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Research article

Event-triggered impulsive control with time delays for input-to-state stabilization of nonlinear systems

Biwen Li and Yu Gu*

School of Mathematics and Statistics, Hubei Normal University, Huangshi 435002, China

* **Correspondence:** Email: 19971226018@163.com.

Abstract: The paper examines the input-to-state stability (ISS) in nonlinear impulsive systems under Event-triggered impulsive control (ETIC), which is a method that differs from traditional approaches by activating the controller only when state-dependent event conditions are met and maintains no control action between consecutive impulse instants. The event-triggered mechanism (ETM) generates state-dependent impulses, and time delays in impulses are comprehensively accounted for. The derived sufficient conditions guarantee the exclusion of Zeno behavior and the assurance of the ISS, while considering external inputs in both continuous and impulsive dynamics. A new event-triggered delayed impulsive control (ETDIC) strategy is presented to reveal the relationship among relevant parameters, namely that time delays in impulses may actually promote stabilization in terms of the ISS. Lyapunov-based criteria are established to prevent fast triggering. These results are applied to nonlinear systems to obtain the ETM and control gains via linear matrix inequalities (LMIs). Two numerical examples validate the proposed theory and strategies.

Keywords: input-to-state stability; event-triggered impulse control; Zeno behavior; delayed impulse; nonlinear impulsive systems
Mathematics Subject Classification: 93C30

1. Introduction

Impulsive systems constitute a prominent subclass of hybrid dynamical systems, characterized by the existence of discrete dynamics, continuous dynamics, and the impulse condition that governs instantaneous state transitions triggered upon meeting specific criteria [1]. The salient advantage of impulsive control strategies lies in their discrete actuation mechanism, where control inputs are only transmitted at a sequence of times [2]. In recent years, considerable research efforts have been devoted to impulsive systems, and have led to significant theoretical developments and practical applications [3, 4].

The input-to-state stability (ISS) is a fundamental concept in the analysis of robust dynamic behaviors of continuous time systems under exogenous disturbances, which guarantees bounded system states for a bounded input and asymptotically converges to zero when the inputs vanish. Originating from the seminal works of [5] and [6], these notions have been extensively developed and applied across various dynamical systems, including discrete-time systems [7], time delay systems [8], networked control systems [9], switched systems [10], and impulsive systems [11]. In particular, the extension of ISS properties to impulsive systems has garnered significant attention. Hespanha et al. [12] pioneered the study of the ISS for impulsive systems and established sufficient conditions to ensure these properties. Significant progress was achieved by Dashkovskiy and Feketa [13], who explored the ISS for a crucial category of impulsive systems characterized by time dependent impulses. However, a notable limitation of these studies is their reliance on time-triggered control strategies, where impulse sequences are predetermined and executed at fixed intervals, irrespective of the system's actual state. Although theoretically sound, this approach often induces unnecessary actuation in practice, thereby compromising the resource efficiency. Consequently, there is a growing interest in exploring event-triggered or state-dependent impulsive control strategies, which could potentially enhance the efficiency and applicability of ISS frameworks in real world scenarios.

Event-triggered impulsive control (ETIC) combines event-triggered strategies with impulsive control advantages, thus emerging as a key focus in control systems to enhance both the resource efficiency and the performance. In contrast to conventional time-triggered methods, event-triggered control mechanisms activate control exclusively actions when the system states satisfy predetermined conditions, thereby effectively eliminating redundant control operations and optimizing resource allocation. This control paradigm is particularly effective in scenarios where the system states or external disturbances vary unpredictably, as it ensures that the control efforts are precisely applied when needed. Nevertheless, a significant issue in applying ETIC is the risk of Zeno behavior, where triggering events happen infinitely often in a finite time frame [14]. Such behavior often induced by external perturbations or measurement noise [15] must be carefully excluded to ensure practical applicability. Recent studies have demonstrated the effectiveness of ETIC in various applications, including multi-agent consensus [16], asymptotic stability of impulsive systems [17], and synchronization of neural networks [18]. Nevertheless, many existing results are limited to specific system types and often neglect the influence of exogenous disturbances, which are ubiquitous in real-world scenarios. To address these limitations, the integration of the ISS theory with ETIC has gained increasing attention [19]. The ISS provides a robust framework to analyze that system behavior under external disturbances, thus ensuring that the system remains stable despite such influences. Recent advancements have explored ISS properties under ETIC with efforts to exclude Zeno behavior and incorporate time delays in impulse transmissions [20, 21]. However, a significant limitation in these studies is the lack of consideration for impulse delays, which are inherently unavoidable in practical implementations. In particular, time delays frequently arise during impulse transmission, which implies that the impulse transient behavior not only depends on the present system state but also on its past states. In reference [22], three levels of event triggering schemes were proposed to investigate the impact of time delayed event-triggered impulse control on ISS stabilization. While this work addresses the issue of delayed impulses, it does not account for the effects of external inputs on the discrete dynamics of the system. This oversight limits the applicability of the proposed framework in scenarios where external disturbances play a critical role in the system's behavior, and ignores any interference of external input in discrete dynamics. To date, there has been very little research that addresses the ISS of impulsive systems using the ETIC strategy. Recent studies have increasingly focused on impulsive systems with stabilizing delays in their impulse effects [23–25]. However, it is noteworthy that nearly all existing results in this area were developed within the conventional time triggered framework, rather than the investigated event-triggered approach. More critically, these studies failed to account for the presence of exogenous disturbances. Consequently, their theoretical analyses became invalid when external perturbations were present in the system. Therefore, we aim to investigate the ISS properties of impulsive systems using the event-triggered delayed impulsive control (ETDIC) strategy. References [26–28] provided the ideas for the methodology in this paper.

Inspired by the aforementioned discussions. This paper investigates the ISS of nonlinear systems with delayed impulses under ETIC strategies, specifically focusing on ETDIC. An impulsive control approach is proposed to achieve the ISS for the considered system through event-triggered mechanism (ETM) driven forced impulses while guaranteeing Zeno-behavior exclusion. The requirements for impulsive control gains and ETM design are derived by solving linear matrix inequalities (LMIs). This work makes contributions in two distinct aspects. In contrast to previous studies, this study makes significant advancements over prior works by explicitly incorporating time delay information in the impulsive dynamics. Specifically, we introduce a novel time delay formulation and integrate it into the system's dynamic analysis. The proposed framework establishes quantitative relationships among the impulse strength, event-triggering parameters, and impulse time delays, which fully illustrates the stabilizing effect of time delays in impulses on the ISS; additionally, the impact of parameter variations on the system stability is analyzed. Moreover, the ETDIC approach proposes multiple Lyapunov-based conditions to ensure ISS properties without Zeno behavior and considering external inputs in the impulsive dynamics, where the ETM that involves forced impulse sequence is proposed.

2. Preliminaries

Notions: Let $\mathbb{R}_+$ and $\mathbb{N}_+$ be a set of positive real number and positive integers, respectively. $\mathbb{R}$ denotes a set of real numbers. $\mathbb{R}^n$ represents n-dimensional spaces equipped with the Euclidean norm $|\cdot|$. Let $|\cdot|_J$ represent the supremum norm defined on the interval J. The symbol $\bullet$ represents a symmetric block within a symmetric matrix. I is an identity matrix of an appropriate dimension. Π^T and Π^{-1} represent the inverse and transpose of matrix Π, respectively. The continuous function α: $\mathbb{R}^+ \to \mathbb{R}^+$ with $\alpha(0) = 0$ is said to be of class $\mathcal{K}$ if it is strictly increasing. Especially, $\mathcal{K}_\infty$ means α is unbounded. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{KL}$ for every fixed t. $A \wedge B$ and $A \vee B$ are the $\min\{A, B\}$ and $\max\{A, B\}$, respectively. Consider the following impulsive system with delayed impulses:

$$
\begin{aligned}
&\dot{z}(t) = g(z(t), u(t)), \qquad t \neq t_r,\ t \geq t_0, \\
&z(t) = h_r(z(s^-), u(s^-)),\ t = t_r,\ s = s_r = \Upsilon_{t_r},
\end{aligned}
\tag{2.1}
$$

where $k \in \mathbb{Z}_+$, z$(t) \in \mathbb{R}^n$, $\dot{z}(t)$ represents the upper right derivative of the function $z(t)$, and $u(t) \in \mathbb{R}^n$ is the input of an exogenous disturbance bounded locally. The continuous function g, h: $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ satisfies $\Xi(0, 0) = \Psi(0, 0) = 0$. Using the impulse sequence $\{t_r\}$ and the parameter $\alpha \in (0, 1], \beta \in (0, 1]$, we define the set $\Upsilon_{\{t_r\}} = \{s_r : s_r = t_r - \alpha e^{\beta}(t_r - t_{r-1})\}$, where the state jump at the impulse time t_r depends on $z(t)$, with s_r being a delayed instant that satisfies $s_r \in (t_{r-1}, t_r]$. Suppose that the solution of system (2.1) is right-continuous with left limits at impulse instants t_r and under the disturbance input $u(t)$, that is, $z(t) = z(t^+)$, $u(t) = u(t^+)$.

Remark 2.1. *By introducing the auxiliary set Υ_{t_r}, one can express time delays in impulses using $\tau = t_r - s_r = \alpha e^{\beta}(t_r - t_{r-1})$, which depends on the impulse sequence $\{t_r\}$ and the parameters α, β. These time delays are variable over time because the time interval between impulses is uncertain. As a result, the magnitude of these time delays can be either relatively small or quite large. For example, $\tau_r = \alpha e^{\beta}/r \to 0$ as $r \to \infty$, when $\{t_r\} = \{\sum_{i=1}^{r} 1/i\}$, while $\tau_r = \alpha e^{\beta}(2r) \to \infty$ as $r \to \infty$, when $\{t_r\} = \{\sum_{i=1}^{r} 2i\}$. In fact, introducing the auxiliary set Υ_{t_r} is crucial to extract time delay information from the impulses.*

Definition 2.1. *System (2.1) is considered ISS if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for any initial states (t_0, z_0) and input $u(t)$, the system's solution satisfies the following inequality:*

$$|z(t)| \leq \beta(x_0, t - t_0) + \gamma(|u|_{[t_0,t]}), \ \forall t \geq t_0.$$

Definition 2.2. *Given a function V: $\mathbb{R}^n \to \mathbb{R}_+$, that is locally Lipschitz, its upper right-hand derivative with respect to system (2.1) is defined as follows:*

$$D^+V[x] = \lim_{h \to 0^+} \frac{1}{h}[V(x + hf(x) - V(x)].$$

3. Main results

Using the ETDIC framework, we develop Lyapunov-based criteria in this section to exclude Zeno phenomena and prove the ISS of system (2.1). To achieve this, the following ETM is considered:

$$\begin{aligned} t_r &= \min\{t_r^*, t_{r-1} + \sigma_r\}, \\ t_r^* &= \inf\{t \geq t_{r-1} : h(t) \geq 0\}, \end{aligned} \tag{3.1}$$

with the event generator function

$$h(t) = V(z(t)) - \exp(a_r)V(z(t_{r-1})) - \exp(b_r)\varsigma(|u|_{[t_0,t]}),$$

where $\varsigma \in \mathcal{K}_\infty$, $r \in \mathbb{Z}_+$, and $V(z(t))$ depend on the solution $z(t)$ of system (2.1) at instant t and the triggered impulse instant t_{r-1}. The forced impulse parameter $\sigma_r \in \mathbb{R}_+$ and the event-triggering parameters $a_r, b_r \in \mathbb{R}_+$ satisfy the following:

$$\sum_{r=1}^{n} (a_r \wedge b_r) \to \infty \text{ as } n \to \infty \text{ and } \inf_{n \in \mathbb{Z}_+}\{\sigma_r\} > 0. \tag{3.2}$$

First, we derive conditions to exclude Zeno behavior in system (2.1).

Theorem 3.1. *Assume there exists function V: $\mathbb{R}^n \to \mathbb{R}_+$, function $\varsigma \in K_\infty$, and positive constants a_r, b_r, σ_r, that satisfy (3.1) as follows:*

$$D^+V(z(t)) \leq \delta V(z(t)), \ V(z(t)) \geq \varsigma(|u|), \tag{3.3}$$

where $\delta > 0$ is a constant, $z \in \mathbb{R}^n$, and $u \in \mathbb{R}^n$. Under the ETM in (3.1), system (2.1) avoids Zeno behavior. Additionally, the event-triggered impulse sequence t_r satisfies the following:

$$t_r - t_{r-1} \geq \frac{a_r \wedge b_r}{\delta} \wedge \sigma_r, \ \forall r \in \mathbb{Z}_+. \tag{3.4}$$

Proof. Let $z(t) = z(t, t_0)$ denote the solution to system (2.1) with the initial condition (t_0, z_0). According to the definition of the ETM in (3.1), events will occur infinitely many times. Thus, it can be assumed that the trigger instants are arranged as $t_1 < t_2 < \cdots < t_r$. Take into account that there exist two alternatives at each impulse instant. One is the type event-triggered impulse instant produced by (3.2), while the other is the forced impulse instant. Consequently, for the instant t_r, where $\forall r \in \mathbb{Z}_+$, the following two cases are taken into account:

Case (a). The impulse instant t_r is the event triggered by (3.1), that is, $t_r = t_r{}^*$. From this, it follows that

$$V(z(t_r^-)) = \exp(a_r)V(z(t_{r-1})) + \exp(b_r)\varsigma(|u|_{[t_0,t_r]}), \tag{3.5}$$

where $V(z(t_r^-)) > \varsigma(\left|u(t_r^-\right|)$. On the one side, if $V(z(t)) > \varsigma(|u(t|)$ holds for all $t \in [t_{r-1}, t_r)$, from condition (3.3), then we derive the following:

$$V(z(t_r^-)) \leq \exp(\delta(t_r - t_{r-1})V(z(t_{r-1})). \tag{3.6}$$

By integrating (3.6) and (3.7), it is possible to derive the following:

$$t_r - t_{r-1} \geq \frac{a_r}{\delta}.$$

On the other side, suppose there are certain instants $t \in [t_{r-1}, t_r)$ for which $V(z(t)) < \varsigma(|u(t|)$. Given that $V(z(t_r^-)) > \varsigma(\left|u(t_r^-\right|)$, there has to be an instant $\hat{t}_r = \sup\{t \in [t_{r-1}, t_r) : V(z(t)) \leq \varsigma(|u(t|)\}$ that satisfies $V(z(\hat{t})) = \varsigma(\left|u(\hat{t})\right|)$ and $V(z(t)) \geq \varsigma(|u(t)|), \forall t \in [\hat{t}_r, t_r)$. Next, when we consider the interval $[\hat{t}_r, t_r)$, based on (3.3), we can deduce

$$V(z(t_r^-)) \leq \exp(\delta(t_r - \hat{t}_r)\varsigma(\left|u(\hat{t})\right|) \leq \exp(\delta(t_r - t_{r-1})\varsigma(|u|_{[t_0,t_r]}),$$

which, together with (3.5), concludes that

$$t_r - t_{r-1} \geq \frac{b_r}{\delta}.$$

Case (b). The impulse instant t_r is the forced impulse instant, that is, $t_r = t_{r-1} + \sigma_r$. It is obvious that

$$t_r - t_{r-1} = \sigma_r.$$

Therefore, from the above two cases, we can infer that

$$t_r - t_{r-1} \geq \frac{a_r \wedge b_r}{\delta} \wedge \sigma_r,$$

which can be derived that

$$t_r \geq \frac{\sum_{i=1}^{r}(a_r \wedge b_r)}{\delta} \wedge r\overline{\sigma} + t_0, \ \forall r \in \mathbb{Z}_+,$$

where $\overline{\sigma} = \inf_{r\in\mathbb{Z}_+}\{\sigma_r\}$. Consider the condition (3.2), where $t_r \to \infty$ as $r \to \infty$; this indicates that system (2.1) avoids Zeno behavior. The proof is completed.

Remark 3.1. *Zeno behavior means that there is an accumulation point where events will keep being triggered. The authors in references [25–27] established a consistent positive lower bound for the inter-impulse time to rule out Zeno behavior. It can be noticed that (3.4) is less strict compared to* *this claim. This is because (3.4) might allow for a situation where t_r approaches infinity, yet $t_r - t_{r-1}$ approaches zero simultaneously. It is reasonable to ensure that there is a uniformly positive lower bound for the inter-impulse time. Therefore, under the conditions of Theorem 3.1, we assume that triggering the parameters a_r and b_r satisfy* $\inf\{a_r\} = \tilde{a} > 0$ *and* $\inf\{b_r\} = \tilde{b} > 0$*, respectively. Then, there is no Zeno behavior for system (2.1) under the ETM (3.1). Moreover, the event-triggered impulse sequence t_r satisfies the following:*

$$t_r - t_{r-1} \geq \frac{\tilde{a} \wedge \tilde{b}}{\delta} \wedge \tilde{\sigma}_r, \ \forall r \in \mathbb{Z}_{+.}$$

In fact, selecting larger values for the triggering parameters a_r and b_r may potentially increase the lower bound of triggering intervals, thereby reducing the frequency of triggering instants. In contrast, choosing smaller values for the triggering parameters a_r and b_r may potentially decrease the lower bound of triggering intervals, thus resulting in an increased frequency of triggering instants.

Remark 3.2. *It should be noted that in Theorem 3.1, the ETM (3.1) consists of both an event-triggered sequence and a forced impulse sequence, where the forced impulse sequence is introduced to prevent the scenario where the pre-designed event condition fails to trigger. Therefore, it must be emphasized that the forced impulse sequence plays a crucial role in ETIC to ensure the ISS property. In fact, although the forced impulse sequence is involved in the ETM (3.1), no upper bound restriction is imposed on this sequence, that is, the forced impulse sequence in this paper can be arbitrary, as long as it satisfies* $\inf_{k\in z_+}\{\sigma_r - \sigma_{r-1}\} > 0$.

Theorem 3.2. *Theorem 3.2 guarantees the ISS for system (2.1) with the ETM (3.1) if there exist the function $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and positive constants $\alpha, \beta, \rho, h_r, q_r$, with $\alpha, \beta \in$* $(0, 1]$ and $h_r + q_r < 1$ as following:

(i) $\alpha_1(|z|) \leq V(t, z) \leq \alpha_2(|z|)$;

(ii) $V(g_k(z, u) \leq h_k V(z) + q_k \varsigma(|u|)$; *and*

(iii) *there exists a positive constant ε, for any given integer $k \in \mathbb{Z}_+$, it holds that*

$$\sum_{r=m}^{k} [a_r - d_r + \delta[(1 - \alpha e^{\beta})\sigma_r] + b_m + \rho k \leq \varepsilon,$$

where $d_r = -\ln(h_r + q_r)$.

Proof. From the ETM (3.1), we obtain the following:

$$V(z(t)) \leq \exp(a_1)V(z_0) + \exp(b_1)\varsigma(|u|_{[t_0,t_1]}), \forall t \in [t_0, t_1); \tag{3.7}$$

in addition, from condition (ii), for the first impulse instant t_1, we have the following:

$$V(z(t_1)) \leq h_1 V(z(s_1^-)) + q_1 \varsigma(\left|u(s_1^-)\right|.$$

On the one side, if $V(z(s_1^-)) \leq \varsigma(\left|u(s_1^-)\right|$, then we conclude that

$$V(z(t_1)) \leq \exp(-d_1)\varsigma(\left|u(s_1^-)\right|.$$

On the other side, if $V(z(s_1^-)) > \varsigma(\left|u(s_1^-)\right|$, similar to the approach in case (a) of Theorem 3.1, then the information can be acquired by making use of Eq (3.7) as follows:

$$V(z(t_1)) \leq \exp(-d_1)V(z(s_1^-))$$

$$\leq \exp(-d_1 + \delta(s_1 - t_1 + t_1 - t_0)(\exp(a_1)V(z_0) + \exp(b_1)\varsigma(|u|_{[t_0,t_1]}))$$
$$= \exp(a_1 - d_1 + \delta(s_1 - t_1 + t_1 - t_0))V(z_0) + (a_1 - d_1 + \delta(s_1 - t_1 + t_1 - t_0) + b_1)\varsigma(|u|_{[t_0,t_1]}).$$

Therefore, for the initial impulse instant t_1, we have the following:

$$V(z(t_1)) \leq \exp(a_1 - d_1 + \delta(s_1 - t_1 + t_1 - t_0))V(z_0) + (-d_1 + \delta(s_1 - t_1 + t_1 - t_0) + b_1)\varsigma(|u|_{[t_0,t_1]}) + \exp(-d_1)\varsigma(\left|u(r_1^-)\right|.$$

Repeating the process in a recursive manner, at the $(k-1)$ -th impulse instant t_{k-1} for $k > 2$, the following holds:

$$\begin{aligned}
V(z(t_{k-1})) \leq{}& \exp\left(\sum_{r=1}^{k-1}[a_r - d_r + \delta(s_r - t_r + t_r - t_{r-1})]\right)V(z_0) \\
&+ \left(\sum_{r=2}^{k-1} a_r + \sum_{r=1}^{k-1}[-d_r + \delta(s_r - t_r + t_r - t_{r-1})] + b_1\right)\varsigma(|u|_{[t_0,t_1]}) \\
&+ \left(\sum_{r=3}^{k-1}[a_r + \delta(s_r - t_r + t_r - t_{r-1})] - \sum_{r=1}^{k-1} d_r\right)\varsigma(|u(s_1^-)|) \\
&+ \cdots \\
&+ \left(a_{k-1} + \sum_{r=k-2}^{k-1}[-d_r + \delta(s_r - t_r + t_r - t_{r-1})] + b_{r-2}\right)\varsigma(|u|_{[t_0,t_{k-2}]}) \\
&+ \left(a_{k-1} + \delta(s_{k-1} - t_{k-1} + t_{k-2} - t_{k-2})] - \sum_{r=k-2}^{k-1} d_r\right)\varsigma(|u(s_{k-2}^-)|) \\
&+ \exp\left(-d_{k-1} + \delta(s_{k-1} - t_{k-1} + t_{k-1} - t_{k-2}) + b_{r-1}\right)\varsigma(|u|_{[t_0,t_{k-1}]}) \\
&+ \exp\left(-d_{k-1}\right)\varsigma(|u(s_{k-1})|) \\
\leq{}& \exp\left(\sum_{r=1}^{k-1}(a_r - d_r + \delta[(1-\alpha e^{\beta})\sigma_r] + b_1)\right)V(z_0) \\
&+ 2\left[\exp\left(\sum_{r=1}^{k-1}(a_r - d_r + \delta[(1-\alpha e^{\beta})\sigma_r] + b_1)\right)\varsigma(|u|_{[t_0,t_1]})\right. \\
&+ \exp\left(\sum_{r=2}^{k-1}(a_r - d_r + \delta[(1-\alpha e^{\beta})\sigma_r] + b_1)\right)\varsigma(|u|_{[t_0,t_2]}) \\
&+ \cdots \\
&+ \exp\left(\sum_{r=k-2}^{k-1}(a_r - d_r + \delta[(1-\alpha e^{\beta})\sigma_r] + b_{k-2})\right)\varsigma(|u|_{[t_0,t_2]}) \\
&\left.+ \exp\left(a_{k-1} - d_{k-1} + \delta[(1-\alpha e^{\beta})\sigma_{k-1}] + b_{k-1}\right)\varsigma(|u|_{[t_0,t_{k-1}]})\right].
\end{aligned}$$

After that, one can derive from condition (iii) that

$$V(z(t_{\mathrm{k}-1})) \leq \exp(\varepsilon - \rho(k-1))V(z_0) + 2\exp(\varepsilon)\varsigma(|u|_{[t_0,t_{k-1}]})];$$

this indicates that for all $t \in [t_{k-1}, t_k]$, where $k \in \mathbb{Z}_+$,

$$\begin{aligned} V(z(t) &\leq \exp(a_k)V(z(t_{\mathrm{k}-1})) + \exp(b_k)\varsigma(|u|_{[t_0,t]}) \\ &\leq \exp(\varphi - \rho(k-1))V(z_0) + \phi(|u|_{[t_0,t]}), \end{aligned}$$

where $\varphi = \varepsilon + a$, $\phi(\cdot) = (2\exp(\varphi \vee b))\varsigma(\cdot)$, $a = \sup_{k\in z_+}\{a_k\}$ and $b = \sup_{k\in z_+}\{b_k\}$. Consequently, based on condition (i), we can ultimately conclude that

$$|z(t)| \leq a_1^{-1}(2\exp(\varphi - \rho(k-1))a_2(|z_0|)) + a_1^{-1}(2\phi(|u|_{[t_0,t]})),\ \forall t \in [t_{k-1}, t_k].$$

This demonstrates the ISS of system (2.1) under the ETM (3.1). Thus, the proof is completed.

Remark 3.3. *It is important to note that condition (iii) demonstrates a connection between the divergence rate parameter c, the triggering parameters a_r, b_r and σ_r, the impulse strength d_r, and the delay parameter θ. Prior works have investigated the ISS in general nonlinear systems with impulsive dynamics via the ETIC approach. In their studies, the system's divergence rate parameter c operates without any imposed restrictions. Nevertheless, when considering the impact of time delays in impulses, an excessive divergence of the system can pose certain challenges in characterizing the time delays in impulses. As a result, condition (iii) establishes a connection that links the divergence rate, the ETM, the strength of impulses, and the delays within impulses. However, due to its particular structure, it is evident that this condition is not straightforward to verify in practical applications. Consequently, when applying it to real-world problems, one might need to conduct tests through trial and error methods or utilize some specific techniques for verification.*

Remark 3.4. *According to the condition in Theorem 3.2, condition (iii) can be rewritten as the following when assume that the sequence $\{d_r\}$ is bounded:*

$$a_r - b_r + \delta[(1 - \alpha e^{\beta})\sigma_r] \leq 0,\ \forall r \in \mathbb{Z}_+.$$

Thus, system (2.1) achieves the ISS. Then, assume that parameters a_r, b_r, d_r and σ_r satisfy $\sup\{a_r\} = a > 0$, $\sup\{b_r\} = b > 0$, $\sup\{\sigma_r\} = \sigma > 0$, and $\inf\{d_r\} = d > 0$; condition (iii) can be rewritten as follows:

$$a - d + \delta[(1 - \alpha e^{\beta})\sigma_r] \leq 0.$$

Thus, system (2.1) achieves the ISS.

It is worth noting that the above comment sets up several straightforward size relationships between the parameters and the impulse strength. These relationships are simple to verify in practical applications. However, condition (iii) does not impose any direct limitations on the magnitude. Consequently, we present the following outcome, which provides an increased flexibility in designing the ETM and adjusting the impulse strength.

4. Application

Let us consider a nonlinear control system with an exogenous disturbance input:

$$\dot{z}(t) = \Pi z(t) + \Lambda f(t) + \Gamma u(t),\ t \geq t_0; \tag{4.1}$$

under impulsive control,

$$z(t) = \Omega z(s^-),\ t = t_r,\ s \in \Upsilon_{t_r},$$

where Π, Λ, and $\Gamma \in \mathbb{R}^n$ are given real matrices, $u(t) \in \mathbb{R}^m$ is the locally bounded external disturbance, function $f(\bullet)$ is globally Lipschitz continuous with M as its Lipschitz matrix, $\{t_r\}$ is the triggered impulse sequence, $\alpha, \beta \in (0, 1]$, and the $\Omega \in \mathbb{R}^n$ is the impulse gain matrix. Our goal is to design an impulsed sequence $\{t_r\}$ and an impulse gain M to stabilize (4.1) in the ISS.

Theorem 4.1. *Assuming there exist positive definite matrices* $\aleph \in \mathbb{R}^{n\times n}$ *and* $\mathrm{K} \in \mathbb{R}^{m\times m}$*, the positive diagonal matrix* $\mathrm{P} \in \mathbb{R}^{n\times n}$*, the real matrix* $\Im \in \mathbb{R}^{n\times n}$*, and positive constants* $a, d, \delta, \alpha, \beta, \sigma$*, with* $\alpha, \beta \in (0, 1]$*,* $u \in \mathbb{R}^m$*,*

$$\begin{pmatrix} \Pi^T\aleph + \aleph\Pi + (1-\delta)\aleph + M^T PM & \aleph\Lambda & \aleph\Gamma \\ \bullet & -P & 0 \\ \bullet & \bullet & -\mathrm{K} \end{pmatrix} \leq 0, \tag{4.2}$$

$$\begin{pmatrix} -\exp(-d)\aleph & \Im \\ \bullet & -\aleph \end{pmatrix} \leq 0; \tag{4.3}$$

then, under the IC gain $\Omega = \aleph^{-1}\Im^T$ and the ETM, the system (4.1) is guaranteed to achieve the ISS as follows:

$$\begin{aligned} t_r &= \min\{t_r^*, t_{r-1} + \sigma\}, \\ t_r^* &= \inf\{t \geq t_{r-1} : h(t) \geq 0\}, \end{aligned} \tag{4.4}$$

with the event-triggering function

$$h(t) = z^T(t)\aleph z(t) - \exp(a_r)z^T(t_{r-1})\aleph z(t_{r-1}) - \exp(b_r)\lambda_{\max}(\mathrm{K})\,|u|^2_{[t_0,t]}\,.$$

Proof. Select $V(t) = z^T(t)\aleph z(t)$; using Ito formula, we derive the following:

$$\begin{aligned} D^+V(t) &= 2z^T(t)\aleph(\Pi z(t) + \Lambda f(t) + \Gamma u(t)) \\ &= z^T(t)(\aleph\Pi + \Pi\aleph^T)z^T(t) + 2z^T(t)\aleph\Lambda f(z(t)) + 2z^T(t)\aleph\Gamma u(t). \end{aligned}$$

Then, we can conclude the following:

$$\begin{aligned} 2z^T(t)\aleph\Lambda f(z(t)) &= z^T(t)\aleph\Lambda f(t) + f^T(z(t))\Lambda^T\aleph z(t) \\ &\leq z^T(t)\aleph\Lambda P^{-1}\Lambda^T\aleph z(t) + z^T(t)M^T PMz(t), \\ 2z^T(t)\aleph\Gamma u(t) &= z^T(t)\aleph\Gamma u(t) + u^T(t)\Gamma^T\aleph z(t) \\ &\leq z^T(t)\aleph\Gamma\mathrm{K}^{-1}\Gamma^T\aleph z(t) + u^T(t)\mathrm{K}u(t). \end{aligned}$$

Using (4.2) and (4.3), when $t \neq t_r, z^T(t)\aleph z(t) \geq \lambda_{\max}(\mathrm{K})\,|u|^2_{[t_0,t]}$, we derive

$$D^+V(t) \leq \delta z^T(t)\aleph z(t) = \delta V(z(t));$$

when $t = t_r$, the following is implied:

$$\begin{aligned} V(z(t_r)) &= z^T(t_r)\aleph z(t_r) \\ &\leq z^T(s_r^-)\Omega^T\aleph\Omega z(s_r^-) \\ &\leq \exp(-d)z^T(s_r^-)\aleph z(s_r^-) \\ &= \exp(-d)V(z(s_r^-)). \end{aligned}$$

Consequently, it is straightforward to verify that all the conditions specified in Theorem 3.2 are satisfied, thus ensuring that system (4.1) can achieve the ISS under the ETM (4.4). This concludes the proof.

Remark 4.1. *To the best of our knowledge, the existing literature has scarcely investigated the ISS problem using the ETDIC approach in the existing literature [29]. The authors [30] explored the stabilization of continuous-time dynamical systems for the ISS using ETDIC approaches. However, in their analysis, which predominantly centered on the robustness pertaining to minimal time delays, the potentially beneficial effects that time delays in impulses could exert were not taken into account. In contrast, in this paper, Theorem 3.2 not only formulates effective ISS conditions using the ETDIC approach, but also comprehensively incorporates time delay characteristics in impulses to analyze dynamical systems. Furthermore, it is established that the impulse delays considered in this research can play a favorable role in stabilizing ISS-based impulsive systems.*

5. Examples

Example 5.1. Consider the following underlying system:

$$\begin{aligned} \dot{z}(t) &= 1.5\sin(t)z(t) + \omega(t), \ \ t \neq t_r, \ \ t \geq 0, \\ z(t) &= \exp(-0.2)x(s^-), \ \ t = t_r, \ \ s = s_r \in \Upsilon_{t_r}, \end{aligned} \tag{5.1}$$

where $\omega(t) = \sin(t)$ is a bound exogenous input. We select $V(z(t)) = z^2(t)$, $\alpha = 0.3$, $\beta = 0.2$, $a_r = 0.09$ and $b_r = 0.2$. Next, let is a design an ETDIC strategy to implement the ISS system (5.1). Choose the Lyapunov function $V(z(t)) = z^2(t)$; when $t \neq t_r$ and $V(z(t_1)) \leq hV(z(s_1^-))$ with $h = exp(-0.2)$, the parameters $\delta = 0.06$ and $d = 0.22$ can be computed. First, we study when the forced impulse instant is not present, and the ETM is as follows:

$$t_r^* = \inf\{t \geq t_{r-1} : |z(t)| \geq \exp(0.09)\,|z(t_{r-1})| + \exp(0.2)|\sin(t)|_{[t_0,t]}\}. \tag{5.2}$$

By simulation (Figure 1), this shows that without the forced impulse sequence, the ISS property of system (5.1) under the ETM (5.2) cannot be ensured. Hence, the forced impulse sequence is essential to maintain the ISS property of the system. Consequently, we will introduce a forced impulse sequence with $t_r = t_{r-1} + 1.5$, which satisfies Remark 3.5: $0 < a + 0.06[(1 - 0.3e^{0.2})\sigma] \leq 0.22$. Then, the ETM will be designed as follows:

$$\begin{aligned} t_r &= \min\{t_r^*, t_{r-1} + 1.5\}, \\ t_r^* &= \inf\{t \geq t_{r-1} : |z(t)| \geq \exp(0.09)\,|z(t_{r-1})| + \exp(0.2)|\sin(t)|_{[t_0,t]}\}. \end{aligned} \tag{5.3}$$

According to Theorem 3.1, system (5.1) satisfies the ISS property under the ETM (5.3), and this conclusion is validated by the results presented in Figure 2. Next, to validate the time delay conclusion presented in Remark 3.5, we set $\alpha = 1.1$ and $\beta = 0.8$ such that $0 < a + 0.06[(1 - 1.1e^{0.8})\sigma] \leq 0.22$ is not satisfied. Figure 3 demonstrates that the system fails to maintain the ISS under the simulation results under the ETM (5.3). This observation confirms the correctness of Remark 3.5. Next, when impulse time delays are eliminated ($\alpha = 0$) in system (5.1) under identical conditions, the simulation evidence from Figure 4 indicates a failure to attain the ISS via the ETM (5.3). This implies that time delays in impulses may potentially contribute to the ISS of system (5.1).

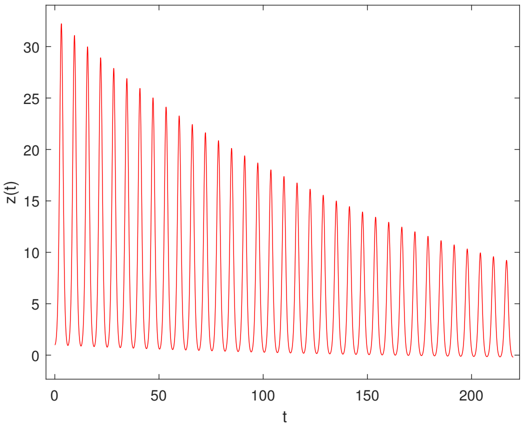

Figure 1. State trajectories of system (5.1) without forced impulse instant.

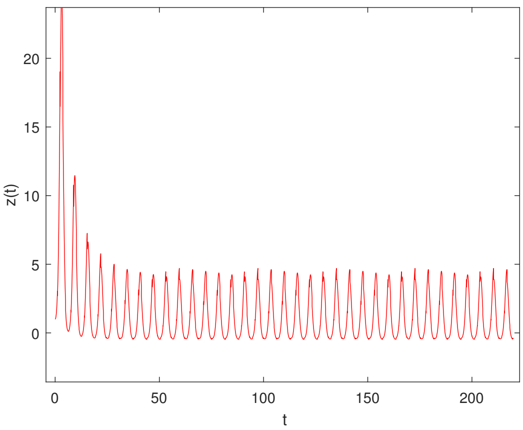

Figure 2. State trajectories of system (5.1) under ETM(5.3).

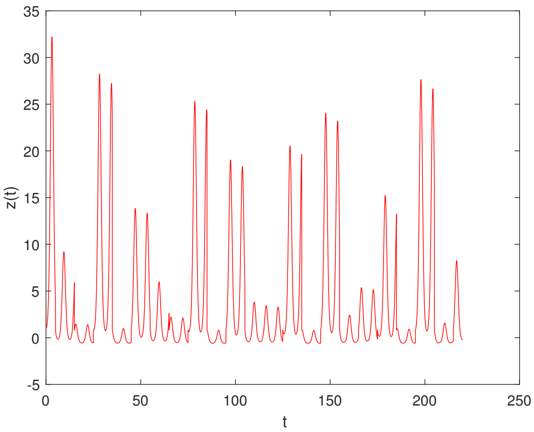

Figure 3. State trajectories of system (5.1) under ETM(5.3) when Remark 3.5 not achiece.

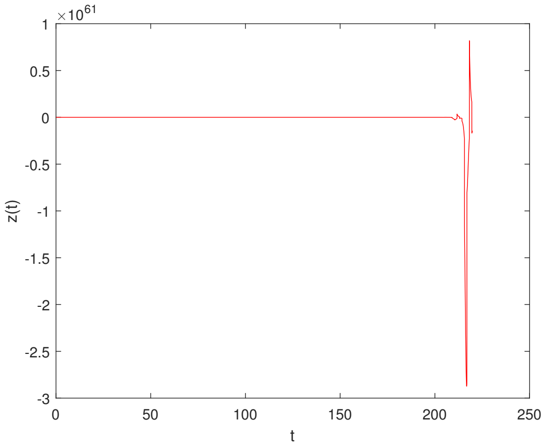

Figure 4. State trajectories of system (5.1) under ETM(5.3) when impulse time delays are eliminated.

Example 5.2. Now, we consider the nonlinear system (4.1) with

$$\Pi = \begin{pmatrix} 0.68 & -1.2 \\ -0.8 & 0.21 \end{pmatrix}, \ \Lambda = \begin{pmatrix} 0.2 & 1.3 \\ 0 & 0.19 \end{pmatrix}, \ \Gamma = \begin{pmatrix} 0.1 & 0.2 \\ 0.15 & 0.25 \end{pmatrix},$$

$f_1(x) = f_2(x) = \tanh 2x$, $u(t) = (\sin t, \cos t)^T$, and $\alpha = 0.15$, $\beta = 0.4$. Figure 5 shows that when the impulse effect is not present, system (4.1) can not reach the ISS. Therefore, we will devise an ETM to enable system (4.1) to satisfy the ISS criterion. We select $a_r = 0.1$, $b_r = 0.1$, $\delta = 0.03$, $d = 0.2$, and $\sigma_r = 3$. Based on Theorem 4.1, system (4.1) can be the ISS under the ETM with $0 < a + 0.03[(1 - 0.15e^{0.4})\sigma] \leq 0.2$. Using Matlab to solve LMI (4.2) and (4.3), then we can design the ETM by the following:

$$\begin{aligned} t_r &= \min\{t_r^*, t_{r-1} + 3\}, \\ t_r^* &= \inf\{t \geq t_{r-1} : z^T(t)\aleph z(t) \geq \exp(0.1) z^T(t_{r-1})\aleph z(t_{r-1}) + 30.8221 \exp(0.1)|u(t)|_{[0,t]}\}, \end{aligned} \tag{5.4}$$

where $\aleph = \begin{pmatrix} 18.8385 & -4.0832 \\ -4.0832 & 23.3975 \end{pmatrix}$, $P = \begin{pmatrix} 8.3587 & 5.8879 \\ 5.8879 & 15.7232 \end{pmatrix}$, $K = \begin{pmatrix} 30.8221 & 0 \\ 0 & 30.8221 \end{pmatrix}$, $\Im = \begin{pmatrix} -3.7677 & -11.4488 \\ -11.4488 & 0.8627 \end{pmatrix}$, and the impulsive control gain $\Omega = \aleph^{-1}\Im^T = \begin{pmatrix} -0.3181 & -0.6233 \\ -0.5448 & -0.0719 \end{pmatrix}$.

According to Theorem 4.1, system (4.1) is the ISS, which is visualized in Figure 6. Based on the given parameters, we introduce a slight modification to the control gain ω, such as $\bar{\Omega} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$. As can be seen from Figure 7, system (4.1) is non-ISS, which shows the feasibility of our proposed event-triggered impulse control method.

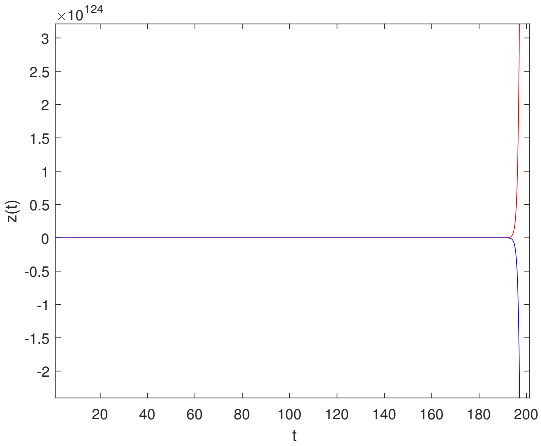

Figure 5. State trajectory of system (4.1) without impulse.

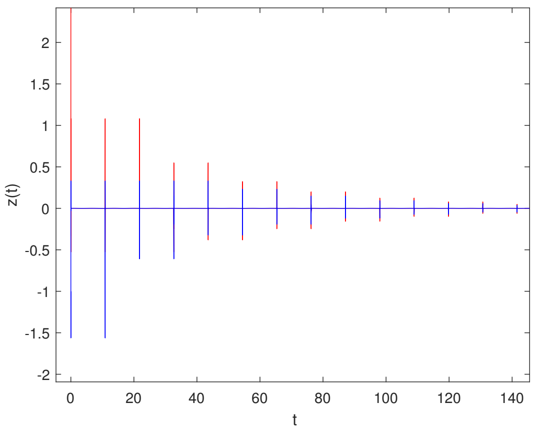

Figure 6. State trajectory of system (4.1) under (5.4).

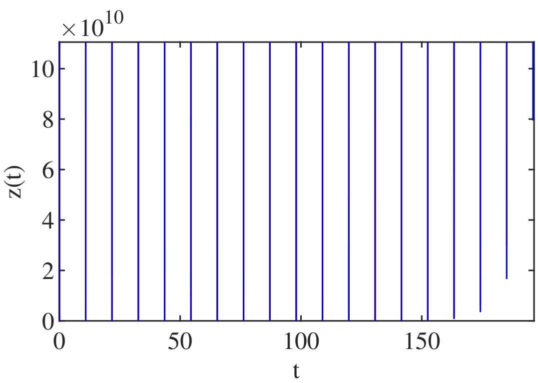

Figure 7. State trajectory of system (4.1) under (5.4) with $\bar{\Omega}$.

6. Conclusions

In this study, the ETDIC strategy was employed to derive sufficient criteria for the ISS properties of nonlinear impulsive systems, where our approach comprehensively accounted for external inputs in the system design and analysis. This paper established explicit relationships among the impulse strength, triggering parameters, and impulse delays, which effectively demonstrated the beneficial effects of time delays in impulsive control systems. Under the ETM, the issue of infinitely fast triggering was effectively mitigated and the ISS characteristics of the impulsive systems were ensured. The developed ETDIC approach was further applied to nonlinear control systems, with a set of ETMs and impulsive control gains designed via LMI techniques. Finally, two numerical examples were provided to demonstrate the effectiveness of the ETDIC strategies. Furthermore, recognizing that the system states may not always be fully measurable, our future research will focus on investigating the ISS properties of nonlinear systems under an output-based ETDIC. Another interesting research would be to apply our proposed impulsive delay design methodology to nonlinear stochastic systems.

Author contributions

Yu Gu: Writing–original draft; Biwen Li: Supervision, Writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

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