AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(8): 18183–18194.
DOI:10.3934/math.2025810
Received: 01 May 2025
Revised: 24 July 2025
Accepted: 01 August 2025
Published: 12 August 2025

Theory article

The Laplacian spectra of the RG-join weighted graphs and related asymptotic network indices

Da Huang[1], **Xing Chen**[1,*], **Cheng Yan**[2] **and Zhiyong Yu**[3]

[1] School of Mathematics and Physics, Xinjiang Institute of Engineering, Urumqi 830023, Xinjiang, China
[2] College of Mathematics and System Science, Shandong University of Science and Technology, Qingdao 266590, Shandong, China
[3] College of Mathematics and System Science, Xinjiang University, Urumqi 830017, Xinjiang, China

* **Correspondence:** Email: chenxingxjnu@163.com.

Abstract: In this article, the Laplacian spectra of the RG-join weighted graphs and the related network indices named network coherence, kirchhoff index, and Laplacian-energy-like invariant are studied via algebraic graph theory and analysis approach. First, the Laplacian spectrum of the weighted RG-join graph is derived, then the union of graphs together with the RG-join operation are applied to form the weighted RG-join graphs with several classic substructures, and the mathematical characterizations for the indices are derived by the L-spectra. In addition, the related asymptotic results are also derived. It is found that, based on the RG-join weighted structure, when the cardinalities of all copy sets inner G_2 are large enough, the network coherence and the Kirchhoff index will be irrelevant with the quantity of copies in G_2, and also irrelevant with the edge weight d_1 in the other subgraph G_1.

Keywords: networked system; topological index; weighted graph; RG-join; Laplacian spectra; Kirchhoff index
Mathematics Subject Classification: 15A18, 93A16, 93B60

1. Introduction

Topological indices derived by algebraic graph theory have broad significance and are applied in a wide range of fields such as analytical chemistry and materials physics, and also have potential values in the networked system models such as coordination problems of complex networks.

In the consensus or synchronization problems of the networked system ([1–8]), the communication relations of the network can always be described by a graph. There exist lots of important researches on coordination related fields associated with the methods of algebraic graph theory ([9–16]). In [4], the necessary conditions are given by the estimated bounds of the eigenvalues of the coupled Laplacian. Ref. [5] characterized the robustness of the networked system with classic and commonly used graphs by Laplacian spectra. The first-order consensus robustness of the system with disturbance is described by the network coherence ([6, 7]), and the significant researches mention that the coherence has a form characterized by Laplacian eigenvalues. Reference [10] studies the connection between the index of symmetric trees and the cardinalities of the leader nodes. Reference [11] obtains the recursive expressions of the Laplacian spectrum of the nested network and then obtains the mathematical expression of the consensus-related index.

Another topological index with a similar expression named Kirchhoff index [17–19], is applied to interpret the graphical properties of molecules. In view of electrical networks, it can imply the resistance distance and the average electrical energy. Another interesting graphical indicator which can be conveyed by the L-spectrum is Laplacian-energy-like invariant(LEL) [20,21], which has similar features to graph energy, and describes the properties related to molecular descriptors [20–22].

During the past decades, the network that owns the composite-like structures composed by graph operations ([23–29]), such as join graph [23,24,29], corona graph [24–26], and product graph [27,28], has become a significant research branch thanks to its wide applications and practical possibility. However, as a field related to coordination problems, the articles that connect algebraic graph theory on the L-spectrum for the asymptotic graphical indicators of composite weighted structures are comparatively not that much.

Inspired by the enlightening articles, our paper mainly studies the L-spectrum of the weighted RG-join graph and their related network indices that can be characterized by the spectra; furthermore, three sorts of weighted graphs with classic substructure generated by the union and RG-join operations have been considered and the indices have been derived, then the related asymptotic results of the FONC (first-order network coherence), Kirchhoff index, and the LEL invariant have been studied.

The main novelties of the article are listed as follows:

I. The L-spectrum of the weighted RG-join generated graph is obtained, and three novel non-isomorphic weighted composite networks with classic subgraphs are designed by the RG-join together with the union graph operator; in addition, their specific corresponding weighted L-spectra are derived.

II. Novel results for the performance indices on the weighted RG-join networks are derived, the analysis method with multivariable parameters is applied to acquire the asymptotic results, and the method of elliptic integral is employed to derive a novel LEL asymptotic result.

III. It is found that if the number of vertices of one copy subgraph in G_2 is large enough, the changing trend of the FONC and Kirchhoff index are not relevant with the quantity of subgraph copies in G_2, and the edge weight d_1 in the other subgraph G_1 according to the framework considered.

Some basic concepts in graph theory are given in the second part, and the formal expression on the indices and weighted L-spectra are characterized. In Section 3, by the methods of algebraic graph theory, the RG-join graphs are designed and the L-spectra are derived, the concrete expression on the indices are obtained, and then the corresponding asymptotic results are acquired. In addition, the expression and asymptotic results on the LEL invariant of the RG-join structure are derived.

2. Preliminaries

2.1. Basic notations

Let G be an undirected graph with the vertex set $\mathcal{V}(G) = \{v_1, v_2, ..., v_N\}$, the edge set $\mathcal{E}(G) = \{(v_i, v_j)|i, j = 1, 2, ..., N; i \neq j\}$ and the adjacent matrix $\mathcal{A}(G) = [a_{ij}]_N$, where a_{ij} satisfies $a_{ij} = a_{ji}$. The Laplacian matrix $\mathfrak{L}(G) = \mathfrak{D}(G) - \mathfrak{A}(G)$, where $\mathfrak{D}(G) := diag(d_1, d_2, ..., d_N)$ and $d_i = \sum_{j \neq i} a_{ij}$. The Laplacian spectrum of G has the following form: $SL(G) = \begin{pmatrix} \vartheta_1(G) & \vartheta_2(G) & ... & \vartheta_r(G) \\ k_1 & k_2 & ... & k_r \end{pmatrix}$, where $\vartheta_1(G) < \vartheta_2(G) < ... < \vartheta_r(G)$ are the eigenvalues of $\mathfrak{L}(G)$, and $k_1, k_2, ..., k_r$ are the multiplicities. Denote the cycle with q vertices by C_q, the fan-graph with θ_2 vertices by F_{θ_2}, and the path with θ_3 vertices by P_{θ_3}.

To construct the weighted RG-join graph, denote the RG-join operation by '$\boxtimes$', and denote the union of two graphs by '$\cup$'. The following definitions and lemmas will be needed in Section 3:

Definition 1. (The RG-join weighted graphs) The RG-join of G_1 and G_2 is denoted by $G_1 \boxtimes G_2$ (see Figure 1 as an example), and it has the vertex set $V(R(G_1)) \cup V(G_2)$ and the weighted edge set $E(R(G_1)) \cup E(G_2) \cup \{(v_{1i}, v_{2j})|\forall v_{1i} \in V_1, \forall v_{2j} \in V_2\}$ (in Figure 1, the green edges, orange edges, and blue edges represent the weighted edges of G_1, the weighted edges of G_2, and the ones between G_1 and G_2, respectively), where R is a graph operation that will add a new vertex to each edge ([19]), and its generated graph $R(G)$ is the graph obtained from G by adding a new vertex e^* corresponding to each edge e of G and by joining each new vertex to the end of the edge e (see Figure 2),

Definition 2. ([25, 26]) Let G be a graph on n vertices, with the adjacency matrix A. Let $\mathbf{1}_n$ be the vector with each entry equal to 1. Define the A-coronal by $\Gamma_A(x) = \mathbf{1}_n^T(xI - A)^{-1}\mathbf{1}_n$.

Lemma 1. [29] Let A be a real matrix of order n, I_n is the identity matrix, J_n denotes the matrix with each entry eaquals to 1, then $\det(xI_n - A - \mu J_n) = (1 - \mu\Gamma_A(x))\det(xI_n - A)$.

Lemma 2. ([25, 26]) Let G be an r-regular graph with n vertices. Then $\Gamma_A(x) = \frac{n}{x-r}$.

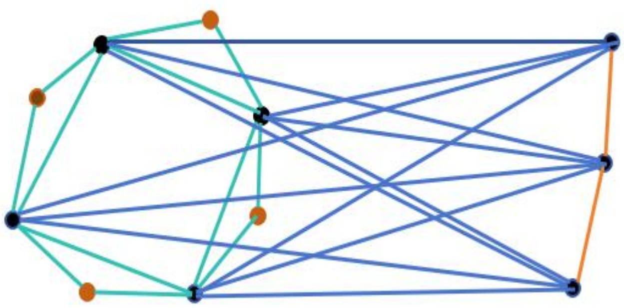

Figure 1. An example of the RG-join graph: $C_4 \boxtimes P_3$.

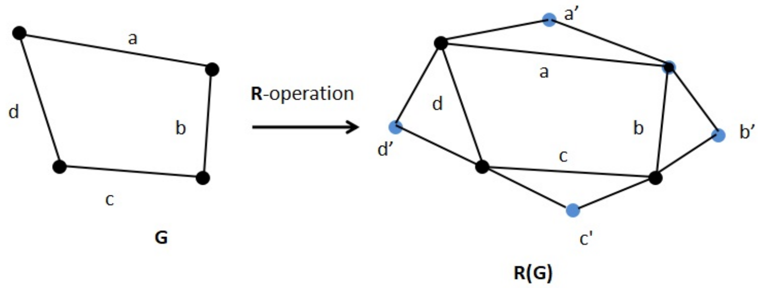

Figure 2. An example of the R-operation, where a,b,c,d are the edges of G.

2.2. The mathematical characterizations for the L-spectrum related indices

Referring to [6, 7], the FONC can be described as: the mean steady-state variance of the deviation from the average of all nodes, and it has the Laplacian spectrum related expression:

$$\mathcal{H} = \frac{1}{2N}\sum_{i=2}^{N}\frac{1}{\rho_i}, \tag{2.1}$$

Another topological index related to the sum of reciprocal of eigenvalues is Kirchhoff index [17–19], which is defined by the sum of resistance distance [17, 18, 30] between all pairs of vertices of the network, and the following equation has been proved.

$$\mathrm{Kf}(G) = N\sum_{i=2}^{N}\frac{1}{\lambda_i}, \tag{2.2}$$

and the index of LEL [20, 21] has the expression by the L-spectrum:

$$\mathrm{LEL}(G) = \sum_{i=2}^{N}\sqrt{\mu_i}, \tag{2.3}$$

where $\rho_i, \lambda_i, \mu_i,\ (i = 1, 2, ..., N)$ are the Laplacian eigenvalues, and N is the order of the graph.

3. Main results

3.1. The L-spectra of the RG-join weighted graph

Theorem 1. Suppose that a graph G_1 is r_1-regular, and has n_1 vertices and m_1 edges, with each edge weighted d_1, and G_2 is an arbitrary graph on n_2 vertices, with each edge weighted d_2; set each edge linking between G_1 and G_2 has weight $\bar{d}$, then the weighted L-spectrum of $G_1 \boxtimes G_2$ has the characterization:

(1). $0 \in SL(G_1 \boxtimes G_2)$ with multiplicity 1;

(2). $2d_1 \in SL(G_1 \boxtimes G_2)$ repeated $m_1 - n_1$ times;

(3). $\frac{(n_1\bar{d}+r_1d_1+2d_1+n_2\bar{d})\pm\sqrt{(n_1\bar{d}+r_1d_1+2d_1+n_2\bar{d})^2-4(n_1\bar{d}r_1d_1+2n_1d_1\bar{d}+2n_2d_1\bar{d})}}{2} \in SL(G_1 \boxtimes G_2)$ with multiplicity 1;

(4). $\frac{(r_1d_1+2d_1+n_2\bar{d}+\mu_i^1)\pm\sqrt{(r_1d_1+2d_1+n_2\bar{d}+\mu_i^1)^2-4(2n_2d_1\bar{d}+3d_1^2\mu_i^{(1)})}}{2} \in SL(G_1 \boxtimes G_2)$ with multiplicity 1, where $i = 2, 3, ..., n_1$.

(5). $n_1\bar{d} + d_2\mu_i^{(2)} \in SL(G_1 \boxtimes G_2)$ with multiplicity 1, where $i = 2, 3, ..., n_2$.

Proof. The Laplacian matrix of $G_1 \boxtimes G_2$ is:

$$L = \begin{pmatrix} (r_1d_1 + n_2\bar{d})I_{n_1} + d_1L_1 & -d_1M & -\bar{d}J_{n_1\times n_2} \\ -d_1M^T & 2d_1I_{m_1} & O_{m_1\times n_2} \\ -\bar{d}J_{n_2\times n_1} & O_{n_2\times m_1} & n_1\bar{d}I_{n_2} + d_2L_2 \end{pmatrix},$$

where L_1, L_2 are the Laplacian matrix of G_1 and G_2, respectively, and M is the incidence matrix of G_1. The Laplacian polynomial of weighted RG-join $G_1 \boxtimes G_2$ is:

$$\begin{aligned} f_{G_1\boxtimes G_2}(L:x) &= \begin{vmatrix} (x - r_1d_1 - n_2\bar{d})I_{n_1} - d_1L_1 & d_1M & \bar{d}J_{n_1\times n_2} \\ d_1M^T & (x - 2d_1)I_{m_1} & O_{m_1\times n_2} \\ \bar{d}J_{n_2\times n_1} & O_{n_2\times m_1} & (x - n_1\bar{d})I_{n_2} - d_2L_2 \end{vmatrix} \\ &= \det((x - n_1\bar{d})I_{n_2} - d_2L_2)\det\Phi, \end{aligned}$$

where

$$\begin{aligned} \Phi =& \begin{pmatrix} (x - r_1d_1 - n_2d_2)I_{n_1} - d_1L_1 & d_1M \\ d_1M^T & (x - 2d_1)I_{m_1} \end{pmatrix} \\ &- \begin{pmatrix} d_2J_{n_1\times n_2} \\ O \end{pmatrix}((x - n_1d_2)I_{n_2} - d_2L_2)^{-1} \begin{pmatrix} d_2J_{n_2\times n_1} & O \end{pmatrix} \\ =& \begin{pmatrix} (x - r_1d_1 - n_2d_2)I_{n_1} - d_1L_1 & d_1M \\ d_1M^T & (x - 2d_1)I_{m_1} \end{pmatrix} - \begin{pmatrix} \bar{d}^2\Gamma_{d_2L_2}(x - n_1d_2)J_{n_1\times n_1} & O \\ O & O \end{pmatrix} \\ =& \begin{pmatrix} (x - r_1d_1 - n_2\bar{d})I_{n_1} - d_1L_1 - \bar{d}^2\Gamma_{d_2L_2}(x - n_1d_2)J_{n_1\times n_1} & d_1M \\ d_1M^T & (x - 2d_1)I_{m_1} \end{pmatrix} \end{aligned}$$

Therefore, by Lemma 1,

$$\begin{aligned} \det\Phi =& (x - 2d_1)^{m_1}\det\left((x - r_1d_1 - n_2\bar{d})I_{n_1} - d_1L_1 - \bar{d}^2\Gamma_{d_2L_2}(x - n_1\bar{d})J - \frac{d_1^2MM^T}{x - 2d_1}\right) \\ =& (x - 2d_1)^{m_1}(1 - \bar{d}^2\Gamma_{d_2L_2}(x - n_1\bar{d})\Gamma_{d_1L_1+\frac{d_1^2MM^T}{x-2d_1}}(x - r_1d_1 - n_2\bar{d}))\cdot \\ & \det((x - r_1d_1 - n_2\bar{d})I_{n_1} - d_1L_1 - \frac{d_1^2MM^T}{x - 2d_1}), \end{aligned}$$

by Lemma 2,

$$\begin{aligned} \bar{d}^2\Gamma_{d_2L_2}(x - n_1\bar{d}) &= \frac{\bar{d}^2n_2}{x - n_1\bar{d}}; \\ \Gamma_{d_1L_1+\frac{d_1^2MM^T}{x-2d_1}}(x - r_1d_1 - n_2\bar{d}) &= \frac{n_1(x - 2d_1)}{x^2 - (r_1d_1 + 2d_1 + n_2\bar{d})x + 2n_2d_1\bar{d}}. \end{aligned}$$

Therefore,

$$\det\Phi = \frac{x(x-2d_1)^{m_1-n_1}}{x-n_1\bar{d}} \cdot \Big(\frac{x^2-(n_1\bar{d}+r_1d_1+2d_1+n_2\bar{d})x+(n_1\bar{d}r_1d_1+2n_1d_1\bar{d}+n_1n_2\bar{d}^2+2n_2\bar{d}d_1-n_2n_1\bar{d}^2)}{x^2-(r_1d_1+2d_1+n_2\bar{d})x+2n_2d_1\bar{d}}\Big) \cdot \prod_{i=1}^{n_1}\Big(x^2-(r_1d_1+2d_1+n_2\bar{d}+\mu_i^{(1)}d_1)x+2n_2\bar{d}d_1+3d_1^2\mu_i^{(1)}\Big).$$

Hence, the Laplacian polynomial of weighted RG-join of G_1 and G_2 is:

$$f_{G_1\boxtimes G_2}(L:x) = x(x-2d_1)^{m_1-n_1} \cdot \Big(x^2-(n_1\bar{d}+r_1d_1+2d_1+n_2\bar{d})x+(n_1\bar{d}r_1d_1+2n_1d_1\bar{d}+n_1n_2\bar{d}^2+2n_2\bar{d}d_1-n_2n_1\bar{d}^2)\Big) \cdot \prod_{i=2}^{n_2}(x-n_1\bar{d}-d_2\mu_i^{(2)}) \cdot \prod_{i=2}^{n_1}\Big(x^2-(r_1d_1+2d_1+n_2\bar{d}+\mu_i^{(1)}d_1)x+2n_2\bar{d}d_1+3d_1^2\mu_i^{(1)}\Big),$$

thus, the result in Theorem 1 holds. □

3.2. Related applications of weighted L-spectrum for the RG-join network

According to the theorem in Section 3.1, the weighted RG-join structures in this section are designed and interpreted as follows. Three types of regular graph are selected for G_1, these are: complete graph, complete balanced k-partite graph, and cycle. Define and denote the three types of structure by the graph operation with RG-join as: $RJ_1 := K_{n_1} \boxtimes (\cup_a P_{n_2})$; $RJ_2 := \Re(p,\theta_1) \boxtimes (\cup_a F_{\theta_2})$; $RJ_3 := C_q \boxtimes (\cup_a P_{\theta_3})$, where K_{n_1} is the complete graph with n_1 vertices; $\Re(p,\theta_1)$ is the p-partite graph with each partition having θ_1 vertices.

3.2.1. The network indices for $RJ_1 := K_{n_1} \boxtimes (\cup_a P_{n_2})$

Here, the networked system that owns the graph $K_{n_1} \boxtimes (\cup_a P_{n_2})$ is also simplified by RJ_1, and so is RJ_2, RJ_3. The weighted L-spectrum of RJ_1 can be written as:

(1). $0 \in SL(K_{n_1} \boxtimes (\cup_a P_{n_2}))$ with multiplicity 1;

(2). $2d_1 \in SL(K_{n_1} \boxtimes (\cup_a P_{n_2}))$ with multiplicity $m_1 - n_1$;

(3). $\frac{(n_1\bar{d}+r_1d_1+2d_1+an_2\bar{d})\pm\sqrt{(n_1\bar{d}+r_1d_1+2d_1+an_2\bar{d})^2-4(n_1\bar{d}r_1d_1+2n_1d_1\bar{d}+2an_2d_1\bar{d})}}{2} \in SL(K_{n_1} \boxtimes (\cup_a P_{n_2}))$ with multiplicity 1;

(4). $x = \frac{((n_1-1)d_1+2d_1+an_2d_2+n_1d_1)\pm\sqrt{((n_1-1)d_1+2d_1+an_2d_2+n_1d_1)^2-4(2an_2d_1\bar{d}+3d_1^2n_1)}}{2} \in SL(K_{n_1} \boxtimes (\cup_a P_{n_2}))$ with multiplicity $n_1 - 1$;

(5). $n_1\bar{d} \in SL(K_{n_1} \boxtimes (\cup_a P_{n_2}))$ with multiplicity $a - 1$;

(6). $n_1\bar{d} + d_2 4sin^2(\frac{k\pi}{2n_2}) \in SL(K_{n_1} \boxtimes (\cup_a P_{n_2}))$ repeated a times, where $k = 1, 2, ..., n_2 - 1$.

Therefore, we have

$$H(RJ_1) = \frac{1}{2(n_1+m_1+an_2)}\Big(\frac{m_1-n_1}{2d_1} + \frac{(n_1\bar{d}+r_1d_1+2d_1+an_2\bar{d})}{(n_1\bar{d}r_1d_1+2n_1d_1\bar{d}+2an_2d_1\bar{d})}$$

$$+ (n_1 - 1)\frac{((n_1 - 1)d_1 + 2d_1 + an_2d_2 + n_1d_1)}{(2an_2d_1\bar{d} + 3d_1^2n_1)} + \frac{(a-1)}{n_1\bar{d}} + a\sum_{k=1}^{n_2-1}\frac{1}{n_1\bar{d} + d_2 4sin^2(\frac{k\pi}{2n_2})}\Big),$$

then

$$\lim_{n_2\to\infty} H(RJ_1) = \frac{1}{2\sqrt{n_1^2\bar{d}^2 + 4d_2n_1\bar{d}}}.$$

Hence, one can see that the asymptotic result is irrelevant with d_1, i.e., the weight of G_1. Therefore, the Kirchhoff index of the structure has the asymptotic relation: $K_1 \sim \frac{(n_1+m_1+an_2)^2}{\sqrt{n_1^2\bar{d}^2+4d_2n_1\bar{d}}}$, when $n_2 \to \infty$.

It can be derived that the asymptotic result based on the infinity of n_2 is irrelevant to the parameters: d_1, m_1, a, and is only relevant with the linking edge weight between G_1 and G_2, the number of vertices of G_1 and the edge weight of G_2.

3.2.2. The network indices for $RJ_2 := \Re(p, \theta_1) \boxtimes (\cup_a F_{\theta_2})$

The weighted L-spectrum of $\Re(p, \theta_1) \boxtimes (\cup_a F_{\theta_2})$ can be characterized as:

(1). $0 \in SL(\Re(p, \theta_1) \boxtimes (\cup_a F_{\theta_2}))$ with multiplicity 1;

(2). $2d_1 \in SL(\Re(p, \theta_1) \boxtimes (\cup_a F_{\theta_2}))$ with multiplicity $p(p-1)\theta_1^2/2 - p\theta_1$;

(3). $\frac{(p\theta_1\bar{d}+(p-1)\theta_1d_1+2d_1+a\theta_2\bar{d})\pm\sqrt{(p\theta_1\bar{d}+(p-1)\theta_1d_1+2d_1+a\theta_2\bar{d})^2-4(p\theta_1\bar{d}(p-1)\theta_1d_1+2p\theta_1d_1\bar{d}+2a\theta_2d_1\bar{d})}}{2} \in SL(\Re(p, \theta_1) \boxtimes (\cup_a F_{\theta_2}))$ with multiplicity 1;

(4). $\frac{((p-1)\theta_1d_1+2d_1+a\theta_2\bar{d}+p\theta_1d_1)\pm\sqrt{((p-1)\theta_1d_1+2d_1+a\theta_2\bar{d}+p\theta_1d_1)^2-4(2a\theta_2d_1\bar{d}+3d_1^2p\theta_1)}}{2} \in SL(K(p, \theta_1) \boxtimes (\cup_a F_{\theta_2}))$ repeated $p-1$ times;

(5). $\frac{((p-1)\theta_1d_1+2d_1+a\theta_2\bar{d}+(p-1)\theta_1d_1)\pm\sqrt{((p-1)\theta_1d_1+2d_1+a\theta_2\bar{d}+(p-1)\theta_1d_1)^2-4(2a\theta_2d_1\bar{d}+3d_1^2(p-1)\theta_1)}}{2} \in SL(K(p, \theta_1) \boxtimes (\cup_a F_{\theta_2}))$ with multiplicity $p(\theta_1 - 1)$;

(6). $p\theta_1\bar{d} \in$ with multiplicity $a - 1$;

(7). $p\theta_1\bar{d} + d_2\theta_2 \in$ with multiplicity a;

(8). $p\theta_1\bar{d} + d_2(1 + 4sin^2(\frac{j\pi}{2(\theta_2-1)}))$ with multiplicity a, where $j = 1, 2, ..., \theta_2 - 2$.

Therefore, the index of network coherence can be derived by:

$$H(RJ_2) = \frac{1}{2p\theta_1 + p^2\theta_1^2 - p\theta_1^2 + 2a\theta_2}\Big(\frac{1}{4d_1}(p(p-1)\theta_1^2 - 2p\theta_1) + \frac{p\theta_1\bar{d} + (p-1)\theta_1d_1 + 2d_1 + a\theta_2\bar{d}}{p\theta_1\bar{d}(p-1)\theta_1d_1 + 2p\theta_1d_1\bar{d} + 2a\theta_2d_1\bar{d}} + \frac{[(p-1)\theta_1d_1 + 2d_1 + a\theta_2\bar{d} + p\theta_1d_1](p-1)}{2a\theta_2d_1\bar{d} + 3d_1^2p\theta_1} + \frac{[(p-1)\theta_1d_1 + 2d_1 + a\theta_2\bar{d} + (p-1)\theta_1d_1]p(\theta_1 - 1)}{2a\theta_2d_1\bar{d} + 3d_1^2(p-1)\theta_1} + \frac{a-1}{p\theta_1\bar{d}} + \frac{a}{p\theta_1\bar{d} + d_2\theta_2} + \sum_{j=1}^{\theta_2-2}\frac{a}{p\theta_1\bar{d} + d_2(1 + 4sin^2(\frac{j\pi}{2(\theta_2-1)}))}\Big)$$

Therefore, when $\theta_2 \to \infty$, we have

$$\lim_{\theta_2\to\infty} H(RJ_2) = \frac{1}{2a}\int_0^1 \frac{a}{p\theta_1\bar{d} + d_2(1+4sin^2(\frac{\pi x}{2}))}dx$$
$$= \frac{1}{2\sqrt{(p\theta_1\bar{d}+d_2)(p\theta_1\bar{d}+5d_2)}}.$$

Therefore, the asymptotic result is irrelevant with d_1, i.e., the weight of the edges in G_1, and it is also irrelevant with a, i.e., the numbers of copies of the fan-subgraphs in G_2. Thus, the Kirchhoff index of the structure has the relation with respect to the graph parameters: $K_2 \sim \frac{(2p\theta_1+p\theta_1^2(p-1)+2a\theta_2)^2}{4\sqrt{(p\theta_1\bar{d}+d_2)(p\theta_1\bar{d}+5d_2)}}$, as $\theta_2 \to \infty$.

3.2.3. The network indices for $RJ_3 := C_q \boxtimes (\cup_a P_{\theta_3})$

The weighted L-spectrum of $C_q \boxtimes (\cup_a P_{\theta_3})$ can be characterized as:

(1) $0 \in SL(C_q \boxtimes (\cup_a P_{\theta_3}))$ with multiplicity 1;

(2) $\frac{(q\bar{d}+4d_1+a\theta_3\bar{d})\pm\sqrt{(q\bar{d}+4d_1+a\theta_3\bar{d})^2-4(4q\bar{d}d_1+2a\theta_3d_1\bar{d})}}{2} \in SL(C_q \boxtimes (\cup_a P_{\theta_3}))$ with multiplicity 1;

(3) $\frac{(4d_1+a\theta_3\bar{d}+4d_1sin^2(\frac{k\pi}{q}))\pm\sqrt{(4d_1+a\theta_3\bar{d}+4d_1sin^2(\frac{k\pi}{q}))^2-4(2a\theta_3d_1\bar{d}+3d_1^2 4sin^2(\frac{k\pi}{q}))}}{2} \in SL(C_q \boxtimes (\cup_a P_{\theta_3}))$ is the single root, where $k = 1, 2, ..., q-1$.

(4). $q\bar{d} \in SL(C_q \boxtimes (\cup_a P_{\theta_3}))$ repeated $a-1$ times;

(5). $q\bar{d} + 4d_2sin^2(\frac{k\pi}{2\theta_3}) \in SL(C_q \boxtimes (\cup_a P_{\theta_3}))$ repeated a times, where $k = 1, 2, ..., \theta_3 - 1$.

Therefore, in this case, the coherence has the expression:

$$H(RJ_3) = \frac{1}{2(2q+a\theta_3)}\Big[\frac{q\bar{d}+4d_1+a\theta_3\bar{d}}{4q\bar{d}d_1+2a\theta_3d_1\bar{d}} + \sum_{k=1}^{q-1}\frac{4d_1+a\theta_3\bar{d}+4d_1sin^2(\frac{k\pi}{q})}{2a\theta_3d_1\bar{d}+12d_1^2sin^2(\frac{k\pi}{q})} + \frac{a-1}{q\bar{d}}$$
$$+ \sum_{k=1}^{\theta_3-1}\frac{a}{q\bar{d}+4d_2sin^2(\frac{k\pi}{2\theta_3})}\Big]$$

Thus, when

(i). $q \to \infty$, one has,

$$\lim_{q\to\infty} H(RJ_3) = \frac{1}{4}\int_0^1 \frac{4d_1+a\theta_3\bar{d}+4d_1sin^2(\pi x)}{2a\theta_3d_1\bar{d}+12d_1^2sin^2(\pi x)}dx = \frac{12d_1+a\theta_3\bar{d}}{12d_1\sqrt{2a\theta_3\bar{d}(2a\theta_3\bar{d}+12d_1)}} + \frac{1}{12d_1}$$

(ii). $\theta_3 \to \infty$,

$$\lim_{\theta_3\to\infty} H(RJ_3) = \frac{1}{2}\int_0^1 \frac{1}{q\bar{d}+4d_2sin^2(\frac{\pi x}{2})}dx = \frac{1}{2\sqrt{(q\bar{d}+4d_2)q\bar{d}}}.$$

Therefore, from (i) and (ii), it can be obtained that if $q \to \infty$, the asymptotic result is irrelevant with d_2; however, when $\theta_3 \to \infty$, it is irrelevant with d_1, that is, the edge weight of G_1, and it is also not relevant with a, i.e.; the number of copies of path subgraphs in G_2. Thus, it can be derived that the Kirchhoff index with the same structure has the asymptotic property: $K_3 \sim \frac{(2q+a\theta_3)^2}{\sqrt{(q\bar{d}+4d_2)q\bar{d}}}$, as $\theta_3 \to \infty$.

Corollary 1. Suppose that G_1 is r_1-regular on n_1 vertices with each edge weighted d_1, and G_2 is an arbitrary graph on n_2 vertices weighted with d_2, and they have the L-spectrum $SL(G_1) = \{\mu_1^{(1)}, \mu_2^{(1)}, ..., \mu_{n_1}^{(1)}\}$ and $SL(G_2) = \{\mu_1^{(2)}, \mu_2^{(2)}, ..., \mu_{n_2}^{(2)}\}$, respectively. Then by Eq (2.3) and Theorem 1, the LEL of the RG-join graph has the following characterization:

$$\begin{aligned}\mathrm{LEL}(G_1 \boxtimes G_2) =& (m_1 - n_1)\sqrt{2d_1} \\ &+ \sqrt{(n_1\bar{d} + r_1 d_1 + 2d_1 + n_2\bar{d}) + 2\sqrt{n_1\bar{d}r_1 d_1 + 2n_1 d_1\bar{d} + 2n_2 d_1\bar{d}}} \\ &+ \sum_{i=2}^{n_1}\sqrt{(r_1 d_1 + 2d_1 + n_2\bar{d} + \mu_i^{(1)} d_1) + 2\sqrt{2n_2 d_1\bar{d} + 3d_1^2\mu_i^{(1)}}} \\ &+ \sum_{i=2}^{n_2}\sqrt{n_1\bar{d} + d_2\mu_i^{(2)}}\end{aligned}$$

Corollary 2. The LEL of the RG-join weighted graph $RJ_1 := K_{n_1} \boxtimes (\cup_a P_{n_2})$ has the following asymptotic result:

$$\begin{aligned}\lim_{n_2\to\infty}\frac{\mathrm{LEL}(RJ_1)}{V(G)} = \lim_{n_2\to\infty}\frac{\mathrm{LEL}(RJ_1)}{n_1 + m_1 + an_2} &= \int_0^1 \sqrt{n_1\bar{d} + 4d_2 sin^2(\frac{\pi x}{2})}\,dx \\ &= \frac{2}{\pi}\sqrt{n_1\bar{d}}\sqrt{1 + \frac{4n_2}{n_1\bar{d}}}E(\frac{4n_2}{n_1\bar{d} + 4n_2}),\end{aligned}$$

where $E(\rho) = \int_0^{\frac{\pi}{2}} \sqrt{1 - \rho^2 sin^2 t}\,dt$ is the second kind of elliptic integral.

Remark 3. The RG-join weighted structure of two graphs might be extended to general layered structures in one's future research. In fact, the weigthed RG-join structure might be seen as a kind of two-layered structure, since the subgraphs G_1 and G_2 both have their own weights, and the edges between them have another identical weight. The mathematical expression of the indices can be regarded as an enlightening reference for the similar topics on composite networks with the join-like graph operations, and the asymptotic properties can be applied to study and improve the topological indices [31] of network performance.

4. Conclusions

This research mainly studies the L-spectrum of RG-join weighted graphs, three types of novel weighted composite networks generated by classic graphs are constructed by the graph operation of RG-join, in addition, their corresponding weighted L-spectra are derived. Novel results for the indices of the weighted RG-join networks are derived, and analysis methods with multivariable parameters are applied to acquire the asymptotic results; in addition, the method of elliptic integral is employed to derive a novel LEL asymptotic result. It is found that if the number of vertices of one copy subgraph in G_2 is large enough, the changing trends of the FONC and Kirhoff index are not relevant with some sort of subgraph copies' quantity in G_2, and also irrelevant with the edge weight d_1 in G_1 based on the considered framework.

Author contributions

Da Huang and Xing Chen: Methodology, Formal analysis, Writing–original draft, Writing–review and editing, Project administration; Cheng Yan: Methodology, Formal analysis, Writing-review and editing; Zhiyong Yu: Writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by the Natural Science Foundation of Xinjiang Uygur Autonomous Region (NSFXJ) (No. 2022D01A247), the National Natural Science Foundation of the People's Republic of China (NSFC) (No. 12361110), and the Research Foundation of Xinjiang Institute of Engineering (No. 2024xgy092605).

Conflict of interest

The authors declare no conflict of interest.

References

1. R. Olfati-Saber, R. M. Murray, Consensus problems in networks of agents with switching topology and time-delays, *IEEE Trans. Automat. Contr.*, **49** (2004), 1520–1533. http://doi.org/10.1109/TAC.2004.834113
2. W. Yu, G. Chen, M. Cao, J. Kurths, Second-order consensus for multiagent systems with directed topologies and nonlinear dynamics, *IEEE Trans. Syst. Man Cybern. B*, **40** (2010), 881–891. http://doi.org/10.1109/TSMCB.2009.2031624
3. G. Wen, Z. Duan, W. Yu, G. Chen, Consensus of second-order multi-agent systems with delayed nonlinear dynamics and intermittent communications, *Int. J. Control*, **86** (2013), 322–331. http://doi.org/10.1080/00207179.2012.727473
4. J. Wu, H. Qin, Y. Hong, Synchronization of netowrked systems and Laplacian-spectrum modification, In: *Proceedings of the 26th Chinese control conference*, Zhangjiajie, China, July 26–31 2007, 573–576. http://doi.org/10.1109/CHICC.2006.4347116
5. G. F. Young, L. Scardovi, N. E. Leonard, Robustness of noisy consensus dynamics with directed communication, In: *Proceedings of the 2010 American control conference*, Baltimore, MD, USA, 30 June–2 July 2010, 6312–6317. http://doi.org/10.1109/ACC.2010.5531506
6. B. Bamieh, M. R. Jovanovi, P. Mitra, S. Patterson, Coherence in large-scale networks: dimension-dependent limitations of local feedback, *IEEE Trans. Automat. Contr.*, **57** (2012), 2235–2249. http://doi.org/10.1109/TAC.2012.2202052

7. S. Patterson, B. Bamieh, Consensus and coherence in fractal networks, *IEEE Trans. Control Netw. Syst.*, **1** (2014), 338–348. http://doi.org/10.1109/TCNS.2014.2357552
8. S. Patterson, Y. Yi, Z. Zhang, A resistance-distance-based approach for optimal leader selection in noisy consensus networks, *IEEE Trans. Control Netw. Syst.*, **6** (2019), 191–201. http://doi.org/10.1109/TCNS.2018.2805639
9. Y. Yi, Z. Zhang, L. Shan, G. Chen, Robustness of first-and second-order consensus algorithms for a noisy scale-free small-world Koch network, *IEEE Trans. Control Syst. Tech.*, **25** (2017), 342–350. http://doi.org/10.1109/TCST.2016.2550582
10. W. Sun, M. Hong, S. Liu, K. Fan, Leader-follower coherence in noisy ring-trees networks, *Nonlinear Dyn.*, **102** (2020), 1657–1665. http://doi.org/10.1007/s11071-020-06011-9
11. J.-B. Liu, Y. Bao, W. Zheng, S. Hayat, Network coherence analysis on a family of nested weighted n-polygon networks, *Fractals*, **29** (2021), 2150260. http://doi.org/10.1142/S0218348X21502601
12. J.-B. Liu, X. Wang, J. Cao, The coherence and properties analysis of balanced 2^p-Ary tree networks, *IEEE Trans. Netw. Sci. Eng.*, **11** (2024), 4719–4728. http://doi.org/10.1109/TNSE.2024.3395710
13. Y. Wan, K. Namuduri, S. Akula, M. Varanasi, The impact of multi-group multi-layer network structure on the performance of distributed consensus building strategies, *Int. J. Robust Nonlinear Control*, **23** (2013), 653–662. http://doi.org/10.1002/rnc.2783
14. J. Zhu, D. Huang, H. Gao, X. Li, The Laplacian spectrum of weighted composite networks and the applications, *AIP Adv.*, **14** (2024), 035229. http://doi.org/10.1063/5.0194325
15. M.-M. Xu, J.-A. Lu, J. Zhou, Synchronizability and eigenvalues of two-layer star networks, *Acta Phys. Sin.*, **65** (2016), 028902. http://doi.org/10.7498/aps.65.028902
16. C. Hu, H. He, H. Jiang, Edge-based adaptive distributed method for synchronization of intermittently coupled spatiotemporal networks, *IEEE Trans. Automat. Contr.*, **67** (2022), 2597–2604. http://doi.org/10.1109/TAC.2021.3088805
17. Y. Yang, D. J. Klein, Comparison theorems on the resistance distances and Kirchhoff indices of S,T-isomers, *Discrete Appl. Math.*, **175** (2014), 87–93. http://doi.org/10.1016/j.dam.2014.05.014
18. Y. Yang, Y. Yu, Resistance distances and Kirchhoff indices under graph operations, *IEEE Access*, **8** (2020), 95650–95656. http://doi.org/10.1109/ACCESS.2020.2995935
19. W. Wang, D. Yang, Y. Luo, The Laplacian polynomial and Kirchhoff index of graphs derived from regular graphs, *Discrete Appl. Math.*, **161** (2013), 3063–3071. http://doi.org/10.1016/j.dam.2013.06.010
20. I. Gutman, B. Zhou, B. Furtila, The Laplacian-energy-like invariant is an energy like invariant, *MATCH Commun. Math. Comput. Chem.*, **64** (2010), 85–96.
21. J.-B. Liu, X.-F. Pan, F.-T. Hu, F.-F. Hu, Asymptotic Laplacian-energy-like invariant of lattice, *Appl. Math. Comput.*, **253** (2015), 205–214. http://doi.org/10.1016/j.amc.2014.12.035
22. J.-B. Liu, X.-F. Pan, Asymptotic incidence energy of lattice, *Physica A*, **422** (2015), 193–202. https://doi.org/10.1016/j.physa.2014.12.006
23. R. P. Varghese, K. R. Kumar, Spectra of a new join of two graphs, *Advances in the Theoretical and Applied Mathematics*, **11** (2016), 459–470.

24. H. Zhang, Y. Yang, C. Li, Kirchhoff index of composite graphs, *Discrete Appl. Math.*, **157** (2009), 2918–2927. http://doi.org/10.1016/j.dam.2009.03.007
25. C. McLeman, E. McNicholas, Spectra of coronae, *Linear Algebra Appl.*, **435** (2011), 998–1007. http://doi.org/10.1016/j.laa.2011.02.007
26. S.-Y. Cui, G.-T. Tian, The spectrum and the signless Laplacian spectrum of coronae, *Linear Algebra Appl.*, **437** (2012), 1692–1703. http://doi.org/10.1016/j.laa.2012.05.019
27. M. H. Khalifeh, H. Yousefi-Azari, A. R. Ashrafi, The hyper-Wiener index of graph operations, *Comput. Math. Appl.*, **56** (2008), 1402–1407. http://doi.org/10.1016/j.camwa.2008.03.003
28. R. Hammack, W. Imrich, S. Klavzar, *Handbook of product graphs*, 2 Eds., New York: CRC Press, 2000. https://doi.org/10.1201/b10959
29. X. Liu, Z. Zhang, Spectra of subdivision-vertex and subdivision-edge joins of graphs, *Bull. Malays. Math. Sci. Soc.*, **42** (2019), 15–31. http://doi.org/10.1007/s40840-017-0466-z
30. D. J. Klein, M. Randić, Resistance distance, *J. Math. Chem.*, **12** (1993), 81–95. http://doi.org/10.1007/BF01164627
31. J.-B. Liu, X.-F. Pan, A unified approach to the asymptotic topological indices of various lattices, *Appl. Math. Comput.*, **270** (2015), 62–73. http://doi.org/10.1016/j.amc.2015.08.008