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Research article

On the global behavior and the periodicity of the solutions of a k-dimensional system of difference equations

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Abstract: In this work, we provide a detailed study on the behavior of solutions of a k-dimensional system of rational difference equations, extending some existing results in the literature. We use the linearized stability theory to establish conditions for the local stability of the corresponding unique equilibrium point, and to show its global stability, we prove that the equilibrium point is a global attractor. Also, conditions for the existence of periodic solutions are provided. Our obtained results are confirmed by some examples.

Keywords: system of difference equations; equilibrium points; stability; boundedness; periodicity
Mathematics Subject Classification: 39A05, 39A23, 39A30

1. Introduction

Modeling natural phenomena with difference equations and their systems has been inevitable for a long time. This fact motivated researchers to investigate in an intense manner the behavior of the solutions of different models of such equations and systems. First, we note that some contributions focused on finding explicit solutions and then used the obtained formulas to provide detailed analysis of the dynamical behavior of the corresponding equations and systems. Solving in closed form non-linear models of difference equations is not an easy task, and to do this, generally the researchers provide some changes of variables to transform the initial models to some known solvable ones, and then they deduced the solutions of the original models. For instance, the authors [1] solved explicitly the following higher-order system of nonlinear difference equations:

$$x_n = \frac{x_{n-k}y_{n-k-l}}{y_{n-l}\left(a_n + b_n x_{n-k}y_{n-k-l}\right)},\ y_n = \frac{y_{n-k}x_{n-k-l}}{x_{n-l}\left(\alpha_n + \beta_n y_{n-k}x_{n-k-l}\right)},$$

where $n \in \mathbb{N}_0 = 0 \cup \mathbb{N}$, $k, l \in \mathbb{N}$, $(a_n)_{n\in\mathbb{N}_0}, (b_n)_{n\in\mathbb{N}_0}, (\alpha_n)_{n\in\mathbb{N}_0}, (\beta_n)_{n\in\mathbb{N}_0}$ and the initial values x_{-i}, y_{-i}, $i = \overline{1, k+l}$, are real numbers. Gumus et al. [2] gave the formula of the solutions of the following difference equations:

$$x_{n+1} = \frac{a x_n x_{n-k+1}}{b x_{n-k+1} + c x_{n-k}},\ n \in \mathbb{N}_0,$$

where a, $b > 0$, the initial values $x_{-k}, ..., x_0$ are real numbers, and k is a positive integer. For some other contributions on the solvability, we refer to [3–5]. It is interesting to note that to deal with the solvability of some types of difference equations, some authors used the method of Lie symmetry; see the following two references [6, 7]. Second, and for understanding the nature of the studied discrete models, many researchers concentrate their interest on some other essential aspects like attractivity, stability, periodicity, and oscillation. For instance, we can consult the following contributions and the references cited in [8–10].

Solving differential equations (ODE, PDE) numerically by using the finite difference method leads to difference equations. For example, it is not hard to see that using the forward Euler scheme with the step size h to solve numerically the following continuous problem

$$\dot{u}(t) = \alpha u(t)\left(1 - \frac{u(t)}{\beta}\right),\ t \in [0, +\infty),\ \alpha, \beta \in (0, +\infty)$$

known as the growth population model of Verhulst, we get the following nonlinear difference equation

$$u_{n+1} = u_n + \alpha h u_n\left(1 - \frac{u_n}{\beta}\right),\ n \in \mathbb{N}_0,$$

this last equation is known as the logistic difference equation.

The choice of a standard or non standard scheme depends on the problem treated. Recently, different schemes for the ADI compact difference method have been proposed to solve numerically some concrete problems. For instance, we refer to the following interesting papers [11–13].

The present work is a contribution to this subject. More precisely, we provide a detailed study on the global stability and the existence of periodic solutions for the following k-dimensional system of difference equations

$$x^1_{n+1} = a_1 + \frac{x^2_n}{b_1 x^2_n + c_1 x^2_{n-1}},\ x^2_{n+1} = a_2 + \frac{x^3_n}{b_2 x^3_n + c_2 x^3_{n-1}}, ...,$$

$$x^{k-1}_{n+1} = a_{k-1} + \frac{x^k_n}{b_{k-1} x^k_n + c_{k-1} x^k_{n-1}},\ x^k_{n+1} = a_k + \frac{x^1_n}{b_k x^1_n + c_k x^1_{n-1}}, \tag{1.1}$$

where $k \in \mathbb{N}$, $n \in \mathbb{N}_0$, the parameters a_i, b_i, c_i, $i = 1, ..., k$, and the initial values x^i_{-1}, x^i_0, $i = 1, ..., k$ are positive real numbers. Our obtained results are confirmed by some examples.

We note that the author [14] investigated the following two-dimensional system of difference equations defined by

$$x_{n+1} = a_1 + \frac{y_n}{b_1 y_n + c_1 y_{n-1}},\ y_{n+1} = a_2 + \frac{x_n}{b_2 x_n + c_2 x_{n-1}},\ n = 0, 1, ..., \tag{1.2}$$

where the parameters a_i, b_i, c_i, $i = 1, 2$ and the initial values x_{-1}, x_0, y_{-1}, y_0 are positive real numbers. In [14], only conditions for the global stability of the unique equilibrium point $(a_1 + \frac{1}{b_1+c_1}, a_2 + \frac{1}{b_2+c_2})$ were established, but no results on the existence of periodic solutions are provided there. Clearly system (1.2) is a particular case of system (1.1). So, the results obtained here for system (1.1) extend those obtained in [14] in relation to system (1.2). Therefore, to complete the study in [14] on the existence of periodic solutions for system (1.2), it is enough to take $k = 2$ in the corresponding results for system (1.1). Finally, we note that in this study we are mainly inspired by [14] and the two following works [15, 16].

2. Stability of the unique equilibrium point

In this section, we establish conditions for global stability of the unique equilibrium point of system (1.1).

It is not hard to see that for system (1.1), we have only one equilibrium point, and it is given by

$$E = (\overline{x^1}, \overline{x^2}, ..., \overline{x^k}) = \left(a_1 + \frac{1}{b_1 + c_1}, a_2 + \frac{1}{b_2 + c_2}, ..., a_k + \frac{1}{b_k + c_k}\right).$$

Let consider the functions

$$f_1,\ f_2, ..., f_k,\ : (0, +\infty)^2 \to (0, +\infty)$$

defined by

$$f_1(u_2, v_2) = a_1 + \frac{u_2}{b_1 u_2 + c_1 v_2},\ f_2(u_3, v_3) = a_2 + \frac{u_3}{b_2 u_3 + c_2 v_3}, ...,$$

$$f_{k-1}(u_k, v_k) = a_{k-1} + \frac{u_k}{b_{k-1} u_k + c_{k-1} v_k},\ f_k(u_1, v_1) = a_k + \frac{u_1}{b_k u_1 + c_k v_1}.$$

Using the functions $f_i(i = 1, ..., k)$, system (1.1) can be written as

$$\begin{aligned} &x^1_{n+1} = f_1(x^2_n, x^2_{n-1}),\ x^2_{n+1} = f_2(x^3_n, x^3_{n-1}), ..., \\ &x^{k-1}_{n+1} = f_{k-1}(x^k_n, x^k_{n-1}),\ x^k_{n+1} = f_k(x^1_n, x^1_{n-1}), \end{aligned} \tag{2.1}$$

and in the vector form

$$X_{n+1} = F(X_n),\ X_n = (x^1_n, x^1_{n-1}, x^2_n, x^2_{n-1}, ..., x^k_n, x^k_{n-1})^T, \tag{2.2}$$

where the function $F : (0, +\infty)^{2k} \to (0, +\infty)^{2k}$ is defined by

$$F(u_1, v_1, u_2, ..., u_k, v_k) = (f_1(u_2, v_2), u_1, f_2(u_3, v_3), u_2, ..., f_{k-1}(u_k, v_k), u_{k-1}, f_k(u_1, v_1), u_k).$$

As a result of the uniqueness of the equilibrium point E, it follows that system (2.2) also has only one equilibrium point

$$\overline{X} = (\overline{x^1}, \overline{x^1}, \overline{x^2}, \overline{x^2}, ..., \overline{x^k}, \overline{x^k}).$$

The functions $f_i (i = 1, ..., k)$ are C^1 on $(0, +\infty)^2$, and the corresponding linearized system about the equilibrium $\overline{X}$ is

$$V_{n+1} = AV_n,\ V_n = X_n - \overline{X} \in \mathbb{N}_0,$$

where the $2k \times 2k$ matrix A is the Jacobian matrix of F evaluated at $\overline{X}$.

In order to study the stability of the equilibrium point E of system (1.1), we need the following known theorem (linearized stability theorem). See, for example, [17].

Theorem 2.1. *The following statements are true:*

(i) If all the eigenvalues of A lie in the open unit disk $|\lambda| < 1$*, then the equilibrium E of system* (1.1) *is asymptotically stable.*
(ii) If at least one eigenvalue of A has an absolute value greater than one, then the equilibrium E of system (1.1) *is unstable.*

In the following result, we give conditions for which the unique equilibrium point E of system (1.1) is locally asymptotically stable.

Theorem 2.2. *Assume that*

$$2^k \prod_{i=1}^{k} c_i < \prod_{i=1}^{k} (b_i + c_i)(a_i(b_i + c_i) + 1), \tag{2.3}$$

then the equilibrium point $\overline{E}$ *of system (1.1) is locally asymptotically stable.*

Proof. Let $A = (A_{ij})_{i,j=1,...,2k}$, be the Jacobian matrix associated with system (1.1) evaluated at $\overline{E}$. Then it is not hard to see that

$$A_{2i-1\ 2i+2} = -A_{2i-1\ 2i+1} = \frac{c_i(b_{i+1} + c_{i+1})}{(b_i + c_i)(a_{i+1}((b_{i+1} + c_{i+1}) + 1)},\ i = 1, ..., k-1,$$

$$A_{2k-1\ 2} = -A_{2k-1\ 1} = \frac{c_k(b_1 + c_1)}{(b_k + c_k)(a_{i+1}((b_1 + c_1) + 1)},$$

$$A_{2i\ 2i-1} = 1,\ i = 1, ..., k$$

and the remaining terms are zero, and its characteristics polynomial is given by

$$P(\lambda) = \lambda^{2k} - q(\lambda - 1)^k,$$

where

$$q = \prod_{i=1}^{k} \left(\frac{c_i}{(b_i + c_i)(a_i(b_i + c_i) + 1)} \right).$$

Consider the two polynomials

$$P_1(\lambda) = \lambda^{2k},\ P_2(\lambda) = q(\lambda - 1)^k,$$

so, it follows that

$$P(\lambda) = P_1(\lambda) - P_2(\lambda).$$

Now, for every $\lambda \in \mathbb{C} : |\lambda| = 1$, we obtain

$$|P_2(\lambda)| = q|(\lambda - 1)^k| = q|\sum_{j=0}^{k} \binom{k}{j} \lambda^j (-1)^{k-j}|, \binom{k}{j} = \frac{k!}{j!(k-j)!}.$$

So,

$$|P_2(\lambda)| \le q \sum_{j=0}^{k} \binom{k}{j}.$$

Using the following combinatorial formula

$$2^k = \sum_{j=0}^{k} \binom{k}{j},$$

we obtain

$$|P_2(\lambda)| \le 2^k q.$$

So, from the assumption (2.3), we obtain

$$|P_2(\lambda)| < 1 = |P_1(\lambda)|, \ \lambda \in \mathbb{C} : |\lambda| = 1.$$

So, by Rouché's Theorem, it follows that all roots of the characteristic polynomial $P(\lambda)$ lie inside the unit disk. By virtue of Theorem 2.1, the equilibrium point $\overline{E}$ is locally asymptotically stable. □

In the following result, we will show that every solution of system (1.1) is bounded.

Lemma 2.1. *Let $\left\{(x_n^1, x_n^2, ..., x_n^k)\right\}_{n \ge -1}$ be a solution of system (1.1); then for $n = 1, 2, ...,$ we have*

$$a_i \le x_n^i \le a_i + \frac{1}{b_i}, \ i = 1, 2, ..., k.$$

Moreover, if we choose

$$x_{-1}^i, x_0^i \in I_i := \left[a_i, a_i + \frac{1}{b_i}\right], \ i = 1, 2, ..., k,$$

then $I_1 \times I_2 \times ... \times I_k$ will be an invariant set for the solutions of system (1.1).

Proof. Let $\left\{(x_n^1, x_n^2, ..., x_n^k)\right\}_{n \ge -1}$ be a solution of system (1.1). It is clear that due to the fact that the parameters and the initial values of our system are positive, it follows that all $x_n^i (i = 1, ..., k)$ are positive. Form system (1.1), with the convention that $x_n^{k+1} = x_n^1$, we get for $i = 1, ..., k$

$$a_i \le x_{n+1}^i \le a_i + \frac{x_n^{i+1}}{b_i x_n^{i+1}}, \ n = 0, 1, ...,$$

from which we obtain

$$a_i \le x_{n+1}^i \le a_i + \frac{1}{b_i}, \ n = 0, 1, ...,$$

that is

$$a_i \le x_n^i \le a_i + \frac{1}{b_i}, \ n = 1, 2, ... \tag{2.4}$$

Now, if we choose $x^i_{-1}, x^i_0 \in I_i$, $i = 1, ..., k$, it follows from (2.4) that,

$$(x^1_n,\ x^2_n,\ ...,\ x^k_n) \in I_1 \times I_2 \times ... \times I_k,\ n \in \mathbb{N}_{-1},$$

where $\mathbb{N}_{-1} = \{-1, 0\} \cup \mathbb{N}$. □

The following result is on the monotony of the functions $f_i (i = 1, ..., k)$.

Lemma 2.2. *The functions f_i, $i = 1, ..., k$ are increasing in the first variable and decreasing in the second variable.*

Proof. The result follows from the fact that for $i = 1, ..., k$, we have

$$\frac{\partial f_i}{\partial u_{i+1}}(u_{i+1}, v_{i+1}) = \frac{c_i v_{i+1}}{(b_i u_{i+1} + c_i v_{i+1})^2} > 0,$$

$$\frac{\partial f_i}{\partial v_{i+1}}(u_{i+1}, v_{i+1}) = -\frac{c_i u_{i+1}}{(b_i u_{i+1} + c_i v_{i+1})^2} < 0$$

with the convention $u_{k+1} = u_1, v_{k+1} = v_1$. □

In the following theorem, we prove that the equilibrium point E of system (1.1) is a global attractor.

Theorem 2.3. *Assume that*

$$\prod_{i=1}^{k} a_i b_i > 2^k. \tag{2.5}$$

Then, the unique equilibrium point E of system (1.1) is global attractor, that is

$$\lim_{n\to+\infty} x^i_n = a_i + \frac{1}{b_i + c_i},\ i = 1, ..., k.$$

Proof. Let $\left\{(x^1_n, x^2_n, ..., x^k_n)\right\}_{n\geq -1}$ be a solution of system (1.1). From Lemma 2.1, we have the solution is bounded. So, let

$$m_i := \liminf_{n\to+\infty} x^i_n,\ M_i := \limsup_{n\to+\infty} x^i_n,\ i = 1, ..., k.$$

We have

$$\forall \epsilon \in (0, \min_{i=1,...,k}(m_i)), \exists n_i \in \mathbb{N}_0 : \forall n \geq n_i, m_i - \epsilon \leq x^i_n \leq M_i + \epsilon,\ i = 1, ..., k. \tag{2.6}$$

Let

$$n_0 = \max_{i=1,...,k}(n_i),$$

it follows that for all $n \geq n_0$, statements in (2.6) are true. Now, using the monotony of the functions $f_i (i = 1, ..., k)$ (Lemma 2.2), we get for $n \geq n_0 + 1$

$$f_i(m_{i+1} + \epsilon, M_{i+1} - \epsilon) \leq x^i_{n+1} \leq f_i(M_{i+1} - \epsilon, m_{i+1} + \epsilon),\ i = 1, ..., k$$

with the convention that $m_{k+1} = m_1,\ M_{k+1} = M_1$. So,

$$f_i(m_{i+1} + \epsilon, M_{i+1} - \epsilon) \leq m_i \leq M_i \leq f_i(M_{i+1} - \epsilon, m_{i+1} + \epsilon),\ i = 1, ..., k. \tag{2.7}$$

Taking $\epsilon \to 0$ and using the continuity of the functions$f_i (i = 1, ..., k)$, it follows from (2.7) that

$$f_i(m_{i+1}, M_{i+1}) \le m_i \le M_i \le f_i(M_{i+1}, m_{i+1}),\ i = 1, ..., k. \tag{2.8}$$

From (2.8), we obtain

$$M_i - m_i \le f_i(M_{i+1}, m_{i+1}) - f_i(m_{i+1}, M_{i+1}),\ i = 1, ..., k, \tag{2.9}$$

and from (2.9) and the fact that $m_i, M_i \in I_i,\ i = 1, ..., k.$, we obtain

$$\begin{aligned} M_i - m_i &\le \frac{c_i(M_{i+1} + m_{i+1})(M_{i+1} - m_{i+1})}{(b_i M_{i+1} + c_i m_{i+1})(b_i m_{i+1} + c_i M_{i+1})} \\ &\le \frac{2c_i M_{i+1}(M_{i+1} - m_{i+1})}{b_i c_i M_{i+1}^2} \\ &= \frac{2}{b_i M_{i+1}}(M_{i+1} - m_{i+1}). \end{aligned}$$

Using the fact that $M_i \ge a_i,\ i = 1, ..., k$, we obtain

$$M_i - m_i \le \frac{2}{b_i a_{i+1}}(M_{i+1} - m_{i+1}),\ i = 1, ..., k, \tag{2.10}$$

with the convention $a_{k+1} = a_1$. Now, from (2.10), we obtain

$$M_1 - m_1 \le \frac{2^k}{a_1 b_1 a_2 b_2 ... a_k b_k}(M_1 - m_1),$$

that is

$$(1 - \frac{2^k}{a_1 b_1 a_2 b_2 ... a_k b_k})(M_i - m_i) \le 0.$$

Now, by condition (2.5), we obtain $M_1 \le m_1$, so $M_1 = m_1$, and from (2.10) (we take first $i = k$, and then $i = k - 1$,...,$i = 2$), and the fact that $m_i \le M_i,\ i = 2, ...k$, it follows that

$$M_k = m_k,\ M_{k-1} = m_{k-1},\ ...,\ M_2 = m_2.$$

So, we obtain

$$\lim_{n \to +\infty} x_n^i = M_i,\ i = 1, ..., k.$$

By the continuity of the functions$f_i,\ i = 1, ..., k$, we obtain

$$\lim_{n \to +\infty} x_n^i = f_i(M_{i+1}, M_{i+1}) = a_i + \frac{1}{b_i + c_i},\ i = 1, ..., k.$$

□

In the following result, which is a consequence of Theorems 2.2 and 2.3, we summarize conditions that guarantee the global stability of the equilibrium point E of system (1.1).

Theorem 2.4. *Assume that*

$$2^k \prod_{i=1}^{k} c_i < \prod_{i=1}^{k} (b_i + c_i)(a_i(b_i + c_i) + 1), \prod_{i=1}^{k} a_i b_i > 2^k.$$

Then, the unique equilibrium point E of system (1.1) is globally stable.

Example 2.1. *Now, we provide an example in which all conditions for the global stability of the equilibrium point E are satisfied. For this aim, let us choose in system* (1.1), $k = 3$. *So, we get the following system:*

$$x_{n+1} = a_1 + \frac{y_n}{b_1 y_n + c_1 y_{n-1}},\ y_{n+1} = a_2 + \frac{z_n}{b_2 z_n + c_2 z_{n-1}},\ z_{n+1} = a_3 + \frac{z_n}{b_3 z_n + c_3 z_{n-1}}, \tag{2.11}$$

where we have written x_n instead of x_n^1, y_n instead of x_n^2, and z_n instead of x_n^3.
From the above theorem, conditions for the global stability of the equilibrium point

$$E = (a_1 + \frac{1}{b_1 + c_1}, a_2 + \frac{1}{b_2 + c_2}, a_3 + \frac{1}{b_3 + c_3})$$

are given by

$$8c_1c_2c_3 < (b_1 + c_1)(b_2 + c_2)(b_3 + c_3)(a_1(b_1 + c_1) + 1)(a_2(b_2 + c_2) + 1)(a_3(b_3 + c_3) + 1),$$

and

$$a_1a_2a_3b_1b_2b_3 > 8.$$

The above conditions are satisfied for the following choice of the parameters:

$$a_1 = \frac{1}{4},\ a_2 = \frac{1}{8},\ a_3 = \frac{1}{2},\ b_1 = 10,\ b_2 = 15,\ b_3 = 20,\ c_1 = 3,\ c_2 = 5,\ c_3 = 7.$$

For this choice, the equilibrium point is $E = (0.326, 0.175, 0.537)$.

We will check the convergence of the solutions for two sets of initial values.

1. *Consider the following initial values:*

$$x_{-1} = 11,\ x_0 = 2,\ y_{-1} = 200,\ y_0 = 7,\ z_{-1} = 3,\ z_0 = 70, \tag{2.12}$$

then using Maple, we obtain

$$\lim_{n\to+\infty} (x_n, y_n, z_n) = (0.326, 0.175, 0.537)$$

as shown in Figures 1–3.

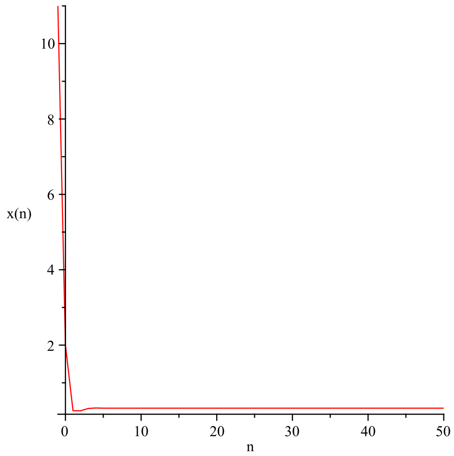

Figure 1. For the initial values in (2.12): $x_n \to 0.326$.

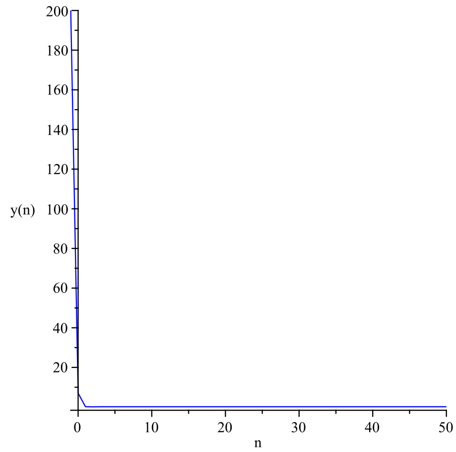

Figure 2. For the initial values in (2.12): $y_n \to 0.175$.

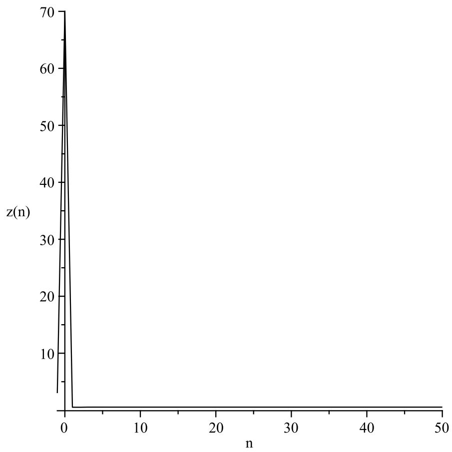

Figure 3. For the initial values in (2.12): $z_n \to 0.537$.

2. *Consider the following initial values*

$$x_{-1} = 500,\ x_0 = 700,\ y_{-1} = 230,\ y_0 = 23,\ z_{-1} = 77,\ z_0 = 624, \tag{2.13}$$

then using Maple, we obtain

$$\lim_{n\to+\infty} (x_n, y_n, z_n) = (0.326, 0.175, 0.537)$$

as shown in Figures 4– 6.

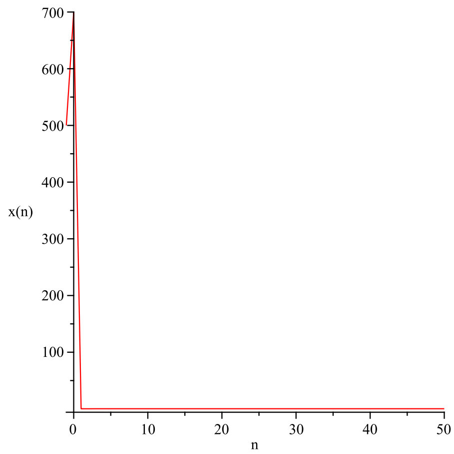

Figure 4. For the initial values in (2.13): $x_n \to 0.326$.

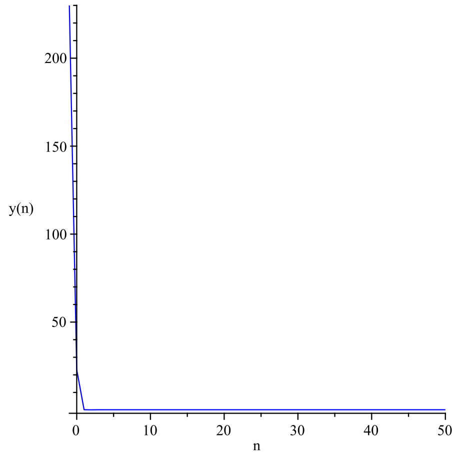

Figure 5. For the initial values in (2.13): $y_n \to 0.175$.

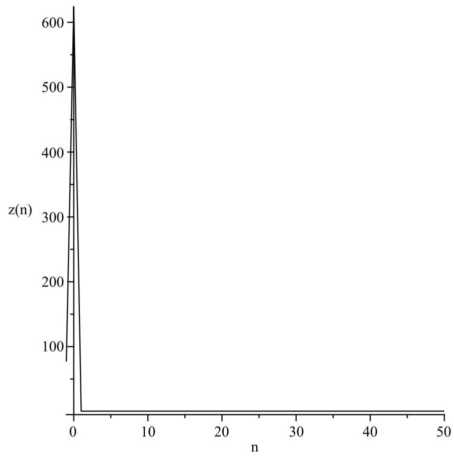

Figure 6. For the initial values in (2.13): $z_n \to 0.537$.

3. Existence of periodic solutions

In this section, we are interested in the existence of prime periodic solutions of period two for system (1.1).

Theorem 3.1. *Assume that* $p_i (i = 1, ..., k)$ *are real numbers such that* $(p_1 - 1)(p_2 - 1)...(p_k - 1) \neq 0$. *System (1.1) has a prime period two solution of the form*

$$x^i_{2n-1} = p_i\alpha_i,\ x^i_{2n} = \alpha_i,\ i = 1, ..., k,\ n = 0, 1, ..., \tag{3.1}$$

if and only if

$$a_i + \frac{1}{b_i + p_{i+1}c_i} = p_i\left(a_i + \frac{p_{i+1}}{p_{i+1}b_i + c_i}\right),\ i = 1, ..., k, \tag{3.2}$$

where

$$\alpha_i = a_i + \frac{p_{i+1}}{p_{i+1}b_i + c_i},\ i = 1, ..., k$$

and with the convention that $p_{k+1} = p_1$.

Proof. 1) Assume that system (1.1) has a prime solution of period two as in (3.1). Then from (1.1), we get

$$p_i\alpha_i = a_i + \frac{\alpha_{i+1}}{b_i\alpha_{i+1} + c_ip_{i+1}\alpha_{i+1}},\ \alpha_i = a_i + \frac{p_{i+1}\alpha_{i+1}}{b_ip_{i+1}\alpha_{i+1} + c_i\alpha_{i+1}},\ i = 1, ..., k. \tag{3.3}$$

From (3.3), we obtain

$$p_i\alpha_i = a_i + \frac{1}{b_i + c_ip_{i+1}},\ \alpha_i = a_i + \frac{p_{i+1}}{b_ip_{i+1} + c_i},\ i = 1, ..., k, \tag{3.4}$$

with the convention that $\alpha_{k+1} = \alpha_1$. Now from (3.4), we get equalities (3.2).

2) Suppose that equalities (3.2) are satisfied, and let

$$x^i_{-1} = p_i\alpha_i,\ x^i_0 = \alpha_i,\ i = 1, ..., k.$$

Then from (1.1) and (3.2), we obtain

$$x^i_1 = a_i + \frac{\alpha_{i+1}}{b_i\alpha_{i+1} + c_ip_{i+1}\alpha_{i+1}} = a_i + \frac{1}{b_i + c_ip_{i+1}} = p_i\left(a_i + \frac{p_{i+1}}{p_{i+1}b_i + c_i}\right) = p_i\alpha_i = x^i_{-1},$$

$$x^i_2 = a_i + \frac{p_{i+1}\alpha_{i+1}}{b_ip_{i+1}\alpha_{i+1} + c_i\alpha_{i+1}} = a_i + \frac{p_{i+1}}{b_ip_{i+1} + c_i} = \alpha_i = x^i_0,\ i = 1, ..., k.$$

By induction we obtain

$$x^i_{2n-1} = p_i\alpha_i,\ x^i_{2n} = \alpha_i,\ i = 1, ..., k,\ n = 0, 1, ...$$

□

Example 3.1. *To illustrate the previous result, we give the following example. Let* $k = 2$ *in* (1.1)*, and writing* x_n *instead of* x^1_n *and* y_n *instead of* x^2_n*, then we obtain the following system:*

$$x_{n+1} = a_1 + \frac{y_n}{b_1y_n + c_1y_{n-1}},\ y_{n+1} = a_2 + \frac{x_n}{b_2x_n + c_2x_{n-1}},\ n = 0, 1, ...,$$

which is exactly system (1.2) *studied in [14], but no results on the existence of periodic solutions are provided. From Theorem 3.1, to get a prime period two solution of the form*

$$x_{2n-1} = p_1\alpha_1,\ x_{2n} = \alpha_1,\ y_{2n-1} = p_2\alpha_2,\ y_{2n} = \alpha_2,$$

we must have

$$a_1 + \frac{1}{b_1 + p_2c_1} = p_1\left(a_1 + \frac{p_2}{p_2b_1 + c_1}\right),\ a_2 + \frac{1}{b_2 + p_1c_2} = p_2\left(a_2 + \frac{p_1}{p_1b_2 + c_2}\right),$$

where

$$\alpha_1 = a_1 + \frac{p_2}{p_2b_1 + c_1},\ \alpha_2 = a_2 + \frac{p_1}{p_1b_2 + c_2}.$$

The above conditions are satisfied for the following choice of the parameters:

$$a_1 = \frac{169}{8686},\ a_2 = \frac{93}{2795},\ p_1 = \frac{1}{7},\ p_2 = 16,\ b_1 = 5,\ c_1 = 6,\ b_2 = \frac{1}{3},\ c_2 = 2.$$

In this case, we get $\alpha_1 = \frac{1785}{8686}$, $\alpha_2 = \frac{288}{8686}$ *and the two periodic solution is*

$$x_{2n-1} = \frac{1}{7}\alpha_1 = \frac{255}{8686},\ x_{2n} = \alpha_1 = \frac{1785}{8686},\ y_{2n-1} = 16\alpha_2 = \frac{4608}{2795},\ y_{2n} = \alpha_2 = \frac{288}{2795},\ n \in \mathbb{N}_0.$$

4. Conclusions

In this study, we have investigated in detail the behavior of the solutions of the k-dimensional system of difference equations given in (1.1). Assumptions on the parameters that guarantee the global stability of the unique equilibrium point E of system (1.1) are established. Also, conditions for the existence of prime two periodic solutions for system (1.1) are shown. Examples of the obtained results are provided. As system (1.2) is a particular case of system (1.1), our obtained results on the stability and the periodicity related to (1.1) extend and complete those for (1.2).

In [14], the author studied the following system of difference equations:

$$x_{n+1} = f(y_n, y_{n-1}),\ y_{n+1} = g(x_n, x_{n-1}),\ n \in \mathbb{N}_0, \tag{4.1}$$

where $f,\ g\ :\ (0,+\infty)^2 \to (0,+\infty)$ are homogeneous functions of degree zero. The system in (4.1) has been extended to the three-dimensional case in [15]. Noting that our system (1.1) can be written as follows

$$x^1_{n+1} = f_1(x^2_n, x^2_{n-1}),\ x^2_{n+1} = f_2(x^3_n, x^3_{n-1}), ..., x^{k-1}_{n+1} = f_{k-1}(x^k_n, x^k_{n-1}),\ x^k_{n+1} = f_1(x^1_n, x^1_{n-1}),\ n \in \mathbb{N}_0, \tag{4.2}$$

where

$$f_i(u, v) = a_i + \frac{u}{b_i u + c_i v},\ i = 1, ..., k.$$

Clearly, $f_i (i = 1, ..., k)$ are particular examples of homogeneous functions of degree zero. So for interested readers, we propose to extend the studies in [14, 15] to a the k-dimensional system (4.2), with $k \geq 4$), and the functions $f_i (i = 1, ..., k)$ are general homogeneous functions of degree zero.

Author contributions

Mouataz Billah Mesmouli: Writing–review and editing; Nouressadat Touafek: Methodology, investigation, software, writing–review and editing, visualization, supervision; Ioan-Lucian Popa: Writing–review and editing, supervision; Abdelkader Moumen: Writing–review and editing, project administration; Taher S. Hassan: Writing–review and editing, project administration. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

There are no conflicts of interest.

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