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Research article

Riemann solitons on spacetimes with pure radiation metrics

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Abstract: We investigate Riemann solitons within spacetimes characterized by pure radiation metrics that are conformally related to a vacuum spacetime. Additionally, it is shown that the Riemann solitons on spacetimes with pure radiation metrics are gradient Riemann solitons with a certain potential function. Moreover, we classify the potential vector fields of Riemann solitons as Ricci collineation, Killing, and Ricci bi-conformal.

Keywords: pure radiation metrics; Riemann soliton; Ricci collineation
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1. Introduction

In the context of general relativity, a four-dimensional, time-oriented Lorentzian manifold is identified as a pure radiation field spacetime when its energy-momentum tensor satisfies a specific condition: $T = al \otimes l$. Here, l represents a null vector, while a is a positive scalar function. This tensor encapsulates essential information regarding the energy density, momentum density, mass density, and other related quantities. The Einstein field equations (EFEs), which are formulated as a set of nonlinear partial differential equations, set a relationship between the geometry of spacetime and the distribution of mass-energy, momentum, and stress. This relationship allows one to define the metric tensor for the structure of spacetime, which is influenced by the arrangement of stress-energy-momentum within that spacetime.

Solutions to EFEs have been systematically categorized through algebraically symmetric frameworks such as Segre types or Petrov, as well as through motion groups. These solutions are frequently examined due to their significant contributions to contemporary research in gravitational physics. They effectively represent various gravitational phenomena, including the dynamics of an expanding universe and rotating black holes. The solutions to EFEs have been extensively researched, particularly in the context of pure radiation, with a special emphasis on conformally Ricci-flat (CRF) metrics. This concept serves as a natural extension of conformally flat metrics.

In [1], all conformally Ricci-flat Einstein-Maxwell null fields were obtained. These solutions represent a specific subset within the broader category of pp-waves, which are Lorentzian manifolds characterized by the presence of a parallel null vector field that meets particular curvature criteria. Wils [2] investigated the presence of pure radiation metrics (PRMs), which are conformal to a vacuum spacetime and do not lie in the Einstein-Maxwell framework. He specifically derived a family of PRMs of a CRF type that were distinct from plane waves. Wils asserted that no additional PRMs of a CRF type existed beyond those he had already identified. However, Ludwig et al. demonstrated that the class of PRMs of type CRF that are not plane waves is indeed broader than what Wils had proposed. They provided proof through two different methodologies that all PRMs of a CRF type lies within the Kundt class [3] and are classified as Petrov type N (or O) when the spin coefficient is non-zero, and as Petrov type III, N, or O when the spin coefficient is zero [4].

The physical characteristics of these PRMs of a CRF type are well defined [2, 4]. Regarding the geometric dimensions, Shaikh et al. explored various geometric structures associated with the PRMs introduced in [5].

The interplay between the geometry of spacetime and its relationship with physical phenomena is a compelling area of study that has significant interest from researchers in both physics and mathematics. In his theory of general relativity, Einstein characterized gravity as a force that causes the curvature of spacetime, which is mathematically represented as the metric tensor. Consequently, geometric vector fields, including Ricci soliton vector fields, possess physical significance and interpretations within this framework. This paper aims to classify all Riemann solitons (RISs) within spacetimes conformal to a vacuum spacetime with PRMs, thereby identifying which of these are under the category of gradient solitons. Our work's classification highlights the geometric structure underlying pure radiation spacetimes that accommodate RIS solutions. These spacetimes correspond to null fluid or scenarios dominated by radiation within the framework of general relativity. The presence of RISs, especially steady forms, suggests the potential for self-similar or equilibrium geometries that arise through Riemann flow. This can be physically understood as stable configurations of radiating systems or gravitational wave-like fields. These insights have particular significance in scenarios that involve gravitational collapse or radiation emitted from black holes.

The notion of "soliton" was first introduced by Kruskal and Zabusky to explain the characteristics of solitary waves. Since then, our understanding of solitons has significantly progressed, thus leading to their application across various domains. A comprehensive physical and mathematical framework has been established for solitons, thus highlighting their importance in contemporary physics [6, 7]. Additionally, the symmetry metric is often simplified to classify solutions to EFEs, with solitons playing a crucial role as a symmetry linked to the geometric evolution of spacetime. Hamilton's work on particular geometric flows, such as Ricci flow [8], has deepened our understanding of kinematics, and these geometric flows have been pivotal in the study of gravitational phenomena [9].

Ricci flow, a concept introduced by Hamilton [8] in the late 20th century, serves as a generalization of the Einstein metric. Specifically, the Ricci flow on a pseudo-Riemannian manifold denoted as $(M, g(t))$, characterized by the Ricci curvature tensor $S_{g(t)}$, is mathematically expressed by the following equation:

$$\frac{\partial}{\partial t}g(t) = 2S_{g(t)}.$$

A self-similar solution to the Ricci flow is referred to as a "Ricci soliton". In this context, a manifold (M, g) admits a Ricci soliton if it satisfies the following equation:

$$L_Y g + 2S = 2\lambda g,$$

for some vector field Y and a real constant λ. Notably, when $Y = 0$, the Ricci solitons correspond to Einstein manifolds. Initially explored in the Riemannian context, Ricci solitons have also been examined under pseudo-Riemannian conditions, particularly focusing on the Lorentzian case [10]. Various physical implications and applications of both the Ricci flow and Ricci solitons have been discussed in the literature [11–13]. In 2024, the authors have studied Hyperbolic Ricci solitons on perfect fluid spacetimes [14].

The concept of Riemann flow on the manifold (M, g), characterized by the Riemann curvature tensor R, was proposed by Udrişte [15, 16] and is expressed by the following equation:

$$\frac{\partial}{\partial t}\mathcal{G}(t) = -2R(g(t)),$$

where $\mathcal{G} = \frac{1}{2}g \odot g$. For two $(0, 2)$-tensors ω and ϑ, the symbol $\odot$ is defined by the following:

$$\begin{aligned}(\omega \odot \vartheta)(W_1, W_2, U_1, U_2) &= \omega(W_1, U_2)\vartheta(W_2, U_1) + \omega(W_2, U_1)\vartheta(W_1, U_2) \\ &\quad -\omega(W_1, U_1)\vartheta(W_2, U_2) - \omega(W_2, U_2)\vartheta(W_1, U_1),\end{aligned}$$

for all vector fields W_1, W_2, U_1 and U_2 on M. A manifold (M^n, g) is classified as a RIS [17] and is represented by (M^n, g, μ, V) when

$$2R + 2\mu\mathcal{G} + g \odot \mathcal{L}_V g = 0, \tag{1.1}$$

for some specific vector field V and a constant μ. The RIS is classified as expanding, shrinking, or steady by characterizing the sign of μ, where μ is positive, negative, or zero, respectively. When V is expressed as grad$f$, the soliton is referred to as a gradient RIS; thus, we have the following:

$$2R + \mu g \odot g + 2g \odot \nabla^2 f = 0.$$

When μ is smooth function, a RIS and a gradient RIS are referred to as an almost RIS and an almost gradient RIS, respectively. The RIS is associated with the Riemann flow and serves as a fixed point in this context.

From Eq (1.1), we derive the following expression:

$$\begin{aligned}2R(W_1, W_2, U_1, U_2) &= -2\mu\left[g(W_1, U_2)g(W_2, U_1) - g(W_1, U_1)g(W_2, U_2)\right] \\ &\quad -\left[g(W_1, U_2)\mathcal{L}_V g(W_2, U_1) + g(W_2, U_1)\mathcal{L}_V g(W_1, U_2)\right] \qquad (1.2)\end{aligned}$$

$$+\left[g(W_1,U_1)\mathcal{L}_V g(W_2,U_2)+g(W_2,U_2)\mathcal{L}_V g(W_1,U_1)\right].$$

By contracting over the vectors W_1 and U_2 in the preceding equation, we arrive at the following result:

$$2S(W_2,U_1)=-2((n-1)\mu+\mathrm{div}V)g(W_2,U_1)-(n-2)\mathcal{L}_V g(W_2,U_1).$$

Numerous studies have explored the concept of RISs on various types of manifolds. For example, Biswas et al. [18] examined RISs on a three-dimensional almost co-Kahler manifold, while Venkatesha et al. [19,20] focused on RISs in contact geometry. Additionally, K. De and U. C. De [21] investigated almost RISs on para-Sasakian manifolds. For more detailed studies on RISs, see [22–25]. In most of the papers published so far, when RISs are studied on spacetime, they have been considered from two viewpoints. One viewpoint is that if spacetime has a vector field that satisfies the RIS equation, then it will have some properties. Another viewpoint regarding the existence of such fields on the spacetimes is that the system formed through RIS equations must be solved, which sometimes is impossible to solve and the system does not have a solution. Here, we examine the existence of RIS vector fields.

The structure of this paper is outlined as follows: Section 2 revisits fundamental notations and important formulas related to spacetimes with pure radiation metrics, which will be referenced throughout the paper; and in Section 3, we categorize RISs within the context of spacetimes with PRMs which are conformal to a vacuum spacetime.

2. Preliminaries

The metric for a CRF pure radiation spacetime with the coordinates $(u;v;x;y)$, where $x>0$, is expressed as follows:

$$g=(xf(u,x,y)-p^2\frac{v^2}{x^2})du^2+2dudv-\frac{4v}{x}dudx-\frac{1}{p^2}(dx^2+dy^2), \tag{2.1}$$

where p is a non-zero constant, and $f(u,x,y)$ is an arbitrary smooth function. In this paper, we assume $\partial_z=\frac{\partial}{\partial z}$ for $z\in\{u,v,x,y\}$.

Let ∇ represent the Levi-Civita connection of g. The curvature tensor R that corresponds to g is determined by the equation

$$R(V,X)=\nabla_{[V,X]}-[\nabla_V,\nabla_X],$$

while the Ricci tensor S of g is expressed as $S(V,W)=tr(X\to R(V,X)W)$. The non-zero components of ∇ are specified by the following:

$$\begin{aligned}
\nabla_{\partial_u}\partial_u &= \frac{p^2v}{x^2}\partial_u+\left(p^2v\partial_x f-\frac{p^4v^3}{x^4}+\frac{1}{2}x\partial_u f\right)\partial_v+\frac{p^2}{2}\left(x\partial_x f+f-\frac{2p^2v^2}{x^3}\right)\partial_x+\frac{1}{2}p^2x\partial_y f\partial_y,\\
\nabla_{\partial_u}\partial_v &= \frac{p^2}{x}\left(\frac{v}{x}\partial_v+\partial_x\right),\\
\nabla_{\partial_u}\partial_x &= \frac{1}{x}\partial_u+\frac{1}{2}\left(x\partial_x f-f-\frac{4p^2v^2}{x^3}\right)\partial_v-\frac{2p^2v}{x^2}\partial_x,\\
\nabla_{\partial_u}\partial_y &= \frac{x\partial_y f}{2}\partial_v,
\end{aligned}$$

$$\nabla_{\partial_v}\partial_x = -\frac{1}{x}\partial_v,$$
$$\nabla_{\partial_x}\partial_x = \frac{2v}{x^2}\partial_v.$$

The only non-zero components of R associated with (2.1) are as follows [26]:

$$R_{uxux} = -\frac{x}{2}\partial^2_{xx}f, \qquad R_{uxuy} = -\frac{x}{2}\partial^2_{xy}f, \qquad R_{uyuy} = -\frac{x}{2}\partial^2_{yy}f, \tag{2.2}$$

where, for instance, $R_{uxux} = R(\partial_u, \partial_x, \partial_u, \partial_x)$. Additionally, the Ricci tensor S is obtained as follows:

$$S = \begin{pmatrix} \frac{p^2}{2}(\partial^2_{xx}f + \partial^2_{yy}f) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.3}$$

with respect to the basis $\{\partial_u, \partial_v, \partial_x, \partial_y\}$.

Let $Y = Y_1\partial_u + Y_2\partial_v + Y_3\partial_x + Y_4\partial_y$ be an arbitrary vector field, where $Y_i = Y_i(u, v, x, y)$ and $i = 1, 2, 3, 4$ are smooth functions. Using (2.1) and direct computation, we obtain the following:

$$\begin{aligned}
(\mathcal{L}_Y g)(\partial_u, \partial_u) &= x\left(\partial_u f Y_1 + \partial_x f Y_3 + \partial_y f Y_4\right) + 2\left(xf - \frac{p^2v^2}{x^2}\right)\partial_u Y_1 + 2\partial_u Y_2 - \frac{4v}{x}\partial_u Y_3 \\
&\quad + \frac{1}{x^3}\left(-2p^2vxY_2 + (x^3 f + 2p^2v^2)Y_3\right), \\
(\mathcal{L}_Y g)(\partial_u, \partial_v) &= \partial_u Y_1 + \left(xf - \frac{p^2v^2}{x^2}\right)\partial_v Y_1 + \partial_v Y_2 - \frac{2v}{x}\partial_v Y_3, \\
(\mathcal{L}_Y g)(\partial_u, \partial_x) &= \frac{2}{x^2}\left(vY_3 - xY_2\right) - \frac{2v}{x}(\partial_u Y_1 + \partial_x Y_3) - \frac{1}{p^2}\partial_u Y_3 + \left(xf - \frac{p^2v^2}{x^2}\right)\partial_x Y_1 + \partial_x Y_2, \\
(\mathcal{L}_Y g)(\partial_u, \partial_y) &= -\frac{2v}{x}\partial_y Y_3 - \frac{1}{p^2}\partial_u Y_4 + \left(xf - \frac{p^2v^2}{x^2}\right)\partial_y Y_1 + \partial_y Y_2, \\
(\mathcal{L}_Y g)(\partial_v, \partial_v) &= 2\partial_v Y_1, \\
(\mathcal{L}_Y g)(\partial_v, \partial_x) &= -\frac{2v}{x}\partial_v Y_1 - \frac{1}{p^2}\partial_v Y_3 + \partial_x Y_1, \\
(\mathcal{L}_Y g)(\partial_v, \partial_y) &= -\frac{1}{p^2}\partial_v Y_4 + \partial_y Y_1, \\
(\mathcal{L}_Y g)(\partial_x, \partial_x) &= -\frac{4v}{x}\partial_x Y_1 - \frac{2}{p^2}\partial_x Y_3, \\
(\mathcal{L}_Y g)(\partial_x, \partial_y) &= -\frac{2v}{x}\partial_y Y_1 - \frac{1}{p^2}(\partial_x Y_4 + \partial_y Y_3), \\
(\mathcal{L}_Y g)(\partial_y, \partial_y) &= -\frac{2}{p^2}\partial_y Y_4.
\end{aligned} \tag{2.4}$$

Additionally, $\mathcal{L}_Y S$ is expressed as follows:

$$(\mathcal{L}_Y S)(\partial_u, \partial_u) = p^2x(\partial^2_{xx}f + \partial^2_{yy}f)\partial_u Y_1 + \frac{1}{2}p^2x\left(\partial_u(\partial^2_{xx}f + \partial^2_{yy}f)\right)Y_1 + \frac{1}{2}p^2(\partial^2_{xx}f + \partial^2_{yy}f)Y_3$$

$$
\begin{aligned}
&+\frac{1}{2}p^2x\left(\partial_x(\partial^2_{xx}f+\partial^2_{yy}f)\right)Y_3+\frac{1}{2}p^2x\left(\partial_y(\partial^2_{xx}f+\partial^2_{yy}f)\right)Y_4,\\
(\mathcal{L}_YS)(\partial_u,\partial_v)&=\frac{1}{2}p^2x\left(\partial^2_{xx}f+\partial^2_{yy}f)\right)\partial_vY_1,\\
(\mathcal{L}_YS)(\partial_u,\partial_x)&=\frac{1}{2}p^2x\left(\partial^2_{xx}f+\partial^2_{yy}f)\right)\partial_xY_1,\\
(\mathcal{L}_YS)(\partial_u,\partial_y)&=\frac{1}{2}p^2x\left(\partial^2_{xx}f+\partial^2_{yy}f)\right)\partial_yY_1,\\
(\mathcal{L}_YS)(\partial_v,\partial_v)&=(\mathcal{L}_YS)(\partial_v,\partial_x)=(\mathcal{L}_YS)(\partial_v,\partial_y)=(\mathcal{L}_YS)(\partial_x,\partial_x)=0,\\
(\mathcal{L}_YS)(\partial_x,\partial_y)&=(\mathcal{L}_YS)(\partial_y,\partial_y)=0.
\end{aligned}
\tag{2.5}
$$

3. RISs of PRMs of type CRF

Now, we study the existence of RISs of PRMs of a CRF type. Using Eq (1.2), if a spacetime with PRMs of a CRF type is a RIS (M^n, g, μ, Y), then the following hold:

$$
\begin{aligned}
2R_{uxvy}&=g(\partial_u,\partial_v)(\mathcal{L}_Yg)(\partial_y,\partial_x),\\
2R_{uxxy}&=-g(\partial_x,\partial_x)(\mathcal{L}_Yg)(\partial_u,\partial_y)+g(\partial_u,\partial_x)(\mathcal{L}_Yg)(\partial_y,\partial_x),\\
2R_{uxuy}&=-g(\partial_u,\partial_x)(\mathcal{L}_Yg)(\partial_u,\partial_y)+g(\partial_u,\partial_u)(\mathcal{L}_Yg)(\partial_y,\partial_x),\\
2R_{uvuv}&=-2\mu g(\partial_u,\partial_v)^2-2g(\partial_u,\partial_v)(\mathcal{L}_Yg)(\partial_u,\partial_v)+g(\partial_u,\partial_u)(\mathcal{L}_Yg)(\partial_v,\partial_v),\\
2R_{uvux}&=-2\mu g(\partial_u,\partial_v)g(\partial_u,\partial_x)-g(\partial_u,\partial_x)(\mathcal{L}_Yg)(\partial_u,\partial_v)-g(\partial_u,\partial_v)(\mathcal{L}_Yg)(\partial_u,\partial_x)\\
&\quad+g(\partial_u,\partial_u)(\mathcal{L}_Yg)(\partial_v,\partial_v),\\
2R_{uvuy}&=-g(\partial_u,\partial_v)(\mathcal{L}_Yg)(\partial_u,\partial_y)+g(\partial_u,\partial_u)(\mathcal{L}_Yg)(\partial_y,\partial_v),\\
2R_{uvvx}&=-g(\partial_u,\partial_x)(\mathcal{L}_Yg)(\partial_v,\partial_v)+g(\partial_u,\partial_v)(\mathcal{L}_Yg)(\partial_x,\partial_v),\\
2R_{uvvy}&=g(\partial_u,\partial_v)(\mathcal{L}_Yg)(\partial_y,\partial_v),\\
2R_{uvxy}&=g(\partial_u,\partial_x)(\mathcal{L}_Yg)(\partial_v,\partial_y),\\
2R_{uxux}&=-2\mu g(\partial_u,\partial_x)^2+2\mu g(\partial_u,\partial_u)g(\partial_x,\partial_x)-2g(\partial_u,\partial_x)(\mathcal{L}_Yg)(\partial_u,\partial_x)\\
&\quad+g(\partial_u,\partial_u)(\mathcal{L}_Yg)(\partial_x,\partial_x)+g(\partial_x,\partial_x)(\mathcal{L}_Yg)(\partial_u,\partial_u),\\
2R_{vxvx}&=g(\partial_x,\partial_x)(\mathcal{L}_Yg)(\partial_v,\partial_y),\\
2R_{vxxy}&=-g(\partial_x,\partial_x)(\mathcal{L}_Yg)(\partial_v,\partial_y),\\
2R_{uxvx}&=2\mu g(\partial_u,\partial_v)g(\partial_x,\partial_x)-g(\partial_u,\partial_x)(\mathcal{L}_Yg)(\partial_v,\partial_x)\\
&\quad+g(\partial_u,\partial_v)(\mathcal{L}_Yg)(\partial_x,\partial_x)+g(\partial_x,\partial_x)(\mathcal{L}_Yg)(\partial_u,\partial_v),\\
2R_{vyxy}&=g(\partial_y,\partial_y)(\mathcal{L}_Yg)(\partial_v,\partial_x),\\
2R_{xyyx}&=-2\mu g(\partial_x,\partial_x)g(\partial_y,\partial_y)-g(\partial_x,\partial_x)(\mathcal{L}_Yg)(\partial_y,\partial_y)-g(\partial_y,\partial_y)(\mathcal{L}_Yg)(\partial_x,\partial_x),\\
2R_{uyuy}&=-2\mu g(\partial_u,\partial_y)^2+2\mu g(\partial_u,\partial_u)g(\partial_y,\partial_y)-2g(\partial_u,\partial_y)(\mathcal{L}_Yg)(\partial_u,\partial_y)\\
&\quad+g(\partial_u,\partial_u)(\mathcal{L}_Yg)(\partial_y,\partial_y)+g(\partial_y,\partial_y)(\mathcal{L}_Yg)(\partial_u,\partial_u),\\
2R_{uyvy}&=2\mu g(\partial_u,\partial_v)g(\partial_y,\partial_y)+g(\partial_u,\partial_v)(\mathcal{L}_Yg)(\partial_y,\partial_y)+g(\partial_y,\partial_y)(\mathcal{L}_Yg)(\partial_v,\partial_u),\\
2R_{uyxy}&=2\mu g(\partial_u,\partial_x)g(\partial_y,\partial_y)+g(\partial_u,\partial_x)(\mathcal{L}_Yg)(\partial_y,\partial_y)+g(\partial_y,\partial_y)(\mathcal{L}_Yg)(\partial_u,\partial_x).
\end{aligned}
$$

Applying (2.1) and (2.2) in the above equations, we have the following:

$$
\begin{aligned}
0 &= (\mathcal{L}_Y g)(\partial_x, \partial_y),\\
0 &= \frac{1}{p^2}(\mathcal{L}_Y g)(\partial_u, \partial_y) - \frac{2v}{x}(\mathcal{L}_Y g)(\partial_y, \partial_x),\\
-x\partial^2_{xy} f &= \frac{2v}{x}(\mathcal{L}_Y g)(\partial_u, \partial_y) + (xf - \frac{p^2v^2}{x^2})(\mathcal{L}_Y g)(\partial_y, \partial_x),\\
0 &= -2\mu - 2(\mathcal{L}_Y g)(\partial_u, \partial_v) + (xf - \frac{p^2v^2}{x^2})(\mathcal{L}_Y g)(\partial_v, \partial_v),\\
0 &= 4\mu\frac{v}{x} + \frac{2v}{x}(\mathcal{L}_Y g)(\partial_u, \partial_v) - (\mathcal{L}_Y g)(\partial_u, \partial_x) + (xf - \frac{p^2v^2}{x^2})(\mathcal{L}_Y g)(\partial_v, \partial_v),\\
0 &= -(\mathcal{L}_Y g)(\partial_u, \partial_y) + (xf - \frac{p^2v^2}{x^2})(\mathcal{L}_Y g)(\partial_v, \partial_y),\\
0 &= \frac{2v}{x}(\mathcal{L}_Y g)(\partial_v, \partial_v) + (\mathcal{L}_Y g)(\partial_v, \partial_x),\\
0 &= (\mathcal{L}_Y g)(\partial_v, \partial_y),\\
-x\partial^2_{xx} f &= -2\mu\frac{4v^2}{x^2} - \frac{2}{p^2}\mu(xf - \frac{p^2v^2}{x^2}) + 4\frac{v}{x}(\mathcal{L}_Y g)(\partial_u, \partial_x)\\
&\quad + (xf - \frac{p^2v^2}{x^2})(\mathcal{L}_Y g)(\partial_x, \partial_x) - \frac{1}{p^2}(\mathcal{L}_Y g)(\partial_u, \partial_u),\\
0 &= -\frac{2}{p^2}\mu + \frac{2v}{x}(\mathcal{L}_Y g)(\partial_v, \partial_x) + (\mathcal{L}_Y g)(\partial_x, \partial_x) - \frac{1}{p^2}(\mathcal{L}_Y g)(\partial_u, \partial_v),\\
0 &= (\mathcal{L}_Y g)(\partial_v, \partial_x),\\
0 &= -\frac{2}{p^4}\mu + \frac{1}{p^2}(\mathcal{L}_Y g)(\partial_y, \partial_y) + \frac{1}{p^2}(\mathcal{L}_Y g)(\partial_x, \partial_x),\\
-x\partial^2_{yy} f &= -\frac{2}{p^2}\mu(xf - \frac{p^2v^2}{x^2}) + (xf - \frac{p^2v^2}{x^2})(\mathcal{L}_Y g)(\partial_y, \partial_y) - \frac{1}{p^2}(\mathcal{L}_Y g)(\partial_u, \partial_u),\\
0 &= -2\mu\frac{1}{p^2} + (\mathcal{L}_Y g)(\partial_y, \partial_y) - \frac{1}{p^2}(\mathcal{L}_Y g)(\partial_u, \partial_v),\\
0 &= 4\mu\frac{v}{xp^2} - \frac{2v}{x}(\mathcal{L}_Y g)(\partial_y, \partial_y) - \frac{1}{p^2}(\mathcal{L}_Y g)(\partial_u, \partial_x).
\end{aligned}
$$

From the above equations, we conclude $\partial^2_{xx} f = \partial^2_{yy} f$, $\partial^2_{xy} f = 0$ and

$$
\begin{aligned}
&(\mathcal{L}_Y g)(\partial_v, \partial_v) = (\mathcal{L}_Y g)(\partial_x, \partial_v) = (\mathcal{L}_Y g)(\partial_y, \partial_v) = (\mathcal{L}_Y g)(\partial_y, \partial_x) = (\mathcal{L}_Y g)(\partial_u, \partial_y) = 0,\\
&(\mathcal{L}_Y g)(\partial_u, \partial_v) = -\mu,\\
&(\mathcal{L}_Y g)(\partial_x, \partial_x) = (\mathcal{L}_Y g)(\partial_y, \partial_y) = \frac{\mu}{p^2},\\
&(\mathcal{L}_Y g)(\partial_u, \partial_x) = \frac{2\mu v}{x},\\
&(\mathcal{L}_Y g)(\partial_u, \partial_u) = -\mu(xf - p^2\frac{v^2}{x^2}) + p^2 x\partial^2_{xx} f.
\end{aligned}
\tag{3.1}
$$

By utilizing Eq (2.4) in the Eq (3.1), we obtain the following:

$$x\left(\partial_u f Y_1 + \partial_x f Y_3 + \partial_y f Y_4\right) + 2\left(xf - \frac{p^2 v^2}{x^2}\right)\partial_u Y_1 + 2\partial_u Y_2 - \frac{4v}{x}\partial_u Y_3 \tag{3.2}$$

$$+\frac{1}{x^3}\left(-2p^2 v x Y_2 + (x^3 f + 2p^2 v^2)Y_3\right) = -\mu(xf - p^2\frac{v^2}{x^2}) + p^2 x \partial^2_{xx} f,$$

$$\partial_u Y_1 + \left(xf - \frac{p^2 v^2}{x^2}\right)\partial_v Y_1 + \partial_v Y_2 - \frac{2v}{x}\partial_v Y_3 = -\mu, \tag{3.3}$$

$$\frac{2}{x^2}(vY_3 - xY_2) - \frac{2v}{x}(\partial_u Y_1 + \partial_x Y_3) - \frac{1}{p^2}\partial_u Y_3 + \left(xf - \frac{p^2 v^2}{x^2}\right)\partial_x Y_1 + \partial_x Y_2 = \frac{2\mu v}{x}, \tag{3.4}$$

$$-\frac{2v}{x}\partial_y Y_3 - \frac{1}{p^2}\partial_u Y_4 + \left(xf - \frac{p^2 v^2}{x^2}\right)\partial_y Y_1 + \partial_y Y_2 = 0, \tag{3.5}$$

$$2\partial_v Y_1 = 0, \tag{3.6}$$

$$-\frac{2v}{x}\partial_v Y_1 - \frac{1}{p^2}\partial_v Y_3 + \partial_x Y_1 = 0, \tag{3.7}$$

$$-\frac{1}{p^2}\partial_v Y_4 + \partial_y Y_1 = 0 \tag{3.8}$$

$$-\frac{4v}{x}\partial_x Y_1 - \frac{2}{p^2}\partial_x Y_3 = \frac{\mu}{p^2}, \tag{3.9}$$

$$-\frac{2v}{x}\partial_y Y_1 - \frac{1}{p^2}(\partial_x Y_4 + \partial_y Y_3) = 0, \tag{3.10}$$

$$-\frac{2}{p^2}\partial_y Y_4 = \frac{\mu}{p^2}. \tag{3.11}$$

From Eq (3.6), we obtain the following:

$$Y_1 = F(u, x, y), \tag{3.12}$$

where F is a smooth function. Integrating (3.11) yields the following:

$$Y_4 = -\frac{1}{2}\mu y + G(u, v, x), \tag{3.13}$$

where G is a smooth function. Inserting (3.12) and (3.13) in (3.8), we deduce $-\frac{1}{p^2}\partial_v G - \partial_y F = 0$, and its integration yields the following:

$$F = \frac{1}{p^2}(\partial_v G)y + F_1(u, x), \tag{3.14}$$

for some smooth function F_1. Equation (3.14) leads to $\partial^2_{vv} G = 0$. Hence, $G(u, v, x) = G_1(u, x)v + G_2(u, x)$,

$$Y_1 = \frac{1}{p^2}G_1 y + F_1$$

and

$$Y_4 = -\frac{1}{2}\mu y + G_1 v + G_2,$$

for some smooth functions G_1 and G_2. By putting Y_1 into Eq (3.7) and the integration Y_3 with respect to v, we conclude the following:

$$Y_3 = (\partial_x G_1)yv + p^2(\partial_x F_1)v + H(u, x, y),$$

for some smooth function H. By replacing Y_1, Y_3, and Y_4 in Eq (3.9), we obtain the following:

$$-\frac{4v}{x}\left(\frac{1}{p^2}(\partial_x G_1)y + (\partial_x F_1)\right) - \frac{2}{p^2}\left((\partial^2_{xx}G_1)yv + p^2(\partial^2_{xx}F_1)v + \partial_x H\right) = \frac{\mu}{p^2}. \tag{3.15}$$

Equation (3.15) is a polynomial with respect to v. Then,

$$\partial_x H = -\frac{\mu}{2}, \ \ \frac{2}{x}(\partial_x G_1) + (\partial^2_{xx}G_1) = 0, \ \ \frac{2}{x}(\partial_x F_1) + (\partial^2_{xx}F_1) = 0,$$

and consequently, $H = -\frac{\mu}{2}x + H_1(u, y)$ and $F_1 = -\frac{F_2(u)}{x} + F_3(u)$ for some smooth functions H_1, F_2, and F_3. Substituting Y_1, Y_3, and Y_4 in Eq (3.10), we acquire the following:

$$\frac{2v}{x}G_1 + 2(\partial_x G_1)v + \partial_x G_2 + \partial_y H_1 = 0,$$

which is a polynomial with respect to v. Then, $\frac{1}{x}G_1 + (\partial_x G_1) = 0$ and $\partial_x G_2 + \partial_y H_1 = 0$. Consequently, by integration, we find the following:

$$G_1 = \frac{G_3(u)}{x}, \ \ G_2 = -G_4(u)x + G_5(u), \ \ H_1(u, y) = G_4(u)y + H_2(u),$$

for some smooth functions G_3, G_4, G_5, and H_2. Therefore, Y_1, Y_3, and Y_4 become the following:

$$\begin{aligned} Y_1 &= \frac{y}{xp^2}G_3(u) - \frac{F_2(u)}{x} + F_3(u), \\ Y_3 &= -\frac{yv}{x^2}G_3(u) + \frac{p^2v}{x^2}F_2(u) - \frac{1}{2}\mu x + G_4(u)y + H_2(u), \\ Y_4 &= -\frac{1}{2}\mu y + \frac{v}{x}G_3(u) - G_4(u)x + G_5(u). \end{aligned}$$

Equation (3.3) yields

$$\frac{y}{xp^2}G_3'(u) - \frac{F_2'(u)}{x} + F_3'(u) + \partial_v Y_2 + 2\frac{yv}{x^3}G_3(u) - \frac{2p^2v}{x^3}F_2(u) = -\mu,$$

which, by integration, gives

$$Y_2 = -\mu v - \frac{yv}{xp^2}G_3'(u) + \frac{v}{x}F_2'(u) - F_3'(u) - \frac{yv^2}{x^3}G_3(u) + \frac{p^2v^2}{x^3}F_2(u) + K(u, x, y),$$

where K is a smooth function. By applying Y_1, Y_2, Y_3, and Y_4 in (3.5), it follows that

$$-\frac{2v}{x}G_4(u) - \frac{1}{p^2}\left(2\frac{v}{x}G_3'(u) - G_4'(u)x + G_5'(u)\right) + \frac{f}{p^2}G_3(u) + \partial_y K = 0.$$

Since v is arbitrary, we find $G_4(u) = -\frac{1}{p^2}G_3'(u)$ and

$$K(u,x,y) = \frac{1}{p^2}\left(-G_4'(u)xy + G_5'(u)y - G_3(u)\int f dy\right) + K_1(u,x),$$

for some smooth function K_1. Equation (3.4) implies that

$$\frac{2v}{x^2}(H_2(u) - F_2'(u)) - \frac{2}{xp^2}\left(G_5'(u)y - G_3\int f dy\right) - \frac{2}{x}K_1 - \frac{1}{p^2}H_2'(u)$$
$$+\frac{f}{x}(-\frac{y}{p^2}G_3 + F_2) - \frac{1}{p^2}G_3(u)\int \partial_x f dy + \partial_x K_1 = 0.$$

Since v is arbitrary, we obtain $H_2(u) = F_2'(u)$ and

$$-\frac{2}{xp^2}\left(G_5'(u)y - G_3\int f dy\right) - \frac{2}{x}K_1 - \frac{1}{p^2}H_2'(u) + \frac{f}{x}(-\frac{y}{p^2}G_3 + F_2) - \frac{1}{p^2}G_3(u)\int \partial_x f dy + \partial_x K_1 = 0.$$

Since $\partial_v Y_1 = 0$, $\partial^3_{vvv}Y_2 = 0$, $\partial^2_{vv}Y_3 = 0$, and $\partial_{vv}Y_4 = 0$, by taking the third derivative of (3.2) with respect to v, we conclude

$$-G_3(u)y + p^2F_2(u) = 0,$$

which is a polynomial with respect to y. Then, $G_3(u) = F_2(u) = 0$. Additionally, we deduce $H_2(u) = G_4(u) = 0$. Again, by the derivative of (3.2), it follows that

$$F_3''(u) + \frac{1}{x^2}G_5'(u)y + \frac{p^2}{x^2}K_1 = 0,$$

which is a polynomial with respect to y. Then, $G_5(u) = a$ and $K_1(u) = -\frac{1}{p^2}F_3''(u)x^2$ for some constant a. Therefore, Y_1, Y_2, Y_3, and Y_4 become

$$Y_1 = F_3(u),\quad Y_2 = -(\mu + F_3'(u))v - \frac{1}{p^2}F_3''(u)x^2,\quad Y_3 = -\frac{1}{2}\mu x,\quad Y_4 = -\frac{1}{2}\mu y + a_1, \tag{3.16}$$

and Eq (3.2) reduces to

$$x\left((\partial_u f)F_3(u) - \frac{1}{2}\mu x(\partial_x f) + (\partial_y f)(-\frac{1}{2}\mu y + a)\right) + 2xfF_3'(u) - \frac{2}{p^2}F_3'''(u)x^2 + \frac{1}{2}\mu xf - p^2x\partial^2_{xx}f = 0. \tag{3.17}$$

Theorem 3.1. *A non-flat PRM g given by (2.1) is a RIS (M^n, g, μ, Y) if and only if Y admits (3.16) such that $\partial^2_{xx}f = \partial^2_{yy}f$, $\partial^2_{xy}f = 0$, and (3.17) is true.*

We have

$$g^{-1} = 2dudv - (xf + 3\frac{v^2p^2}{x^2})dv^2 - 4\frac{vp^2}{x}dvdx - p^2(dx^2 + dy^2).$$

From (2.1) and (3.16), the RIS is the gradient RIS with

$$Y = \nabla h = \partial_v h\partial_u + \left(\partial_u h - (xf + 3\frac{v^2p^2}{x^2})\partial_v h - 2\frac{vp^2}{x}\partial_x h\right)\partial_v + (-2\frac{vp^2}{x}\partial_v h - p^2\partial_x h)\partial_x - p^2\partial_y h\partial_y,$$

for some smooth function h if

$$\begin{cases}\partial_v h = Y_1 = F_3(u),\\ \partial_u h - (xf + 3\frac{v^2p^2}{x^2})\partial_v h - 2\frac{vp^2}{x}\partial_x h = Y_2 = -(\mu + F_3'(u))v - \frac{1}{p^2}F_3''(u)x^2,\\ -2\frac{vp^2}{x}\partial_v h - p^2\partial_x h = Y_3 = -\frac{1}{2}\mu x,\\ -p^2\partial_y h = Y_4 = -\frac{1}{2}\mu y + a_1.\end{cases}$$

Thus,

$$\begin{cases}\partial_u h = (xf - \frac{v^2p^2}{x^2})F_3(u) - F_3'(u)v - \frac{1}{p^2}F_3''(u)x^2,\\ \partial_v h = F_3(u),\\ \partial_x h = \frac{\mu x}{2p^2} - \frac{2v}{x}F_3(u),\\ \partial_y h = \frac{1}{2p^2}\mu y - \frac{a_1}{p^2}.\end{cases} \tag{3.18}$$

By taking the derivative of first and second equations of (3.18), we conclude that

$$\partial^2_{uv}h = -2\frac{vp^2}{x^2}F_3(u) - F_3'(u), \quad \partial^2_{vu}h = F_3'(u).$$

Then, $F_3(u) = 0$ and

$$\partial_u h = 0, \quad \partial_v h = -0, \quad \partial_x h = \frac{\mu x}{2p^2}, \quad \partial_y h = \frac{1}{2p^2}\mu y - \frac{a_1}{p^2}.$$

Therefore, we conclude the following corollary and remark.

Corollary 3.1. *A non-flat PRM g given by (2.1) is a gradient RIS with the potential function* $h = \frac{\mu x^2}{4p^2} + \frac{1}{4p^2}\mu y^2 - \frac{a_1}{p^2}y + b$ *for some constant b if* $\partial^2_{xx}f = \partial^2_{yy}f$, $\partial^2_{xy}f = 0$, *and (3.17) is true.*

Remark 3.1. *If a non-flat PRM g given by (2.1) is a gradient RIS with the potential function h, then* $S = -(n-1)\mu g - (\Delta h)g - 2(n-1)\mathrm{Hess}h$. *By taking the trace, we obtain the scalar curvature with* $r = -n(n-1)\mu - n^2\Delta h$. *Furthermore, if the spacetime is closed, then* $\int_M r = -n(n-1)\mu Vol(M)$.

In the following remarks, we consider (M, g) to be a pseudo-Riemannian manifold.

Remark 3.2. *A vector field Y defined on* (M, g) *is classified as a conformal vector field when it satisfies the condition* $\mathcal{L}_Y g = 2\psi g$ *for some smooth function* ψ. *In the case where* $\psi = 0$, *Y is referred as a Killing vector field. The study of conformal vector fields is extensively covered in the literature, including works such as [27]. Furthermore, if the second partial derivative* $\partial^2_{xx}f = 0$ *holds and any RIS associated with a non-flat PRM g described in (2.1) is steady, then the potential vector field corresponding to this soliton becomes Killing.*

Remark 3.3. *A vector field Y defined on* (M, g) *is referred as a Ricci collineation if it satisfies the condition* $\mathcal{L}_Y S = 0$. *According to Theorem 3.1 and Eq (2.5), any potential vector field associated with a RIS of a non-flat PRM g described in Eq (2.1) admits a Ricci collineation vector field, provided that the RIS is steady and either* $F_3(u) = 0$ *or* $f = f_1(x, y)e^{-4\int \frac{F_3'(u) - \frac{1}{4}\mu}{F_3(u)}du}$ *for some smooth function* f_1.

Remark 3.4. *On an* (M, g), *a vector field Y is said to be a Ricci bi-conformal [28] when*

$$\mathcal{L}_Y g = \alpha g + \beta S, \quad \mathcal{L}_Y S = \alpha S + \beta g, \tag{3.19}$$

where α *and* β *are smooth functions. For additional examples, see [29–32]. From Theorem 3.1 and Eq (2.5), any potential vector field of RIS of a non-flat PRM g given by (2.1) is Ricci bi-conformal vector field for* $\alpha = \beta = 0$ *and* $\partial^2_{xx}f = \partial^2_{yy}f = 0$.

4. Conclusions

In this study, we investigated RISs within spacetimes defined by pure radiation metrics that are conformally related to a vacuum spacetime. We demonstrated that non-flat spacetimes with pure radiation metrics can support steady, shrinking, and expanding RISs. Furthermore, we provided a classification of all RISs present in these spaces and examined the gradient RISs. Our findings indicate that any potential vector field associated with a RIS in spacetimes with pure radiation metrics admits a Killing vector field when the RIS is steady. Additionally, we conclude that any potential vector field of a RIS in these spacetimes acts as a Ricci collineation vector field and a Ricci bi-conformal vector field under some specific conditions.

Author contributions

Rawan Bossly: Conceptualization, investigation, methodology, writing-review & editing; Shahroud Azami: Conceptualization, investigation, methodology, writing-original draft; Dhriti Sundar Patra: Conceptualization, methodology, writing-review & editing; Abdul Haseeb: Conceptualization, investigation, methodology, writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest in this paper.

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