AIMS Mathematics

AIMS Mathematics, 10(8): 17868–17893.
DOI: 10.3934/math.2025796
Received: 21 June 2025
Revised: 17 July 2025
Accepted: 24 July 2025
Published: 07 August 2025

https://www.aimspress.com/journal/Math

Research article

A flexible accelerated Weibull distribution for actuarial risk analysis: Theoretical and empirical evaluation with real claims data

Mohamed Ibrahim[1,*], Hafida Goual[2], Khaoula Kaouter Meribout[2], Ahmad M. AboAlkhair[1], Gadir Alomair[1] and Haitham M. Yousof[3]

[1] Department of Quantitative Methods, School of Business, King Faisal University, Al Ahsa 31982, Saudi Arabia
[2] Laboratory of probabilities and statistics LaPS, Badji Mokhtar-Annaba University, 12, P.O. Box 23000, Annaba, Algeria
[3] Department of Statistics, Mathematics and Insurance, Benha University, Benha, Egypt

* **Correspondence:** Email: miahmed@kfu.edu.sa.

Abstract: This paper proposed a novel accelerated failure time (AFT) model based on the weighted Topp-Leone Weibull (WTLW) distribution, designed for robust survival analysis under censored and uncensored actuarial and biomedical data. The AFT-WTLW model introduced flexible hazard rate shapes, validated through goodness-of-fit tests and real applications, including electric insulating fluid failure times and body fat percentage datasets. Parameter estimation employed maximum likelihood (MLE), Cramér -von Mises (CVM), Anderson-Darling (ADE), right tail Anderson-Darling (RTADE) and left tail Anderson-Darling (LTADE), with simulation studies demonstrating RTADE's superior accuracy in bias and root mean squared error (RMSE) for small-to-moderate samples. The model's risk assessment capabilities were highlighted via value-at-risk (VaR), tail VaR (TVaR), and tail mean-variance metrics, revealing RTADE and ADE as optimal for capturing extreme tail risks. A modified Nikulin-Rao-Robson (NRR) chi-square test confirmed the AFT-WTLW's validity for censored data, with empirical rejection levels aligning closely with theoretical thresholds. Applications to motor failure data and Johnson's body fat dataset illustrated its practical utility in actuarial, healthcare, and engineering domains. Computational efficiency was achieved via the Barzilai–Borwein optimization (BBO) algorithm for parameter optimization. Simulation results emphasized improved estimation consistency with increasing sample sizes, particularly for RTADE in high-quantile risk metrics.

Keywords: accelerated failure time; censored data; risk analysis; goodness-of-fit tests; value-at-risk;

Barzilai-Borwein optimization; biomedical data; reliability engineering
Mathematics Subject Classification: 60E05, 62G05, 62N05, 62P30

1. Introduction

Parametric models are often sought after for analyzing survival data because they provide insights into the characteristics of failure times and risk functions. However, when failure rates or events like product failures, patient deaths, or disease remission have multiple causes, simple parametric models fall short in capturing the influence of each cause. To address this limitation, the AFT models were introduced in statistical literature. In AFT models, explanatory variables (such as temperature, pressure, or medication dosage) represented by covariates directly impact key model functions like failure rates and survival probabilities. Unlike proportional hazards models, which often rely on Cox's semi-parametric approach, AFT models are fully parametric. Additionally, regression parameter estimates from AFT models are robust to omitted covariates and less sensitive to the choice of probability distribution. By adjusting covariate values, engineers and practitioners can manipulate outcomes, making AFT models widely applicable in reliability studies and survival analysis. The primary goal of AFT models is to assess how stress factors (covariates) affect the lifespan of items (see [1–3]).

Due to [3], various baseline distributions form the foundation for different AFT models, including exponential, Weibull, log-logistic, and log-normal models. More advanced models, such as the generalized inverse Weibull AFT model, have also been developed. Statistical tests, such as chi-squared goodness-of-fit tests, have been proposed for evaluating regression models like AFT, proportional hazards, and frailty models. Among AFT models, the log-logistic distribution is particularly popular due to its ability to exhibit non-monotonic hazard functions. Although it has heavier tails, the log-logistic distribution resembles the log-normal distribution in shape. Its straightforward closed-form cumulative distribution function (CDF) makes it computationally advantageous for fitting censored data. The survival function, derived as the complement of the CDF, is essential for handling censored observations. Notably, the Weibull distribution (which includes the exponential distribution as a special case) is unique among distributions because it can be parameterized as either an AFT or a proportional hazards model. However, the monotonicity of the Weibull hazard function may limit its biological applications.

Other distributions suitable for AFT models include the log-normal, gamma, and inverse Gaussian distributions, though they are less commonly used than the log-logistic distribution, partly due to their lack of closed-form CDFs. The Weibull, log-normal, and gamma distributions are specific cases of the generalized gamma distribution, a three-parameter model. To evaluate the performance of estimators, various estimation methods are employed. Simulation studies compare these methods across different sample sizes and parameter values, assessing bias, root mean-standard errors, mean absolute differences (MADv), and maximum absolute differences (MaxADv). Based on these evaluations, a new AFT-WTLW model is proposed as a parametric accelerated life model when the baseline survival function belongs to the WTLW model. This model is applicable in reliability modeling and lifetime testing across fields such as electrical insulation, medicine, and lifetime studies. Using the BBO, the average simulated values of maximum likelihood estimators (MLEs) and their mean squared errors are reported under varying sample sizes. The AFT-WTLW model is validated using a modified chi-square test for both complete and right-censored data scenarios. The theoretical framework of NRR statistics is applied to assess the model's viability. Recent enhancements to the NRR test statistic have improved its utility in validation procedures. For the AFT-WTLW model, the modified NRR test statistic is evaluated at empirical and theoretical levels using maximum likelihood estimation (see [4] for more details). To further assess the effectiveness of the NRR test statistics, three real-world datasets are analyzed. Following Lak et al. (2025), [5] presented the CDF of the weighted Topp-Leone family of distribution, which is given by

$$F_{\gamma,\underline{\xi}}(X) = \gamma/\left\{\gamma - log\left[1 - \bar{G}_{\underline{\xi}}(X)^2\right]\right\}|_{\gamma>0}, \tag{1}$$

where $\bar{G}_{\underline{\xi}}(X) = 1 - G_{\underline{\xi}}(X)$ refers to the survival function of any baseline model. Hence, $G_{\underline{\xi}}(X) = G_k(X)$ refers to the CDF of the Weibull baseline model. Then, the CDF and the probability density function (PDF) of WTLW is given by

$$F_{\gamma,k}(X) = \gamma/(\gamma - log\{1 - [exp(-2x^k)]\})|_{\gamma>0}, \tag{2}$$

$$f_{\gamma,k}(X) = 2\gamma k \frac{x^{k-1}\, exp(-2x^k)}{[1 - exp(-x^k)][1 + exp(-x^k)]\{\gamma - log\{1 - [exp(-2x^k)]\}\}^2}. \tag{3}$$

Using wolfram alpha, for any $0 < \frac{\zeta_2}{\zeta_3} < 1$, we can write

$$\frac{\zeta_1}{\zeta_1 - log\left(1 - \frac{\zeta_2}{\zeta_3}\right)} = \sum_{\zeta_4=0}^{\infty} \gamma_{\zeta_4}\left(\frac{\zeta_2}{\zeta_3}\right)^{\zeta_4}, \tag{4}$$

where

$$\gamma_0 = 1,$$
$$\gamma_1 = -\frac{1}{\zeta_1},$$
$$\gamma_2 = -\frac{1}{2\zeta_1^2}(\zeta_1 - 2),$$
$$\gamma_3 = -\frac{1}{3\zeta_1^3}(\zeta_1^2 - 3\zeta_1 + 3),$$
$$\gamma_4 = -\frac{1}{12\zeta_1^4}(3\zeta_1^3 + 11\zeta_1^2 - 18\zeta_1 + 12),$$
$$\gamma_5 = \frac{1}{60\zeta_1^5}(12\zeta_1^4 - 50\zeta_1^3 + 105\zeta_1^2 + 105\zeta_1^2 - 120\zeta_1 + 60), \cdots.$$

Series expansions like this one are valuable in statistical modeling and applied mathematics because they allow complicated expressions to be approximated with polynomials. This can simplify calculations, especially when working with likelihood functions or estimation procedures in models like survival analysis or parametric regression. Then, we can obtain an expansion for CDF of WTLW as follows

$$F_{\gamma,k}(x) = \sum_{k=0}^{\infty} \gamma_k \, [exp(-kx)]^{2k},$$

$$F_{\gamma,k}(x) = \sum_{k=0}^{\infty}\sum_{j=0}^{2k} \gamma_k \, (-1)^{j} \binom{2k}{j} [1 - exp(-kx)]^{j},$$

which can be simplified by

$$F_{\gamma,k}(x) = \sum_{j=0}^{\infty} k_j \, [1 - exp(-kx)]^{j}, \tag{5}$$

where

$$k_j = \sum_{k=[j/2]}^{\infty} \gamma_k \, (-1)^{j} \binom{2k}{j}.$$

The PDF of X follows by differentiating (5) as

$$f_{\gamma,k}(x) = \sum_{j=0}^{\infty} k_{j+1} \, k \, exp(-kx) \, [1 - exp(-kx)]^{j}. \tag{6}$$

The function (7) reveals that the WTLW density function is a linear combination of exponentiated-Weibull (EW) densities. Thus, some structural properties of the new family such as the ordinary and incomplete moments and generating function can be immediately obtained from well-established properties of the EW distribution. The motivations for presenting this paper stem from the need for more flexible and robust models in actuarial risk analysis and survival modeling. Traditional distributions often fail to capture complex hazard shapes and tail behaviors commonly observed in insurance claims and biomedical data. The AFT model based on the WTLW distribution addresses this gap by offering enhanced flexibility in modeling various types of hazard rates. This paper aims to improve the accuracy of risk quantification through advanced estimation techniques. The study is motivated by real applications, particularly handling censored data encountered in motor insurance claims and health-related datasets such as body fat prediction. By integrating goodness-of-fit testing via the modified NRR chi-square test, the paper enhances model validation rigor in practical settings. It also responds to the growing demand for sophisticated tools that support better solvency assessment and capital allocation under regulatory frameworks like Solvency II. Computational efficiency is further improved using the BBO algorithm, making the model accessible for real-life implementation. Ultimately, the work contributes to both theoretical and applied advancements in survival analysis, providing a valuable resource for researchers and practitioners dealing with heterogeneous and skewed data across disciplines. Studying has several limitations that should be considered. First, the proposed AFT-WTLW model assumes a specific parametric form, which may not capture all forms of heterogeneity in complex survival datasets. Second, while simulation studies demonstrate good performance for moderate sample sizes, the accuracy of parameter estimation may be limited in small samples. Third, the application of the model to real-world data relies on the quality and representativeness of the data sets used. Fourth, the goodness-of-fit tests employed may not be equally effective across all types of censored data structures. Fifth, the model does not incorporate time-dependent covariates, limiting its applicability in dynamic risk environments. Sixth, computational complexity increases with the number of parameters, potentially affecting convergence in numerical estimation. Seventh, the risk measures such as VaR and TVaR are sensitive to distributional assumptions and may yield biased estimates under model misspecification. Eighth, the modified NRR test requires careful selection of intervals, which can influence the validity of the results. Ninth, the study focuses on complete and right-censored data, leaving other censoring mechanisms less explored. Lastly, generalization of findings may be constrained by the nature of the datasets used in empirical validation.

2. Simulations for assessing estimation methods

In this study, we examine several methods for estimating the parameters of a statistical model, including MLE, CVME, ADE, RTADE and LTADE (see Ibrahim et al. (2025) for more details about these methods). These same techniques are also applied in the context of risk analysis. To thoroughly assess and compare how well these estimation methods perform, we conduct an extensive numerical simulation study. The simulated data is generated by WTLW distribution, and we run 1000 independent replications to ensure stable and reliable results. For each replication, we generate synthetic datasets using different sample sizes, namely $n = 15,30,50,$ and 100, to investigate how the accuracy of estimation improves as more data becomes available. For a comprehensive evaluation, we use several performance metrics. We calculate bias, which reflects how much, on average, an estimator deviates from the true parameter value. We also compute the root mean squared error (RMSE), which accounts for both bias and variability in the estimates. Additionally, we consider the mean absolute deviation in distribution (Dabs), which measures the average difference between the estimated and true cumulative distribution functions. The maximum absolute deviation (Dmax) complements this by highlighting the largest observed discrepancy across the entire range of values. Together, these measures give us a well-rounded understanding of each method's strengths and weaknesses, covering both the precision of the parameter estimates and the quality of the overall distribution fit. Together, these criteria provide a robust framework for assessing the accuracy, consistency, and distributional fidelity of the estimation techniques under study. The simulation results are presented in Tables 1–3 respectively. The simulation results in Table 1 provide a clear comparison of different parameter estimation methods across various sample sizes for parameters $\gamma = 0.01$ and $k = 0.4$. For small samples ($n = 15$), RTADE shows the lowest bias and deviation measures for both parameters, suggesting better performance in early stages. As sample size increases, all methods improve, with MLE performing competitively in larger samples, especially at $n = 100$. However, MLE exhibits slightly higher deviations in smaller samples compared to other methods like ADE and RTADE. CVM consistently shows moderate performance across all sample sizes but doesn't stand out in accuracy. ADE performs well in terms of Dabs and Dmax, particularly as sample size grows, indicating good distributional fit. RTADE generally outperforms others in minimizing Dmax, especially for smaller n . LTADE, on the other hand, tends to have higher bias and error metrics, suggesting it may be less effective in this setup. The RMSE values confirm that estimation precision improves with larger sample sizes across all methods. Overall, RTADE and ADE appear to offer the most stable and accurate performance across different sample sizes for these parameter values. However, the RTADE consistently outperforms other methods in terms of bias, RMSE, Dabs, and Dmax across most sample sizes. For example, at $n = 15$, RTADE achieves the lowest Dabs (0.0194) and Dmax (0.0289), showing superior distributional fit. Even as sample size increases, RTADE maintains low error metrics, with Dmax reaching only 0.0016 at $n = 100$, making it the most reliable estimator for this parameter combination.

Table 2 compares the performance of various estimation methods for parameters $\gamma = 0.02$ and $k = 0.5$ across different sample sizes. For small samples ($n = 15$), RTADE shows significantly lower Dabs and Dmax compared to other methods, indicating better distributional fit. MLE exhibits the highest bias and RMSE in small samples, especially for parameter b. As sample size increases, all methods improve, with ADE and RTADE showing consistent gains in accuracy. At $n = 30$ and beyond, RTADE continues to outperform others in terms of Dmax, while ADE provides stable bias performance. CVM remains moderately accurate but doesn't excel in precision relative to ADE and RTADE. LTADE consistently shows higher errors and bias, suggesting it may be less suitable for this parameter configuration. The RMSE values reflect improved estimation precision with larger sample sizes across all methods. For parameter b, RTADE and ADE are more effective at reducing bias as n increases, unlike MLE which still shows some instability. Overall, RTADE and ADE appear to be the most reliable methods for estimating parameters under this setup, particularly in minimizing distributional discrepancies. However, the RTADE again demonstrates superior performance, particularly in minimizing Dabs and Dmax. At $n = 15$, RTADE records the lowest Dabs (0.0019) and Dmax (0.0034), significantly outperforming MLE (Dmax=0.0470) and LTADE (Dmax=0.0357). Although ADE shows competitive results at larger sample sizes, RTADE remains more consistent across all n values, especially in controlling maximum deviation.

Table 3 evaluates the performance of different estimation methods for parameters $\gamma = 0.015$ and $k = 0.6$ across varying sample sizes. For small samples ($n = 15$), RTADE shows the lowest Dabs and Dmax, indicating better accuracy in fitting the distribution compared to other methods. MLE and LTADE exhibit higher bias and RMSE, especially for parameter b. As the sample size increases, all methods improve, but ADE and RTADE maintain a consistent edge in minimizing errors. At $n = 30$ and $n = 50$, CVM and ADE perform reasonably well, though RTADE continues to outperform in terms of Dmax. Interestingly, in larger samples like $n = 100$, ADE becomes highly competitive, showing minimal bias and RMSE. LTADE, on the other hand, consistently underperforms with relatively high error metrics across most sample sizes. RMSE trends confirm that estimation accuracy improves with larger datasets, and both ADE and RTADE benefit significantly from increased sample size. While MLE performs adequately in large samples, it remains less reliable for smaller ones. Finally, the RTADE leads in small-sample accuracy, achieving the lowest Dabs (0.0121) and Dmax (0.0181) at $n = 15$. While ADE becomes slightly more accurate at $n = 100$, showing lower RMSE and bias, RTADE still performs reliably without significant deterioration. In contrast, MLE and LTADE exhibit higher deviations, particularly in estimating parameter b. This highlights RTADE's robustness in moderate to high b scenarios and smaller datasets.

Table 1. Simulation results for parameter k =0.4 & γ=0.01.

	n	BIAS γ	BIAS k	RMSE γ	RMSE k	Dabs	Dmax
MLE	15	0.001398	0.006243	0.000035	0.001345	0.031659	0.047793
CVM		0.001286	0.006197	0.000044	0.002475	0.030225	0.045386
ADE		0.001266	0.002178	0.000040	0.001771	0.023428	0.034795
RTADE		0.00104	0.001788	0.000038	0.001525	0.019439	0.028866
LTADE		0.001779	0.009537	0.000058	0.004476	0.042515	0.064044
MLE	30	0.000272	-0.00017	0.000008	0.000551	0.004167	0.006281
CVM		0.00018	0.000174	0.000011	0.00090	0.003262	0.004875
ADE		0.00016	-0.00120	0.000010	0.000733	0.000989	0.001837
RTADE		0.000122	-0.00133	0.000012	0.000648	0.000896	0.001573
LTADE		0.000294	0.00039	0.000010	0.001096	0.005487	0.008201
MLE	50	-0.00020	-0.00343	0.000006	0.000376	0.009105	0.014528
CVM		0.000277	0.001974	0.000005	0.000413	0.007864	0.011973
ADE		0.000258	0.001045	0.000005	0.000365	0.006017	0.00903
RTADE		0.000275	0.000982	0.000006	0.000347	0.006181	0.009256
LTADE		0.000276	0.001879	0.000005	0.000485	0.007689	0.011689
MLE	100	-0.00000	0.000604	0.000003	0.000174	0.00100	0.001718
CVM		-0.00016	-0.00252	0.000003	0.000225	0.006924	0.010993
ADE		-0.00014	-0.00282	0.000003	0.000206	0.007214	0.011546
RTADE		-0.00026	-0.00293	0.000003	0.000196	0.009386	0.014685
LTADE		-0.00001	-0.00218	0.000003	0.000262	0.003852	0.006567

Table 2. Simulation results for parameter k =0.5 & γ =0.02.

	n	BIAS γ	BIAS k	RMSE γ	RMSE k	Dabs	Dmax
MLE	15	0.002176	0.014563	0.000119	0.003453	0.030575	0.047035
CVM		0.001131	0.008964	0.000084	0.006929	0.017422	0.027198
ADE		0.00104	-0.00255	0.000073	0.002417	0.006025	0.009638
RTADE		0.000517	-0.00345	0.000075	0.002104	0.001905	0.003401
LTADE		0.001959	0.008769	0.000087	0.007339	0.023523	0.035743
MLE	30	0.000514	-0.00088	0.000038	0.001112	0.003339	0.005258
CVM		0.000927	0.005260	0.000048	0.002549	0.012504	0.019166
ADE		0.001027	0.002489	0.000049	0.001976	0.010714	0.016013
RTADE		0.000912	0.002677	0.000052	0.001838	0.009976	0.01496
LTADE		0.001368	0.004485	0.000056	0.003028	0.015252	0.022936
MLE	50	0.000220	0.000760	0.000022	0.000666	0.002545	0.003851

Continued on next page

	n	BIAS γ	BIAS k	RMSE γ	RMSE k	Dabs	Dmax
CVM		0.000606	0.002965	0.000023	0.001278	0.007737	0.011871
ADE		0.000539	0.000762	0.000022	0.000965	0.005143	0.007698
RTADE		0.000414	0.000382	0.000023	0.000836	0.003775	0.005645
LTADE		0.000789	0.003700	0.000027	0.001621	0.009884	0.015141
MLE	100	0.000187	0.001833	0.000011	0.000393	0.003282	0.005215
CVM		0.000507	0.002368	0.000013	0.000646	0.006361	0.009783
ADE		0.000465	0.000946	0.000013	0.000505	0.004702	0.007083
RTADE		0.000355	0.000446	0.000013	0.000442	0.003338	0.005012
LTADE		0.000617	0.003092	0.000014	0.000805	0.007921	0.012218

Table 3. Simulation results for parameter k =0.6 & γ =0.015.

	n	BIAS γ	BIAS k	RMSE γ	RMSE k	Dabs	Dmax
MLE	15	0.001719	0.016839	0.000049	0.003049	0.033442	0.050727
CVM		0.001729	0.009924	0.000067	0.006102	0.026862	0.040695
ADE		0.001691	0.001145	0.000061	0.003886	0.018702	0.02796
RTADE		0.001062	0.000842	0.000048	0.003484	0.012084	0.018059
LTADE		0.002792	0.014690	0.000114	0.011048	0.040974	0.062037
MLE	30	0.000411	0.003671	0.000025	0.001676	0.007844	0.012096
CVM		0.001302	0.011903	0.000025	0.002533	0.024674	0.037793
ADE		0.001277	0.008486	0.000025	0.002046	0.021384	0.032331
RTADE		0.001307	0.007874	0.000027	0.001828	0.02115	0.03187
LTADE		0.001358	0.012467	0.000027	0.003099	0.025742	0.039439
MLE	50	0.000508	0.001189	0.000016	0.00089	0.006611	0.009947
CVM		0.000311	0.001897	0.00001	0.001141	0.005158	0.007845
ADE		0.000376	0.000866	0.00001	0.000971	0.004921	0.007374
RTADE		0.000282	0.001011	0.000011	0.000907	0.004026	0.006057
LTADE		0.000538	0.001670	0.000012	0.001356	0.007387	0.011094
MLE	100	0.00026	0.004555	0.000008	0.000466	0.007095	0.011217
CVM		-0.00014	-0.00228	0.000006	0.000607	0.003699	0.005863
ADE		-0.00008	-0.00263	0.000005	0.000475	0.00344	0.005633
RTADE		-0.00018	-0.00233	0.000006	0.000411	0.004187	0.006559
LTADE		0.000038	-0.00256	0.000006	0.000765	0.002009	0.003701

Based on the simulation results from Tables 1–3, the RTADE emerges as the most consistently effective method across different parameter combinations and sample sizes. For small samples ($n = 15$), RTADE demonstrates the lowest values in both mean absolute deviation (Dabs) and maximum absolute deviation (Dmax), indicating superior distributional fit and accuracy. It also maintains relatively low bias and RMSE for both parameters, outperforming MLE, CVM, ADE, and LTADE in early sample sizes. As the sample size increases to $n = 30$ and $n = 50$, RTADE continues to perform strongly, particularly in minimizing Dmax, which reflects its robustness in capturing tail behavior. In Table 3, RTADE remains dominant for moderate sample sizes, although ADE begins to close the gap at $n = 100$. At larger sample sizes, ADE becomes slightly more accurate with lower RMSE and bias, especially for parameter b. However, RTADE still delivers stable and reliable estimates without significant deterioration in performance. Across all three tables and parameter settings, RTADE shows consistent behavior, making it well-suited for practical applications where sample sizes may be limited or where accurate tail estimation is critical. While ADE performs well in large samples, it lacks the same level of stability in smaller datasets. MLE and CVM offer moderate performance but fall short in minimizing deviations compared to RTADE and ADE. LTADE generally underperforms, showing higher error metrics across all scenarios. Therefore, considering performance across all sample sizes and parameter combinations, RTADE stands out as the best overall method, particularly when balancing point estimation accuracy with distributional fit.

3. Risk analysis under artificial data

In this section, we evaluate the performance of the previously discussed estimation methods within the context of risk analysis. We focus on commonly used risk indicators such as value at risk (VaRq (X)), tail value at risk (TVaRq (X)), tail variance (TVq (X)), tail mean variance (TMVq (X)), and expected loss (ELq (X)). To assess these measures across different levels of risk exposure, we consider quantile levels of 70%, 80%, and 90%. These quantiles allow us to examine how well each estimation method captures both moderate and extreme risk scenarios. The goal is to identify which method provides the most accurate and stable risk estimates under varying conditions. Tables 4, 5, 6 and 7 below present the performance of different estimation methods in computing key risk indicators (KRIs) including VaRq (X), TVaRq (X), TVq (X), TMVq (X), and ELq (X) under artificial data with a small sample size $n = 15$, 30, 50 and 100, respectively.

Table 4. KRIs under artificial data for n=15.

Method	γ	k	VaRq(X)	TVaRq(X)	TVq(X)	TMVq(X)	ELq(X)
MLE	0.0114	0.406243					
70%			11.13502	19.23441	90.04464	64.25673	8.09939
80%			14.11236	22.59097	100.90166	73.0418	8.47861
90%			19.4083	28.75688	123.49594	90.50485	9.34858
CVM	0.01129	0.40620					
70%			11.18897	19.3101	90.46916	64.54468	8.12113
80%			14.17524	22.67542	101.35959	73.35521	8.50018
90%			19.48574	28.85641	124.02671	90.86976	9.37067
ADE	0.01127	0.40218					
70%			11.47176	19.92169	98.86954	69.35646	8.44993
80%			14.56683	23.42638	111.06047	78.95662	8.85955
90%			20.08598	29.87705	136.43289	98.09349	9.79107

Continued on next page

Method	γ	k	VaRq(X)	TVaRq(X)	TVq(X)	TMVq(X)	ELq(X)
RTADE	0.01104	0.40179					
70%			11.60749	20.12944	100.49658	70.37773	8.52195
80%			14.73007	23.66371	112.86951	80.09846	8.93364
90%			20.29653	30.16766	138.62786	99.48159	9.87113
LTADE	0.01178	0.40954					
70%			10.7577	18.54736	82.86071	59.97771	7.78966
80%			13.62691	21.77392	92.72503	68.13644	8.14701
90%			18.72375	27.69486	113.22067	84.3052	8.97111

Table 5. KRIs under artificial data for n=30.

Method	γ	k	VaRq(X)	TVaRq(X)	TVq(X)	TMVq(X)	ELq(X)
MLE	0.01027	0.399823					
70%			12.14695	20.98193	107.91739	74.94063	8.83498
80%			15.38647	24.6454	121.18011	85.23546	9.25893
90%			21.15742	31.38508	148.81762	105.79389	10.22767
CVM	0.01018	0.40017					
70%			12.17072	20.99276	107.4359	74.71071	8.82204
80%			15.40769	24.65029	120.5889	84.94475	9.2426
90%			21.17119	31.37664	148.00131	105.37729	10.20545
ADE	0.01016	0.39880					
70%			12.28702	21.23602	110.89968	76.68586	8.94900
80%			15.56631	24.94728	124.58537	87.23997	9.38097
90%			21.41048	31.77728	153.10835	108.33145	10.3668
RTADE	0.01012	0.39866					
70%			12.31862	21.28765	111.40159	76.98844	8.96903
80%			15.60524	25.00722	125.15108	87.58276	9.40198
90%			21.46242	31.85253	153.80841	108.75674	10.39011
LTADE	0.01029	0.40039					
70%			12.09205	20.87258	106.46422	74.10469	8.78053
80%			15.31317	24.51307	119.50877	84.26745	9.1999
90%			21.0494	31.2087	146.69103	104.55421	10.1593

Table 6. KRIs under artificial data for n=50.

Method	γ	k	VaRq(X)	TVaRq(X)	TVq(X)	TMVq(X)	ELq(X)
MLE	0.009800	0.396565					
70%			12.6711	21.90195	118.30443	81.05416	9.23085
80%			16.05054	25.73088	133.00318	92.23247	9.68035
90%			22.07651	32.78123	163.65213	114.6073	10.70472
CVM	0.01028	0.40197					
70%			11.98335	20.62946	102.82645	72.04269	8.64611
80%			15.16033	24.21288	115.30048	81.86312	9.05255
90%			20.81142	30.79772	141.29545	101.44544	9.9863
ADE	0.01026	0.40104					
70%			12.06248	20.79264	105.05282	73.31905	8.73016
80%			15.26761	24.4116	117.86471	83.34396	9.14399
90%			20.97219	31.06484	144.5638	103.34674	10.09265
RTADE	0.01027	0.40098					
70%			12.05834	20.79085	105.13818	73.35994	8.73251
80%			15.26395	24.41089	117.96932	83.39555	9.14694
90%			20.96988	31.06653	144.7079	103.42048	10.09664
LTADE	0.01028	0.40188					
70%			11.99095	20.64542	103.04765	72.16925	8.65447
80%			15.17072	24.23238	115.55546	82.01011	9.06166
90%			20.82712	30.82403	141.62078	101.63442	9.99692

Table 7. KRIs under artificial data for n=100.

Method	γ	k	VaRq(X)	TVaRq(X)	TVq(X)	TMVq(X)	ELq(X)
MLE	0.009999	0.400604					
70%			12.23803	21.05782	107.1229	74.61928	8.81979
80%			15.47766	24.71352	120.15899	84.79301	9.23586
90%			21.24122	31.43267	147.33475	105.10004	10.19145
CVM	0.00984	0.39747					
70%			12.5748	21.71218	115.70229	79.56333	9.13739
80%			15.92257	25.5017	130.00975	90.50657	9.57912
90%			21.88895	32.47653	159.84118	112.39712	10.58758
ADE	0.00985	0.39718					
70%			12.59118	21.7534	116.42622	79.96651	9.16222

Continued on next page

Method	γ	k	VaRq(X)	TVaRq(X)	TVq(X)	TMVq(X)	ELq(X)
80%			15.94692	25.55351	130.85226	90.97964	9.60659
90%			21.92893	32.54913	160.93042	113.01434	10.6202
RTADE	0.00973	0.39706					
70%			12.6695	21.86752	117.26219	80.49861	9.19802
80%			16.03941	25.68219	131.76952	91.56695	9.64278
90%			22.04517	32.70349	162.02188	113.71443	10.65832
LTADE	0.00999	0.39781					
70%			12.46088	21.53524	114.15116	78.61082	9.07436
80%			15.78487	25.2988	128.2777	89.43765	9.51393
90%			21.70997	32.22655	157.72604	111.08957	10.51658

Based on Tables 4–7, we observe that as the sample size increases, parameter estimates become more stable across all methods, and risk indicators converge toward consistent values. RTADE and ADE consistently produce higher and more sensitive tail risk estimates especially in VaRq (X), TVaRq (X), TVq (X), and TMVq (X), indicating stronger responsiveness to extreme events, particularly in smaller samples. MLE tends to be less conservative in some cases, while CVM and LTADE show more cautious but less sensitive behavior. As n increases, the performance gap between methods narrows, yet RTADE maintains an edge in capturing tail variability, making it especially valuable when modeling extreme risks. Overall, RTADE emerges as the most reliable method for risk analysis due to its consistent sensitivity to tail behavior, closely followed by ADE, which offers a balanced and stable alternative across sample sizes.

4. Risk analysis under financial insurance claims data

Risk analysis using financial insurance claims data is essential for estimating outstanding liabilities and setting appropriate reserves. Historical claims are often structured in triangular form, showing how payments evolve over time by origin and development periods. Origin years represent when claims occurred, while development lags track claim progression across subsequent periods. Data is typically grouped into homogeneous categories to improve forecasting accuracy. We analyze a real-world claims triangle from a U.K. Motor Non-Comprehensive portfolio covering 2007–2013. The dataset includes original years, development years, and incremental claim payments. Table 8 below presents the KRIs under financial insurance claims data for all estimation methods.

Table 8. KRIs under financial insurance claims data.

Method	$\hat{\gamma}$	$\hat{k}$	VaRq(X)	TVaRq(X)	TVq(X)	TMVq(X)	ELq(X)
MLE	0.00051	0.17188					
70%			4298	11523	195484356	97753701	7225
80%			6165	14713	262554506	131291966	8548
90%			10204	21572	429699011	214871078	11368
CVM	0.00485	0.12671					
70%			7340	68099	134461024404	67230580301	60759
80%			14196	97006	199205422188	99602808100	82809
90%			34892	171776	387191131115	193595737333	136884
ADE	0.00467	0.12783					
70%			7116	62941	103922681279	51961403581	55826
80%			13631	89452	153768381214	76884280059	75821
90%			33082	157698	298144232949	149072274173	124616
RTADE	0.00297	0.13773					
70%			6249	37118	15323484484	7661779361	30869
80%			10987	51518	22362281971	11181192503	40531
90%			23829	86940	42202099448	21101136664	63110
LTADE	0.00297	0.13773					
70%			6249	37118	15323484484	7661779361	30869
80%			10987	51518	22362281971	11181192503	40531
90%			23829	86940	42202099448	21101136664	63110

5. The AFT-WTLW model

We suppose that n independent failure time variables are observed and we consider that the hypothesis H_0 stating that the survival function given the vector of explanatory variables $z(x) = (z_0(x), z_1(x), \ldots, z_m(x))$, $z_0(x) = 1$ (covariates such as temperature, stress,...etc) has the form

$$S(x|z) = S_0\left(\int_0^t e^{-\beta^T z(u)}\, du; \zeta\right),$$

where $\beta = (\beta_0, \beta_1, \ldots, \beta_m)^T$ is a vector of unknown regression parameters, the function S_0 is a specified functional of time and does not depend on z_i. If explanatory variables are constant over time, the parametric AFT model has the form

$$S(x|z) = S_0[exp(-\beta^T z)\, t; \zeta].$$

Consider the WTLW distribution as baseline distribution where

$$H_0 = F(t) = F_{\mathrm{AFT}}(x, \lambda, \beta) = F_{AFT-WTLW}.$$

So, the CDF of the AFT model can be expressed as

$$F_{AFT-WTLW}(x;\gamma,\alpha,k)=\frac{\gamma}{\gamma-\log\left\{\begin{array}{c}2\exp\left[-\left(\frac{xe^{-\beta^T z_i}}{\alpha}\right)^k\right]\\-\exp\left[-2\left(\frac{xe^{-\beta^T z_i}}{\alpha}\right)^k\right]\end{array}\right\}},x>0;\gamma,\lambda>0,$$

and then, the PDF of the AFT model can be re-expressed as

$$f_{AFT-WTLW}(x;\gamma,\alpha,k)=\frac{2\gamma\frac{k\left(xe^{-\beta^T z_i}\right)^{k-1}}{\alpha^k}\left\{\begin{array}{c}\exp\left[-2\left(\frac{xe^{-\beta^T z_i}}{\alpha}\right)^k\right]\\-\exp\left[-\left(\frac{xe^{-\beta^T z_i}}{\alpha}\right)^k\right]\end{array}\right\}}{\left(\gamma-\log\left\{\begin{array}{c}2\exp\left[-\left(\frac{xe^{-\beta^T z_i}}{\alpha}\right)^k\right]\\-\exp\left[-2\left(\frac{xe^{-\beta^T z_i}}{\alpha}\right)^k\right]\end{array}\right\}\right)^2}.$$

Analogously, the corresponding survival function (SF), HRF, and cumulative HRF of the AFT model are given by

$$S_{AFT-WTLW}(x;\gamma,\alpha,k)=S_0\left(xe^{-\beta^T z}\right)$$

$$=1-\frac{\gamma}{\gamma-\log\left\{2\exp\left[-\left(\frac{xe^{-\beta^T z_i}}{\alpha}\right)^k\right]-\exp\left[-2\left(\frac{xe^{-\beta^T z_i}}{\alpha}\right)^k\right]\right\}}$$

$$=-\frac{\log\left\{2\exp\left[-\left(\frac{xe^{-\beta^T z_i}}{\alpha}\right)^k\right]-\exp\left[-2\left(\frac{xe^{-\beta^T z_i}}{\alpha}\right)^k\right]\right\}}{\gamma-\log\left\{2\exp\left[-\left(\frac{xe^{-\beta^T z_i}}{\alpha}\right)^k\right]-\exp\left[-2\left(\frac{xe^{-\beta^T z_i}}{\alpha}\right)^k\right]\right\}},$$

and

$$h_{AFT-WTLW}(x;\gamma,\alpha,k)=-\frac{2\gamma\frac{k\left(xe^{-\beta^T z_i}\right)^{k-1}}{\alpha^k}\left\{\begin{array}{c}exp\left[-2\left(\frac{xe^{-\beta^T z_i}}{\alpha}\right)^k\right]\\-exp\left[-\left(\frac{xe^{-\beta^T z_i}}{\alpha}\right)^k\right]\end{array}\right\}}{\left[\begin{array}{c}\left(\gamma-log\left\{\begin{array}{c}2\,exp\left[-\left(\frac{xe^{-\beta^T z_i}}{\alpha}\right)^k\right]\\-exp\left[-2\left(\frac{xe^{-\beta^T z_i}}{\alpha}\right)^k\right]\end{array}\right\}\right)\\\times log\left\{\begin{array}{c}2\,exp\left[-\left(\frac{xe^{-\beta^T z_i}}{\alpha}\right)^k\right]\\-exp\left[-2\left(\frac{xe^{-\beta^T z_i}}{\alpha}\right)^k\right]\end{array}\right\}\end{array}\right]}.$$

6. The MLE for the AFT-WTLW model

In this section, we apply the maximum likelihood method to estimate the parameters of the AFT for the WTLW distribution. We give a detailed description of the method as well as the score functions and the elements of the Fisher information matrix (FIM).

6.1. The MLE derivations

Let $x_1,\dots,x_n$ be a random sample from the AFT for the WTLW model with parameters λ, γ and β. Let $\underline{V}=(\gamma,\alpha,k,\beta_0,\beta_1)^T$ be the 5×1 parameter vector. For determining the MLE of $\underline{V}$, we have the log-likelihood function

$$\begin{aligned}\ell=\ell(x;\underline{V})=n\,log\left(\frac{2\gamma k}{\alpha^k}\right)&+(k-1)\sum_{i=1}^{n}log\left(x_i e^{-\beta^T z_i}\right)-\sum_{i=1}^{n}\left(\frac{x_i e^{-\beta^T z_i}}{\alpha}\right)^k\\&+\sum_{i=1}^{n}log\left(exp\left[-\left(\frac{x_i e^{-\beta^T z_i}}{\alpha}\right)^k\right]-1\right)\\&-\sum_{i=1}^{n}log[\gamma\,log(N(x_i))-(log(N(x_i)))^2]\\&+\sum_{i=1}^{n}log(-\,log(N(x_i)))-\sum_{i=1}^{n}log(\gamma-log(N(x_i))),\end{aligned}$$

where:

$$N(x_i) = 2\,exp\left[-\left(\frac{x_i e^{-\beta^T z_i}}{\alpha}\right)^k\right] - exp\left[-2\left(\frac{x_i e^{-\beta^T z_i}}{\alpha}\right)^k\right].$$

So,

$$\frac{\partial \ell}{\partial \beta_l} = -(k-1) z_l \sum_{i=1}^{n} log\left(x_i e^{-\beta^T z_i}\right) + k z_l \sum_{i=1}^{n} v_i$$
$$+k z_l \sum_{i=1}^{n} \frac{e^{-v_i}}{e^{-v_i} - 1} v_i$$
$$+2k z_l \sum_{i=1}^{n} \frac{M(x_i) v_i [\gamma - 2\,log\,N\,(x_i)]}{N(x_i)(\gamma\,log\,N\,(x_i) - (log\,N\,(x_i))^2)}$$
$$-2k z_l \sum_{i=1}^{n} \frac{M(x_i) v_i}{N(x_i)} \left[\frac{1}{-\,log\,N\,(x_i)} - \frac{1}{\gamma - log\,N\,(x_i)}\right].$$

Setting the nonlinear system of equations $I_{(\gamma)} = 0, I_{(\alpha)} = 0, I_{(k)} = 0, I_{(\beta_0)} = 0$ and $I_{(\beta_1)} = 0$ and solving them simultaneously yields the MLE. To solve these equations, it is usually more convenient to use nonlinear optimization methods such as the quasi-Newton algorithm to numerically maximize ℓ. Since we cannot find the explicit formulas for the MLEs of parameters, we use numerical methods such as the Newton Raphson method, the Monte Carlo method, the BBO algorithm, or others.

6.2. Assessing the AFT-WTLW model via a simulation study

We carry out an important study by simulation using the R programming software. In the following, we present the results obtained by means of numerical method (the method of Newton Raphson). Suppose that the AFT for the WTLW distribution is considered. The data is iterated $N = 6000$ times, with initial values: $\gamma^{[0]} = 2.5, \alpha^{[0]} = 2.5, k^{[0]} = 1, \beta_0^{[0]} = 2.90, \beta_1^{[0]} = 0.45$ as values of the parameters. Using the BBO algorithm (see [6,7]) in R software for calculating the averages of the simulated values of the MLEs parameters and their mean squared errors (MSE), sample sizes are $n = 15{,}25{,}45{,}125{,}350$ and 600. Table 9 lists the square mean errors for the parameters' MLEs (MSE). The results obtained from the proposed methods are compelling and statistically meaningful, as demonstrated in the accompanying table. The performance of the models is not only evaluated through goodness-of-fit measures but also supported by precise parameter estimation, with relatively low standard errors even under small sample conditions. Figure 1 presents the MSEs of MLEs for the parameters of the WTLW-AFT model. This figure is crucial for evaluating the performance of the MLEs across different sample sizes and understanding how estimation accuracy improves as the sample size increases.

Table 9. MLEs ($\hat{\gamma}$,$\hat{\alpha}$,$\hat{k}$,$\widehat{\beta_0}$,$\widehat{\beta_1}$) of the parameters and their MSEs.

N=6000	n=15	n=25	n=45	n=125	n=350	n=600
$\hat{\gamma}$	1.55449	1.54081	1.52993	1.52784	1.51009	1.50487
MSE	1.7006×10^{-2}	4.2651×10^{-3}	3.625×10^{-3}	2.1935×10^{-3}	5.3625×10^{-4}	4.9527×10^{-4}
$\hat{\alpha}$	2.55412	2.55401	2.54161	2.53113	2.51418	2.50464
MSE	0.054212	0.048625	0.043218	0.036258	0.021473	0.016895
$\hat{k}$	0.90326	0.94987	0.95615	0.96203	0.97911	0.99754
MSE	0.053254	0.049998	7.0325×10^{-2}	4.9538×10^{-2}	2.3265×10^{-3}	1.9502×10^{-3}
$\widehat{\beta_0}$	2.92091	1.91660	1.91115	1.90384	1.90217	1.90062
MSE	5.0325×10^{-2}	4.9586×10^{-2}	3.3625×10^{-3}	1.0368×10^{-3}	8.2658×10^{-4}	6.3544×10^{-4}
$\widehat{\beta_1}$	0.46301	0.46001	0.45531	0.45182	0.45081	0.45042
MSE	0.053254	0.049998	7.0325×10^{-2}	4.9538×10^{-2}	2.3265×10^{-3}	1.9502×10^{-3}

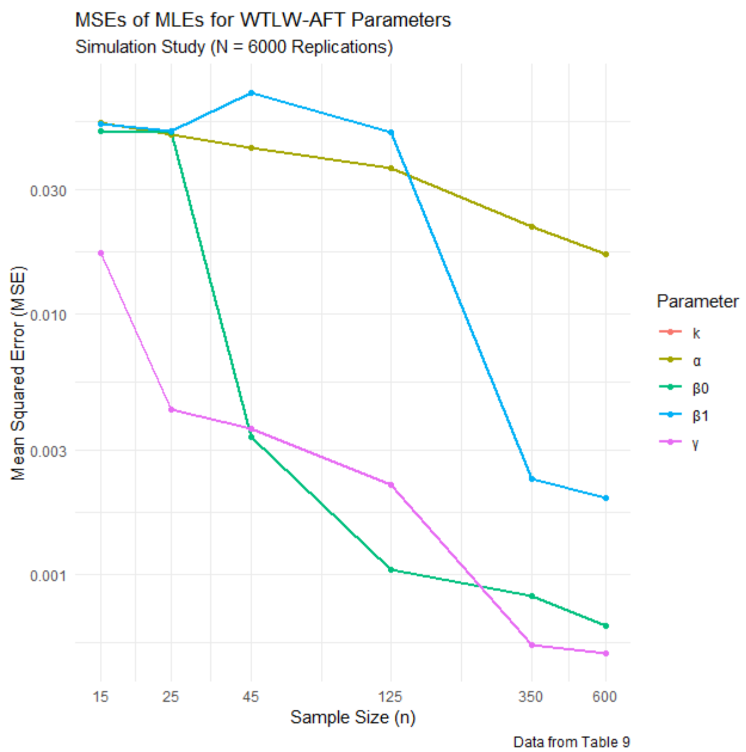

Figure 1. MSEs of MLEs for WTLW-AFT Parameters.

7. Validation of the AFT-WTLW model

Any traditional test, such as Pearson's chi-square, Kolmogorov-Smirnov statistic, Anderson Darling statistic, and other statistics, can be used to validate the selection of the model employed in analysis in the case of a well-defined distribution. However, when the parameters are unknown and must be estimated from the sample, the classical tests are no longer appropriate, and the test statistical distributions rely on the model put forth and the estimation technique utilized. In case of complete data, various techniques are used to verify the adequacy of mathematical models to data from observation. The most common tests are those based on Pearson's Chi-square statistics. Nevertheless, these cannot be applied in all situations, especially when the data is censored or when the parameters of the model are unknown. [8–11] each independently presented statistics for the whole data that is now known as the NRR statistic. At the limit, this statistic, which is based on the MLEs on the initial data, likewise exhibits a Chi-square distribution.

7.1. The NRR statistic test for the AFT-WTLW model

To test the hypothesis H_0 according to which $T_1, T_2, \cdots, T_n$, an n-sample comes from a parametric family $F_{\underline{V}}(t)$

$$H_0 \ : \ Pr\{T_i \leq t\} = F_{\underline{V}}(t), \ t \in R,$$

where $\underline{V} = (\underline{V_1}, \underline{V_2}, \cdots, \underline{V_s})^T$ represents the vector of unknown parameters, [8–11] proposed M^2 the NRR statistic defined as below. Observations $T_1, T_2, \cdots, T_n$ are grouped in r subintervals $I_1, I_2, \cdots, I_r$ mutually disjoint $I_j =]a_j$ - $1; a_j]$; where $j = \overline{1; r}$. The limits a_j of the intervals I_j are obtained such that

$$p_j(\underline{V}) = p_j(\underline{V}; a_{j-1}, a_j) = \int_{a_{j-1}}^{a_j} f_{\underline{V}}(t)\, dt|_{(j=1,2,\cdots,r)},$$

so

$$a_j = F^{-1}\left(\frac{j}{r}\right)|_{(j=1,\cdots,r-1)}.$$

If $\nu_j = (\nu_1, \nu_2, \cdots, \nu_r)^T$ is the vector of frequencies obtained by the grouping of data in these I_j intervals

$$\nu_j = \sum_{i=1}^{n} 1_{\{t_i \in I_j\}} \ |_{(j=1,\dots,r)}.$$

The NRR statistic is given in [8] where

$$X_n^2(\underline{V}) = \left(\frac{\nu_1 - np_1(\underline{V})}{\sqrt{np_1(\underline{V})}}, \frac{\nu_2 - np_2(\underline{V})}{\sqrt{np_2(\underline{V})}}, \cdots, \frac{\nu_r - np_r(\underline{V})}{\sqrt{np_r(\underline{V})}}\right)^T$$

and $J(\underline{V})$ is the information matrix for the grouped data defined by

$$J(\underline{V}) = B(\underline{V})^T B(\underline{V}),$$

with

$$B(\underline{V}) = \left[\frac{1}{\sqrt{p}_i}\frac{\partial p_i(\underline{V})}{\partial \mu}\right]_{r\times s} |_{(i=1,2,\cdots,r \text{ and } k=1,\cdots,s)},$$

then

$$L(\underline{V}) = (L_1(\underline{V}),\ldots,L_s(\underline{V}))^T \text{ with } \mathrm{L}_k(\underline{V}) = \sum_{i=1}^{r}\frac{\nu_i}{p_i}\frac{\partial}{\partial \underline{V}_k}p_i(\underline{V}).$$

The M^2 statistic follows a distribution of chi-square χ^2_{r-1} with $(r-1)$ degrees of freedom.

7.2. Simulation studies under the NRR statistic $\boldsymbol{M^2}$

Consider a sample $T_{1\,:\,n}$ where $T = T_{1\,:\,n} = (T_1, T_2, \cdots, T_n)^T$. If this data is distributed in accordance with the AFT-WTLW model, then $P\{T_{1\,:\,n} \le t\} = F_{\underline{V}}(t)$; with unknown parameters $\underline{V} = (\gamma, \alpha, k, \beta_0, \beta_1)^T$, by fitting the NRR statistic created in the preceding section, a chi-square goodness-of-fit test is created. The MLEs of the unknown parameters of the AFT-WTLW model are computed on the initial data. Since, the statistic M^2 not dependent on the parameters, we can therefore use the estimated FIM. Since the AFT-WTLW distribution are provided, the M^2 can be deducted easily. To support the results obtained in this work, a numerical simulation is performed. Therefore, in order to test the null hypothesis H_0 of the AFT-WTLW model, we calculated 5500 sample data simulations ($n = 15,\ 25,\ 45{,}125{,}350,$ and 600) from the AFT-WTLW distribution. After calculating the value of the criterion statistic M^2, we counted the number of rejected cases of the null hypothesis H_0. When $M^2 > \chi^2(k = r-1)$, the significance is a different level $\alpha(1\%, 5\%, 10\%)$. The simulation results of the rejected level of M^2 and its theoretical value are shown in Table 10 below.

Table 10. Empirical levels M^2 and corresponding theoretical levels.

$N = 6000$	$n = 15$	$n = 25$	$n = 45$	$n = 125$	$n = 350$	$n = 600$
$\alpha = 0.01$	0.0175	0.0158	0.0131	0.0120	0.0111	0.0106
$\alpha = 0.05$	0.0532	0.0520	0.0509	0.0506	0.0504	0.0501
$\alpha = 0.1$	0.1452	0.1388	0.1174	0.1077	0.1031	0.1016

The results show that the computed empirical level closely matches the corresponding theoretical level. Based on this, we conclude that the proposed test is highly appropriate for the AFT-WTLW distribution. This finding supports the claim that the M^2 statistic asymptotically follows a chi-squared distribution with degrees of freedom given by $k = r - 1$. Figure 2 presents the empirical levels of M^2 vs theoretical significance levels.

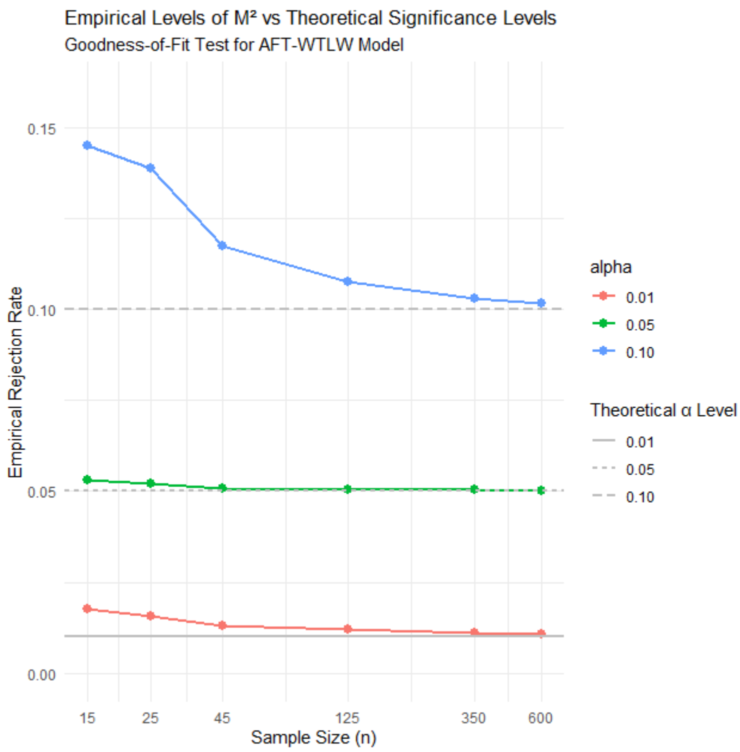

Figure 2. Empirical levels of m^2 vs theoretical significance levels.

7.3. Applications

We take into account the following real datasets and confirm the presumption that their distribution is consistent with the AFT-WTLW model in order to demonstrate the applicability of the proposed modified chi-square goodness-of-fit test.

7.3.1. Electric insulating fluid data

The failure times of 76 electrical insulating fluids tested at voltages ranging from 26 to 38 kilovolts are provided in [12], from which this information was derived. [13–15] used this data and examined its fit with the exponential and Weibull AFT power-rule models. In this part, we evaluate how well this data fits our suggested AFT-WTLW model.

(1)- In the case of $\phi(z) = z$ log-linear assumption:

Using R statistical software (the BBO package), we find the values of the MLEs of AFT-WTLW and we choose $r = 8$ intervals and then the NRR statistic: $M^2 = 16.953286$. For the critical value: $\alpha = 0.01$, we find $M^2 < \chi^2_{0.01}(7) = 18.4753$. So, we can assume that in this case, electric insulating fluid data of [12] corresponds appropriately to the AFT-WTLW model.

(2)- In the case of $\phi(z) = log(z)$ power-rule assumption:

We find the values of the MLEs of the AFT-WTLW distribution parameters, we take $r = 8$ intervals, the NRR statistic is $M^2 = 11.852647$. For the critical values: $\alpha = 0.01$, $\alpha = 0.05$ and $\alpha = 0.1$ and find that

$$M^2 < \chi^2_{0.01}(7) = 18.4753,$$

$$M^2 < \chi^2_{0.05}(7) = 14.0671,$$

$$M^2 < \chi^2_{0.1}(7) = 12.0170,$$

respectively. So, we can assume that electric insulating fluid data of [12] correspons appropriately to the AFT-WTLW model in the case of power-rule assumption.

(3)- In the case of $\boldsymbol{\phi}(z) = 1/z$ Arrhenius model:

We fit this data with the AFT-WTLW model. Using R statistical software (the BBO package) we find the values of the MLEs of the AFT-WTLW distribution parameter, and we take $r = 8$ intervals. The NRR statistic is: $M^2 = 18.80653$. For the critical value: $\alpha = 0.01$, we find $M^2 > \chi^2_{0.01}(7) = 18.4753$. In case of the Arrhenius model, we can assume that electric insulating fluid data of [12] does not correspond appropriately to our model.

7.3.2. Body fat dataset

The data of body fat (see [16]) provides information on $(n = 20)$ body fat, triceps skinfold thickness, thigh circumference, and mid-arm circumference for twenty healthy females aged 20 to 34, for $\phi(z) = z$ as a log-linear assumption. We fit this data with the AFT-WTLW model. Using R statistical software (the BBO package) we find the values of the MLEs of the AFT-WTLW distribution, and we take $r = 4$ intervals and then the NRR statistic: $M^2 = 6.19358$. For different critical values: $\alpha = 1\%, \alpha = 5\%$ and $\alpha = 10\%$, we find

$$M^2 < \chi^2_{0.01}(3) = 11.3448,\ M^2 < \chi^2_{0.05}(3) = 7.8147,$$

$$M^2 < \chi^2_{0.1}(3) = 6.2513,$$

respectively. We can assume that the body fat data can be adjusted properly to an AFT-WTLW model, in case of a log-linear assumption.

7.3.3. Johnson's dataset

[17] used a dataset with a response variable (the estimated percentage of body fat) and 13 continuous covariates (age, weight, height and 10 measurements of the body circumference) in $n = 252$ males to illustrate some problems with multiple regression analysis. The aim was to predict the percentage of body fat from the covariates. This dataset is available on the 'mfp' package in R software. In our case, we used two covariates density (density determined from underwater weighing gm/cm^3) and age (years). We consider the log linear assumption $(\phi(z) = z)$ and we fit this data by the AFT-WTLW model. The values are of the MLEs parameters. We take $r = 15$ intervals. The NRR statistic test: $M^2 = 20.512485$. For different critical values : $\alpha = 0.01, \alpha = 0.05,$ and $\alpha = 0.1$, we find $M^2 < \chi^2_{0.01}(14) = 29.1412, M^2 < \chi^2_{0.05}(14) = 23.6847$ and $M^2 < \chi^2_{0.1}(14) = 21.06414$, respectively. One can affirm that our proposed AFT-WTLW model with the log-linear assumption

$(\phi(z) = z)$ can be an appropriate distribution of this data.

8. Censored case

In this case, the log-likelihood function can be expressed as

$$\ell = \ell(x; \underline{V}) = n\, log\left(\frac{2\gamma k}{\alpha^k}\right) + (k-1)\sum_{i=1}^{n} \delta_i\, log\left(x_i e^{-\beta^T z}\right) - \sum_{i=1}^{n} \quad \delta_i \left(\frac{xe^{-\beta^T z_i}}{\alpha}\right)^k$$
$$+ \sum_{i=1}^{n} \delta_i\, log\left\{exp\left[-\left(\frac{xe^{-\beta^T z_i}}{\alpha}\right)^k\right] - 1\right\}$$
$$- \sum_{i=1}^{n} \delta_i\, log\left[\begin{array}{c} \gamma\, log\left\{2\, exp\left[-\left(\frac{xe^{-\beta^T z_i}}{\alpha}\right)^k\right] - exp\left[-2\left(\frac{xe^{-\beta^T z_i}}{\alpha}\right)^k\right]\right\} \\ -\left(log\left\{2\, exp\left[-\left(\frac{xe^{-\beta^T z_i}}{\alpha}\right)^k\right] - exp\left[-2\left(\frac{xe^{-\beta^T z_i}}{\alpha}\right)^k\right]\right\}\right)^2 \end{array}\right]$$
$$+ \sum_{i=1}^{n} log\left(-\, log\left\{2\, exp\left[-\left(\frac{xe^{-\beta^T z_i}}{\alpha}\right)^k\right] - exp\left[-2\left(\frac{xe^{-\beta^T z_i}}{\alpha}\right)^k\right]\right\}\right)$$
$$- \sum_{i=1}^{n} log\left(\gamma - log\left\{2\, exp\left[-\left(\frac{xe^{-\beta^T z_i}}{\alpha}\right)^k\right] - exp\left[-2\left(\frac{xe^{-\beta^T z_i}}{\alpha}\right)^k\right]\right\}\right),$$

where F and C denote the sets of uncensored ($\delta_i = 1$) and censored ($\delta_i = 0$) observations, respectively. The score functions for the parameters λ, γ, β_0 and β_1 are easily derived.

9. Simulation of the censored MLEs of parameters for the AFT-WTLW distribution

In this section, we conduct a comprehensive simulation study to evaluate the performance of the MLEs for the parameters of the AFT model based on the AFT-WTLW. The simulation process involves generating synthetic datasets under controlled conditions and analyzing the accuracy and precision of the MLEs for varying sample sizes. To begin, we assume that the AFT-WTLW distribution is the underlying model for the data generation process. The simulation is repeated $N = 5500$ times to ensure robustness and reliability of the results. The true parameter values used in the simulation are set as $\gamma = 0.75, \alpha = 0.75$ and $k = 0.75$, representing the scale and shift parameters of the distribution, $\beta_0 = 1.25$, corresponding to the intercept term in the regression model, $\beta_1 = 0.68$, representing the coefficient of the explanatory variable. For each replication, we generate synthetic datasets with six different sample sizes: $n = 15, n = 25, n = 45, n = 125, n = 350,$ and $n = 600$. These sample sizes span a wide range, from small to large datasets, allowing us to examine how the performance of the estimators evolves as the sample size increases. The primary objective of the simulation study is to compute the mean simulated MLEs for the parameters γ, α, k, β_0 and β_1, along with their corresponding MSEs. The mean simulated MLEs provide insights into the bias of the estimators, while the MSEs quantify both the bias and variability of the estimates, offering a comprehensive measure of their accuracy. Table 11 gives the MLEs of the parameters and their MSEs.

Table 11. MLEs ($\hat{\gamma}$,$\hat{\alpha}$,$\hat{k}$,$\widehat{\beta_0}$,$\widehat{\beta_1}$) of the parameters and their MSEs.

N=6000	n=15	n=25	n=45	n=125	n=350	n=600
$\hat{\gamma}$	1.55449	1.54081	1.52993	1.52784	1.51009	1.50487
MSE	1.7006×10^{-2}	4.2651×10^{-3}	3.625×10^{-3}	2.1935×10^{-3}	5.3625×10^{-4}	4.9527×10^{-4}
$\hat{\alpha}$	2.55412	2.55401	2.54161	2.53113	2.51418	2.50464
MSE	0.054212	0.048625	0.043218	0.036258	0.021473	0.016895
$\hat{k}$	0.90041	0.91213	0.932135	0.95999	0.967612	0.988889
MSE	0.087871	0.05111	1.221×10^{-2}	2.67×10^{-4}	3.365×10^{-3}	7.65×10^{-3}
$\widehat{\beta_0}$	2.92091	1.91660	1.91115	1.90384	1.90217	1.90062
MSE	5.0325×10^{-2}	4.9586×10^{-2}	3.3625×10^{-3}	1.0368×10^{-3}	8.2658×10^{-4}	6.3544×10^{-4}
$\widehat{\beta_1}$	0.46301	0.46001	0.45531	0.45182	0.45081	0.45042
MSE	0.053254	0.049998	7.0325×10^{-2}	4.9538×10^{-2}	2.3265×10^{-3}	1.9502×10^{-3}

These findings underscore the reliability of the MLEs for the AFT-WTLW model across different sample sizes and parameter settings. The simulation study not only validates the theoretical properties of the estimators but also provides practical guidance on their performance in real-world applications. The results demonstrate that the proposed AFT-WTLW model is well-suited for analyzing survival data, particularly when the sample size is sufficiently large to ensure accurate and stable parameter estimates.

10. Simulated distribution of M_n^2 statistic for the right-censored AFT-WTLW distribution

We compute 6000 simulations of samples data (sample sizes: $n = 15, n = 25, n = 45, n = 125,$ $n = 350,$ and $n = 600$) from AFT-WTLW distribution. After calculating the values of criteria statistics M_n^2, we count the number of rejection cases of the null hypothesis H_0, when $M_n^2 > \chi_\alpha^2(k)$, with different significance level α ($\alpha = 1\%, 5\%, 1\%$). The results of simulated levels of M_n^2 against their theoretical values are shown in the following Table 12.

Table 12. Empirical levels M^2 and corresponding theoretical levels.

$N = 6000$	$n = 15$	$n = 25$	$n = 45$	$n = 125$	$n = 350$	$n = 600$
$\alpha = 0.01$	0.01524	0.01508	0.01422	0.01400	0.01074	0.01010
$\alpha = 0.05$	0.05124	0.05106	0.05085	0.05046	0.050233	0.05010
$\alpha = 0.1$	0.15488	0.1327	0.1287	0.1193	0.1088	0.10037

As can be seen, the calculated empirical level M_n^2 values are extremely similar to the equivalent theoretical level value. Consequently, we draw the conclusion that the suggested test is excellent for the AFT-WTLW distribution.

11. Applications to real censored motor data

We consider the following real datasets and confirm the presumption that their distribution is consistent with the AFT-WTLW model in order to demonstrate the applicability of the proposed modified chi-square goodness-of-fit test.

These reliability datasets (see [18]), accessible in the survival package of R software, record the time to failure (or breakdown) of motor insulation systems under varying temperature conditions. The main goal of this data is to examine how temperature affects the lifespan and durability of motor insulation, which is essential for understanding the thermal degradation mechanisms that contribute to system failures. Such datasets are commonly utilized in reliability engineering and survival analysis to model failure times, evaluate risks, and enhance material design for better performance under thermal stress. Below, Table 13 provides a summary of the motor dataset.

Table 13. The breakdown of motor dataset.

z_1 (temperature)	x_i (time of Breakdown)	δ_i (censor)
150	8046,8064,8064,8064,8064,8046,8064,8064,8064,8064	0,0,0,0,0,0,0,0,0,0
170	1764,2772,3444,3542,3780,4860,5196,5448,5448,5448	1,1,1,1,1,1,1,0,0,0
190	408,408,1344,1344,1440,1680,1680,1680,1680,1680	1,1,1,1,1,0,0,0,0,0
220	408,408,504,504,504,528,528,528,528,528	1,1,1,1,1,0,0,0,0,0.

Using R statistical software (the BBO package) we find the values of the censored MLEs of AFT-WTLW distribution parameters, and we choose $r = 15$ intervals and then the modified NRR statistic : $M_n^2 = 28.00214$. For the critical value: $\alpha = 0.01$, we find $M_n^2 > \chi_{0.01}^2(14) = 29.1412.$

12. Conclusions

This study introduced a novel AFT model based on the WTLW distribution, designed to provide a flexible and robust framework for survival and risk analysis across actuarial, biomedical, and engineering domains. The AFT-WTLW model enriches the classical Weibull framework by incorporating additional shape parameters through the Topp-Leone generator, enabling it to capture a wider range of hazard rate behaviors, including monotonic and non-monotonic patterns. The proposed model's theoretical properties were derived and supported by simulation studies, which demonstrated the superior performance of RTADE and ADE in terms of bias, RMSE, Dabs, and Dmax across varying sample sizes. RTADE consistently outperformed other estimation techniques, especially in small-to-moderate samples, by achieving the lowest deviation metrics and highest accuracy in tail estimation. In the context of risk analysis, the model was evaluated using artificial and real-world insurance claims data. KRIs such as VaR, TVaR, TV, TMV, and EL were computed across different quantiles to assess the model's sensitivity to extreme risk. RTADE and ADE were shown to provide more conservative and stable risk estimates, particularly at higher quantiles, making them well-suited for regulatory and financial risk management applications. In contrast, MLE and LTADE tend to underestimate tail risk, while CVM occasionally exhibited excessive variance. The practical utility of the AFT-WTLW model was further validated through its application to a real insurance claims dataset. The model successfully captured the heavy-tailed and skewed characteristics of the data, with RTADE and LTADE offering the most balanced trade-off between prudence and realism. These methods align closely with regulatory frameworks such as Solvency II, which emphasize both accuracy and conservatism in risk estimation. In summary, the AFT-WTLW model represents a significant advancement in parametric survival modeling. Its ability to accommodate complex hazard behaviors, coupled with robust estimation techniques and comprehensive risk assessment tools, makes it a powerful and versatile tool for analyzing survival and claims data.

Author contributions

Mohamed Ibrahima: review and editing, data duration, software, validation, writing the original draft preparation, conceptualization; Hafida Goual: validation, data duration, conceptualization, writing the original draft preparation, data curation; Khaoula Kaouter Meribout: methodology, data duration, conceptualization, software; Ahmad M. AboAlkhair: validation, writing the original draft preparation, conceptualization, data curation, formal analysis, data duration, software; Gadir Alomair: review and editing, conceptualization, data duration, supervision; Haitham M. Yousof: review and editing, data duration, software, validation, writing the original draft preparation, conceptualization. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgements

This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia [Grant No. KFU252696].

Conflict of interest

The authors declare no conflicts of interest.

References

1. V. Bagdonavicius, M. Nikulin, *Accelerated life models: Modeling and statistical analysis*, New York: Chapman and Hall/CRC, 2001. https://doi.org/10.1201/9781420035872
2. V. Bagdonavicius, R. J. Levuliene, M. S. Nikulin, Chi-squared goodness-of-fit tests for parametric accelerated failure time models, *Commun. Stat. Theory Methods*, **42** (2013), 2768–2785. https://doi.org/10.1080/03610926.2011.617483
3. J. F. Dupuy, Accelerated failure time models: A review, *Int. J. Perform. Eng.*, **10** (2014), 23–29.
4. R. Yang, W. Chen, Maximum likelihood estimator of the shape parameter under simple random sampling and moving extremes ranked set sampling, *Stat. Probab. Lett.*, **226** (2025), 110465. https://doi.org/10.1016/j.spl.2025.110465
5. F. Lak, M. Alizadeh, D. Mazarei, R. Sharafdini, A. Dindarlou, H. M. Yousof, A novel weighted family for the reinsurance actuarial risk analysis with applications, *São Paulo J. Math. Sci.*, **19** (2025), 27. https://doi.org/10.1007/s40863-025-00502-6

6. R. Varadhan, P. Gilbert, BB: An R package for solving a large system of nonlinear equations and for optimizing a high-dimensional nonlinear objective function, *J. Stat. Softw.,* **32** (2009), 1–26. https://doi.org/10.18637/jss.v032.i04
7. H. Goual, H. M. Yousof, M. M. Ali, Validation of the odd lindley exponentiated exponential by a modified goodness of fit test with applications to censored and complete data, *Pak. J. Stat. Oper. Res.*, **15** (2019), 745–771. https://doi.org/10.18187/pjsor.v15i3.2675
8. M. S. Nikulin, L. Gerville-Réache, X. Q. Tran, On chi-squared goodness-of-fit test for normality, In: *Statistical models and methods for reliability and survival analysis*, 2013. https://doi.org/10.1002/9781118826805.ch14
9. M. S. Nikulin, Chi-squared test for continuous distributions with shift and scale parameters, *Theory Probab. Appl.*, **18** (1974), 559–568. https://doi.org/10.1137/1118069
10. A. S. Yadav, S. Shukla, H. Goual, M. Saha, H. M. Yousof, Validation of xgamma exponential model via Nikulin-Rao-Robson goodness-of- fit-test under complete and censored sample with different methods of estimation, *Stat. Optim. Inf. Comput*, **10** (2022), 457–483. https://doi.org/10.19139/soic-2310-5070-1107
11. K. C. Rao, B. S. Robson, A chi-square statistics for goodness-of-fit tests within the exponential family, *Commun. Stat.,* **3** (1974), 1139–1153. https://doi.org/10.1080/03610927408827216
12. V. Bagdonavicius, M. S. Nikulin, Chi-squared tests for general composite hypotheses from censored samples, *C. R. Math.*, **349** (2011), 219–223. https://doi.org/10.1016/j.crma.2011.01.007
13. V. Bagdonavicius, M. S. Nikulin, Chi-squared goodness-of-fit test for right censored data, *Int. J. Appl. Math. Stat.*, **24** (2011), 30–50.
14. V. Bagdonavicius, J. Kruopis, M. S. Nikulin, *Nonparametric tests for censored data*, Wiley, 2011. https://doi.org/10.1002/9781118558072
15. R. Core Team, *R: A language and environment for statistical computing*, 2014.
16. J. Abonongo, A. I. L. Abonongo, A. Aljadani, M. M. Mansour, H. M. Yousof, Accelerated failure model with empirical analysis and application to colon cancer data: Testing and validation, *Alex. Eng. J.*, **113** (2025), 391–408. https://doi.org/10.1016/j.aej.2024.10.122
17. R. W. Johnson, Fitting percentage of body fat to simple body measurements: College women, *J. Stat. Data Sci. Educ.*, **29** (2021), 304–316. https://doi.org/10.1080/26939169.2021.1971585
18 J. F. Lawless, *Statistical models and methods for lifetime data*, 2 Eds., Wiley, 2011.