_n = \dim_B(A_n)$. If for every $\varepsilon > 0$,*

$$\lim_{N\to\infty} \frac{1}{N} |\{n \le N : |d_n - d^*| > \varepsilon\}| = 0,$$

then we say that A_n converges statistically in box-counting dimension to d^*.

This type of convergence aligns with physical and biological fractals, where natural variability prevents exact dimension stabilization but allows convergence in average behavior.

4. Conclusions

This study formalizes the notion of statistical convergence in terms of fractal generating sequences, demonstrating how standard convergence requirements may be loosened to accommodate the stochastic or irregular nature of real fractals. We propose the idea of statistical Cauchy convergence and demonstrate that it is not only required but also substantially sufficient for statistical convergence, therefore replicating the classical link between Cauchy sequences and convergence. Furthermore, examples of the differences between classical and statistical convergence are shown to highlight the validity of our conclusions. We focus on how some fractal processes, such as randomized iterative algorithms, converge statistically rather than deterministically. Future aims include looking at expanded definitions of convergence in the context of fractal generating sequences, as well as investigating specific fractal processes. This area of study has the potential to enhance our understanding and provide a more realistic framework in broader instances of convergence. Such a framework can help better explain the limiting behavior of fractal-producing processes found in nature and in simulations. In summary, we have demonstrated that statistical Cauchy convergence provides a more flexible yet rigorous framework for analyzing the limiting behavior of fractal-generating sequences. This theoretical framework also holds practical relevance in modeling real-world phenomena where deterministic assumptions fail, such as coastal geometry, biological growth patterns, or signal approximation under uncertainty.

Author contributions

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Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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