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Research article

Positive solutions to the discrete boundary value problem involving the singular ϕ-Laplacian

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Abstract: In this paper, we investigated the existence of solutions to the discrete boundary value problem involving the singular ϕ-Laplacian. First, we extended the domain of the singular operator to the entire real line, which leads to an auxiliary problem corresponding to the original one. Then, by using critical point theory combined with the strong maximum principle, we obtained a series of conditions for the existence of one positive solution or multiple positive solutions for the original problem. Finally, three numerical examples were provided to illustrate the effectiveness of our results.

Keywords: singular ϕ-Laplacian; boundary value problem; critical point theory; strong maximum principle; positive solutions
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1. Introduction

Let $\mathbb{R}$ be set of all real number and $\mathbb{Z}$ is a positive integer. For $a, b \in \mathbb{Z}$, define $\mathbb{Z}(a, b) = \{a, a+1, \cdots, b\}$ when $a \leq b$. In this paper, we consider the following discrete Dirichlet problem involving the singular ϕ-Laplacian:

$$\begin{cases} -\nabla\phi\left(\Delta u_k\right) = \lambda f\left(k, u_k\right), & k \in \mathbb{Z}(1, T), \\ u_0 = u_{T+1} = 0, \end{cases} \tag{1.1}$$

where T is a given positive integer and the singular operator $\phi(s) = \frac{s}{\sqrt{1-s^2}}$, $s \in (-1, 1)$. ∇ is the backward difference operator defined by $\nabla u_k = u_k - u_{k-1}$. Δ is the forward difference operator defined by $\Delta u_k = u_{k+1} - u_k$. $f(k, \cdot) \in C(\mathbb{R}, \mathbb{R})$ for each $k \in \mathbb{Z}(1, T)$ and λ is a positive real parameter.

Difference equations are widely applied in various fields, including biology [1, 2], ecology [3–5], computer science [6], and so on. Boundary value problems with the Laplacian operator are widely applied in practical areas such as heat conduction and elasticity, and play a key role in the development of mathematical theories such as critical point theory and nonlinear functional analysis. Researchers have carried out extensive studies on this topic [7–9]. In the study of the boundary value problem for difference equations, researchers have derived many results using fixed-point theory [10, 11], the method of upper and lower solution techniques [12,13], and a topological approach [14]. In 2003, Guo and Yu [15] first used critical point theory to prove the existence of periodic and subharmonic solutions for the following second-order difference equation:

$$\Delta^2 x_{n-1} + f(n, x_n) = 0, \quad n \in \mathbb{Z},$$

where $\Delta^2 x_n = \Delta(\Delta x_n)$. Since then, many researchers have utilized critical point theory to study difference equations and obtained many other meaningful and interesting results, such as periodic solutions [16–18], homoclinic solutions [19–21], heteroclinic solutions [22], and boundary value problems [23–25].

In [26], Zhou and Ling considered the following Dirichlet problem of the second-order nonlinear difference equation:

$$\begin{cases} -\Delta(\phi_c(\Delta u_{k-1})) = \lambda f(k, u_k), & k \in \mathbb{Z}(1, T), \\ u_0 = u_{T+1} = 0, \end{cases} \qquad (D_\lambda^f)$$

where the ϕ_c-Laplacian operator is definded by $\phi_c(s) = s/\sqrt{1+s^2}$. Using variational methods and critical point theorems, the authors obtained some sufficient conditions for the existence of infinitely many positive solutions to problem (D_λ^f). Furthermore, Zhou et al. [27] used critical point theory to study the following discrete Dirichlet problem:

$$\begin{cases} -\Delta(\phi_c(\Delta u_{k-1})) + q_k \phi_c(u_k) = \lambda f(k, u_k), & k \in \mathbb{Z}(1, T), \\ u_0 = u_{T+1} = 0, \end{cases}$$

where $q_k \geq 0$ for all $k \in \mathbb{Z}(1, T)$. The authors obtained sufficient conditions for the existence of at least two positive solutions to this problem.

In [28], Chen et al. considered the following Robin problem with singular ϕ-Laplacian:

$$\begin{cases} \nabla\phi(\Delta u_k) + \lambda\mu_k (u_k)^q = 0, & k \in \mathbb{Z}(1, T), \\ \Delta u_0 = u_{T+1} = 0. \end{cases}$$

By using upper and lower solutions, topological methods, and Szulkin's critical point theory, the authors demonstrated the existence of positive solutions to this problem.

In [29], Qiu used critical point theory to study the following boundary value problem with a singular ϕ-Laplacian:

$$\begin{cases} \nabla\phi(\Delta u_k) + f(k, u_k) = 0, & k \in \mathbb{Z}(1, T), \\ u_0 = \alpha u_1, \quad u_{T+1} = 0. \end{cases}$$

By extending the domian of the singular operator ϕ and applying the variational principle, the author obtained the existence of infinitely many solutions to the problem according to the oscillatory behavior of the nonlinear term f at the zero point.

To best our knowledge, there is little literature on the existence conditions for one or multiple positive solutions to the Dirichlet problem involving a singular ϕ-Laplacian operator. Inspired by the previous conclusions, this paper intends to establish the existence of solutions to the discrete Dirichlet boundary value problem with the singular operator ϕ, based on the behavior of the nonlinear term f at several positive points. In contrast to previous studies, the difficulty of this problem is to guarantee that the solution u obtained from the existence conditions lies within the domain of the ϕ-Laplacian operator, that is, $|\Delta u_k| < 1$, $k \in \mathbb{Z}(0, T)$. Our main tools are variational methods, critical point theory, and the strong maximum principle.

This paper is organized as follows. In Section 2, we provide some relevant definitions, concepts, and lemmas. We extend the domain of the singular ϕ-Laplacian operator to $(-\infty, +\infty)$ and subsequently derive an auxiliary problem corresponding to problem (1.1). Moreover, we establish the variational framework for the auxiliary problem. In Section 3, we use Lemma 1 to prove Theorem 1. To address the issue of the possible trivial solution for the single solution in Theorem 1, we provide the conditions for the existence of a single positive solution and multiple positive solutions in Corollaries 1 and 2, respectively. Next, we prove Lemma 6 using Lemma 2. Then, by defining a new function $\tilde{f}$ and applying Lemma 6, we establish in Theorem 2 the existence conditions for solutions to problem (1.1), which rely on the properties of the nonlinear term f at several positive points. Furthermore, by the strong maximum principle, we obtain a positive solution for problem (1.1) in Corollary 3. In Section 4, three concrete examples are given to illustrate our results. In Section 5, we share the main conclusions of the paper.

2. Preliminaries

For convenience, we first introduce two crucial lemmas that will be used to investigate problem (1.1) in this paper.

Lemma 1. *[30, Lemma 2.1] Let X denote a real finite-dimensional Banach space and $I_\lambda : X \to \mathbb{R}$ be a functional satisfying the following structure hypothesis:*
(A_1) $I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$ for $u \in X$, where $\lambda > 0$, $\Phi, \Psi : X \to \mathbb{R}$ are two continuous functionals of class C^1 on X where Φ is coercive, which means $\lim_{\|u\|\to+\infty} \Phi(u) = +\infty$.
(A_2) Φ is convex and $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$.
(A_3) If x_1, x_2 are local minima for the functional $I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$ such that $\Psi(x_1) \geq 0$ and $\Psi(x_2) \geq 0$, then

$$\inf_{t\in[0,1]} \Psi(tx_1 + (1-t)x_2) \geq 0.$$

Further, assume that there are two positive constants ρ_1, ρ_2 and $\overline{u} \in X$ with

$$\rho_1 < \Phi(\overline{u}) < \frac{\rho_2}{2},$$

such that

$$(i)\ \frac{\sup\limits_{u\in\Phi^{-1}(-\infty,\rho_1)} \Psi(u)}{\rho_1} < \frac{1}{2}\frac{\Psi(\overline{u})}{\Phi(\overline{u})}, \quad (ii)\ \frac{\sup\limits_{u\in\Phi^{-1}(-\infty,\rho_2)} \Psi(u)}{\rho_2} < \frac{1}{4}\frac{\Psi(\overline{u})}{\Phi(\overline{u})}.$$

Then, for $\lambda \in \left(\frac{2\Phi(\overline{u})}{\Psi(\overline{u})}, \min\left\{\frac{\rho_1}{\sup_{u\in\Phi^{-1}(-\infty,\rho_1)}\Psi(u)}, \frac{\rho_2/2}{\sup_{u\in\Phi^{-1}(-\infty,\rho_2)}\Psi(u)}\right\}\right)$, *the functional* I_λ *admits at least three distint critical points* u_1, u_2, u_3 *such that* $u_1 \in \Phi^{-1}(-\infty, \rho_1)$, $u_2 \in \Phi^{-1}(\rho_1, \rho_2/2)$, *and* $u_3 \in \Phi^{-1}(-\infty, \rho_2)$.

Let

$$\varphi(r) = \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\left(\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u)\right) - \Psi(u)}{r - \Phi(u)}.$$

Now, we introduce the second lemma.

Lemma 2. *[31, Theorem 5.2] Let* X *be a real Banach space and let* $\Phi, \Psi : X \to \mathbb{R}$ *be two continuously* $\widehat{Ga}$*teaux differentiable functionals with* Φ *bounded from below. Fix* $r > \inf_X \Phi$ *and assume that for each* $\lambda \in \left(0, \frac{1}{\varphi(r)}\right)$, *the function* $I_\lambda = \Phi - \lambda\Psi$ *satisfies the* $(PS)^{[r]}$*-condition.*

Then for each $\lambda \in \left(0, \frac{1}{\varphi(r)}\right)$, *there is* $u_{0,\lambda} \in \Phi^{-1}(-\infty, r)$ *such that* $I_\lambda(u_{0,\lambda}) \leq I_\lambda(u)$ *for all* $u \in \Phi^{-1}(-\infty, r)$ *and* $I_\lambda'(u_{0,\lambda}) = 0$.

The Palais-Smale condition ((PS)-condition) is used in Lemma 2, so we recall the definition of the (PS)-condition. Assume that $(X, \|\cdot\|)$ is a real Banach space. X^* is the space of all bounded linear functionals from X to $\mathbb{R}$ and the norm defined on it is given by:

$$\|f\|_{X^*} = \sup_{x\in X, \|x\|\leq 1} |f(x)|, \quad f \in X^*.$$

Let $I \in C^1(X, \mathbb{R})$, and we say I satisfies the (PS)-condition if any sequence $\{u_n\}$ is such that
(α) $\{I(u_n)\}$ is bounded,
(β) $\lim_{n\to+\infty} \left\|I'(u_n)\right\|_{X^*} = 0$ has a convergent subsequence.

Fix $r \in (-\infty, +\infty)$, and we say that I satisfies the $(PS)^{[r]}$-condition if any sequence (u_n) is such that (α), (β), and
(γ) $\Phi(u_n) < r, \forall n \in \mathbb{N}$, has a convergent subsequence.

Remark 1. *If* I *satisfies the* (PS)*-condition, then it satisfies the* $(PS)^{[r]}$*-condition.*

Next, we establish the variational framework associated with problem (1.1). Let $S = \{u : \mathbb{Z}(0, T+1) \to \mathbb{R} : u_0 = u_{T+1} = 0\}$ be a T-dimensional Banach space with the norm as follows:

$$\|u\| = \left(\sum_{k=0}^{T} |\Delta u_k|^2\right)^{1/2}.$$

We note that the singular operator $\phi(s) = s/\sqrt{1-s^2}$ in problem (1.1) is defined on the interval $(-1, 1)$. In order to use Lemmas 1 and 2, it is necessary to extend the domain of the singular operator ϕ to $(-\infty, +\infty)$. Let a be a constant such that $0 < a < 1$. Take

$$h(s) = \begin{cases} \dfrac{s}{\sqrt{1-s^2}}, & |s| \leq a, \\ \dfrac{1}{\sqrt{1-a^2}} s, & |s| > a. \end{cases}$$

Then, h is a continuous and monotonically increasing function in $(-\infty, +\infty)$, and the primary function $H(s) = \int_0^s h(\tau)d\tau$ is given by

$$H(s) = \begin{cases} 1 - \sqrt{1 - s^2}, & |s| \le a, \\ \dfrac{1}{2\sqrt{1 - a^2}} s^2 + a^*, & |s| > a, \end{cases}$$

where $a^* = 1 - \frac{2-a^2}{2\sqrt{1-a^2}}$ and we find that $a^* \le 0$. It is evident that H is convex.

We define

$$\Phi(u) = \sum_{k=0}^{T} H(\Delta u_k), \quad \Psi(u) = \sum_{k=1}^{T} F(k, u_k),$$

for each $u \in S$, where $F(k, \xi) = \int_0^{\xi} f(k, \tau)d\tau$ for every $k \in \mathbb{Z}(1, T)$.

Remark 2. *Clearly, $H(s) \ge (1/2)s^2$ for all $s \in \mathbb{R}$. Thus, for each $u \in S$, we obtain $\sum_{k=0}^{T} H(\Delta u_k) \ge \frac{1}{2}\sum_{k=0}^{T} |\Delta u_k|^2$. That is $\Phi(u) \ge \frac{1}{2}\|u\|^2$.*

Set

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u),$$

for $u \in S$. Owing to $\Phi, \Psi \in C^1(S, \mathbb{R})$, I_λ is also a $C^1(S, \mathbb{R})$ functional. Using the boundary condition, one has

$$\begin{aligned} I'_\lambda(u)(v) &= \lim_{t\to 0} \frac{I_\lambda(u + tv) - I_\lambda(u)}{t} \\ &= \lim_{t\to 0}\left[\frac{\Phi(u + tv) - \Phi(u)}{t} - \lambda\frac{\Psi(u + tv) - \Psi(u)}{t}\right] \\ &= \sum_{k=0}^{T} h(\Delta u_k)\Delta v_k - \lambda\sum_{k=1}^{T} f(k, u_k)v_k \\ &= \sum_{k=1}^{T} h(\Delta u_{k-1})v_k - \sum_{k=1}^{T} h(\Delta u_k)v_k - \lambda\sum_{k=1}^{T} f(k, u_k)v_k \\ &= -\sum_{k=1}^{T}\left[\nabla h(\Delta u_k) + \lambda\sum_{k=1}^{T} f(k, u_k)\right]v_k, \end{aligned}$$

for all $u, v \in S$.

Therefore, u is a critical point of I_λ in S if and only if u is a solution to the following boundary value problem:

$$\begin{cases} -\nabla h(\Delta u_k) = \lambda f(k, u_k), & k \in \mathbb{Z}(1, T), \\ u_0 = u_{T+1} = 0. \end{cases} \tag{2.1}$$

Remark 3. *If u is a solution of problem (2.1) in S satisfying $|\Delta u_k| \le a$ for $k \in \mathbb{Z}(0, T)$, then u is also a solution of problem (1.1).*

Remark 4. *If u is a solution of problem (2.1) in S such that $\Phi(u) \le 1 - \sqrt{1-a^2}$, then u satisfies problem (1.1). In fact, since $\Phi(u) = \sum_{k=0}^{T} H(\Delta u_k) \le 1 - \sqrt{1-a^2}$, it follows that $H(\Delta u_k) \le 1 - \sqrt{1-a^2}$ for $k \in \mathbb{Z}(0,T)$. This means $|\Delta u_k| \le a$ for $k \in \mathbb{Z}(0,T)$.*

We consider two different norms in S, given by:

$$\|u\|_\infty = \max\{|u_k| : k \in \mathbb{Z}(1,T)\},$$

$$\|u\|_2 = \left(\sum_{k=1}^{T} |u_k|^2\right)^{\frac{1}{2}}.$$

According to the inequality in [32, Lemma 2.2]:

$$\max_{k\in\mathbb{Z}(1,T)} \{|u_k|\} \le \frac{(T+1)^{(p-1)/p}}{2} \|u\|,$$

where $u \in S$ and $p > 1$. Thus, we have:

$$\|u\|_\infty \le \frac{\sqrt{T+1}}{2} \|u\|, \forall u \in S. \tag{2.2}$$

Lemma 3. *[33, (2.2)] For any $u \in S$, the following relation holds:*

$$\sqrt{\lambda_1}\|u\|_2 \le \|u\| \le \sqrt{\lambda_T}\|u\|_2,$$

where $\lambda_1 = 4\sin^2\frac{\pi}{2(T+1)}$ and $\lambda_T = 4\sin^2\frac{T\pi}{2(T+1)}$.

In order to obtain positive solutions of problem (2.1), we need the following two lemmas. Similar to in [29, Theorem 6], we can establish the following strong maximum principle.

Lemma 4. *Assume $u \in S$ such that either*

$$u_k > 0 \quad or \quad -\nabla(h(\Delta u_k)) \ge 0, \tag{2.3}$$

for all $k \in \mathbb{Z}(1,T)$. Then, either $u_k > 0$ for all $k \in \mathbb{Z}(1,T)$ or $u \equiv 0$.

Proof. There exist $\tau \in Z(1,T)$ such that

$$u_\tau = \min\{u_k : k \in Z(1,T)\}.$$

If $u_\tau > 0$, then the lemma holds.

If $u_\tau \le 0$, then $u_\tau = \min\{u_k : k \in Z(0,T+1)\}$. Because $\Delta u_{\tau-1} = u_\tau - u_{\tau-1} \le 0$, $\Delta u_\tau = u_{\tau+1} - u_\tau \ge 0$, and the functional $h(s)$ is monotonically increasing with $h(0) = 0$, we have

$$h(\Delta u_\tau) \ge 0 \ge h(\Delta u_{\tau-1}).$$

On the other hand, (2.3) implies

$$h(\Delta u_\tau) \le h(\Delta u_{\tau-1}).$$

Therefore, we can obtain $h(\Delta u_\tau) = 0 = h(\Delta u_{\tau-1})$. Additionally, it follows that $u_{\tau-1} = u_\tau = u_{\tau+1}$. If $\tau + 1 = T + 1$, we have $u_\tau = u_{\tau+1} = 0$. Otherwise, replacing τ by $\tau + 1$, we get $u_{\tau+2} = u_{\tau+1}$. Repeat this process and we have $u_\tau = u_{\tau+1} = u_{\tau+2} = \cdots = u_{T+1} = 0$. Similarly, we have $u_\tau = u_{\tau-1} = u_{\tau-2} = \cdots = u_0 = 0$. Thus $u \equiv 0$ and the lemma holds. □

Let

$$F^{+}(k,\xi)=\int_{0}^{\xi} f(k,t^{+})dt, \quad (k,\xi)\in\mathbb{Z}(1,T)\times\mathbb{R},$$

where $t^{+}=\max\{0,t\}$. Now we define

$$I_{\lambda}^{+}(u)=\Phi(u)-\lambda\Psi^{+}(u),$$

where $\Psi^{+}(u)=\sum_{k=1}^{T}F^{+}(k,u_k)$ and Φ is defined as before.

Similarly, critical points of $I_{\lambda}^{+}(u)$ are solutions of the following problem:

$$\begin{cases}-\nabla h(\Delta u_k)=\lambda f(k,u_k^{+}), & k\in\mathbb{Z}(1,T),\\ u_0=u_{T+1}=0.\end{cases} \tag{2.4}$$

The following lemma comes from [34, Lemma 3].

Lemma 5. *If $f(k,0)\geq 0$ for each $k\in\mathbb{Z}(1,T)$, then all critical points of I_{λ}^{+} are nonnegative solutions of problem (2.1).*

3. Main results

Let

$$F_x=\sum_{k=1}^{T}F(k,x), \quad \forall x\in\mathbb{R}.$$

Theorem 1. *Assume that for each $k\in\mathbb{Z}(1,T)$, $f(k,t)\geq 0$ when $t\geq 0$. Moreover, there exist three positive constants c_1, c_2, and d $(d\leq a)$ satisfying*
(H_1) $c_1^2\leq\frac{T+1}{2}\left(1-\sqrt{1-a^2}\right)$,
(H_2) $c_1^2<(T+1)\left(1-\sqrt{1-d^2}\right)<\frac{c_2^2}{2}$,
(H_3) $\max\left\{\frac{(T+1)F_{c_1}}{c_1^2},\frac{2(T+1)F_{c_2}}{c_2^2}\right\}<\frac{F_d}{2\left(1-\sqrt{1-d^2}\right)}$.

Then, for each $\lambda\in\Lambda_1:=\left(\frac{4\left(1-\sqrt{1-d^2}\right)}{F_d},\min\left\{\frac{2c_1^2}{(T+1)F_{c_1}},\frac{c_2^2}{(T+1)F_{c_2}}\right\}\right)$, problem (1.1) has at least one solution.

Proof. We use Lemma 1 to prove this conclusion. First, we consider problem (2.4). According to the definitions of S, Φ, Ψ^{+}, and I_{λ}^{+} in Section 2, $\Phi(u)$, $\Psi^{+}(u)$ are two continuously differentiable functions, and one has $\inf_{u\in S}\Phi(u)=\Phi(0)=\Psi^{+}(0)=0$.

Due to the fact that H is convex, we have

$$\begin{aligned}\Phi(tu+(1-t)v)&=\sum_{k=0}^{T}H(t\Delta u_k+(1-t)\Delta v_k)\\ &\leq\sum_{k=0}^{T}\left[tH(\Delta u_k)+(1-t)H(\Delta v_k)\right]\\ &=t\Phi(u)+(1-t)\Phi(v),\end{aligned}$$

for each $t\in[0,1]$, $u,v\in S$. Therefore, Φ is convex.

By Remark 2, we have $\Phi(u) \geq \frac{1}{2}\|u\|^2$. When $\|u\| \to +\infty$, it follows that $\Phi(u) \to +\infty$. Therefore, Φ is coercive.

If x_1, x_2 are local minima for the function I_λ^+ such that $\Psi^+(x_1) \geq 0$, $\Psi^+(x_2) \geq 0$, then by $f(k,t) \geq 0$ for $t \geq 0$ and the strong maximum principle, we conclude that $x_1 \geq 0$, $x_2 \geq 0$. So for any $t \in [0,1]$, we have $tx_1 + (1-t)x_2 \geq 0$. Furthermore, we obtain

$$\Psi^+(tx_1 + (1-t)x_2) \geq 0,$$

and (A_3) is comfirmed. At this point, we have proved (A_1)–(A_3) in Lemma 1.

Let $\rho_i = \frac{2c_i^2}{T+1}$, $i = 1,2$, and $\overline{u} \in S$ be given by

$$\overline{u}_k = \begin{cases} d, & k \in \mathbb{Z}(1,T), \\ 0, & k = 0 \text{ } or \text{ } k = T+1. \end{cases}$$

We obtain

$$\Phi(\overline{u}) = 2\left(1 - \sqrt{1-d^2}\right), \quad \Psi^+(\overline{u}) = F_d.$$

According to (H_2), we acquire

$$\rho_1 < \Phi(\overline{u}) < \frac{\rho_2}{2}.$$

If $\Phi(u) \leq \rho_i$, by Remark 2 and (2.2), we get $\|u\| \leq \sqrt{2\rho_i}$, which further implies

$$\|u\|_\infty \leq \frac{\sqrt{T+1}}{2}\|u\| \leq \sqrt{\frac{T+1}{2}\rho_i} = c_i, \quad (i = 1,2),$$

for $u \in S$. Then, we have

$$\begin{aligned} \frac{\sup\limits_{u \in \Phi^{-1}(-\infty,\rho_i)} \Psi^+(u)}{\rho_i} &\leq \frac{\sup\limits_{\|u\| \leq \sqrt{2\rho_i}} \Psi^+(u)}{\rho_i} \\ &\leq \frac{\sum\limits_{k=1}^{T} \max\limits_{|\xi| \leq c_i} F^+(k,\xi)}{\rho_i} \\ &= \frac{(T+1)F_{c_i}}{2c_i^2}, \qquad (i = 1,2). \end{aligned}$$

Hence, from (H_3), we have verified assumptions (i) and (ii) of Lemma 1.

All the assumptions used in Lemma 1 have been proven. Therefore, for $\lambda \in \Lambda_1$, problem (2.4) has at least three solutions u_1, u_2, and u_3 satisfying $u_1 \in \Phi^{-1}\left(-\infty, \frac{2c_1^2}{T+1}\right)$, $u_2 \in \Phi^{-1}\left(\frac{2c_1^2}{T+1}, \frac{c_2^2}{T+1}\right)$, $u_3 \in \Phi^{-1}\left(-\infty, \frac{2c_2^2}{T+1}\right)$, and u_1, u_2, u_3 are nonnegative. By Lemma 5, they are also solutions to problem (2.1).

Using (H_1), we have

$$\Phi(u_1) < \frac{2c_1^2}{T+1} \leq 1 - \sqrt{1-a^2}.$$

By Remark 4, it follows that u_1 is a solution to problem (1.1). Thus, the proof of Theorem 1 is complete. □

Theorem 1 provides the existence conditions for solutions to problem (1.1). However, it has a limitation in that it only establishes conditions for the existence of a single solution. If $f(k,0) = 0$ for $\forall k \in \mathbb{Z}(1,T)$, then it is clear that $u \equiv 0$ is a solution to problem (1.1). At this point, the existence of a single solution determined by Theorem 1 is meaningless.

To resolve this issue, we improve Theorem 1 and obtain Corollaries 1 and 2, ensuring that the obtained solution is nonzero or establishing the existence of multiple solutions.

Corollary 1. *Assume that there exists a $k_0 \in \mathbb{Z}(1,T)$ such that $f(k_0,0) > 0$ and all the conditions in Theorem 1 are satisfied. Then, for each $\lambda \in \Lambda_1$, problem (1.1) has at least one positive solution.*

Corollary 2. *Replace condition (H_1) with*
(S) $c_2^2 \leq \frac{T+1}{2}\left(1-\sqrt{1-a^2}\right)$,
and the other conditions of Theorem 1 remain. Then for each $\lambda \in \Lambda_1$, problem (1.1) has at least three nonnegative solutions.

Proof. According to the proof in Theorem 1, u_1, u_2, and u_3 are solutions to problem (2.1). From (S) and (H_2), we have

$$\frac{2c_1^2}{T+1} < \frac{c_2^2}{T+1} < \frac{2c_2^2}{T+1} \leq \left(1-\sqrt{1-a^2}\right).$$

By appling Remark 4, we can conclude that u_1, u_2, and u_3 are solutions to problem (1.1). Thus, the corollary holds. □

Before introducing the next theorem, we first present the following lemma.

Lemma 6. *Assume that there exists a $k_0 \in \mathbb{Z}(1,T)$, such that $f(k_0,0) \neq 0$ and there exist two positive constants c and d $(d \leq a)$ satisfying*
(H_4) $(T+1)\left(1-\sqrt{1-d^2}\right) < c^2$,
(H_5) $B := \min_{k\in\mathbb{Z}(1,T)} \liminf_{|\xi|\to+\infty} \frac{F(k,\xi)}{\xi^2}$,
(H_6)

$$0 \leq A := \frac{T+1}{2}\frac{\sum_{k=1}^{T}\max_{|t|\leq c} F(k,t) - F_d}{c^2-(T+1)\left(1-\sqrt{1-d^2}\right)} < \frac{2\sqrt{1-a^2}}{\lambda_T}B.$$

Then, for each $\lambda \in \Lambda_2 := \left(\frac{\lambda_T}{2\sqrt{1-a^2}B}, \frac{1}{A}\right)$, problem (2.1) admits at least one nontrivial solution.

Proof. Let S, Φ, Ψ, and I_λ be defined as in Section 2. Now our goal is to use Lemma 2 to prove our conclusion. Clearly, $\Phi(u)$, $\Psi(u)$ are two continuously Gâteaux differentiable functions and Φ is bounded from below.

Let $r = \frac{2c^2}{T+1}$. Similarly, if $\Phi(u) \leq r$, we obtain $\|u\| \leq \sqrt{2r}$ and

$$\|u\|_\infty \leq \frac{\sqrt{T+1}}{2}\|u\| \leq \frac{\sqrt{T+1}}{2}\sqrt{2r} = c,$$

for $u \in S$. Define $\overline{u} \in S$ as given earlier. Then, we get

$$\varphi(r) \leq \inf_{\Phi(u)<r} \frac{\left(\sup_{\|u\|\leq\sqrt{2r}} \Psi(u)\right) - \Psi(u)}{r-\Phi(u)}$$

$$
\leq \inf_{\Phi(u)<r} \frac{\sum\limits_{k=1}^{T} \max\limits_{|t|\leq c} F(k,t) - \Psi(u)}{r - \Phi(u)}
$$
$$
\leq \frac{T+1}{2} \frac{\sum\limits_{k=1}^{T} \max\limits_{|t|\leq c} F(k,t) - F_d}{c^2 - (T+1)\left(1 - \sqrt{1-d^2}\right)}.
$$

Now, we demonstrate that I_λ satisfies the $(PS)^{[r]}$-condition.

First, assume that $B < +\infty$. Due to $\lambda > \frac{\lambda_T}{2\sqrt{1-a^2}B}$, when we fix $\lambda \in \Lambda_2$, it is evident that there exists a positive constant $\varepsilon\,(\varepsilon < B)$ such that

$$
\lambda > \frac{\lambda_T}{2\sqrt{1-a^2}\,(B-\varepsilon)}. \tag{3.1}
$$

According to (H_5), we can deduce that

$$
\liminf_{|\xi|\to+\infty} \frac{F(k,\xi)}{\xi^2} \geq B > B - \varepsilon, \quad k \in \mathbb{Z}(1,T).
$$

Consequently, there is a positive constant h_1 such that

$$
F(k,\xi) \geq (B-\varepsilon)\,\xi^2 - h_1,
$$

for each $(k,\xi) \in \mathbb{Z}(1,T) \times \mathbb{R}$. Then, from Lemma 3, we have

$$
\begin{aligned}
\lambda \sum_{k=1}^{T} F(k,u_k) &\geq \lambda \sum_{k=1}^{T} \left[(B-\varepsilon)\,{u_k}^2 - h_1\right] \\
&\geq \lambda (B-\varepsilon)\,\|u\|_2^2 - \lambda T h_1 \\
&\geq \frac{\lambda (B-\varepsilon)}{\lambda_T} \|u\|^2 - \lambda T h_1,
\end{aligned}
$$

for $u \in S$. On the other hand, we can observe that

$$
\begin{aligned}
\Phi(u) &= \sum_{k=0}^{T} H(\Delta u_k) \\
&= \sum_{k=0,|\Delta u_k|\leq a}^{T} \left(1 - \sqrt{1-\Delta u_k^2}\right) + \sum_{k=0,|\Delta u_k|>a}^{T} \left(\frac{\Delta u_k^2}{2\sqrt{1-a^2}} + a^*\right) \\
&\leq \sum_{k=0,|\Delta u_k|\leq a}^{T} 1 + \sum_{k=0,|\Delta u_k|>a}^{T} \frac{\Delta u_k^2}{2\sqrt{1-a^2}} \\
&\leq \frac{1}{2\sqrt{1-a^2}} \|u\|^2 + T + 1,
\end{aligned}
$$

for $u \in S$. Thus, we can conclude that

$$
I_\lambda(u) = \Phi(u) - \lambda \Psi(u)
$$

$$\leq \left(\frac{1}{2\sqrt{1-a^2}} - \frac{\lambda (B-\varepsilon)}{\lambda_T}\right)\|u\|^2 + T + 1 + \lambda T h_1,$$

for $u \in S$. From (3.1), it is true that $\left(\frac{1}{2\sqrt{1-a^2}} - \frac{\lambda(B-\varepsilon)}{\lambda_T}\right) < 0$. When $\|u\| \to +\infty$, we have $I_\lambda \to -\infty$, which implies $-I_\lambda$ is coercive. This shows that the functional I_λ satisfies the (PS)-condition. By Remark 1, I_λ satisfies the $(PS)^{[r]}$-condition.

Next, assume that $B = +\infty$. Fix M and let

$$M > \frac{\lambda_T}{2\lambda\sqrt{1-a^2}}. \tag{3.2}$$

From (H_5), it follows that

$$\liminf_{|\xi|\to+\infty} \frac{F(k,\xi)}{\xi^2} > M, \quad k \in \mathbb{Z}(1,T).$$

Then, there exists a positive constant h_2 such that

$$\sum_{k=1}^{T} \lambda F(k,u_k) \geq \frac{\lambda M}{\lambda_T}\|u\|^2 - \lambda T h_2, \quad u \in S.$$

Therefore, we can obtain

$$\begin{aligned} I_\lambda(u) &= \Phi(u) - \lambda\Psi(u) \\ &\leq \left(\frac{1}{2\sqrt{1-a^2}} - \frac{\lambda M}{\lambda_T}\right)\|u\|^2 + T + 1 + \lambda T h_2, \end{aligned}$$

for $u \in S$. Using (3.2), $\left(\frac{1}{2\sqrt{1-a^2}} - \frac{\lambda M}{\lambda_T}\right) < 0$. Similarly, I_λ satisfies the $(PS)^{[r]}$-condition.

The conditons of Lemma 2 are fulfilled. Clearly $u \equiv 0$ is not a solution to problem (2.1). Therefore, for $\lambda \in \Lambda_2$, problem (2.1) admits least one nontrivial solution u satisfying $u \in \Phi^{-1}\left(-\infty, \frac{2c^2}{T+1}\right)$. Hence, Lemma 6 has been proven. □

We find that when the solution u of problem (2.1) is also a solution of problem (1.1), it satisfies $|\Delta u_k| \leq a$. Clearly, for each $k \in \mathbb{Z}(1,T)$, $|u_k|$ is bounded. Without loss of generality, assume that

$$\|u\|_\infty = \max_{0\leq k\leq T+1} |u_k| \leq u^*.$$

Now, for each $k \in \mathbb{Z}(1,T)$, we define a new function $\tilde{f}$:

$$\tilde{f}(k,t) = \begin{cases} f(k,t), & |t| \leq u^*, \\ g(k,t), & |t| > u^*, \end{cases}$$

where g is chosen such that $\tilde{f}(k,\cdot) \in C(\mathbb{R},\mathbb{R})$.

Let

$$\tilde{F}(k,\xi) = \int_0^\xi \tilde{f}(k,t)\,dt,$$

and define

$$\tilde{I}_\lambda(u) = \Phi(u) - \lambda\tilde{\Psi}(u),$$

for $u \in S$, where $\tilde{\Psi}(u) = \sum_{k=1}^{T} \tilde{F}(k, u_k)$.

Hence, critical points of $\tilde{I}_\lambda(u)$ are solutions of the following problem:

$$\begin{cases} -\nabla h(\Delta u_k) = \lambda \tilde{f}(k, u_k), & k \in \mathbb{Z}(1, T), \\ u_0 = u_{T+1} = 0. \end{cases} \tag{3.3}$$

Remark 5. *If the solution u of problem (3.3) satisfies* $|\Delta u_k| \le a$ *, it is also a solution to problem (2.1), and furthermore, it is a solution to problem (1.1).*

Now, we introduce the next theorem.

Theorem 2. *Assume that there exists a* $k_0 \in \mathbb{Z}(1, T)$*, such that* $f(k_0, 0) \ne 0$ *and there exist two positive constants c and d satisfying*

(H_7) $2\left(1 - \sqrt{1 - d^2}\right) < \frac{2c^2}{T+1} \le \left(1 - \sqrt{1 - a^2}\right)$,

(H_8)

$$A := \frac{T+1}{2} \frac{\sum\limits_{k=1}^{T} \max\limits_{|t| \le c} F(k, t) - F_d}{c^2 - (T+1)\left(1 - \sqrt{1 - d^2}\right)} \ge 0.$$

Then, for each $\lambda \in \left(0, \frac{1}{A}\right)$*, problem (1.1) admits at least one nontrivial solution.*

Proof. Let

$$\tilde{f}(k, t) = \begin{cases} f(k, t), & |t| \le u^*, \\ f(k, u^*) + (t - u^*)^3, & t > u^*, \\ f(k, -u^*) + (t + u^*)^3, & t < -u^*, \end{cases}$$

where $u^* > \max\{c, d\}$. Therefore,

$$\tilde{F}(k, \xi) = \begin{cases} F(k, \xi), & |t| \le u^*, \\ \dfrac{(\xi - u^*)^4}{4} + f(k, u^*)(\xi - u^*) + \displaystyle\int_0^{u^*} f(k, t)\, dt, & t > u^*, \\ \dfrac{(\xi + u^*)^4}{4} + f(k, -u^*)(\xi + u^*) + \displaystyle\int_0^{-u^*} f(k, t)\, dt, & t < -u^*. \end{cases}$$

Let S, Φ, $\tilde{\Psi}$, and $\tilde{I}_\lambda$ be defined as before. It is easy to obtain

$$\min_{k \in \mathbb{Z}(1,T)} \liminf_{|\xi| \to +\infty} \frac{\tilde{F}(k, \xi)}{\xi^2} = +\infty,$$

and (H_7) implies $d \le a$. Based on Lemma 6, we can conclude that problem (3.3) has a nontrivial solution u such that $u \in \Phi^{-1}\left(-\infty, \frac{2c^2}{T+1}\right)$. Using (H_7), we have

$$\Phi(u) < \frac{2c^2}{T+1} \le 1 - \sqrt{1 - a^2}.$$

Thus, $|\Delta u_k| \le a$ for $k \in \mathbb{Z}(1, T)$. By Remarks 4 and 5, u is also a solution to problem (1.1). Hence, Theorem 2 has been proven.

□

To obtain a positive solution to problem (1.1) in Theorem 2, we present the following corollary.

Corollary 3. *Assume that $f(k,0) > 0$, $k \in \mathbb{Z}(1,T)$, and all the conditions in Theorem 2 are satisfied. Then, for each $\lambda \in \Lambda_2$, problem (1.1) has at least one positive solution.*

Proof. Let

$$f(k,\xi^+) = \begin{cases} f(k,\xi), & \xi > 0, \\ f(k,0), & \xi \le 0, \end{cases}$$

and we discuss problem (2.4). It is easy to observe that $u = (u_k) \equiv 0$ is not a solution to problem (2.4) and one has either

$$u_k > 0 \quad or \quad -\nabla(h(\Delta u_k)) = f(k,0) > 0,$$

for all $k \in \mathbb{Z}(1,T)$. Then, we obtain that u is a positive solution to problem (2.4) by Lemma 4. Additonally, by Lemma 5, u is also a positive solution to problem (2.1). Therefore, u is a positive solution to problem (1.1). Then, Corollary 3 holds. □

4. Examples

In this section, we provide three examples to verify our conclusions.

Example 1. We consider problem (1.1) with $T = 3$ and let

$$f(k,t) = f(t) = \begin{cases} (t+\varepsilon)^3, & t < 1-\varepsilon, \\ e^{-4(t-1+\varepsilon)}, & t \ge 1-\varepsilon, \end{cases}$$

for each $k \in \mathbb{Z}(1,3)$ and $\varepsilon > 0$ is sufficiently small. Then, we have

$$F(k,t) = F(t) = \begin{cases} \dfrac{(t+\varepsilon)^4}{4} - \dfrac{\varepsilon^4}{4}, & t < 1-\varepsilon, \\ \dfrac{-e^{-4(t-1+\varepsilon)}}{4} + \dfrac{1}{2} - \dfrac{\varepsilon^4}{4}, & t \ge 1-\varepsilon. \end{cases}$$

Let $a = d = \frac{\sqrt{3}}{2}$, $c_1 = \frac{1}{5}$, $c_2 = 7$. Thus, $1 - \sqrt{1-d^2} = 1 - \sqrt{1-a^2} = \frac{1}{2}$. Obviously, conditions (H_1) and (H_2) in Theorem 1 hold.

In addition, we find that

$$\frac{(T+1)F_{c_1}}{c_1^2} = \frac{4 \times 3 \times \left[\frac{\left(\frac{1}{5}+\varepsilon\right)^4}{4} - \frac{\varepsilon^4}{4}\right]}{\left(\frac{1}{5}\right)^2} = 0.2925,$$

$$\frac{2(T+1)F_{c_2}}{c_2^2} = \frac{2 \times 4 \times 3 \times \left[-\frac{e^{-4\times(7-1+\varepsilon)^4}}{4} + \frac{1}{2} - \frac{\varepsilon^4}{4}\right]}{(7)^2} \approx 0.25,$$

and

$$\frac{F_d}{2\left(1-\sqrt{1-d^2}\right)} = \frac{3 \times \left[\frac{\left(\frac{\sqrt{3}}{2}+\varepsilon\right)^4}{4} - \frac{\varepsilon^4}{4}\right]}{2 \times \frac{1}{2}} \approx 0.528.$$

Therefore, condition (H_3) is satisfied. Then, using Corollary 1, for each $\lambda \in (3.79, 6.83)$, the problem:

$$\begin{cases} -\nabla\phi\left(\Delta u_k\right) = \lambda f\left(t\right), & k \in \mathbb{Z}\left(1,3\right), \\ u_0 = u_4 = 0, \end{cases}$$

has at least one positive solution.

Example 2. Let $T = 4$ and consider the problem (1.1) with

$$f\left(k,t\right) = f\left(t\right) = \begin{cases} -128x^3 + 48x^2, & t < \dfrac{1}{4}, \\ e^{1-4x}, & t \geq \dfrac{1}{4}. \end{cases}$$

Then, we have

$$F\left(k,t\right) = F\left(t\right) = \begin{cases} -32x^4 + 16x^3, & t < \dfrac{1}{4}, \\ \dfrac{-e^{1-4x}}{4} + \dfrac{3}{8}, & t \geq \dfrac{1}{4}. \end{cases}$$

Let $a = \frac{3\sqrt{11}}{10}, d = \frac{1}{4}, c_1 = \frac{1}{50}, c_2 = \frac{3}{2}$. Thus, $1 - \sqrt{1-a^2} = \frac{9}{10}$, $1 - \sqrt{1-d^2} \approx 0.032$. It is easy to deduce that (H_2) in Theorem 1 and (S) in Corollary 2 hold.

Moreover, we have

$$\frac{\left(T+1\right)F_{c_1}}{c_1^2} = \frac{-32 \times (0.02)^4 + 16(0.02)^3}{(0.02)^2} \approx 6.15,$$

$$\frac{2\left(T+1\right)F_{c_2}}{c_2^2} = \frac{2 \times 5 \times 4 \times \left(-\frac{e^{1-4\times\frac{3}{2}}}{4} + \frac{3}{8}\right)}{\left(\frac{3}{2}\right)^2} \approx 6.64,$$

and

$$\frac{F_d}{2\left(1 - \sqrt{1-d^2}\right)} = \frac{4 \times \left(-\frac{e^{1-4\times\frac{1}{4}}}{4} + \frac{3}{8}\right)}{2 \times \left(1 - \sqrt{1 - \left(\frac{1}{4}\right)^2}\right)} \approx 7.87.$$

Therefore, (H_3) in Theorem 1 is satisfied. Then, by Corollary 2, for each $\lambda \in \left(\frac{1}{4}, \frac{3}{10}\right)$, the following problem:

$$\begin{cases} -\nabla\phi\left(\Delta u_k\right) = \lambda f\left(t\right), & k \in \mathbb{Z}\left(1,4\right), \\ u_0 = u_5 = 0, \end{cases}$$

has at least three nonnegative solutions.

Example 3. Let $T = 8$, and we consider the boundary value problem (1.1) with

$$f\left(k,t\right) = f\left(t\right) = \frac{1}{2}\cos 2t,$$

for each $k \in \mathbb{Z}\left(1,8\right)$. Then,

$$F\left(k,t\right) = F\left(t\right) = \frac{1}{4}\sin 2t.$$

Let $a = \frac{2\sqrt{2}}{3}$, $d = \frac{\sqrt{2}}{3}$, and $c = \sqrt{2}$. Clearly, condition (H_7) in Theorem 2 holds. Furthermore,

$$A = \frac{9}{2} \times \frac{8 \times \frac{1}{4} - 8 \times \frac{1}{4} \times \sin\left(2 \times \frac{\sqrt{2}}{3}\right)}{\left(\frac{4}{5}\right)^2 - 9 \times \left(1 - \sqrt{1 - \left(\frac{1}{3}\right)^2}\right)} \approx 1.83 > 0.$$

Thus, condition (H_8) is satisfied. Then, using Corollary 3, for each $\lambda \in (0, 0.54)$, the following problem:

$$\begin{cases} -\nabla\phi\left(\Delta u_k\right) = \lambda\left(\dfrac{1}{2}\cos 2u_k\right), & k \in \mathbb{Z}(1, 8), \\ u_0 = u_9 = 0, \end{cases}$$

admits at least one positive solution.

5. Conclusions

In this paper, the existence of solutions to the discrete Dirichlet problem involving the singular ϕ-Laplacian is studied by critical point theory. We first propose an auxiliary problem related to problem (1.1) and establish its variational framework. Then, using Remarks 3 and 4, we connect its solution to the solution of the original problem. Based on the behavior of the nonlinear term f in several positive points, we found a solution to problem (1.1) in Theorem 1 through Lemma 1. In order to make the results more meaningful, we strengthen the assumptions in Theorem 1 and obtain the existence of a positive solution and the existence of multiple positive solutions in Corollaries 1 and 2, respectively. To derive existence conditions for solutions in Theorem 2 that are independent of the behavior of the nonlinear term f at infinity, we establish Lemma 6 using Lemma 2 and introduc a new auxiliary problem (3.3). By the strong maximum principle, a positive solution to problem (1.1) is obtained in Corollary 3.

Author contributions

Zhiqiang Huang: Conceptualization, methodology, formal analysis and investigation, writing-original draft preparation; Zhan Zhou: Formal analysis and investigation, writing-review and editing, funding acquisition, supervision. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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