AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 10(8): 18070–18093.
DOI: 10.3934/math.2025805
Received: 01 July 2025
Revised: 31 July 2025
Accepted: 04 August 2025
Published: 08 August 2025

Research article

Spectral collocation approach for solving the time-fractional Kuramoto-Sivashinsky equation using the Fibonacci coefficient polynomials

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Abstract: This work presents an algorithm that applies the typical collocation method to solve the time-fractional Kuramoto–Sivashinsky equation (TFKSE). Our suggested new basis functions are the Fibonacci coefficient polynomials. We develop new theoretical results of these polynomials, such as their integer and fractional derivatives. The proposed approach efficiently estimates spatial and temporal derivatives using the operational matrices (OMs) of derivatives of the introduced polynomials. The approach converts the TFKSE controlled by their conditions into a non-linear system of equations that may be handled numerically. A thorough error and convergence analysis study for the proposed Fibonacci coefficient expansion is presented. Several illustrative numerical examples, including those with known exact solutions, are presented to confirm the efficiency and applicability of the suggested approach, even with fewer terms of basis functions. In addition, comparisons with some numerical methods are presented to verify our numerical approach's high accuracy.

Keywords: time-fractional differential equations; Fibonacci numbers; collocation method; operational matrices; convergence analysis
Mathematics Subject Classification: 65M60, 11B39, 40A05, 34A08

1. Introduction

In several branches of the practical sciences, fractional differential equations (FDEs) play an essential role. Their reasoning for why standard DEs miss certain events is sound. The reason behind this is their remarkable capacity to imitate the actions of genes and memory. Examples of physiological and biological processes they imitate, including neuronal activity and tumor growth, were presented in [1]. Because of the analytical impossibility of solving such equations, numerical analysis was usually employed in their solution. Several numerical approaches were utilized to solve the different FDEs. Among these methods are the Chebyshev neural network-based numerical scheme [2], homotopy analysis transform method [3], modified Adomian decomposition method [4], Adomian decomposition method [5], Pell polynomial-based numerical approach [6], operational matrix methods [7, 8], collocation methods [9, 10], higher-order predictor-corrector method [11], Laplace transform with residual power series method [12], spatial sixth-order numerical scheme [13], neural networks-based numerical method [14], and Galerkin method [15, 16].

Fibonacci numbers and their corresponding polynomials are significant in theoretical and applied mathematics. Fibonacci numbers are famous for their natural appearance, such as in the arrangement of leaves, flowers, and shells, and their deep connection to the golden ratio. They are widely used in computer science (e.g., algorithms and data structures), combinatorics, and even in financial market analysis through Fibonacci retracement levels. Fibonacci polynomials are generalizations of Fibonacci numbers. They have several applications in areas like algebra, approximation theory, digital signal processing, and numerical analysis. In particular, the Fibonacci polynomials and their generalized ones are utilized to solve DEs of different types. For example, the authors in [17] used a convolved Fibonacci collocation approach to treat the FitzHugh-Nagumo equation. The fractional Burgers equation was addressed using a shifted Fibonacci polynomial method presented in [18]. Fibonacci polynomials were applied in [19] to address FDEs. For the 2D Sobolev equation, a combined Lucas-Fibonacci polynomial technique was proposed in [20]. For solving the variable-order fractional Burgers-Huxley equations, the authors of [21] used the generalized shifted Vieta-Fibonacci polynomials.

Spectral methods are numerical techniques that have gained prominence in various domains. These methods produce precise approximations that converge at exponential rates [22]. Furthermore, unlike the finite element or finite difference methods, they offer global solutions. These techniques may be employed to handle many kinds of DEs. There are primarily three approaches to spectral methods. We can find applications and benefits for any approach. For examples of linear and certain nonlinear problems that may be effectively solved using the Galerkin and Petrov-Galerkin techniques, see, for instance, [23–26]. The Tau method can handle more complicated boundary conditions, which gives it a broader variety of applications compared to the Galerkin method; see, for example, [27, 28]. One benefit of the collocation approach is that it can deal with DEs governed by any underlying conditions; see [29, 30].

Using operational matrices of derivatives (OMDs) significantly influences the numerical treatment of many DEs with spectral techniques. It allows one to convert DEs into systems of algebraic equations, which are more manageable computationally. OMDs have been utilized in many publications to solve different types of DEs. In [31], the Jacobi Galerkin OMDs of derivatives were established and used for the numerical treatment of the multi-term variable-order time-fractional diffusion-wave equations. In [32], a collocation technique using an operational matrix of fractional-order Lagrange polynomials was utilized for solving the space-time fractional-order equation. The authors in [33] used a Haar wavelet operational matrix method to treat pantograph FDEs. Another matrix method was used in [34] to treat the time fractional diffusion equations. Other OMDs of the eighth kind, Chebyshev polynomials, were employed to solve the Kawahara equation. In [35], an operational matrix approach was followed to treat the generalized Caputo FDEs.

The Kuramoto-Sivashinsky equation (KSE) is a fourth-order nonlinear partial differential equation recognized for its application in simulating intricate spatiotemporal dynamics, such as turbulence and chaos. Initially developed in the late 1970s by Yoshiki Kuramoto and Gregory Sivashinsky, it has evolved into a crucial instrument across several scientific and technical fields. Several physical and chemical phenomena are described by the Kuramoto-Sivashinsky equation, such as chemical reaction-diffusion, plasma instability, issues with viscous flow, propagation of flame fronts, and magnetized plasmas [36, 37]. Due to the importance of the KSE and its modified versions, many authors were interested in using different algorithms to treat them. In [38], the authors treated the KSE via the Galerkin method. The authors in [39] solved the KSE equation using a combination of two quintic B-splines and the differential quadrature method. A fractional power series approach was proposed in [40] to handle the nonlinear KSE. In [41], the soliton behavior of generalized KS-type equations was studied using Hermite splines. The septic Hermite collocation method was applied in [42] to analyze the KSE. A compact two-level implicit exponential scheme for the KSE and Fisher-Kolmogorov equations was introduced in [43]. Furthermore, many studies were devoted to treating the TFKSE. In [44], the authors developed an enhanced numerical technique using Morgan-Voyce polynomials to address the TFKSE. A computational algorithm was presented in [45] to treat the TFKSE, while [46] proposed a robust compact difference scheme on graded meshes for solving the TFKSE. In [47], another study for the TFKSE was presented. In [48], the authors investigated dynamic behavior and presented a semi-analytical solution of the nonlinear fractional-order KSE. Other notable contributions include a Galerkin mixed finite element approach for the two-dimensional KSE using tensor cubic B-splines [49], and an analytical solution of the stochastic fractional KS equation via the Riccati method [50].

This article's main aim is to propose a numerical algorithm using the collocation method to numerically treat the TFKSE.

Our main contributions, including the novelty of our work, can be listed in the following items:

- As far as we know, this is the first time that the Fibonacci coefficient polynomials have been used in numerical analysis.
- Many new theoretical results are established in this paper.
- The new theoretical results regarding the introduced polynomials are used to design a new collocation procedure for the numerical treatment of the TFKSE.
- We think that these polynomials may be utilized in other contributions in different applied sciences.

We also comment here that the theoretical background of the Fibonacci coefficient polynomials was the basis for deriving our proposed numerical methods and applying the collocation method, as well as studying the convergence analysis of the introduced expansion.

The paper is structured as follows: The next section overviews some elementary properties of the fractional calculus. Moreover, some properties of Fibonacci numbers are given in this section.

Section 3 introduces polynomials whose power form coefficients are Fibonacci numbers and develops some new formulas for them. Section 4 proposes a numerical approach for solving the TFKSE using the collocation method. Section 5 investigates the convergence and error analysis of the proposed double expansion. Some numerical experiments are displayed in Section 6 to demonstrate the efficiency and accuracy of our proposed numerical scheme. Finally, we end the paper with concluding remarks in Section 7.

2. Some essential definitions and formulas

This section gives an account of the fractional calculus theory. In addition, we provide an account of Fibonacci polynomials and some of their properties.

2.1. Caputo's fractional derivative

The fractional-order derivative in the sense of the Caputo is defined as:

$$D_t^{\nu}\theta(s) = \frac{1}{\Gamma(k-\nu)}\int_0^s (s-t)^{k-\nu-1}\theta^{(k)}(t)dt, \quad \nu > 0, \quad s > 0,\ k-1 < \nu \leq k, \quad k \in \mathbb{N}_0. \tag{2.1}$$

For D_t^{ν} with $k-1 < \nu \leq k, \quad k \in \mathbb{N}_0$, the following identities are valid:

$$D_t^{\nu}C = 0, \quad C \text{ is a constant}, \tag{2.2}$$

$$D_t^{\nu}\, s^k = \begin{cases} 0, & \text{if } k \in \mathbb{N} \quad \textit{and} \quad k < \lceil \nu \rceil, \\ \frac{k!}{\Gamma(k-\nu+1)}\, s^{k-\nu}, & \text{if } k \in \mathbb{N} \quad \textit{and} \quad k \geq \lceil \nu \rceil, \end{cases} \tag{2.3}$$

where $\mathbb{N}_0 = \{1, 2, ...\}$ and $\mathbb{N} = \{0, 1, 2, \ldots\}$, and $\lceil \nu \rceil$ is the ceiling function.

2.2. An overview of Fibonacci numbers

Fibonacci numbers $\{F_n\}_{n\geq 0}$ satisfy the following recurrence relation:

$$F_{n+1} = F_n + F_{n-1}, \quad F_0 = 0,\ F_1 = 1. \tag{2.4}$$

The Binet form of Fibonacci numbers is

$$F_n = \frac{\left(\frac{1}{2}\left(1+\sqrt{5}\right)\right)^n - \left(\frac{1}{2}\left(1-\sqrt{5}\right)\right)^n}{\sqrt{5}}.$$

In addition, these numbers can be written explicitly in the following combinatorial form:

$$F_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k}.$$

3. Introducing polynomials with Fibonacci coefficients

In this section, we introduce polynomials whose coefficients are Fibonacci numbers. They are defined as [51]

$$\psi_i(y) = \sum_{r=0}^{i} F_{r+1}\, y^{i-r}, \tag{3.1}$$

or alternatively as

$$\psi_i(y) = \sum_{r=0}^{i} F_{i-r+1}\, y^{r}, \tag{3.2}$$

where F_i are the standard Fibonacci numbers.

Now, it is easy to derive the recurrence relation that is satisfied by $\psi_i(y)$. The following lemma displays this relation.

Lemma 3.1. *The polynomials $\psi_i(y)$ satisfy the following first-order non-homogeneous recurrence relation:*

$$\psi_i(y) = y\,\psi_{i-1}(y) + F_{i+1}. \tag{3.3}$$

Proof. Making use of the expression in (3.1), we get

$$\begin{aligned} \psi_i(y) - y\,\psi_{i-1}(y) &= \sum_{r=0}^{i} F_{r+1}\, y^{i-r} - y \sum_{r=0}^{i-1} F_{r+1}\, y^{i-r-1} \\ &= \sum_{r=0}^{i} F_{r+1}\, y^{i-r} - \sum_{r=0}^{i-1} F_{r+1}\, y^{i-r}, \end{aligned}$$

which clearly gives

$$\psi_i(y) - y\,\psi_{i-1}(y) = F_{i+1}.$$

This proves Lemma 3.1. □

Now, we will prove a useful inequality regarding the Fibonacci numbers in the following lemma.

Lemma 3.2. *The following inequality holds for every non-negative integer i:*

$$F_{i+2} \le \left(\sqrt{5}\right)^{i}. \tag{3.4}$$

Proof. We prove by induction. For $i = 0$, each side is equal to 1. Now, assume that (3.4) holds for F_{k+2} for all $k < i$. Then to complete the proof, we show the inequality in (3.4) itself. If we start from the following recurrence relation:

$$F_{i+2} = F_{i+1} + F_i,$$

then the induction hypothesis leads to

$$F_{i+2} \le \left(\sqrt{5}\right)^{i-1} + \left(\sqrt{5}\right)^{i-2},$$

which, accordingly, implies that

$$F_{i+2} \le \left(\sqrt{5}\right)^{i-2}\left(\sqrt{5}+1\right) < \left(\sqrt{5}\right)^{i}.$$

The proof is now complete. □

We will use these polynomials to solve the TFKSE. To this end, we will establish some theoretical results of these polynomials and develop the following formulas:

- The inversion formula of these polynomials.
- The higher-order derivative formula of these polynomials as a combination of their original polynomials.
- The OMDs of the integer derivatives of these polynomials.
- The operational matrix of fractional derivatives of these polynomials.

Now, we will state and prove the inversion formula of $\psi_i(y)$. This formula will be useful to derive further formulas for these polynomials.

Theorem 3.1. *The inversion formula of $\psi_i(y)$ is*

$$y^i = \psi_i(y) - \psi_{i-1}(y) - \psi_{i-2}(y), \; i \geq 0. \tag{3.5}$$

Proof. Assume the following notation:

$$\theta_i(y) = \psi_i(y) - \psi_{i-1}(y) - \psi_{i-2}(y), \tag{3.6}$$

and we have to show that

$$\theta_i(y) = y^i.$$

Based on (3.1), and (3.6), we can write

$$\theta_i(y) = \sum_{r=0}^{i} F_{r+1} y^{i-r} - \sum_{r=0}^{i-1} F_{r+1} y^{i-r-1} - \sum_{r=0}^{i-2} F_{r+1} y^{i-r-2}. \tag{3.7}$$

Noting the special values $F_1 = F_2 = 1, \theta_i(y)$ can be written as

$$\theta_i(y) = y^i + \sum_{r=2}^{i} \left(F_{r+1} - F_r - F_{r-1}\right) y^{i-r}. \tag{3.8}$$

Based on the recurrence relation (2.4), it is easy to see that

$$\theta_i(y) = y^i.$$

This ends the proof. □

Remark 3.1. *It is useful to write the inversion formula* (3.5) *in the following form:*

$$y^i = \sum_{r=0}^{i} a_{r,i} \, \psi_r(y), \tag{3.9}$$

where

$$a_{r,i} = \begin{cases} 1, & \text{if } r = i, \\ -1, & \text{if } r = i-1, \\ -1, & \text{if } r = i-2, \\ 0, & \text{otherwise}. \end{cases} \tag{3.10}$$

In the following theorem, we give an explicit expression for the high-order derivatives of $\psi_i(y)$.

Theorem 3.2. *For any two positive integers n, q with $n \geq q$, one has*

$$\begin{aligned} D^q\psi_n(y) &= (1+n-q)_q\,\psi_{n-q}(y) \\ &+ \sum_{L=1}^{n-q}\Big(F_{L+1}\,(1-L+n-q)_q - F_L\,(2-L+n-q)_q - F_{L-1}\,(3-L+n-q)_q\Big)\,\psi_{n-q-L}(y), \end{aligned} \tag{3.11}$$

where the notation $(z)_q$ represents the Pochhammer function defined as

$$(z)_q = \frac{\Gamma(z+q)}{\Gamma(z)}. \tag{3.12}$$

Proof. Based on Formula (3.1), we can express $D^q\psi_n(y)$ in the form

$$D^q\psi_n(y) = \sum_{k=0}^{n-q} F_{k+1}\,(n-k-q+1)_q\,y^{n-k-q}. \tag{3.13}$$

Inserting the inversion formula (3.5), the last formula turns into the following formula:

$$D^q\psi_n(y) = \sum_{k=0}^{n-q} F_{k+1}\,(n-k-q+1)_q\,\Big(\psi_{n-k-q}(y) - \psi_{n-k-q-1}(y) - \psi_{n-k-q-2}(y)\Big), \tag{3.14}$$

which can be expressed as

$$\begin{aligned} D^q\psi_n(y) &= (1+n-q)_q\,\psi_{n-q}(y) - \frac{q(n-1)!}{(n-q)!}\,\psi_{n-q-1}(y) \\ &+ \sum_{L=2}^{n-q}\Big(F_{L+1}(1-L+n-q)_q - F_L\,(2-L+n-q)_q - F_{L-1}\,(3-L+n-q)_q\Big)\,\psi_{n-q-L}(y), \end{aligned} \tag{3.15}$$

which is equal to

$$\begin{aligned} D^q\psi_n(y) &= (1+n-q)_q\,\psi_{n-q}(y) \\ &+ \sum_{L=1}^{n-q}\Big(F_{L+1}\,(1-L+n-q)_q - F_L\,(2-L+n-q)_q - F_{L-1}\,(3-L+n-q)_q\Big)\,\psi_{n-q-L}(y). \end{aligned} \tag{3.16}$$

This completes the proof. □

Remark 3.2. *It is helpful to write an alternative formula for* (3.11) *in the following form:*

$$D^q\psi_n(y) = \sum_{p=0}^{n-q} U_{p,n,q}\,\psi_p(y), \tag{3.17}$$

where

$$U_{p,n,q} = \begin{cases} (p+1)_q\,F_{n-p-q+1} - (p+2)_q\,F_{n-p-q} - (p+3)_q\,F_{n-p-q-1}, & 0 \leq p \leq n-q-1, \\ (n-q+1)_q, & p = n-q. \end{cases}$$

The following corollary exhibits some specific derivatives of the polynomials $\psi_m(y)$.

Corollary 3.1. *The first-, second-, and fourth-order derivatives of* $\psi_m(y)$ *can be represented as*

$$\begin{aligned}\frac{d\,\psi_n(y)}{d\,y} &= \sum_{p=0}^{n-1} \mathcal{Z}^1_{p,n}\,\psi_p(y), \quad n \ge 1,\\ \frac{d^2\,\psi_n(y)}{d\,y^2} &= \sum_{p=0}^{n-2} \mathcal{Z}^2_{p,n}\,\psi_p(y), \quad n \ge 2,\\ \frac{d^4\,\psi_n(y)}{d\,y^4} &= \sum_{p=0}^{n-4} \mathcal{Z}^4_{p,n}\,\psi_p(y), \quad n \ge 4,\end{aligned} \tag{3.18}$$

where

$$\mathcal{Z}^1_{p,n} = \begin{cases} -(p+3)F_{n-p-2} - (p+2)F_{n-p-1} + (p+1)F_{n-p}, & 0 \le p \le n-2,\\ n, & p = n-1,\end{cases}$$

$$\mathcal{Z}^2_{p,n} = \begin{cases} (p+2)\Big((p+1)F_{n-p-1} - (p+3)F_{n-p-2}\Big) - (p+3)(p+4)F_{n-p-3}, & 0 \le p \le n-3,\\ (n-1)\,n, & p = n-2,\end{cases}$$

$$\mathcal{Z}^4_{p,n} = \begin{cases} (p+3)(p+4)\Big((p+2)\Big((p+1)F_{n-p-3} - (p+5)F_{n-p-4}\Big) - (p+5)(p+6)F_{n-p-5}\Big),\\ 0 \le p \le n-5,\\ (n-3)(n-2)(n-1)n, \ p = n-4.\end{cases}$$

Proof. The results of Corollary 3.1 are special ones of Theorem 3.2, setting, $q = 1, 2$, and 4. □

Corollary 3.2. *If we define the following vector:*

$$\boldsymbol{\psi}(y) = [\psi_0(y), \psi_1(y), \ldots, \psi_{\mathcal{M}}(y)]^T, \tag{3.19}$$

then, we have the following matrix form expressions:

$$\begin{aligned}\frac{d\,\boldsymbol{\psi}(y)}{d\,y} &= \boldsymbol{U}\,\boldsymbol{\psi}(y),\\ \frac{d^2\,\boldsymbol{\psi}(y)}{d\,yx^2} &= \boldsymbol{V}\,\boldsymbol{\psi}(y),\\ \frac{d^4\,\boldsymbol{\psi}(y)}{d\,y^5} &= \boldsymbol{K}\,\boldsymbol{\psi}(y),\end{aligned} \tag{3.20}$$

where $\boldsymbol{U} = (\mathcal{Z}^1_{p,n})$, $\boldsymbol{V} = (\mathcal{Z}^2_{p,n})$, *and* $\boldsymbol{K} = (\mathcal{Z}^4_{p,n})$ *are the OMDs of order* $(\mathcal{M}+1)^2$.

Proof. This is an immediate consequence of Corollary 3.1. □

The following theorem gives the formula for the fractional derivatives of $\psi_i(y)$.

Theorem 3.3. *$D_t^{\gamma}\,\psi_j(t)$ has the following representation for $\gamma \in (0,1)$:*

$$D_t^{\gamma}\,\psi_j(t) = t^{-\gamma}\left(\sum_{p=0}^{j} Q_{p,j}\,\psi_p(t) + \lambda\right), \tag{3.21}$$

where

$$Q_{p,j} = \sum_{L=p}^{j} \frac{L!\,F_{j-L+1}\,a_{p,L}}{\Gamma(L-\gamma+1)}, \tag{3.22}$$

and $\lambda = -\frac{F_{j+1}}{\Gamma(1-\gamma)}$.

Proof. If we apply the Caputo fractional derivative to (3.2), then after making use of formula (2.3), we get

$$D_t^{\gamma}\,\psi_j(t) = t^{-\gamma}\sum_{p=1}^{j} \frac{p!\,F_{j-p+1}}{\Gamma(p-\gamma+1)}\,t^{p}, \tag{3.23}$$

which can be rewritten after using the inversion formula (3.9) as

$$D_t^{\gamma}\,\psi_j(t) = t^{-\gamma}\sum_{p=1}^{j}\sum_{L=0}^{p} \frac{p!\,F_{j-p+1}\,a_{L,p}}{\Gamma(p-\gamma+1)}\,\psi_L(t). \tag{3.24}$$

Therefore, the previous equation can be rewritten after rearranging the terms as

$$D_t^{\gamma}\,\psi_j(t) = t^{-\gamma}\left(\sum_{p=0}^{j} Q_{p,j}\,\psi_p(t) + \lambda\right), \tag{3.25}$$

where

$$Q_{p,j} = \sum_{L=p}^{j} \frac{L!\,F_{j-L+1}\,a_{p,L}}{\Gamma(L-\gamma+1)}, \tag{3.26}$$

and $\lambda = -\frac{F_{j+1}}{\Gamma(1-\gamma)}$. Thus, the proof is now complete. □

Corollary 3.3. *The fractional derivative of the vector form $\boldsymbol{\psi}(t)$ can be expressed as*

$$\frac{d^{\gamma}\,\boldsymbol{\psi}(t)}{d\,t^{\gamma}} = t^{-\gamma}\,\left(\boldsymbol{D}^{\gamma}\,\boldsymbol{\psi}(t) + \lambda\right), \tag{3.27}$$

where $\boldsymbol{D}^{\gamma} = (Q_{p,j})$.

Proof. An immediate result of Theorem 3.3. □

4. A collocation approach for the TFKSE

This section is interested in proposing a numerical algorithm to treat the TFKSE. We will design a numerical algorithm to solve this equation by applying the collocation method. We will use the Fibonacci coefficient polynomials as basis functions.
Consider the following TFKSE [52]:

$$\frac{\partial^{\gamma}\chi(y,t)}{\partial t^{\gamma}}+a_1(y,t)\chi(y,t)\frac{\partial\chi(y,t)}{\partial y}+a_2(y,t)\frac{\partial^2\chi(y,t)}{\partial y^2}$$
$$+a_3(y,t)\frac{\partial^4\chi(y,t)}{\partial y^4}=f(y,t),\quad (y,t)\in(0,1)\times(0,1], \tag{4.1}$$

constrained by the following conditions:

$$\chi(y,0)=g(y), \tag{4.2}$$
$$\chi(0,t)=,g_0(t),\quad \chi(1,t)=g_1(t), \tag{4.3}$$
$$\left.\frac{\partial\chi(y,t)}{\partial y}\right|_{y=0}=0,\quad \left.\frac{\partial\chi(y,t)}{\partial y}\right|_{y=1}=0, \tag{4.4}$$

or

$$\chi(y,0)=g(y), \tag{4.5}$$
$$\chi(0,t)=g_0(t),\quad \chi(1,t)=g_1(t), \tag{4.6}$$
$$\left.\frac{\partial^2\chi(y,t)}{\partial y^2}\right|_{y=0}=0,\quad \left.\frac{\partial^2\chi(y,t)}{\partial y^2}\right|_{y=1}=0, \tag{4.7}$$

where $a_1(y,t)$, $a_2(y,t)$, and $a_3(y,t)$ are real-valued functions of y and t, $a_2(y,t)$, and $a_3(y,t)$ are connected to the growth of linear stability and surface tension [53], respectively. It is assumed that $a_1(y,t)$, $a_2(y,t)$, $a_3(y,t)$, $f(y,t)$, $g(y)$, $g_0(t)$, and $g_1(t)$ are sufficiently smooth functions.

Now, consider the following space:

$$\mathcal{W}^{\mathcal{M}}=\text{span}\{\psi_m(y)\,\psi_n(t):0\le m,n\le\mathcal{M}\}. \tag{4.8}$$

Any function $\chi^{\mathcal{M}}(y,t)\in\mathcal{W}^{N}$ can be expressed as

$$\chi^{\mathcal{M}}(y,t)=\sum_{m=0}^{\mathcal{M}}\sum_{n=0}^{\mathcal{M}}\hat{\chi}_{mn}\,\psi_m(y)\,\psi_n(t)=\boldsymbol{\psi}(y)^T\,\hat{\boldsymbol{\chi}}\,\boldsymbol{\psi}(t), \tag{4.9}$$

where $\boldsymbol{\psi}(y)$ is the vector defined in (3.19), and $\hat{\boldsymbol{\chi}}=(\hat{\chi}_{mn})_{0\le m,n\le\mathcal{M}}$ is the matrix of unknowns, whose order is $(\mathcal{M}+1)^2$.

Now, the residual $\mathcal{R}_{\mathcal{M}}(y,t)$ of Eq (4.1) can be expressed as

$$\mathcal{R}_{\mathcal{M}}(y,t)=\frac{\partial^{\gamma}\chi^{\mathcal{M}}(y,t)}{\partial t^{\gamma}}+a_1(y,t)\chi^{\mathcal{M}}(y,t)\frac{\partial\chi^{\mathcal{M}}(y,t)}{\partial y}+a_2(y,t)\frac{\partial^2\chi^{\mathcal{M}}(y,t)}{\partial y^2}$$
$$+a_3(y,t)\frac{\partial^4\chi^{\mathcal{M}}(y,t)}{\partial y^4}-f(y,t). \tag{4.10}$$

The application of Corollaries 3.2 and 3.3 aids to express $\mathcal{R}_{\mathcal{M}}(y,t)$ as

$$\mathcal{R}_{\mathcal{M}}(y,t)=\boldsymbol{\psi}(y)^T\,\hat{\boldsymbol{\chi}}\,(t^{-\gamma}\,(\boldsymbol{D}^{\gamma}\,\boldsymbol{\psi}(t)+\lambda))+a_1(y,t)\left[\boldsymbol{\psi}(y)^T\,\hat{\boldsymbol{\chi}}\,\boldsymbol{\psi}(t)\right]\left[(\boldsymbol{U}\,\boldsymbol{\psi}(y))^T\,\hat{\boldsymbol{\chi}}\,\boldsymbol{\psi}(t)\right]$$
$$+a_2(y,t)\left[(\boldsymbol{V}\,\boldsymbol{\psi}(y))^T\,\hat{\boldsymbol{\chi}}\,\boldsymbol{\psi}(t)\right]+a_3(y,t)\left[(\boldsymbol{K}\,\boldsymbol{\psi}(y))^T\,\hat{\boldsymbol{\chi}}\,\boldsymbol{\psi}(t)\right]-f(y,t). \tag{4.11}$$

Now, to obtain the expansion coefficients $\hat{\chi}_{mn}$, we apply the spectral collocation method by forcing the residual $\mathcal{R}_{\mathcal{M}}(y,t)$ to be zero at some collocation points $\left(\frac{m+1}{\mathcal{M}+2}, \frac{n+1}{\mathcal{M}+2}\right)$, as follows:

$$\mathcal{R}_{\mathcal{M}}\left(\frac{m+1}{\mathcal{M}+2}, \frac{n+1}{\mathcal{M}+2}\right) = 0, \quad 1 \le m \le \mathcal{M}-3, \quad 1 \le n \le \mathcal{M}. \tag{4.12}$$

Moreover, the initial and boundary conditions (4.2)–(4.4) or (4.5)–(4.7) imply the following equations:

$$\boldsymbol{\psi}\left(\frac{m+1}{\mathcal{M}+2}\right)^T \hat{\boldsymbol{\chi}}\, \boldsymbol{\psi}(0) = g\left(\frac{m+1}{\mathcal{M}+2}\right), \quad 1 \le m \le \mathcal{M}+1, \tag{4.13}$$

$$\boldsymbol{\psi}(0)^T \hat{\boldsymbol{\chi}}\, \boldsymbol{\psi}\left(\frac{n+1}{\mathcal{M}+2}\right) = g_0\left(\frac{n+1}{\mathcal{M}+2}\right), \quad 1 \le n \le \mathcal{M}, \tag{4.14}$$

$$\boldsymbol{\psi}(1)^T \hat{\boldsymbol{\chi}}\, \boldsymbol{\psi}\left(\frac{n+1}{\mathcal{M}+2}\right) = g_1\left(\frac{n+1}{\mathcal{M}+2}\right), \quad 1 \le n \le \mathcal{M}, \tag{4.15}$$

$$(\boldsymbol{U}\, \boldsymbol{\psi}(0))^T \hat{\boldsymbol{\chi}}\, \boldsymbol{\psi}\left(\frac{n+1}{\mathcal{M}+2}\right) = 0, \quad 1 \le n \le \mathcal{M}, \tag{4.16}$$

$$(\boldsymbol{U}\, \boldsymbol{\psi}(1))^T \hat{\boldsymbol{\chi}}\, \boldsymbol{\psi}\left(\frac{n+1}{\mathcal{M}+2}\right) = 0, \quad 1 \le n \le \mathcal{M}, \tag{4.17}$$

or

$$\boldsymbol{\psi}\left(\frac{m+1}{\mathcal{M}+2}\right)^T \hat{\boldsymbol{\chi}}\, \boldsymbol{\psi}(0) = g\left(\frac{m+1}{\mathcal{M}+2}\right), \quad 1 \le m \le \mathcal{M}+1, \tag{4.18}$$

$$\boldsymbol{\psi}(0)^T \hat{\boldsymbol{\chi}}\, \boldsymbol{\psi}\left(\frac{n+1}{\mathcal{M}+2}\right) = g_0\left(\frac{n+1}{\mathcal{M}+2}\right), \quad 1 \le n \le \mathcal{M}, \tag{4.19}$$

$$\boldsymbol{\psi}(1)^T \hat{\boldsymbol{\chi}}\, \boldsymbol{\psi}\left(\frac{n+1}{\mathcal{M}+2}\right) = g_1\left(\frac{n+1}{\mathcal{M}+2}\right), \quad 1 \le n \le \mathcal{M}, \tag{4.20}$$

$$(\boldsymbol{V}\, \boldsymbol{\psi}(0))^T \hat{\boldsymbol{\chi}}\, \boldsymbol{\psi}\left(\frac{n+1}{\mathcal{M}+2}\right) = 0, \quad 1 \le n \le \mathcal{M}, \tag{4.21}$$

$$(\boldsymbol{V}\, \boldsymbol{\psi}(1))^T \hat{\boldsymbol{\chi}}\, \boldsymbol{\psi}\left(\frac{n+1}{\mathcal{M}+2}\right) = 0, \quad 1 \le n \le \mathcal{M}. \tag{4.22}$$

To get $\hat{\chi}_{mn}$, one can use Newton's iterative approach to solve the $(\mathcal{M}+1)^2$ nonlinear system of equations in (4.13)–(4.17) or (4.18)–(4.22).

Remark 4.1. *We utilized the points* $\left(\frac{m+1}{\mathcal{M}+2}, \frac{n+1}{\mathcal{M}+2}\right)$ *due to their simplicity, uniform distribution in* $(0,1)$*, and compatibility with the structure of our basis functions.*

5. The convergence and error analysis

In this section, we study the convergence of the Fibonacci number expansion. So, the following necessary lemmas and theorems are presented.

- Lemma 5.1 gives an upper estimate for $\psi_i(y)$.

- Lemma 5.2 expresses the infinitely differentiable function $f(y)$ in terms of $\psi_i(y)$.
- Theorem 5.1 gives an upper estimate for the unknown expansion coefficients $\hat{f}_n$.
- Theorem 5.2 gives an upper estimate for the unknown double expansion coefficients $\hat{\chi}_{mn}$.
- Theorem 5.3 gives an upper estimate for the truncation error $|\chi(y,t) - \chi^{\mathcal{M}}(y,t)|$.

Lemma 5.1. *The following inequality holds for $\psi_i(y)$:*

$$\psi_i(y) \leq \left(\sqrt{5}\right)^i, \quad y \in (0,1),\ i \geq 0. \tag{5.1}$$

Proof. We will prove by induction. First, for $i = 0$, since we have $\psi_0(y) = 1$, the equality holds in this case. Now, assume that the Inequality 5.1 holds, and so we have to prove the following inequality:

$$\psi_{i+1}(y) \leq \left(\sqrt{5}\right)^{i+1}, \quad y \in (0,1),\ i \geq 0. \tag{5.2}$$

Now the recurrence relation (3.3) leads to

$$\psi_{i+1}(y) = y\,\psi_i(y) + F_{i+2}. \tag{5.3}$$

In virtue of the inequality (3.4) together with the induction hypothesis (5.1), we get

$$\begin{aligned}\psi_{i+1}(y) &\leq 2\left(\sqrt{5}\right)^i \\ &< \left(\sqrt{5}\right)^{i+1}.\end{aligned}$$

This ends the proof. □

Lemma 5.2. *The infinitely differentiable function $f(y)$ at the origin has the following expansion:*

$$f(y) = \sum_{n=0}^{\infty}\sum_{s=n}^{\infty} \frac{f^s(0)\,a_{n,s}}{s!}\,\psi_n(y), \tag{5.4}$$

where $a_{n,s}$ is defined in Eq (3.10).

Proof. Consider the following expansion for $f(y)$:

$$f(y) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\,y^n. \tag{5.5}$$

Inserting the inversion formula (3.9) into the last formula gives

$$f(y) = \sum_{n=0}^{\infty}\sum_{r=0}^{n} \frac{f^{(n)}(0)\,a_{r,n}}{n!}\,\psi_r(y), \tag{5.6}$$

which can be written alternatively as

$$f(y) = \sum_{n=0}^{\infty}\sum_{s=n}^{\infty} \frac{f^s(0)\,a_{n,s}}{s!}\,\psi_n(y). \tag{5.7}$$

This completes the proof of this lemma. □

Theorem 5.1. *If $f(y)$ is defined on $[0,1]$ and $|g^{(i)}(0)| \leq \lambda^i, \quad i > 0$, where λ is a positive constant and $f(y) = \sum_{n=0}^{\infty} \hat{f}_n \, \psi_n(y)$, then we obtain*

$$|\hat{f}_n| \leq \frac{e^{\lambda}\lambda^n}{n!}. \tag{5.8}$$

Moreover, the series converges absolutely.

Proof. Lemma 5.2 together with the assumptions of the theorem leads to

$$\begin{aligned} |\hat{f}_n| &= \left| \sum_{s=n}^{\infty} \frac{f^s(0)\, a_{n,s}}{s!} \right| \\ &\leq \sum_{s=n}^{\infty} \frac{\lambda^s}{s!} \\ &= e^{\lambda}\left(1 - \frac{\Gamma(n,\lambda)}{\Gamma(n)}\right), \end{aligned} \tag{5.9}$$

where $\Gamma(n,\lambda)$ is the incomplete gamma function.

Now, based on the following inequality that holds for any non-negative integer n:

$$e^{\lambda}\left(1 - \frac{\Gamma(n,\lambda)}{\Gamma(n)}\right) \leq \frac{e^{\lambda}\lambda^n}{n!)}, \quad \forall\, n \geq 0, \tag{5.10}$$

we get

$$|\hat{f}_n| \leq \frac{e^{\lambda}\lambda^n}{n!}. \tag{5.11}$$

The second part of the theorem is easy to obtain based on the following inequality:

$$\sum_{n=0}^{\infty} \left|\hat{f}_n \, \psi_n(y)\right| \leq \sum_{n=0}^{\infty} \frac{e^{\lambda}(\sqrt{5})^n \lambda^n}{\Gamma(n+1)} = e^{\lambda(1+\sqrt{5})}, \tag{5.12}$$

so the series converges absolutely. □

Theorem 5.2. *Let $\chi^{\mathcal{M}}(y,t) = f_1(y)\, f_2(t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \hat{\chi}_{mn}\, \psi_m(y)\, \psi_n(t)$, with $|f_1^{(i)}(0)| \leq \ell_1^i$ and $|f_2^{(i)}(0)| \leq \ell_2^i$, where ℓ_1 and ℓ_2 are positive constants. One has*

$$|\hat{\chi}_{mn}| \leq \frac{e^{\ell_1+\ell_2}\ell_1^m\, \ell_2^n}{m!\, n!}. \tag{5.13}$$

Moreover, the series converges absolutely.

Proof. If we make use of Lemma 5.2 together with the assumption that $\chi^{\mathcal{M}}(y,t) = f_1(y)\, f_2(t)$, then we obtain

$$\hat{\chi}_{mn} = \sum_{p=m}^{\infty}\sum_{q=n}^{\infty} \frac{f_1^p(0)\, f_2^q(0)\, a_{n,q}\, a_{m,p}}{p!\, q!}. \tag{5.14}$$

Using the assumption $|f_1^{(i)}(0)| \leq \ell_1^i$ and $|f_2^{(i)}(0)| \leq \ell_2^i$, one gets

$$|\hat{\chi}_{mn}| \leq \sum_{p=m}^{\infty} \frac{\ell_1^p\, a_{m,p}}{p!} \times \sum_{q=n}^{\infty} \frac{\ell_2^q\, a_{n,q}}{q!}. \tag{5.15}$$

Now performing similar steps as in the proof of Theorem 5.1, we get the desired result. □

Theorem 5.3. *We get the following upper estimate on the truncation error if $\chi(y,t)$ meets the hypothesis of Theorem 5.2.*

$$|\chi(y,t) - \chi^{\mathcal{M}}(y,t)| < \frac{e^{(1+\sqrt{5})(\ell_1+\ell_2)}\ \sqrt{5}^{\mathcal{M}+1}\left(\ell_1^{\mathcal{M}+1} + \ell_2^{\mathcal{M}+1}\right)}{\mathcal{M}!}. \tag{5.16}$$

Proof. The representations of $\chi(y,t)$ and $\chi^{\mathcal{M}}(y,t)$ lead to

$$\begin{aligned}|\chi(y,t) - \chi^{\mathcal{M}}(y,t)| &= \left|\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \hat{\chi}_{mn}\,\psi_m(y)\,\psi_n(t) - \sum_{m=0}^{\mathcal{M}}\sum_{n=0}^{\mathcal{M}} \hat{\chi}_{mn}\,\psi_m(y)\,\psi_n(t)\right| \\ &\leq \left|\sum_{m=0}^{\mathcal{M}}\sum_{n=\mathcal{M}+1}^{\infty} \hat{\chi}_{mn}\,\psi_m(y)\,\psi_n(t)\right| + \left|\sum_{m=\mathcal{M}+1}^{\infty}\sum_{n=0}^{\infty} \hat{\chi}_{mn}\,\psi_m(y)\,\psi_n(t)\right|.\end{aligned} \tag{5.17}$$

If Theorem 5.2 and Lemma 5.1 are used together with the following inequalities:

$$\begin{aligned}&\sum_{m=0}^{\mathcal{M}} \frac{e^{\ell_1}(\sqrt{5})^m \ell_1^m}{m!} = \frac{e^{\sqrt{5}\ell_1+\ell_1}\Gamma\left(\mathcal{M}+1,\ \sqrt{5}\,\ell_1\right)}{\mathcal{M}!} < e^{\ell_1(1+\sqrt{5})}, \\ &\sum_{n=\mathcal{M}+1}^{\infty} \frac{e^{\ell_2}(\sqrt{5})^n \ell_2^n}{n!} = \frac{e^{\sqrt{5}\ell_2+\ell_2}\left(\Gamma(\mathcal{M}+1) - \Gamma\left(\mathcal{M}+1,\ \sqrt{5}\,\ell_2\right)\right)}{\mathcal{M}!} < \frac{e^{\ell_2(1+\sqrt{5})}\left(\sqrt{5}\,\ell_2\right)^{\mathcal{M}+1}}{\mathcal{M}!}, \\ &\sum_{m=\mathcal{M}+1}^{\infty} \frac{e^{\ell_1}(\sqrt{5})^m \ell_1^m}{m!} = \frac{e^{\sqrt{5}\ell_1+\ell_1}\left(\Gamma(\mathcal{M}+1) - \Gamma\left(\mathcal{M}+1,\ \sqrt{5}\ell_1\right)\right)}{\mathcal{M}!} < \frac{e^{\ell_1(1+\sqrt{5})}\left(\sqrt{5}\,\ell_1\right)^{\mathcal{M}+1}}{\mathcal{M}!}, \\ &\sum_{n=0}^{\infty} \frac{e^{\ell_2}(\sqrt{5})^n \ell_2^n}{n!} = e^{\ell_2(1+\sqrt{5})},\end{aligned} \tag{5.18}$$

then, we get the following estimation:

$$|\chi(y,t) - \chi^{\mathcal{M}}(y,t)| < \frac{e^{(1+\sqrt{5})(\ell_1+\ell_2)}\ (\sqrt{5})^{\mathcal{M}+1}\left(\ell_1^{\mathcal{M}+1} + \ell_2^{\mathcal{M}+1}\right)}{\mathcal{M}!}. \tag{5.19}$$

Theorem 5.3 is now proved. □

6. Illustrative examples

This section presents some illustrative examples to test numerically our proposed collocation algorithm. We also compare it with other methods to validate its high accuracy.

Example 6.1. *[52] Consider the following equation:*

$$\frac{\partial^{\gamma} \chi(y,t)}{\partial t^{\gamma}} + e^{-20t} \chi(y,t) \frac{\partial \chi(y,t)}{\partial y} + (1 + 100y) \frac{\partial^2 \chi(y,t)}{\partial y^2} = f(y,t), \quad (y,t) \in (0,1) \times (0,1], \tag{6.1}$$

governed by the following conditions:

$$\chi(y,0) = \chi(0,t) = \chi(1,t) = \left.\frac{\partial \chi(y,t)}{\partial y}\right|_{y=0} = \left.\frac{\partial \chi(y,t)}{\partial y}\right|_{y=1} = 0, \tag{6.2}$$

where $f(y,t)$ is chosen to meet the exact solution given as $\chi(y,t) = t^4 y^3 (1-y)^3$.

Table 1 displays a comparison of L_∞ errors between our method at various values of γ and the method in [52]. Also, the CPU time is reported in this table. Figure 1 shows the absolute errors (AEs) for different γ values with $\mathcal{M} = 6$. Table 2 reports the maximum absolute errors (MAEs) and the L^∞ errors for various values of γ when $\mathcal{M} = 6$.

Table 1. Comparison of L_∞ errors for Example 6.1.

γ	Method in [52] at $N_s = 128$, $M_t = 64$	CPU time	Proposed method at $\mathcal{M} = 6$	CPU time
0.6	4.2262×10^{-10}	2.0861	3.62647×10^{-13}	3.624
0.8	3.9223×10^{-10}	2.0981	3.18499×10^{-13}	3.874
0.95	3.3087×10^{-10}	2.0787	6.66482×10^{-12}	3.421

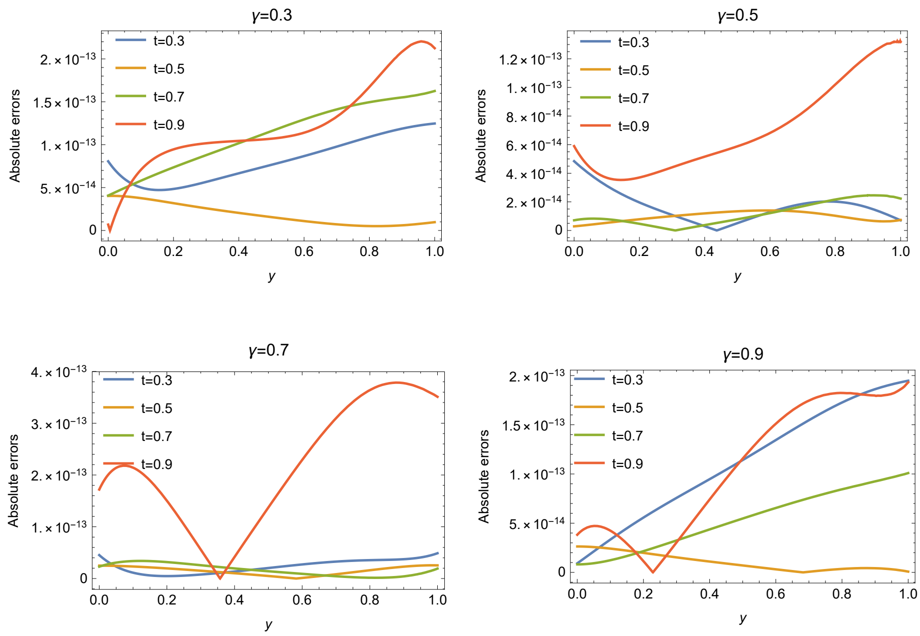

Figure 1. The AEs of Example 6.1.

Table 2. The errors for Example 6.1 at $\mathcal{M} = 6$.

	$\gamma = 0.3$	$\gamma = 0.5$	$\gamma = 0.6$	$\gamma = 0.7$	$\gamma = 0.75$	$\gamma = 0.95$
MAEs	1.15323×10^{-12}	3.58471×10^{-13}	6.72707×10^{-13}	8.95568×10^{-13}	2.65462×10^{-13}	3.4184×10^{-13}
L_∞ errors	1.30037×10^{11}	4.3144×10^{-12}	3.62647×10^{-13}	3.86266×10^{-12}	3.00833×10^{-13}	6.66482×10^{-12}

Example 6.2. *[52] Consider the following equation:*

$$\frac{\partial^{\gamma}\chi(y,t)}{\partial t^{\gamma}} - 2\chi(y,t)\frac{\partial\chi(y,t)}{\partial y} + 4\frac{\partial^{2}\chi(y,t)}{\partial y^{2}} + \frac{\partial^{4}\chi(y,t)}{\partial y^{4}} = f(y,t), \quad (y,t)\in(0,1)\times(0,1], \tag{6.3}$$

governed by the following conditions:

$$\chi(y,0) = \chi(0,t) = \chi(1,t) = \left.\frac{\partial\chi(y,t)}{\partial y}\right|_{y=0} = \left.\frac{\partial\chi(y,t)}{\partial y}\right|_{y=1} = 0, \tag{6.4}$$

where $f(y,t)$ is chosen to meet the exact solution given by

$$\chi(y,t) = t^{4}y^{2}\left(y^{3} - \frac{5y^{2}}{2} + 2y - \frac{1}{2}\right).$$

Table 3 displays a comparison of L^2 and L_∞ errors between our method at various values of γ with the method in [52]. In addition, the CPU time is reported in this table. Figure 2 shows the AEs at various values of γ, where $\mathcal{M} = 5$. Table 4 reports the AEs at different values of t when $\gamma = 0.9$.

Table 3. Comparison of L^2 and L_∞ errors for Example 6.2.

	Method in [52] at $N_s = 160$, $M_t = 320$			Proposed method at $\mathcal{M} = 5$			
γ	L_∞ error	L^2 error	CPU time	L_∞ error	CPU time	L^2 error	CPU time
0.3	4.0055×10^{-12}	8.2604×10^{-13}	17.0300	3.43691×10^{-13}	2.937	2.10446×10^{-13}	40.39
0.5	7.3038×10^{-12}	9.0548×10^{-12}	16.0248	6.39944×10^{-13}	3.061	3.35476×10^{-13}	43.327
0.8	5.0390×10^{-11}	3.1675×10^{-11}	16.1246	8.37401×10^{-13}	3.093	1.41058×10^{-13}	44.406

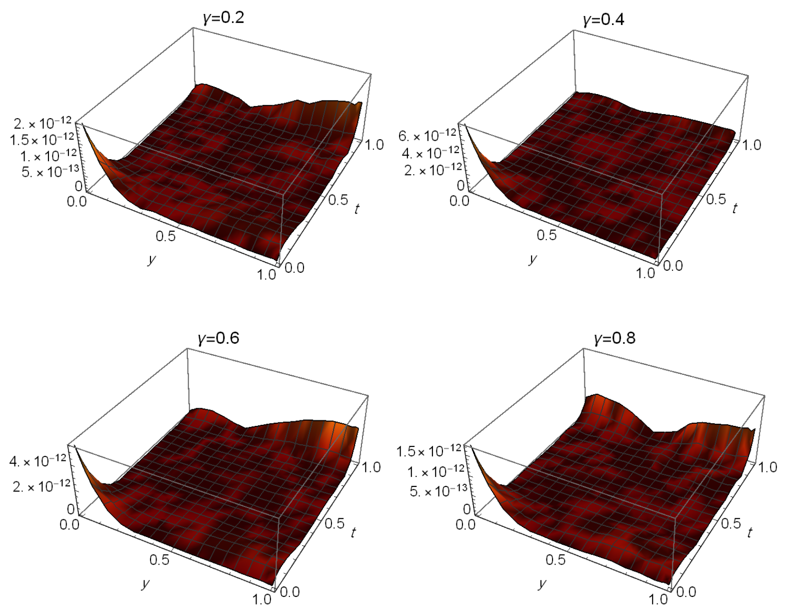

Figure 2. The AEs of Example 6.2.

Table 4. The AEs of Example 6.2 when $\gamma = 0.9$.

y	$t = 0.3$	$t = 0.6$	$t = 0.9$
0.1	2.09825×10^{-15}	1.33865×10^{-14}	4.12812×10^{-14}
0.2	5.53317×10^{-15}	1.22161×10^{-14}	4.79746×10^{-14}
0.3	1.13182×10^{-14}	8.58992×10^{-15}	2.52957×10^{-14}
0.4	1.67923×10^{-14}	3.99377×10^{-15}	1.04227×10^{-14}
0.5	2.25354×10^{-14}	4.44384×10^{-16}	4.88908×10^{-14}
0.6	2.8438×10^{-14}	3.88936×10^{-15}	8.42889×10^{-14}
0.7	3.39674×10^{-14}	5.71309×10^{-15}	1.13609×10^{-13}
0.8	3.84362×10^{-14}	5.39347×10^{-15}	1.34909×10^{-13}
0.9	4.1266×10^{-14}	2.47014×10^{-15}	1.45858×10^{-13}

Example 6.3. *Consider the following equation:*

$$\frac{\partial^\gamma \chi(y,t)}{\partial t^\gamma} + \chi(y,t)\frac{\partial \chi(y,t)}{\partial y} + \frac{\partial^2 \chi(y,t)}{\partial y^2} + \frac{\partial^4 \chi(y,t)}{\partial y^4} = f(y,t), \quad (y,t) \in (0,1) \times (0,1], \tag{6.5}$$

governed by the following initial and boundary conditions:

$$\chi(y,0) = \left.\frac{\partial \chi(y,t)}{\partial y}\right|_{y=0} = \left.\frac{\partial \chi(y,t)}{\partial y}\right|_{y=1} = 0, \tag{6.6}$$

$$\chi(0,t) = t^2, \quad \chi(1,t) = \left(\frac{1}{2}(-1-e) + e\right) t^2, \tag{6.7}$$

where $f(y,t)$ is chosen to meet the exact solution given by $\chi(y,t) = t^2 \left(\frac{1}{2} y(-ey + y - 2) + e^y\right)$.

Figure 3 shows the MAEs and L_∞ errors at various values of $\mathcal{M}$ when $\gamma = 0.5$. Table 5 reports the MAEs and L_∞ errors at various values of $\mathcal{M}$ when $\gamma = 0.9$. In addition, the CPU time of our proposed method is listed in this table. Figure 4 shows the AEs at different values of γ at $\mathcal{M} = 8$. Table 6 reports the AEs at different values of $\mathcal{M}$ when $\gamma = 0.4$.

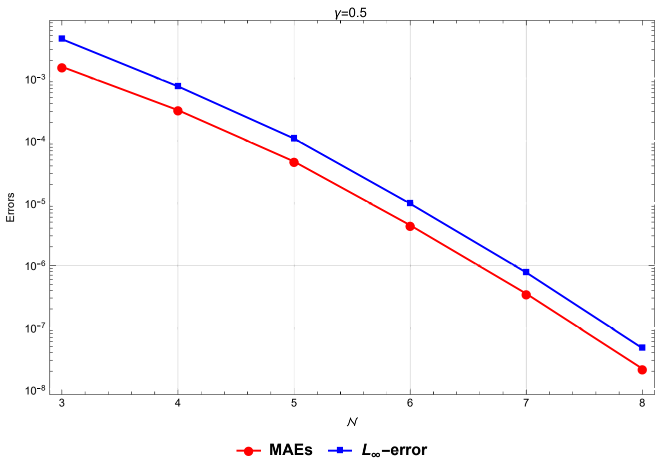

Figure 3. The errors of Example 6.3 at various values of $\mathcal{M}$ when $\gamma = 0.5$.

Table 5. The errors for Example 6.3 at $\gamma = 0.9$.

	$\mathcal{M}=3$	$\mathcal{M}=4$	$\mathcal{M}=5$	$\mathcal{M}=6$	$\mathcal{M}=7$	$\mathcal{M}=8$
MAE	1.57152×10^{-3}	3.1497×10^{-4}	4.70058×10^{-5}	4.44165×10^{-6}	3.52031×10^{-7}	2.20649×10^{-8}
CPU time	2.062	2.328	2.968	3.593	4.298	5.984
L_∞ errors	4.371×10^{3}	7.60422×10^{-4}	1.09901×10^{-4}	9.87333×10^{-6}	7.72464×10^{-7}	4.71035×10^{-8}
CPU time	2.343	2.625	3.28	3.983	4.626	6.312

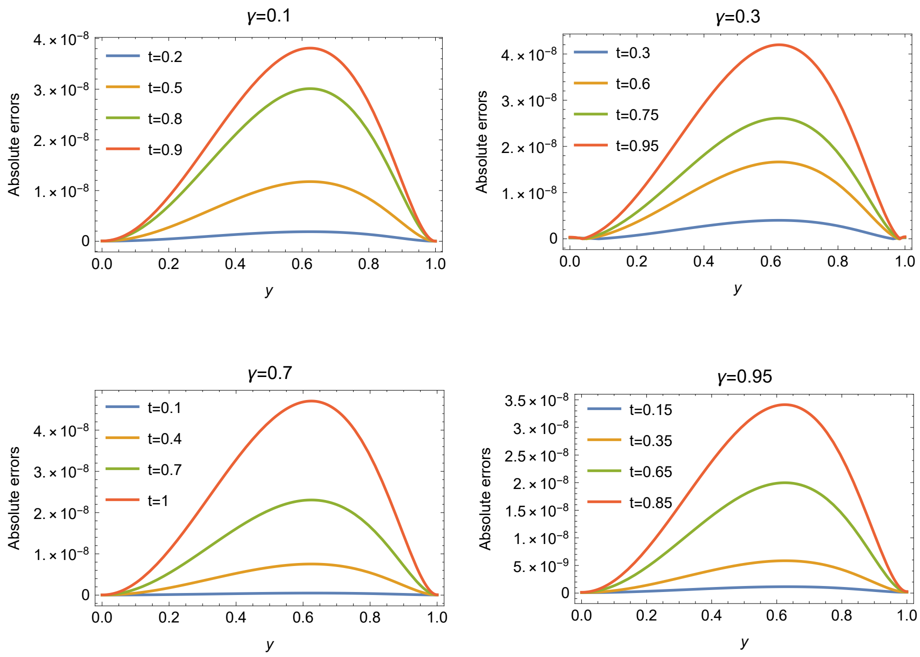

Figure 4. The AEs of Example 6.3.

Table 6. The AEs of Example 6.3 when $\gamma = 0.4$.

$y=t$	$\mathcal{M}=5$	$\mathcal{M}=6$	$\mathcal{M}=7$	$\mathcal{M}=8$
0.1	9.80291×10^{-8}	6.19331×10^{-9}	6.09922×10^{-10}	1.49565×10^{-11}
0.2	1.29712×10^{-6}	9.37314×10^{-8}	8.17104×10^{-9}	4.16195×10^{-10}
0.3	5.4043×10^{-6}	4.23511×10^{-7}	3.475×10^{-8}	1.92459×10^{-9}
0.4	1.38087×10^{-5}	1.14117×10^{-6}	9.07976×10^{-8}	5.2277×10^{-9}
0.5	2.62349×10^{-5}	2.26186×10^{-6}	1.77311×10^{-7}	1.04782×10^{-8}
0.6	3.95184×10^{-5}	3.55522×10^{-6}	2.78133×10^{-7}	1.68099×10^{-8}
0.7	4.71227×10^{-5}	4.45028×10^{-6}	3.52586×10^{-7}	2.18678×10^{-8}
0.8	4.10576×10^{-5}	4.10235×10^{-6}	3.3476×10^{-7}	2.15263×10^{-8}
0.9	1.8921×10^{-5}	2.01446×10^{-6}	1.72181×10^{-7}	1.16294×10^{-8}

Example 6.4. *Consider the following equation:*

$$\frac{\partial^\gamma \chi(y,t)}{\partial t^\gamma} + 2\chi(y,t)\frac{\partial \chi(y,t)}{\partial y} + 4\frac{\partial^2 \chi(y,t)}{\partial y^2} + \frac{\partial^4 \chi(y,t)}{\partial y^4} = f(y,t), \quad (y,t)\in(0,1)\times(0,1], \tag{6.8}$$

governed by the following initial and boundary conditions:

$$\chi(y,0) = \chi(0,t) = \chi(1,t) = \left.\frac{\partial^2 \chi(y,t)}{\partial y^2}\right|_{y=0} = \left.\frac{\partial^2 \chi(y,t)}{\partial y^2}\right|_{y=1} = 0, \tag{6.9}$$

where $f(y,t)$ *is chosen such that the exact solution of this problem is* $\chi(y,t) = t^{\alpha+4}\sin(\pi y)$.

Table 7 presents a comparison of L_∞ *errors between our method at different values of* γ *with the method in [52]. Table 8 reports the AEs at different values of* t *when* $\gamma = 0.9$. *Figure 5 shows the approximate solution, exact solution, and AEs at* $\gamma = 0.4$ *when* $\mathcal{M} = 8$.

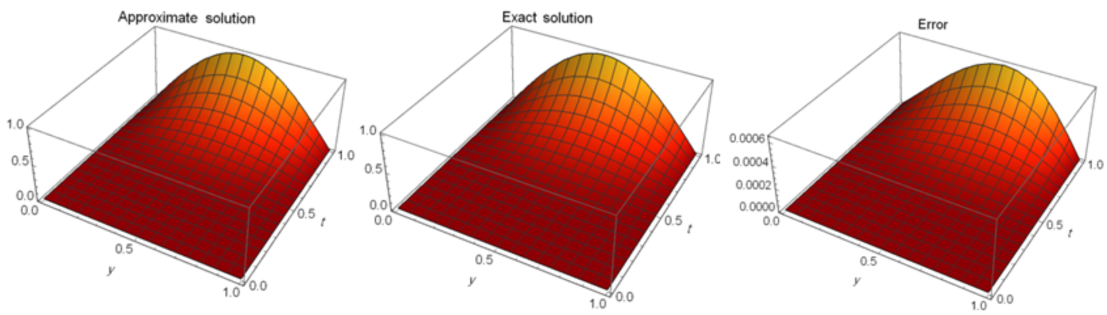

Figure 5. The approximate solution, exact solution, and AEs at $\gamma = 0.4$ of Example 6.4.

Table 7. Comparison of L_∞ errors for Example 6.4.

γ	Method in [52] at $N_s = 32$, $M_t = 1000$	Proposed method at $\mathcal{M} = 8$
0.5	1.2999×10^{-3}	6.3895×10^{-4}
0.6	-	6.3453×10^{-4}
0.7	-	6.2912×10^{-4}
0.8	1.2701×10^{-3}	2.2778×10^{-4}

Table 8. The AEs of Example 6.4 when $\gamma = 0.9$.

y	$t = 0.1$	$t = 0.4$	$t = 0.7$
0.1	1.3909×10^{-7}	1.46229×10^{-6}	2.44068×10^{-5}
0.2	2.64524×10^{-7}	2.96519×10^{-6}	4.92949×10^{-5}
0.3	3.64236×10^{-7}	4.37193×10^{-6}	7.23941×10^{-5}
0.4	4.28438×10^{-7}	5.52204×10^{-6}	9.10903×10^{-5}
0.5	4.50822×10^{-7}	6.26114×10^{-6}	1.02895×10^{-4}
0.6	4.29164×10^{-7}	6.44585×10^{-6}	1.05535×10^{-4}
0.7	3.6552×10^{-7}	5.94869×10^{-6}	9.70369×10^{-5}
0.8	2.66×10^{-7}	4.67173×10^{-6}	7.59479×10^{-5}
0.9	1.40213×10^{-7}	2.61134×10^{-6}	4.23392×10^{-5}

7. Concluding remarks

A numerical scheme based on the Fibonacci collocation method was successfully applied to the TFKSE. The development of this method involved introducing the Fibonacci coefficient polynomials. The collocation method was the method that was utilized to reduce the problem with its governing conditions into a system of non-linear algebraic equations. The numerical experiments in this study validated the effectiveness of the proposed method. The results indicated that the scheme could achieve high accuracy across various test problems, including problems with known analytical solutions. Furthermore, the algorithm maintained its precision even with fewer basis terms, demonstrating its potential as a low-cost yet reliable tool. Future work may consider applying the proposed approach to treat other FDEs. In addition, we aim to introduce other new modified and generalized sequences of polynomials of our proposed sequence to treat other types of DEs.

Author contributions

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Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The researchers wish to extend their sincere gratitude to the Deanship of Scientific Research at the Islamic University of Madinah for the support provided to the Post- Publishing Program.

Conflict of interest

The authors declare that they have no competing interests.

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