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Research article

Coinciding Bishop frames and the geometry of W-Bertrand curves

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Abstract: Three orthogonal unit vectors—the tangent, normal, and binormal vectors—are among the components of a new generation of the Bishop frame that are thoroughly examined in this research. An alternative to the Frenet frame, it is a frame field specified on a curve in Euclidean space. For curves for which the second derivative is unavailable, it is helpful. In addition, the circumstances under which the Bishop frame of one curve and the Bishop frame of another coincide are specified. Replicating such strategies when the Bishop frame of one curve coincides with the Bishop frame of another curve would be beneficial. In our article, we will present the concept of W-Bertrand curves according to the Bishop frame in the Euclidean 3-space and examine several kinds of W-Bertrand curves based on the Bishop frame.

Keywords: Bertrand curves; Bishop frame; Frenet frame; Bertrand W-curve
Mathematics Subject Classification: 53A04, 53A35

1. Introduction

The theory of curves in differential geometry uses calculus techniques to investigate the geometric characteristics of plane and space curves. The most widely used framework for examining the calculus of curves is the Frenet frame. In its most basic form, this structure shows the kinematics of a particle traveling in a curve. The curvature and torsion of the curve, which quantify how a curve bends, allow it to make this movement [6].

Bishop [2] defined a frame known as the Bishop frame in 1975 as a substitute for the Frenet frame. For curves for which the second derivative is unavailable, this frame is helpful. Three orthogonal unit vectors make up the Bishop frame. These vectors are the tangent, normal, and binormal vectors.

The Bishop frame does not necessitate the definition of the second derivative of the curve, in contrast to the Frenet frame [8, 17]. Important details about a curve in theory are provided by the curvature functions [11, 14].

Named for the 19th-century French mathematician Joseph Bertrand (1850), Bertrand curves are a classic topic in differential geometry. These curves are a unique pair of curves with comparable curvature properties and a constant distance between corresponding points. Two curves in space, $\alpha(s)$ and $\alpha^*(s^*)$ (where s, s^* are the arc-length parameters) are referred to as Bertrand curves if the principal normal lines coincide for each corresponding point. This implies that, in a broad sense, one curve can be viewed as the offset or parallel curve of the other. The curvature κ and the torsion τ of the two curves for the Bertrand curves must meet a linear relation: $a\kappa + b\tau = 1$, where a, b are constants, [1, 3].

In [20], the authors defined the Bertrand B-curve and Bertrand B-pair curves and studied the properties of the Bertrand B-curves by using the BishopII frame. Also, they examined the relationships between the Bishop curvatures of the Bertrand B-pair curves with respect to each other. Consequently, it has been crucial for academics to calculate a curve's curvature and torsion and determine a correlation between them in Euclidean; see [5, 12, 13]. Readers can consult the references [9, 15] to learn more about the Mannheim and Bertrand curves.

Provide certain differential geometric features of the Smarandache curves and study specific Smarandache curves in Euclidean 3-space according to the Bishop frame introduced in [4]. Examine the E^3 Bishop spinor equations of the curves. Furthermore, the relationships between Frenet frames and the spinor formulations of Bishop frames are stated in [16].

In [10], the authors defined null Cartan and Pseudo-null Bertrand curves in Minkowski space E^3_1 according to their Bishop frame and obtained the necessary and sufficient conditions for Pseudo-null curves to be Bertrand B-curves in terms of their Bishop curvatures. For more details on Minkowski space, see [7, 18]. Moreover, in [19], the Bertrand B-curve in the three-dimensional sphere $S^3(r)$ is defined, some conclusions about a pair of Bertrand B-curves in $S^3(r)$ are analyzed, and an example is given.

In this study, we define the Bertrand W-curve according to the Bishop frame and give some relations between the corresponding curvatures of the pairs of curves. As far as the writers are aware, the idea of the Bertrand W-curve in the Euclidean 3-space can also be examined in connection with Bertrand curves.

2. Preliminaries

Let $\overrightarrow{A} = (A_1, A_2, A_3)$, and $\overrightarrow{B} = (B_1, B_2, B_3)$ be two vectors in 3-dimensional Euclidean space ($\mathbb{R}^3$), equipped with the standard inner product given by $\langle \overrightarrow{A}, \overrightarrow{B} \rangle = A_1B_1 + A_2B_2 + A_3B_3$. The norm of a vector $\overrightarrow{A} \in \mathbb{R}^3$ is given by $\left\| \overrightarrow{A} \right\| = \sqrt{\langle \overrightarrow{A}, \overrightarrow{A} \rangle}$. The curve $\alpha(s)$ is said to be of a unit speed or parametrized by arc length function s if $\langle \alpha'(s), \alpha'(s) \rangle = 1$.

The Bishop frame, or parallel transport frame, is an alternative approach to defining a moving frame that is well defined even when the curve has a vanishing second derivative. The parallel transport frame is based on $\mathbf{T}(s)$ (the tangent vector field of the unit speed curve $\alpha(s)$, and choosing $\mathbf{N}_1(s)$, and $\mathbf{N}_2(s)$ normal to $\mathbf{T}(s)$ at each point by making $\mathbf{N}_1(s)$ and $\mathbf{N}_2(s)$ vary smoothly throughout the path regardless of the curvature. Let $\alpha(s)$ be an arc length-parametrized C^2 curve where $\langle \mathbf{T}(s), \mathbf{N}_1(s) \rangle = \langle \mathbf{T}(s), \mathbf{N}_2(s) \rangle = \langle \mathbf{N}_1(s), \mathbf{N}_2(s) \rangle = 0$. In our article we will take the condition $\langle \mathbf{N}_1'(s), \mathbf{N}_2(s) \rangle = 0$. $\mathbf{N}_1(s)$ is called the unit first normal vector, which is parallel along the curve $\alpha(s)$, and $\mathbf{N}_2(s) = \mathbf{T}(s) \times \mathbf{N}_1(s)$ is called the second normal vector. The alternative frame equations are as follows:

$$\begin{bmatrix} \mathbf{T}'(s) \\ \mathbf{N}_1'(s) \\ \mathbf{N}_2'(s) \end{bmatrix} = \begin{bmatrix} 0 & k_1(s) & k_2(s) \\ -k_1(s) & 0 & 0 \\ -k_2(s) & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T}(s) \\ \mathbf{N}_1(s) \\ \mathbf{N}_2(s) \end{bmatrix}, \tag{2.1}$$

where $k(s) = \sqrt{k_1^2(s) + k_2^2(s)}$, and $\theta(s) = \tan^{-1}(\frac{k_2(s)}{k_1(s)}), k_1(s) \neq 0, \tau(s) = \frac{-d\theta}{ds}$,

$$\begin{aligned} k_1(s) &= k(s)\cos(\theta), \\ k_2(s) &= k(s)\sin(\theta), \end{aligned}$$

and

$$\begin{aligned} \mathbf{T}(s) &= \mathbf{T}(s), \\ \mathbf{N}_1(s) &= \mathbf{N}(s)\cos(\theta) - \mathbf{B}\sin(\theta), \\ \mathbf{N}_2(s) &= \mathbf{N}(s)\sin(\theta) + \mathbf{B}\cos(\theta), \end{aligned}$$

such that $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ is the Frenet frame and $k(s)$ and $\tau(s)$ are the curvature and torsion of the curve $\alpha(s)$.

3. Main results

In the following section, we give the definition of Bertrand B-mate in Euclidean 3-space according to the Bishop frame, and then introduce the Bertrand W-mate according to the Bishop frame in Euclidean 3-space. Also, we introduce some characterization of the Bertrand W-curve according to the Bishop frame in the Euclidean 3-space.

Definition 3.1. *[20] Let I, and I^* be open intervals in $\mathbb{R}$, and $\beta(s) : I \to \mathbb{R}^3$ be a unit speed curve, with Bishop frame $\{\mathbf{T}(s), \mathbf{N}_1(s), \mathbf{N}_2(s)\}$, and $\beta^*(s^*(s)) : I^* \to \mathbb{R}^3$ be an arbitrary curve with the Bishop frame $\{\mathbf{T}_1^*(s^*), \mathbf{N}_1^*(s^*), \mathbf{N}_2^*(s^*)\}$. If the Bishop vector $\mathbf{N}_1(s)$ is collinear with the Bishop vector $\mathbf{N}_1^*(s^*)$ at the corresponding points of the curves $\beta(s)$, and $\beta^*(s^*(s))$. Then $\beta(s)$ is called the Bertrand B-curve according to the Bishop frame. In particular, $\beta^*(s^*(s))$ is called the Bertrand B-mate according to the Bishop frame of $\beta(s)$, and a pair of curves $(\beta(s), \beta^*(s^*))$ is called the Bertrand B-pair according to the Bishop frame.*

Definition 3.2. *Let $\beta(s) : I \to \mathbb{R}^3 (s \longmapsto \beta(s))$ be a unit speed curve, and $\mathbf{T}(s), \mathbf{N}_1(s)$, $\mathbf{N}_2(s)$, $k_1(s)$, and $k_2(s)$ be Bishop apparatus of the curve $\beta(s)$. Define a curve $\beta^*(s^*(s))$ by:*

$$\beta^*(s^*(s)) = \int W(s)ds + \lambda(s)\mathbf{N}_1(s),$$

where $\lambda(s) : I \to \mathbb{R}(s \mapsto \lambda(s))$ is a differentiable function and $W(s) = \mu_1(s)\mathbf{T}(s) + \mu_2(s)\mathbf{N}_1(s) + \mu_3(s)\mathbf{N}_2(s)$ is a unit vector field and $\mu_1^2(s)+\mu_2^2(s)+\mu_3^2(s) = 1$. Let $\{\mathbf{T}^*(s^*), \mathbf{N}_1^*(s^*), \mathbf{N}_2^*(s^*)\}$ be orthonormal Bishop frame of the curve $\beta^*(s^*(s))$, and let $k_1^*(s^*), k_2^*(s^*)$ be the Bishop curvatures of the curve $\beta^*(s^*)$. If $\{\mathbf{N}_1(s), \mathbf{N}_1^*(s^*)\}$ is linear dependent $(\mathbf{N}_1^*(s^*) = \epsilon\mathbf{N}_1(s)), \epsilon = \pm 1$, we say that the two curves $\beta(s)$, and $\beta^*(s^*)$ are Bertrand W-mate according to the Bishop Frame. If $W(s) = \mathbf{T}(s)$, then $\beta(s)$, and $\beta^*(s^*)$ are Bertrand B-mate according to the Bishop frame.

In the following theorem we obtain the Bishop apparatus of the Bertrand W-curve $\beta^*(s^*(s))$ according to the Bishop frame in terms of the Bishop apparatus of $\beta(s)$.

Theorem 3.1. *Let $\beta(s)$ be a unit speed curve, and $\{\mathbf{T}(s), \mathbf{N}_1(s), \mathbf{N}_2(s), k_1(s), k_2(s)\}$ be its Bishop apparatus. Suppose that $\beta^*(s^*(s))$ is a Bertrand W-mate of $\beta(s)$ according to the Bishop frame, then the Bishop apparatus of $\beta^*(s^*(s))$ is given by the following equations:*

$$\begin{aligned}
\mathbf{T}^*(s^*) &= \delta_1(s)\mathbf{T}(s) + \delta_2(s)\mathbf{N}_2(s), \\
\mathbf{N}_2^*(s^*) &= \epsilon \frac{\left[\Big(\delta_1'(s) - \delta_2(s)k_2(s)\Big)\mathbf{T}(s) + \Big(\delta_2'(s) + \delta_1(s)k_2(s)\Big)\mathbf{N}_2(s)\right]}{\left[\Big(\delta_1'(s) - \delta_2(s)k_2(s)\Big)^2 + \Big(\delta_2'(s) + \delta_1(s)k_2(s)\Big)^2\right]^{\frac{1}{2}}}, \\
k_1^*(s^*) &= \epsilon \frac{\delta_1(s)k_1(s)}{\sqrt{\Big(\mu_1(s) - \lambda(s)k_1(s)\Big)^2 + \mu_3^2(s)}}, \\
k_2^*(s^*) &= \epsilon \frac{\left[\Big(\delta_1'(s) - \delta_2(s)k_2(s)\Big)^2 + \Big(\delta_2'(s) + \delta_1(s)k_2(s)\Big)^2\right]^{\frac{1}{2}}}{\sqrt{\Big(\mu_1(s) - \lambda(s)k_1(s)\Big)^2 + \mu_3^2(s)}},
\end{aligned}$$

such that $\delta_1(s) = \frac{\mu_1(s)-\lambda(s)k_1(s)}{\sqrt{\Big(\mu_1(s)-\lambda(s)k_1(s)\Big)^2+\mu_3^2(s)}}$, *and* $\delta_2(s) = \frac{\mu_3(s)}{\sqrt{\Big(\mu_1(s)-\lambda(s)k_1(s)\Big)^2+\mu_3^2(s)}}$.

Proof. Let $\beta^*(s^*(s))$ be a Bertrand W-curve according to the Bishop frame of the curve $\beta(s)$. Then from Definition 3.2, $\beta^*(s^*(s))$ is defined by the following formula:

$$\beta^*(s^*(s)) = \int W(s)ds + \lambda(s)\mathbf{N}_1(s), \tag{3.1}$$

where $\{\mathbf{N}_1(s), \mathbf{N}_1^*(s^*)\}$ is linearly dependent.

Differentiating Eq (3.1) with respect to the parameter s, we obtain

$$\mathbf{T}^*(s^*)\frac{ds^*}{ds} = (\mu_1(s) - \lambda(s)k_1(s))\mathbf{T}(s) + (\mu_2(s) + \lambda'(s))\mathbf{N}_1(s) + \mu_3(s)\mathbf{N}_2(s). \tag{3.2}$$

Multiplying both sides of Eq (3.2) by the vector $\mathbf{N}_1(s)$, we have $\lambda(s) = -\int \mu_2(s)ds$, and

$$\mathbf{T}^*(s^*)\frac{ds^*}{ds} = \Big(\mu_1(s) - \lambda(s)k_1(s)\Big)\mathbf{T}(s) + \mu_3(s)\mathbf{N}_2(s). \tag{3.3}$$

By taking the scalar product of Eq (3.3) with itself, we obtain the following equation:

$$\Big(\frac{ds^*}{ds}\Big)^2 = \Big(\mu_1(s) - \lambda(s)k_1(s)\Big)^2 + \mu_3^2(s),$$

$$1 = \frac{\Big(\mu_1(s) - \lambda(s)k_1(s)\Big)^2}{\Big(\frac{ds^*}{ds}\Big)^2} + \frac{\mu_3^2(s)}{\Big(\frac{ds^*}{ds}\Big)^2}.$$

Assume that $\frac{\mu_1(s)-\lambda(s)k_1(s)}{\sqrt{\left(\mu_1(s)-\lambda(s)k_1(s)\right)^2+\mu_3^2(s)}} = \delta_1(s)$, and $\frac{\mu_3(s)}{\sqrt{\left(\mu_1(s)-\lambda(s)k_1(s)\right)^2+\mu_3^2(s)}} = \delta_2(s)$, we get

$$\mathbf{T}^*(s^*) = \delta_1(s)\mathbf{T}(s) + \delta_2(s)\mathbf{N}_2(s). \tag{3.4}$$

Differentiating Eq (3.4) with respect to the parameter s, we get the following equation:

$$\frac{d\mathbf{T}^*}{ds^*}\frac{ds^*}{ds} = \left(\delta_1'(s) - \delta_2(s)k_2\right)\mathbf{T}(s) + \delta_1(s)k_1\mathbf{N}_1(s) + \left(\delta_2'(s) + \delta_1(s)k_2\right)\mathbf{N}_2(s). \tag{3.5}$$

Substituting the Bishop Eq (2.1) into Eq (3.5), we have the next equation

$$\begin{aligned} & \left(k_1^*(s^*)\mathbf{N}_1^*(s^*) + k_2^*(s^*)\mathbf{N}_2^*(s^*)\right)\frac{ds^*}{ds} \\ = & \left(\delta_1'(s) - \delta_2(s)k_2(s)\right)\mathbf{T}(s) + \delta_1(s)k_1(s)\mathbf{N}_1(s) + \left(\delta_2'(s) + \delta_1(s)k_2(s)\right)\mathbf{N}_2(s). \end{aligned}$$

Since $\mathbf{N}_1(s)$ and $\mathbf{N}_1^*(s^*)$ are linearly dependent, then

$$k_1^*(s^*) = \epsilon\frac{\delta_1(s)k_1(s)}{\sqrt{\left(\mu_1(s) - \lambda(s)k_1(s)\right)^2 + \mu_3^2(s)}}. \tag{3.6}$$

Therefore,

$$k_2^*(s^*)\mathbf{N}_2^*(s^*)\frac{ds^*}{ds} = \left(\delta_1'(s) - \delta_2(s)k_2(s)\right)\mathbf{T}(s) + \left(\delta_2'(s) + \delta_1(s)k_2(s)\right)\mathbf{N}_2(s). \tag{3.7}$$

Taking the scalar product of Eq (3.7) with itself, we obtain

$$\left(k_2^*(s^*)\right)^2(\frac{ds^*}{ds})^2 = \left(\delta_1'(s) - \delta_2(s)k_2(s)\right)^2 + \left(\delta_2'(s) + \delta_1(s)k_2(s)\right)^2.$$

Hence,

$$\begin{aligned} k_2^*(s^*) &= \epsilon\frac{\left[\left(\delta_1'(s) - \delta_2(s)k_2(s)\right)^2 + \left(\delta_2'(s) + \delta_1(s)k_2(s)\right)^2\right]^{\frac{1}{2}}}{\sqrt{\left(\mu_1(s) - \lambda(s)k_1(s)\right)^2 + \mu_3^2(s)}}, \\ \mathbf{N}_2^*(s^*) &= \epsilon\frac{\left[\left(\delta_1'(s) - \delta_2(s)k_2(s)\right)\mathbf{T}(s) + \left(\delta_2'(s) + \delta_1(s)k_2(s)\right)\mathbf{N}_2(s)\right]}{\left[\left(\delta_1'(s) - \delta_2(s)k_2(s)\right)^2 + \left(\delta_2'(s) + \delta_1(s)k_2(s)\right)^2\right]^{\frac{1}{2}}}. \end{aligned}$$

□

Example 3.1. *Suppose that* $\gamma(s) = \left(\frac{1}{2}\cos s, \frac{1}{2}\sin s, \frac{\sqrt{3}}{2}s\right)$, *then*

$$\mathbf{T}(s) = \frac{1}{2}\left(-\sin s, \cos s, \sqrt{3}\right),$$

$$\begin{aligned}\mathbf{N}(s) &= \Big(-\cos s, -\sin s, 0\Big),\\ \mathbf{B}(s) &= \frac{1}{2}\Big(\sqrt{3}\sin s, -\sqrt{3}\cos s, 1\Big),\\ k(s) &= \frac{1}{2}, \tau(s) = \frac{\sqrt{3}}{2},\end{aligned}$$

and $\theta(s) = \int \tau(s)ds = \frac{\sqrt{3}}{2}s + c_o$*. For simplicity, putting* $c_o = 0$*, and therefore*

$$\mathbf{N}_1(s) = \cos(\frac{\sqrt{3}}{2}s)\Big(-\cos s, -\sin s, 0\Big) + \frac{1}{2}\sin(\frac{\sqrt{3}}{2}s)\Big(\sqrt{3}\sin s, -\sqrt{3}\cos s, 1\Big),$$

$$\mathbf{N}_2(s) = -\sin(\frac{\sqrt{3}}{2}s)\Big(-\cos s, -\sin s, 0\Big) + \frac{1}{2}\cos(\frac{\sqrt{3}}{2}s)\Big(\sqrt{3}\sin s, -\sqrt{3}\cos s, 1\Big),$$

$$k_1(s) = \frac{1}{2}\cos(\frac{\sqrt{3}}{2}s), k_2 = \frac{1}{2}\sin(\frac{\sqrt{3}}{2}s).$$

Suppose that $\mu_1(s) = \mu_2(s) = \mu_3(s) = \frac{1}{\sqrt{3}}$*, then* $\lambda(s) = \Big(c_1 - \frac{1}{\sqrt{3}}s\Big)$*, where* c_1 *is constant. Therefore,*

$$W(s) = \frac{1}{\sqrt{3}}\Big(W_x(s), W_y(s), W_z(s)\Big),$$

where

$$W_x(s) = -\frac{1}{2}\sin s + \Big(\sin(\frac{\sqrt{3}}{2}s) - \cos(\frac{\sqrt{3}}{2}s)\Big)\cos s + \frac{\sqrt{3}}{2}\Big(\sin(\frac{\sqrt{3}}{2}s) - \cos(\frac{\sqrt{3}}{2}s)\Big)\sin s,$$

$$W_y(s) = \frac{1}{2}\cos s + \Big(\sin(\frac{\sqrt{3}}{2}s) - \cos(\frac{\sqrt{3}}{2}s)\Big)\sin s - \frac{\sqrt{3}}{2}\Big(\sin(\frac{\sqrt{3}}{2}s) + \cos(\frac{\sqrt{3}}{2}s)\Big)\cos s,$$

$$W_z(s) = \frac{\sqrt{3}}{2} + \frac{1}{2}\sin(\frac{\sqrt{3}}{2}s) + \frac{1}{2}\cos(\frac{\sqrt{3}}{2}s).$$

Hence,

$$\beta^*(s^*(s)) = \left(\begin{array}{c} \frac{1}{\sqrt{3}}\left(\begin{array}{c} (\frac{1}{2}\cos s - \frac{1}{(\sqrt{3}+2)}\cos((\frac{\sqrt{3}}{2}+1)s) - \frac{1}{(\sqrt{3}-2)}\cos((\frac{\sqrt{3}}{2}-1)s) \\ -\frac{1}{(\sqrt{3}+2)}\sin((\frac{\sqrt{3}}{2}+1)s) - \frac{1}{(\sqrt{3}-2)}\sin((\frac{\sqrt{3}}{2}-1)s) \\ +\frac{\sqrt{3}}{(2\sqrt{3}-4)}\sin((\frac{\sqrt{3}}{2}-1)s) - \frac{\sqrt{3}}{(2\sqrt{3}+4)}\sin((\frac{\sqrt{3}}{2}+1)s) \\ -\frac{\sqrt{3}}{(2\sqrt{3}+4)}\cos((\frac{\sqrt{3}}{2}+1)s) + \frac{\sqrt{3}}{(2\sqrt{3}-4)}\cos((\frac{\sqrt{3}}{2}-1)s) + C_2 \end{array}\right) \\ +(c_1 - \frac{1}{\sqrt{3}}s)(-\cos(\frac{\sqrt{3}}{2}s)\cos s + \frac{1}{2}\sin(\frac{\sqrt{3}}{2}s)(\sqrt{3}\sin s), \\ \frac{1}{\sqrt{3}}\left(\begin{array}{c} \frac{1}{2}\sin s + \frac{1}{(\sqrt{3}+2)}\cos((\frac{\sqrt{3}}{2}+1)s) - \frac{1}{(\sqrt{3}-2)}\cos((\frac{\sqrt{3}}{2}-1)s \\ +\frac{1}{(\sqrt{3}-2)}\sin((\frac{\sqrt{3}}{2}-1)s - \frac{1}{(\sqrt{3}+2)}\sin((\frac{\sqrt{3}}{2}+1)s) \\ +\frac{\sqrt{3}}{(2\sqrt{3}+4)}\cos((\frac{\sqrt{3}}{2}+1)s) + \frac{\sqrt{3}}{(2\sqrt{3}-4)}\cos((\frac{\sqrt{3}}{2}-1)s) \\ -\frac{\sqrt{3}}{(2\sqrt{3}+4)}\sin((\frac{\sqrt{3}}{2}+1)s) - \frac{\sqrt{3}}{(2\sqrt{3}-4)}\sin((\frac{\sqrt{3}}{2}-1)s) + C_3 \end{array}\right) + \\ (c_1 - \frac{1}{\sqrt{3}}s)(-\cos(\frac{\sqrt{3}}{2}s)\sin s + \frac{\sqrt{3}}{2}\sin(\frac{\sqrt{3}}{2}s)\sin s), \\ \frac{1}{\sqrt{3}}\left(\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{3}}\cos(\frac{\sqrt{3}}{2}s) + \frac{1}{\sqrt{3}}\sin(\frac{\sqrt{3}}{2}s) + C_4\right) \\ +(c_1 - \frac{1}{\sqrt{3}}s) \end{array}\right),$$

such that C_2, C_3*, and* C_4 *are constants.*

Theorem 3.2. *Let $\{\beta(s),\ \beta^*(s^*)\}$ be Bertrand W-mate according to Bishop frame with $\{\mathbf{T}(s), \mathbf{N}_1(s), \mathbf{N}_2(s), k_1(s), k_2(s)\}$, $\{\mathbf{T}^*(s^*), \mathbf{N}_1^*(s^*), \mathbf{N}_2^*(s^*), k_1^*(s^*), k_2^*(s^*)\}$ Bishop apparatus, respectively. Then the angle between $\mathbf{T}(s)$ and $\mathbf{T}^*(s^*)$ at the corresponding points is given by:*

$$\psi(s) = \cos^{-1}\left(\frac{\mu_1(s) - \lambda(s)k_1(s)}{\sqrt{\left(\mu_1(s) - \lambda(s)k_1(s)\right)^2 + \mu_3^2(s)}} - c_0\right),$$

where c_0 is constant.

Proof. From the angle definition between two vectors in Euclidean 3-space E^3.

$$\langle \mathbf{T}(s), \mathbf{T}^*(s^*)\rangle = \|\mathbf{T}(s)\|\,\|\mathbf{T}^*(s^*)\| \cos\psi(s), \tag{3.8}$$

where $\psi(s)$ is the angle between the two unit vectors $\mathbf{T}(s)$ and $\mathbf{T}^*(s^*)$.

Differentiating both sides of Eq (3.8) with respect to s, we have

$$\langle \frac{d\mathbf{T}(s)}{ds}, \mathbf{T}^*(s^*)\rangle + \frac{ds^*}{ds}\langle \mathbf{T}(s), \frac{d\mathbf{T}^*(s^*)}{ds^*}\rangle = \frac{d}{ds}\cos\psi(s). \tag{3.9}$$

Substituting from Eq (2.1) into Eq (3.9), we obtain

$$\frac{d}{ds}\cos\psi(s) = \langle k_1(s)\mathbf{N}_1(s) + k_2(s)\mathbf{N}_2(s), \mathbf{T}^*(s^*)\rangle + \frac{ds^*}{ds}\langle \mathbf{T}(s), k_1^*(s)\mathbf{N}_1^*(s^*) + k_2^*(s^*)\mathbf{N}_2^*(s^*)\rangle$$
$$= k_1(s)\langle \mathbf{N}_1(s), \mathbf{T}^*(s^*)\rangle + k_2(s)\langle \mathbf{N}_2(s), \mathbf{T}^*(s^*)\rangle + \frac{ds^*}{ds}k_1^*(s)\langle \mathbf{T}(s), \mathbf{N}_1^*(s^*)\rangle + \frac{ds^*}{ds}k_2^*(s^*)\langle \mathbf{T}(s), \mathbf{N}_2^*(s^*)\rangle.$$

Since $\mathbf{N}_1(s)$ and $\mathbf{N}_1^*(s^*)$ are linearly dependent, then we have the following equation:

$$k_2(s)\langle \mathbf{N}_2(s), \mathbf{T}^*(s^*)\rangle + \frac{ds^*}{ds}k_2^*(s^*)\langle \mathbf{T}(s), \mathbf{N}_2^*(s^*)\rangle = \frac{d}{ds}\cos\psi(s). \tag{3.10}$$

Substituting form Theorem 3.1 about $\mathbf{T}^*(s^*)$ and $\mathbf{N}_2^*(s^*)$ in Eq (3.10), we have

$$k_2(s)\langle \mathbf{N}_2(s), \delta_1(s)\mathbf{T}(s) + \delta_2(s)\mathbf{N}_2(s)\rangle + \frac{ds^*}{ds}k_2^*(s^*)\langle \mathbf{T}(s), \gamma_1(s)\mathbf{T}(s) + \gamma_2(s)\mathbf{N}_2(s)\rangle = \frac{d}{ds}\cos\psi(s),$$

where

$$\delta_1(s) = \frac{\mu_1(s) - \lambda(s)k_1(s)}{\sqrt{\left(\mu_1(s) - \lambda(s)k_1(s)\right)^2 + \mu_3^2(s)}},$$

$$\delta_2(s) = \frac{\mu_3(s)}{\sqrt{\left(\mu_1(s) - \lambda(s)k_1(s)\right)^2 + \mu_3^2(s)}},$$

$$\gamma_1(s) = \frac{\delta_1'(s) - \delta_2(s)k_2(s)}{\left[\left(\delta_1'(s) - \delta_2(s)k_2(s)\right)^2 + \left(\delta_2'(s) + \delta_1(s)k_2(s)\right)^2\right]^{\frac{1}{2}}},$$

and

$$\gamma_2(s) = \frac{\delta_2'(s) + \delta_1(s)k_2(s)}{\left[\left(\delta_1'(s) - \delta_2(s)k_2(s)\right)^2 + \left(\delta_2'(s) + \delta_1(s)k_2(s)\right)^2\right]^{\frac{1}{2}}}.$$

Therefore,

$$k_2(s)\delta_2(s) + \frac{ds^*}{ds}k_2^*(s^*)\gamma_1(s) = \frac{d}{ds}\cos\psi(s).$$

Substituting about $k_2^*(s^*)$ from Theorem 3.1, we get $\delta_1'(s) = \frac{d}{ds}\cos\psi(s)$. This means that

$$\frac{\mu_1(s) - \lambda(s)k_1(s)}{\sqrt{\left(\mu_1(s) - \lambda(s)k_1(s)\right)^2 + \mu_3^2(s)}} = \cos\psi(s) + c_0,$$

where c_0 is constant, and implies that

$$\psi(s) = \cos^{-1}\left(\frac{\mu_1(s) - \lambda(s)k_1(s)}{\sqrt{\left(\mu_1(s) - \lambda(s)k_1(s)\right)^2 + \mu_3^2(s)}} - c_0\right).$$

□

Theorem 3.3. *Let* $\{\beta(s),\ \beta^*(s^*)\}$ *be a Bertrand W-mate according to the Bishop frame with* $\{\mathbf{T}(s), \mathbf{N}_1(s), \mathbf{N}_2(s), k_1(s), k_2(s)\}$, $\{\mathbf{T}^*(s^*), \mathbf{N}_1^*(s^*), \mathbf{N}_2^*(s^*), k_1^*(s^*), k_2^*(s^*)\}$ *Bishop apparatus, respectively. Then the angle between* $\mathbf{N}_2(s)$ *and* $\mathbf{N}_2^*(s^*)$ *at corresponding points is obtained by*

$$\varphi(s) = \cos^{-1}\left(\int\left[-k_2(s)\gamma_1(s) - \sqrt{\left(\mu_1(s) - \lambda(s)k_1(s)\right)^2 + \mu_3^2(s)}\ k_2^*(s^*)\delta_2(s)\right]ds\right),$$

so that $\delta_2(s) = \frac{\mu_3(s)}{\sqrt{\left(\mu_1(s)-\lambda(s)k_1(s)\right)^2 + \mu_3^2(s)}}$, *and* $\gamma_1(s) = \frac{\delta_1'(s)-\delta_2(s)k_2(s)}{\left[\left(\delta_1'(s)-\delta_2(s)k_2(s)\right)^2+\left(\delta_2'(s)+\delta_1(s)k_2(s)\right)^2\right]^{\frac{1}{2}}}$.

Proof. Let $\varphi(s)$ be the angle between the two unit vectors $\mathbf{N}_2(s)$ and $\mathbf{N}_2^*(s^*)$. From the angle definition of two vectors in Euclidean 3-space,

$$\langle \mathbf{N}_2(s), \mathbf{N}_2^*(s^*)\rangle = \|\mathbf{N}_2(s)\| \left\|\mathbf{N}_2^*(s^*)\right\| \cos\varphi(s). \tag{3.11}$$

Differentiating both sides of Eq (3.11) with respect to s, we obtain

$$\langle\frac{d\mathbf{N}_2(s)}{ds}, \mathbf{N}_2^*(s^*)\rangle + \frac{ds^*}{ds}\langle \mathbf{N}_2(s), \frac{d\mathbf{N}_2^*(s^*)}{ds^*}\rangle = \frac{d}{ds}\cos\varphi(s). \tag{3.12}$$

Substituting from Eq (2.1) into Eq (3.12), we obtain

$$\langle -k_2(s)\mathbf{T}(s), \mathbf{N}_2^*(s^*)\rangle + \frac{ds^*}{ds}\langle \mathbf{N}_2(s), -k_2^*(s^*)\mathbf{T}^*(s)\rangle = \frac{d}{ds}\cos\varphi(s). \tag{3.13}$$

Substituting form Theorem 3.1 about $\mathbf{T}^*(s^*)$ and $\mathbf{N}_2^*(s^*)$ in Eq (3.13), we find

$$\frac{d}{ds}\cos\varphi(s) = \langle -k_2(s)\mathbf{T}(s), \gamma_1(s)\mathbf{T}(s) + \gamma_2(s)\mathbf{N}_2(s)\rangle + \frac{ds^*}{ds}\langle \mathbf{N}_2(s), -k_2^*(s^*)(\delta_1(s)\mathbf{T}(s) + \delta_2(s)\mathbf{N}_2(s)\rangle,$$

where

$$\delta_1(s) = \frac{\mu_1(s) - \lambda(s)k_1(s)}{\sqrt{\left(\mu_1(s) - \lambda(s)k_1(s)\right)^2 + \mu_3^2(s)}},$$

$$\delta_2(s) = \frac{\mu_3(s)}{\sqrt{\left(\mu_1(s) - \lambda(s)k_1(s)\right)^2 + \mu_3^2(s)}},$$

$$\gamma_1(s) = \frac{\delta_1^{'}(s) - \delta_2(s)k_2(s)}{\left[\left(\delta_1^{'}(s) - \delta_2(s)k_2(s)\right)^2 + \left(\delta_2^{'}(s) + \delta_1(s)k_2(s)\right)^2\right]^{\frac{1}{2}}},$$

and

$$\gamma_2(s) = \frac{\delta_2^{'}(s) + \delta_1(s)k_2(s)}{\left[\left(\delta_1^{'}(s) - \delta_2(s)k_2(s)\right)^2 + \left(\delta_2^{'}(s) + \delta_1(s)k_2(s)\right)^2\right]^{\frac{1}{2}}}.$$

Therefore,

$$-k_2(s)\gamma_1(s) - \sqrt{\left(\mu_1(s) - \lambda(s)k_1(s)\right)^2 + \mu_3^2(s)}\ k_2^*(s^*)\delta_2(s) = \frac{d}{ds}\cos\varphi(s),$$

which implies

$$\varphi(s) = \cos^{-1}\left(\int\left[-k_2(s)\gamma_1(s) - \sqrt{\left(\mu_1(s) - \lambda(s)k_1(s)\right)^2 + \mu_3^2(s)}\ k_2^*(s^*)\delta_2(s)\right]ds\right).$$

□

Theorem 3.4. *Let* $\{\beta(s), \beta^*(s^*)\}$ *be a Bertrand W-mate according to the Bishop frame with* $k_1(s)$ *not equal to zero, then*

$$k_1(s) = \frac{\mu_1(s)}{\lambda(s)},$$

and

$$k_2(s) = \frac{\delta_1^{'}(s)}{\delta_2(s)},$$

where

$$\delta_1(s) = \frac{\mu_1(s) - \lambda(s)k_1(s)}{\sqrt{\left(\mu_1(s) - \lambda(s)k_1(s)\right)^2 + \mu_3^2(s)}},$$

and

$$\delta_2(s) = \frac{\mu_3(s)}{\sqrt{\left(\mu_1(s) - \lambda(s)k_1(s)\right)^2 + \mu_3^2(s)}}.$$

Proof. Since $\{\beta(s), \beta^*(s^*)\}$ is a Bertrand W-mate according to the Bishop frame, then

$$\langle \mathbf{N}_1^*(s^*), \mathbf{N}_2(s)\rangle = 0. \tag{3.14}$$

Differentiating Eq (3.14) with respect to s, and using Eq (2.1), then using Theorem 3.1, we get

$$k_1(s)\delta_1(s)\delta_2(s) = 0.$$

If $\delta_1(s) = 0$, then $k_1(s) = \frac{\mu_1(s)}{\lambda(s)}$, and if $\delta_2(s) = 0$, then $\mu_3(s) = 0$. Also, we have

$$\langle \mathbf{N}_1(s), \mathbf{N}_2^*(s^*)\rangle = 0. \tag{3.15}$$

Differentiating Eq (3.15) with respect to s, and using Eq (2.1), then using Theorem 3.1, we obtain

$$k_1(s)\gamma_1(s) = 0, \quad k_2(s) = \frac{\delta_1'(s)}{\delta_2(s)},$$

where $\delta_1(s) = \frac{\mu_1(s)-\lambda(s)k_1(s)}{\sqrt{\left(\mu_1(s)-\lambda(s)k_1(s)\right)^2+\mu_3^2(s)}}$, and $\delta_2(s) = \frac{\mu_3(s)}{\sqrt{\left(\mu_1(s)-\lambda(s)k_1(s)\right)^2+\mu_3^2(s)}}$. □

Theorem 3.5. *Let $\beta(s)$ be a unit speed curve, and let $\{\mathbf{T}(s), \mathbf{N}_1(s), \mathbf{N}_2(s), k_1(s), k_2(s)\}$ be its Bishop apparatus. Suppose that $\beta^*(s^*(s))$ is a Bertrand W-mate according to the Bishop frame of $\beta(s)$; then the following statement is satisfied:*

$$\mu_3(s) = \left(\mu_1(s) - \lambda(s)k_1(s)\right)\tan\left(\zeta(s)\right),$$

where $\zeta(s)$ is the angle between $\mathbf{T}^*(s^*), \mathbf{T}(s)$.

Proof. Substituting the values of $\delta_1(s)$ and $\delta_2(s)$ in Eq (3.4), we obtain

$$\mathbf{T}^*(s^*) = \frac{\mu_1(s) - \lambda(s)k_1(s)}{\sqrt{\left(\mu_1(s) - \lambda(s)k_1(s)\right)^2 + \mu_3^2(s)}}\mathbf{T}(s) + \frac{\mu_3(s)}{\sqrt{\left(\mu_1(s) - \lambda(s)k_1(s)\right)^2 + \mu_3^2(s)}}\mathbf{N}_2(s).$$

Putting

$$\cos\left(\zeta(s)\right) = \frac{\mu_1(s) - \lambda(s)k_1(s)}{\sqrt{\left(\mu_1(s) - \lambda(s)k_1(s)\right)^2 + \mu_3^2(s)}},$$

and

$$\sin\left(\zeta(s)\right) = \frac{\mu_3(s)}{\sqrt{\left(\mu_1(s) - \lambda(s)k_1(s)\right)^2 + \mu_3^2(s)}}.$$

Then, $\mu_3(s) = \left(\mu_1(s) - \lambda(s)k_1(s)\right)\tan\left(\zeta(s)\right)$. □

Corollary 3.1. *Let $\beta(s)$ be a unit speed curve and let $\{\mathbf{T}(s), \mathbf{N}_1(s), \mathbf{N}_2(s), k_1(s), k_2(s)\}$ be its Bishop apparatus. Suppose that $\beta^*(s^*(s))$ is a Bertrand W-mate according to the Bishop frame of $\beta(s)$, then $k_1^*(s^*) = \epsilon k_1(s)\cos\left(\zeta(s)\right)$, where $\zeta(s)$ is the angle between $\mathbf{T}^*(s^*)$ and $T(s)$.*

Proof. We have

$$\mathbf{T}^*(s^*) = \cos\Big(\zeta(s)\Big)\mathbf{T}(s) + \sin\Big(\zeta(s)\Big)\mathbf{N}_2(s).$$

Differentiating both sides with respect to s, we find

$$\frac{d\mathbf{T}^*}{ds^*}\frac{ds^*}{ds} = -\sin(\zeta(s))\Big(\zeta'(s) + k_2(s)\Big)\mathbf{T}(s) + k_1(s)\cos(\zeta(s))\mathbf{N}_1(s) + \cos(\zeta(s))\Big(k_2(s) - \zeta'(s)\Big)\mathbf{N}_2(s). \quad (3.16)$$

Substituting from Eq (2.1) into Eq (3.16), we obtain

$$\begin{aligned}&\Big(k_1^*(s^*)\mathbf{N}_1^*(s^*) + k_2^*(s^*)\mathbf{N}_2^*(s^*)\Big)\frac{ds^*}{ds}\\ &= -\sin(\zeta(s))\Big(\zeta'(s) + k_2(s)\Big)\mathbf{T}(s) + k_1(s)\cos(\zeta(s))\mathbf{N}_1(s) + \cos(\zeta(s))(k_2(s) - \zeta'(s))\mathbf{N}_2(s).\end{aligned} \quad (3.17)$$

Since $\mathbf{N}_1(s)$ and $\mathbf{N}_1^*(s^*)$ are linearly dependent, and by taking the inner product of both sides of Eq (3.17) with $\mathbf{N}_1(s)$, we obtain

$$\cos\Big(\zeta(s)\Big) = \epsilon\frac{k_1^*(s^*)}{k_1(s)}.$$

□

Corollary 3.2. *Let $\beta(s)$ be a unit speed curve and let $\{\mathbf{T}(s), \mathbf{N}_1(s), \mathbf{N}_2(s), k_1(s), k_2(s)\}$ be its Bishop apparatus. Suppose that $\beta^*(s^*(s))$ is a Bertrand W-mate according to the Bishop frame of $\beta(s)$, then*

- *If $\mu_1(s) = 1, \mu_2(s) = \mu_3(s) = 0$, then $\lambda(s) = c$, where c is a nonzero constant, and $k_1(s) = \frac{1}{c\tan\left(\zeta(s)\right)}, c \neq 0, \tan\left(\zeta(s)\right) \neq 0$.*
- *If $\mu_2(s) = 1, \mu_1(s) = \mu_3(s) = 0$, then $\lambda(s) = c - s$, where c is constant, then $k_1(s)\tan\left(\zeta(s)\right) = 0$.*
- *If $\mu_3(s) = 1, \mu_1(s) = \mu_2(s) = 0$, then $\lambda(s) = c$, where c is a nonzero constant, and $k_1(s) = \frac{-1}{c\tan\left(\zeta(s)\right)}, c \neq 0, \tan\left(\zeta(s)\right) \neq 0$.*

In the following theorem we introduce special cases of Bertrand W-mates according to the Bishop frame.

Theorem 3.6. *Let $\beta(s)$ and $\beta^*(s^*)$ be two curves in $\mathbb{R}^3$ parametrized by arc length parameters s and s^*, respectively, with Bishop curvatures $k_1(s) = k_1^*(s^*) = 0$, $k_2(s)$, and $k_2^*(s^*)$. Suppose that $\beta^*(s^*(s))$ is a Bertrand W-mate of $\beta(s)$ according to the Bishop frame, then*

$$k_2^*(s^*)^2 = \frac{\Big(\upsilon_1'(s) - k_2(s)\upsilon_2(s)\Big)^2 + \Big(\upsilon_2'(s) + \upsilon_1(s)k_2(s)\Big)^2}{1 - \Big(\mu_2(s)\Big)^2},$$

where

$$\upsilon_1(s) = \frac{\mu_1(s)}{\sqrt{1 - \Big(\mu_2(s)\Big)^2}},$$

and

$$\upsilon_2(s) = \frac{\mu_3(s)}{\sqrt{1 - \Big(\mu_2(s)\Big)^2}}.$$

Proof. Assume that $\beta^*(s^*)$ is a Bertrand W-curve of $\beta(s)$ according to the Bishop frame. Then by using Definition 3.2, we can write the curve $\beta^*(s^*)$ as

$$\beta^*(s^*(s)) = \int W(s)ds + \lambda(s)\mathbf{N}_1(s), \tag{3.18}$$

where $\lambda(s)$ is some differentiable function. Differentiating Eq (3.18) with respect to s and using Bishop Eq (2.1), we obtain

$$\mathbf{T}^*(s^*)\frac{ds^*}{ds} = \mu_1(s)\mathbf{T}(s) + \Big(\lambda^{'}(s) + \mu_2(s)\Big)\mathbf{N}_1(s) + \mu_3(s)\mathbf{N}_3(s). \tag{3.19}$$

But $\lambda(s) = -\int \mu_2(s)ds$, then we obtain

$$\mathbf{T}^*(s^*)\frac{ds^*}{ds} = \mu_1(s)\mathbf{T}(s) + \mu_3(s)\mathbf{N}_3(s). \tag{3.20}$$

By taking the scalar product of Eq (3.20) with itself, we have

$$\Big(\frac{ds^*}{ds}\Big)^2 = 1 - \Big(\mu_2(s)\Big)^2.$$

Therefore,

$$\mathbf{T}^*(s^*) = \upsilon_1(s)\mathbf{T}(s) + \upsilon_2(s)\mathbf{N}_2(s), \tag{3.21}$$

where

$$\upsilon_1(s) = \frac{\mu_1(s)}{\sqrt{1 - \Big(\mu_2(s)\Big)^2}},$$

and

$$\upsilon_2(s) = \frac{\mu_3(s)}{\sqrt{1 - \Big(\mu_2(s)\Big)^2}}.$$

Differentiating Eq (3.21) with respect to s and applying Bishop Eq (2.1), we obtain

$$\sqrt{1 - \Big(\mu_2(s)\Big)^2}\, k_2^*(s^*)\mathbf{N}_2^*(s^*) = \Big(\upsilon_1^{'}(s) - k_2(s)\upsilon_2(s)\Big)\mathbf{T}(s) + \Big(\upsilon_2^{'}(s) + \upsilon_1(s)k_2(s)\Big)\mathbf{N}_2(s). \tag{3.22}$$

By taking the scalar product of Eq (3.22) with itself, we reach

$$k_2^*(s^*)^2 = \frac{\Big(\upsilon_1^{'}(s) - k_2(s)\upsilon_2(s)\Big)^2 + \Big(\upsilon_2^{'}(s) + \upsilon_1(s)k_2(s)\Big)^2}{1 - \Big(\mu_2(s)\Big)^2}.$$

□

Theorem 3.7. *Let $\beta(s)$ and $\beta^*(s^*)$ be two curves in $\mathbb{R}^3$ parametrized by arc length parameters s and s^*, respectively, with Bishop curvatures $k_1(s)$, $k_1^*(s^*)$, and $k_2(s) = k_2^*(s^*) = 0$. Suppose that $\beta^*(s^*(s))$ is a Bertrand W-curve according to the Bishop frame of $\beta(s)$, then $k_1^*(s^*) = \frac{k_1(s)\Big(\mu_1(s)-\lambda(s)k_1(s)\Big)}{\Big(\mu_1(s)-\lambda(s)k_1(s)\Big)^2+\Big(\mu_3(s)\Big)^2}$, and $\lambda(s) = \frac{\sqrt{1-c^2}\ \mu_1(s)}{ck_1(s)}$, where c is an integral constant.*

Proof. Assume that $\beta^*(s^*)$ is a Bertrand W-curve according to the Bishop frame of $\beta(s)$. Then by using Definition 3.2, we can write the curve $\beta^*(s^*)$ as

$$\beta^*(s^*(s)) = \int W(s)ds + \lambda(s)\mathbf{N}_1(s), \tag{3.23}$$

where $\lambda(s)$ is some differentiable function. Differentiating Eq (3.23) with respect to s and using Bishop Eq (2.1), we obtain

$$\mathbf{T}^*(s^*)\frac{ds^*}{ds} = \Big(\mu_1(s) - \lambda(s)k_1(s)\Big)\mathbf{T}(s) + \mu_3(s)\mathbf{N}_2(s). \tag{3.24}$$

By taking the scalar product of Eq (3.24) with itself, we obtain

$$\Big(\frac{ds^*}{ds}\Big)^2 = \Big(\mu_1(s) - \lambda(s)k_1(s)\Big)^2 + \Big(\mu_3(s)\Big)^2.$$

Therefore,

$$\mathbf{T}^*(s^*) = \eta_1(s)\mathbf{T}(s) + \eta_2(s)\mathbf{N}_2(s), \tag{3.25}$$

where

$$\eta_1(s) = \frac{\mu_1(s) - \lambda(s)k_1(s)}{\sqrt{\Big(\mu_1(s) - \lambda(s)k_1(s)\Big)^2 + \Big(\mu_3(s)\Big)^2}}, \quad \eta_2(s) = \frac{\mu_3(s)}{\sqrt{\Big(\mu_1(s) - \lambda(s)k_1(s)\Big)^2 + \Big(\mu_3(s)\Big)^2}}.$$

Differentiating Eq (3.25) with respect to s, and using Bishop Eq (2.1), we find

$$k_1^*(s^*)\mathbf{N}_1^*(s^*)\frac{ds^*}{ds} = \eta_1'(s)\mathbf{T}(s) + \eta_1(s)k_1(s)\mathbf{N}_1(s) + \eta_2'(s)\mathbf{N}_2(s). \tag{3.26}$$

Since $\mathbf{N}_1(s)$ and $\mathbf{N}_1^*(s^*)$ are linearly dependent, then

$$k_1^*(s^*) = \epsilon\frac{k_1(s)\Big(\mu_1(s) - \lambda(s)k_1(s)\Big)}{\Big(\mu_1(s) - \lambda(s)k_1(s)\Big)^2 + \Big(\mu_3(s)\Big)^2}.$$

From Eq (3.26), $\eta_2'(s) = 0$, i.e., $\frac{\mu_3(s)}{\sqrt{\Big(\mu_1(s)-\lambda(s)k_1(s)\Big)^2+\Big(\mu_3(s)\Big)^2}} = c$, where c is a constant. By simple calculation we find that $\lambda(s) = \frac{\sqrt{1-c^2}\,\mu_1(s)}{ck_1(s)}$. □

4. Conclusions

This study investigates in detail the components of a new generation of the Bishop frame, including three orthogonal unit vectors: the tangent, normal, and binormal vectors. It is a frame field described on a curve in Euclidean space and is an alternative to the Frenet frame. For curves for which the second derivative is not accessible, it is useful. Furthermore, the conditions under which the Bishop frame of one curve coincides with the Bishop frame of another are described. When the Bishop frame of one curve coincides with the Bishop frame of another, it would be advantageous to replicate such tactics. The idea of W-Bertrand curves according to the Bishop frame in Euclidean 3-space has been introduced, and several types of W-Bertrand curves based on the Bishop frame have been examined.

Author contributions

All authors jointly worked on the results, and they read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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