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Research article

Fourth-order effective approximation of the normalized Riemann-Liouville tempered fractional derivatives and its applications

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Abstract: In this paper, a fourth-order quasi-compact approximation for the normalized Riemann-Liouville tempered fractional derivatives was proposed. Its effectiveness was proved by using the generating function method, and it was applied to the numerical solution of the two-sided space tempered fractional diffusion equation with the time Caputo tempered fractional derivative. For the time Caputo tempered fractional derivative, we transformed the Caputo tempered fractional derivative into the Riemann-Liouville tempered fractional derivative through the relationship between them, and then employed the tempered weighted and shifted Grünwald difference operator to approximate the Riemann-Liouville tempered fractional derivative in the time direction. Thus, an efficient numerical scheme with second-order accuracy in time and fourth-order accuracy in space was derived. The stability and convergence of the numerical scheme were rigorously and elaborately proved, and the effectiveness of the numerical scheme was verified by a series of simulations conducted on numerical examples.

Keywords: normalized Riemann-Liouville tempered fractional derivatives; effective fourth-order quasi-compact approximation; space-time tempered fractional diffusion equation; tempered weighted and shifted Grünwald difference operator; stability and convergence
Mathematics Subject Classification: 65M06, 65M12

1. Introduction

In this paper, the initial and boundary value problem of the following space-time tempered fractional diffusion equation is considered:

$$\begin{cases} {}^{C}D_{-,t}^{\alpha,\kappa}u(x,t) = l(D_{-,x}^{\beta,\lambda_1}u(x,t) - \lambda_1^{\beta}u(x,t) - \beta\lambda_1^{\beta-1}\frac{\partial u(x,t)}{\partial x}) \\ + r(D_{+,x}^{\beta,\lambda_2}u(x,t) - \lambda_2^{\beta}u(x,t) + \beta\lambda_2^{\beta-1}\frac{\partial u(x,t)}{\partial x}) + f(x,t), & (x,t) \in (a,b) \times (0,T], \\ u(x,0) = 0, & x \in [a,b], \\ u(a,t) = 0, u(b,t) = 0, & t \in [0,T], \end{cases} \tag{1.1}$$

where $0 < \alpha < 1$, $1 < \beta < 2$, $\lambda_1, \lambda_2 > 0$, and diffusion coefficients l and r are positive constants. ${}^{C}D_{-,t}^{\alpha,\kappa}u(x,t)$, which denotes the left Caputo tempered fractional derivative, is defined as

$$ {}^{C}D_{-,t}^{\alpha,\kappa}u(x,t) = \frac{e^{-\kappa t}}{\Gamma(1-\alpha)}(\int_0^t \frac{\frac{\partial(e^{\kappa\tau}u(x,\tau))}{\partial\tau}}{(t-\tau)^{\alpha}}d\tau), \tag{1.2}$$

based on the well-established relationship between the Caputo fractional derivative and the Riemann-Liouville fractional derivative, as elaborated in [32]. Through a series of rigorous and concise mathematical derivations, the following results can be obtained

$$ {}^{C}D_{-,t}^{\alpha,\kappa}u(x,t) = D_{-,t}^{\alpha,\kappa}u(x,t) - \frac{u(x,0)}{\Gamma(1-\alpha)}e^{-\kappa t}t^{-\alpha} = D_{-,t}^{\alpha,\kappa}u(x,t). \tag{1.3}$$

The Riemann-Liouville fractional derivatives $D_{-,t}^{\alpha,\kappa}u(x,t)$, $D_{-,x}^{\beta,\lambda_1}u(x,t)$, and $D_{+,x}^{\beta,\lambda_2}u(x,t)$ are defined by the following equations:

$$\begin{cases} D_{-,t}^{\alpha,\kappa}u(x,t) = \dfrac{e^{-\kappa t}}{\Gamma(1-\alpha)}\dfrac{\partial}{\partial t}(\displaystyle\int_0^t \frac{e^{\kappa\tau}u(x,\tau)}{(t-\tau)^{\alpha}}d\tau), \\ D_{-,x}^{\beta,\lambda_1}u(x,t) = \dfrac{e^{-\lambda_1 x}}{\Gamma(2-\beta)}\dfrac{\partial^2}{\partial x^2}(\displaystyle\int_a^x \frac{e^{\lambda_1\tau}u(\tau,t)}{(x-\tau)^{\beta-1}}d\tau), \\ D_{+,x}^{\beta,\lambda_2}u(x,t) = \dfrac{e^{\lambda_2 x}}{\Gamma(2-\beta)}\dfrac{\partial^2}{\partial x^2}(\displaystyle\int_x^b \frac{e^{-\lambda_2\tau}u(\tau,t)}{(\tau-x)^{\beta-1}}d\tau). \end{cases} \tag{1.4}$$

By making substitutions in problem (1.1) using the relationships in Eq (1.3), the following equivalent problem can be obtained:

$$\begin{cases} D_{-,t}^{\alpha,\kappa}u(x,t) = l[D_{-,x}^{\beta,\lambda_1}u(x,t) - \lambda_1^{\beta}u(x,t) - \beta\lambda_1^{\beta-1}\frac{\partial u(x,t)}{\partial x}] + r[D_{+,x}^{\beta,\lambda_2}u(x,t) \\ - \lambda_2^{\beta}u(x,t) + \beta\lambda_2^{\beta-1}\frac{\partial u(x,t)}{\partial x}] + f(x,t), & (x,t) \in (a,b) \times (0,T], \\ u(x,0) = 0, & x \in [a,b], \\ u(a,t) = 0, u(b,t) = 0, & t \in [0,T]. \end{cases} \tag{1.5}$$

In recent decades, numerous scholars have found that fractional derivatives have a wide range of applications in fields such as groundwater hydrology [8, 22, 30, 51], plasma physics [35, 37], finance [12, 44], and biology [14]. This has led to the development of fractional models. However, due to the non-locality of fractional derivatives, it is difficult to obtain the analytical solutions of fractional models. Therefore, many scholars have focused on the numerical research of fractional models [9, 11, 15, 17–26, 28, 30, 31, 34, 38, 39, 42, 45, 46, 48].

Typically, the Caputo fractional derivative is used to describe the fractional derivative in time, and the Riemann-Liouville fractional derivative is used to describe the fractional derivative in space. For the direct approximation formulas of the Caputo fractional derivative, there are the $L1$ [56], $L1-2$ [18], and $L1-2_{\sigma}$ [2] formulas. As for the direct approximation formulas of the Riemann-Liouville fractional derivative, there is the Grünwald difference operator [29] and the Lubich [7] difference operator, as well as the methods derived from their ideas [21, 27, 43, 53].

Recently, a variant of the classical fractional derivative, called the tempered fractional derivative, has attracted the attention of many scholars. The classical fractional derivative describes infinite long-range interactions with extremely slow decay (power-law decay). For the finite range of action in practical scenarios, scholars have obtained the tempered fractional derivative by introducing an exponential factor, which is more consistent with the characteristics of the finite range of action in actual physical systems. For more detailed information, please refer to [1] and [4]. The tempered fractional derivative has been applied to time, space, and space-time fractional differential equations, resulting in time [30], space [1], and space-time [13] tempered fractional differential equations, respectively. For these tempered fractional models, many scholars have done a great deal of work [5, 10, 16, 20, 23–26, 40, 41, 47, 49].

Among them, regarding the treatment of the Caputo tempered fractional derivative in time, some work has already been done. First, through the relationship between the Caputo fractional derivative and the Riemann-Liouville fractional derivative, the Caputo tempered fractional derivative in time is transformed into the Riemann-Liouville tempered fractional derivative. The Lagrange linear interpolation [52], the quasi-compact idea [19], and the tempered weighted and shifted Grünwald difference operator [54] are respectively used to obtain the second-order approximation simultaneously. Regarding the space tempered fractional diffusion equation, Baeumer and Meerschaert [4] introduced first-order tempered shifted Grünwald difference operators for approximating the left and right Riemann-Liouville tempered fractional derivatives. Subsequently, Li and Deng [6] extended these operators, developing second-order tempered weighted and shifted Grünwald difference operators. Yu et al. [50] investigated third-order quasi-compact schemes applicable to one-sided space tempered fractional diffusion equations, while Qiu [55] proposed fourth-order numerical schemes. Guo et al. [44] leveraged the concept of weighted and shifted Lubich difference operators, originally proposed in [27], to devise a fourth-order approximation method for two-sided space tempered fractional diffusion equations. However, this scheme uses points outside the interval, which reduces its applicable scope.

The contributions and novelties of this study, distinguishing it from previous works, are as follows:

(*i*) We propose an effective fourth-order quasi-compact approximation for the left and right normalized Riemann-Liouville tempered fractional derivatives.

(*ii*) The derived approximation formula is applicable to the solution of two-sided space tempered fractional diffusion equations, where the left and right tempering parameters can be distinct.

(*iii*) The developed numerical scheme avoids the utilization of points outside the computational interval.

(*iv*) The approach for achieving fourth-order approximation in the space direction is more straightforward compared to existing methodology.

The organization of the remaining sections of this paper is as follows: The effective fourth-order approximation of the normalized tempered fractional derivatives and the derivation of the numerical scheme are presented in Section 2. In Section 3, a detailed and rigorous theoretical analysis of the numerical scheme is provided. Numerical simulations are carried out in Section 4 to verify the effectiveness of the numerical scheme. In Section 5, this paper is briefly summarized, and an outlook for the follow-up work is provided.

2. Numerical method

In this section, we present an effective fourth-order quasi-compact approximation for the normalized left and right space Riemann-Liouville tempered fractional derivatives. Inspired by the fact that [27] and [44] proved the negative definiteness of matrices by means of the generating function method [36], we attempt to propose an effective fourth-order approximation for the normalized Riemann-Liouville tempered fractional derivatives via the generating function method. Different from the ideas in [44] and [27], our approximation scheme does not make use of points outside the interval. Moreover, the scheme is simpler than the previous methods. For the time Riemann-Liouville tempered fractional derivative, the tempered weighted and shifted Grünwald difference operator [6] is employed for approximation, resulting in an effective second-order approximation [54]. Finally, an efficient numerical scheme with second-order accuracy in time and fourth-order accuracy in space is derived.

2.1. Effective fourth-order approximation of the normalized tempered fractional derivatives

In this subsection, we will first present some preliminary knowledge.

$$\mathfrak{F}_{\lambda}^{n+\xi}(\mathbb{R}) = \{\mu \,|\, \mu \in L_1(\mathbb{R}), \text{and} \int_{\mathbb{R}} (|\lambda| + |\omega|)^{n+\xi} |\hat{\mu}(\omega)| d\omega < \infty\}$$

is a fractional Sobolev space $\mathfrak{F}_{\lambda}^{n+\xi}(\mathbb{R})$, where $\hat{\mu}(\omega) = \int_{\mathbb{R}} \mu(x) e^{-i\omega x} dx$ is the Fourier transform of $\mu(x)$.

To discretize the tempered fractional derivatives, we perform equidistant partitions on the spatial interval $[a, b]$ and the temporal interval $[0, T]$, respectively, obtaining the spatial grid points $x_i = a + ih, h = \frac{b-a}{M}$ $(0 \le i \le M)$, and the temporal grid points $t_n = n\tau, \tau = \frac{T}{N}$ $(0 \le n \le N)$. Now, we introduce the tempered and shifted Grünwald difference operators, as well as the approximation expansions for the tempered fractional derivatives [4, 6].

Lemma 2.1. *[4, 6] Let $m < \xi < m + 1$ ($m \in \mathbb{N}$), $\lambda \ge 0$, the shift number p is an integer, h is the step size, and $\mu(x)$ is defined on the bounded interval $[a, b]$ and belongs to $\mathfrak{F}_{\lambda}^{n+\xi}(\mathbb{R})$ after zero extension on the interval $x \in (-\infty, a) \cup (b, +\infty)$. The tempered and shifted Grünwald difference operators are defined as*

$$\begin{cases} A_{h,p}^{\xi,\lambda}\mu(x) = \dfrac{1}{h^{\xi}} \displaystyle\sum_{k=0}^{[\frac{x-a}{h}]+p} g_k^{(\xi)} e^{-(k-p)\lambda h} \mu(x - (k-p)h), \\ \hat{A}_{h,p}^{\xi,\lambda}\mu(x) = \dfrac{1}{h^{\xi}} \displaystyle\sum_{k=0}^{[\frac{b-x}{h}]+p} g_k^{(\xi)} e^{-(k-p)\lambda h} \mu(x + (k-p)h), \end{cases} \tag{2.1}$$

and then

$$\begin{cases} A_{h,p}^{\xi,\lambda}\mu(x) = D_{-,x}^{\xi,\lambda}\mu(x) + \sum_{k=1}^{n-1} c_k^{\xi,p} D_{-,x}^{k+\xi,\lambda}\mu(x)h^k + O(h^n), \\ \hat{A}_{h,p}^{\xi,\lambda}\mu(x) = D_{+,x}^{\xi,\lambda}\mu(x) + \sum_{k=1}^{n-1} c_k^{\xi,p} D_{+,x}^{k+\xi,\lambda}\mu(x)h^k + O(h^n), \end{cases} \tag{2.2}$$

where $g_k^{(\xi)} = (-1)^k\binom{\xi}{k}(k \geq 0)$ *represents the normalized Grünwald weights, which are derived from*

$$(1-s)^{\xi} = \sum_{k=0}^{+\infty} g_k^{(\xi)} s^k, \tag{2.3}$$

$c_k^{\xi,p}$ *are the power series expansion coefficients of the function* $W_p(s) = e^{ps}(\frac{1-e^{-s}}{s})^{\xi} = \sum_{k=0}^{+\infty} c_k^{\xi,p} s^k$, *and the first four coefficients are given as*

$$\begin{cases} c_0^{\xi,p} = 1, \\ c_1^{\xi,p} = p - \dfrac{\xi}{2}, \\ c_2^{\xi,p} = \dfrac{12p^2 - 12p\xi + \xi + 3\xi^2}{24}, \\ c_3^{\xi,p} = \dfrac{8p^3 - 12p^2\xi + 2p(\xi + 3\xi^2) - \xi^2 - \xi^3}{48}. \end{cases} \tag{2.4}$$

It is worth noting that Li and Deng [6] proposed a second-order approximation of the tempered fractional derivative (the tempered weighted and shifted Grünwald difference operators) via a weighting approach, which opened up a new avenue for effectively enhancing the spatial convergence order. Guo et al. [44] derived a fourth-order approximation of tempered fractional derivatives by adopting the idea of weighted and shifted Lubich difference operators [27], but with a shift number $p = 2$ that exceeds the spatial interval. Additionally, the quasi-compact approximation framework for tempered fractional derivatives was developed [50, 55], yet its application was restricted to a one-sided spatial tempered fractional diffusion equation. In what follows, we will present an effective fourth-order quasi-compact approximation for tempered fractional derivatives, which is applicable to solving a broader class of spatial tempered fractional diffusion equations.

Remark 2.1. *To construct the quasi-compact approximation, we make the system of equations satisfy the following:*

$$\begin{cases} \gamma_1^{(\xi)} + \gamma_0^{(\xi)} + \gamma_{-1}^{(\xi)} = 1, \\ \gamma_1^{(\xi)} c_1^{\xi,1} + \gamma_0^{(\xi)} c_1^{\xi,0} + \gamma_{-1}^{(\xi)} c_1^{\xi,-1} = 0, \\ \gamma_1^{(\xi)} c_3^{\xi,1} + \gamma_0^{(\xi)} c_3^{\xi,0} + \gamma_{-1}^{(\xi)} c_3^{\xi,-1} = 0, \end{cases} \tag{2.5}$$

where $\xi \in \{\beta, \beta+1\}$, and then

$$\begin{cases} \gamma_1^{(\xi)} = \frac{1}{12}(\xi^2 + 3\xi + 2), \\ \gamma_0^{(\xi)} = \frac{1}{6}(-\xi^2 + 4), \\ \gamma_{-1}^{(\xi)} = \frac{1}{12}(\xi^2 - 3\xi + 2). \end{cases}$$

Under the above construction, we can first obtain the following second-order approximation of the tempered fractional derivatives:

$$\begin{cases} B_h^{\xi,\lambda}\mu(x) = \sum_{p=-1}^{1} \gamma_p^{(\xi)} A_{h,p}^{\xi,\lambda}\mu(x) = \frac{1}{h^\xi} \sum_{k=0}^{[\frac{x-a}{h}]+1} w_k^{(\xi,\lambda)}\mu(x-(k-1)h) \\ \qquad = D_{-,x}^{\xi,\lambda}\mu(x) + O(h^2), \\ \hat{B}_h^{\xi,\lambda}\mu(x) = \sum_{p=-1}^{1} \gamma_p^{(\xi)} \hat{A}_{h,p}^{\xi,\lambda}\mu(x) = \frac{1}{h^\xi} \sum_{k=0}^{[\frac{b-x}{h}]+1} w_k^{(\xi,\lambda)}\mu(x+(k-1)h) \\ \qquad = D_{+,x}^{\xi,\lambda}\mu(x) + O(h^2), \end{cases} \tag{2.6}$$

where

$$w_k^{(\xi,\lambda)} = (\gamma_1^{(\xi)} g_k^{(\xi)} + \gamma_0^{(\xi)} g_{k-1}^{(\xi)} + \gamma_{-1}^{(\xi)} g_{k-2}^{(\xi)}) e^{-(k-1)\lambda h} \quad (k \geq 0, g_{-2}^{(\xi)} = g_{-1}^{(\xi)} = 0). \tag{2.7}$$

In particular, when $\xi = 1$ and 2, there are the following approximations:

$$\begin{cases} D_{-,x}^{1,\lambda}\mu(x) = \sum_{p=-1}^{1} \gamma_p^{(1)} A_{h,p}^{1,\lambda}\mu(x) + O(h^2) \\ \qquad = \frac{1}{2h}[e^{\lambda h}\mu(x+h) - e^{-\lambda h}\mu(x-h)] + O(h^2) \\ \qquad = B_h^{1,\lambda}\mu(x) + O(h^2), \\ D_{+,x}^{1,\lambda}\mu(x) = \sum_{p=-1}^{1} \gamma_p^{(1)} \hat{A}_{h,p}^{1,\lambda}\mu(x) + O(h^2) \\ \qquad = \frac{1}{2h}[e^{\lambda h}\mu(x-h) - e^{-\lambda h}\mu(x+h)] + O(h^2) \\ \qquad = \hat{B}_h^{1,\lambda}\mu(x) + O(h^2), \end{cases} \tag{2.8}$$

$$\begin{cases} D^{2,\lambda}_{-,x}\mu(x) = \sum_{p=-1}^{1} \gamma_p^{(2)} A^{2,\lambda}_{h,p}\mu(x) + O(h^2) \\ \qquad = \frac{1}{h^2}[e^{\lambda h}\mu(x+h) - 2\mu(x) + e^{-\lambda h}\mu(x-h)] + O(h^2) \\ \qquad = B_h^{2,\lambda}\mu(x) + O(h^2), \\ D^{2,\lambda}_{+,x}\mu(x) = \sum_{p=-1}^{1} \gamma_p^{(2)} \hat{A}^{2,\lambda}_{h,p}\mu(x) + O(h^2) \\ \qquad = \frac{1}{h^2}[e^{-\lambda h}\mu(x+h) - 2\mu(x) + e^{\lambda h}\mu(x-h)] + O(h^2) \\ \qquad = \hat{B}_h^{2,\lambda}\mu(x) + O(h^2). \end{cases} \tag{2.9}$$

In order to construct an effective fourth-order quasi-compact approximation smoothly, we first give a definition and two lemmas.

Definition 2.1. *[36] Let $n \times n$ Toeplitz matrix T_n be of the form:*

$$T_n = \begin{pmatrix} t_0 & t_{-1} & \dots & t_{2-n} & t_{1-n} \\ t_1 & t_0 & t_{-1} & \dots & t_{2-n} \\ \vdots & t_1 & t_0 & \ddots & \vdots \\ t_{n-2} & \dots & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} & \dots & t_1 & t_0 \end{pmatrix}, \tag{2.10}$$

i.e., $t_{i,j} = t_{i-j}$, and T_n is constant along its diagonals. Assume that the diagonals $\{t_k\}_{k=-n+1}^{n-1}$ are the Fourier coefficients of a function $f(x)$, i.e.,

$$t_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx. \tag{2.11}$$

Then the function $f(x)$ is called the generating function of T_n.

Lemma 2.2. *[36] Let T_n be a Toeplitz matrix with the generating function $f(x)$, being a 2π-periodic continuous real-valued function. Denote $\lambda_{min}(T_n)$ and $\lambda_{max}(T_n)$ as the smallest and largest eigenvalues of T_n, respectively. Then we have*

$$f_{min}(x) \le \lambda_{min}(T_n) \le \lambda_{max}(T_n) \le f_{max}(x), \tag{2.12}$$

where $f_{min}(x)$ and $f_{max}(x)$ denote the minimum and maximum values of $f(x)$, respectively. In particular, if $f(x)$ is a non-positive function and is not always zero, and $f_{min}(x) \neq f_{max}(x)$,

$$f_{min}(x) < \lambda(T_n) < f_{max}(x). \tag{2.13}$$

Lemma 2.3. *[33] A real matrix A of order M is positive definite if and only if $D = \frac{A+A^T}{2}$ is positive definite.*

Next, we present an effective fourth-order quasi-compact approximation of the following normalized tempered fractional derivatives.

Lemma 2.4. *Let* $\mu(x) \in \mathfrak{F}_{\lambda}^{4+\beta}(\mathbb{R})$, $1 < \beta < 2$, $\lambda_1, \lambda_2 > 0$, $c_\beta = \frac{-\beta^2+\beta+4}{24}$, *and the continuous operators* $\Lambda_x^{\beta,\lambda_1} = (I + c_\beta h^2 D_{-,x}^{2,\lambda_1})$ *and* $\Delta_x^{\beta,\lambda_2} = (I + c_\beta h^2 D_{+,x}^{2,\lambda_2})$ *are given to operate on the normalized left and right Riemann-Liouville tempered fractional derivatives, respectively. Then*
(i) *when* $0 < \lambda_2 h \le \lambda_1 h \le 1/5$,

$$\begin{cases} \Lambda_x^{\beta,\lambda_1}[D_{-,x}^{\beta,\lambda_1}\mu(x) - \lambda_1^\beta \mu(x) - \beta\lambda_1^{\beta-1}\dfrac{\partial\mu(x)}{\partial x}] = \Lambda_x^{\beta,\lambda_1} D_{-,x}^{\beta,\lambda_1}\mu(x) + (\beta-1)\lambda_1^\beta \Lambda_x^{\beta,\lambda_1}\mu(x) \\ -\beta\lambda_1^{\beta-1}\Lambda_x^{1,\lambda_1} D_{-,x}^{1,\lambda_1}\mu(x) + \beta\lambda_1^{\beta-1}(\dfrac{1}{6} - c_\beta)h^2 D_{-,x}^{2,\lambda_1}(D_{-,x}^{1,\lambda_1}\mu(x)), \end{cases} \tag{2.14}$$

$$\begin{cases} \Lambda_x^{\beta,\lambda_1}[D_{+,x}^{\beta,\lambda_2}\mu(x) - \lambda_2^\beta \mu(x) + \beta\lambda_2^{\beta-1}\dfrac{\partial\mu(x)}{\partial x}] = \Delta_x^{\beta,\lambda_2} D_{+,x}^{\beta,\lambda_2}\mu(x) + (\beta-1)\lambda_2^\beta \Lambda_x^{\beta,\lambda_1}\mu(x) \\ -\beta\lambda_2^{\beta-1}\Delta_x^{1,\lambda_2} D_{+,x}^{1,\lambda_2}\mu(x) + \beta\lambda_2^{\beta-1}(\dfrac{1}{6} - c_\beta)h^2 D_{+,x}^{2,\lambda_2}(D_{+,x}^{1,\lambda_2}\mu(x)) \\ + c_\beta h^2[(\lambda_1^2 - \lambda_2^2)I][D_{+,x}^{\beta,\lambda_2}\mu(x) - \beta\lambda_2^{\beta-1}D_{+,x}^{1,\lambda_2}\mu(x)] \\ + 2c_\beta h^2(\lambda_1 + \lambda_2)[\lambda_2 I - D_{+,x}^{1,\lambda_2}\mu(x)][D_{+,x}^{\beta,\lambda_2}\mu(x) - \beta\lambda_2^{\beta-1}D_{+,x}^{1,\lambda_2}\mu(x)]; \end{cases} \tag{2.15}$$

(ii) *when* $0 < \lambda_1 h < \lambda_2 h \le 1/5$,

$$\begin{cases} \Delta_x^{\beta,\lambda_2}[D_{-,x}^{\beta,\lambda_1}\mu(x) - \lambda_1^\beta \mu(x) - \beta\lambda_1^{\beta-1}\dfrac{\partial\mu(x)}{\partial x}] = \Lambda_x^{\beta,\lambda_1} D_{-,x}^{\beta,\lambda_1}\mu(x) + (\beta-1)\lambda_1^\beta \Delta_x^{\beta,\lambda_2}\mu(x) \\ -\beta\lambda_1^{\beta-1}\Lambda_x^{1,\lambda_1} D_{-,x}^{1,\lambda_1}\mu(x) + \beta\lambda_1^{\beta-1}(\dfrac{1}{6} - c_\beta)h^2 D_{-,x}^{2,\lambda_1}(D_{-,x}^{1,\lambda_1}\mu(x)) \\ + c_\beta h^2[(\lambda_2^2 - \lambda_1^2)I][D_{-,x}^{\beta,\lambda_1}\mu(x) - \beta\lambda_1^{\beta-1}D_{-,x}^{1,\lambda_1}\mu(x)] \\ + 2c_\beta h^2(\lambda_1 + \lambda_2)[\lambda_1 I - D_{-,x}^{1,\lambda_1}\mu(x)][D_{-,x}^{\beta,\lambda_1}\mu(x) - \beta\lambda_1^{\beta-1}D_{-,x}^{1,\lambda_1}\mu(x)], \end{cases} \tag{2.16}$$

$$\begin{cases} \Delta_x^{\beta,\lambda_2}[D_{+,x}^{\beta,\lambda_2}\mu(x) - \lambda_2^\beta \mu(x) + \beta\lambda_2^{\beta-1}\dfrac{\partial\mu(x)}{\partial x}] = \Delta_x^{\beta,\lambda_2} D_{+,x}^{\beta,\lambda_2}\mu(x) + (\beta-1)\lambda_2^\beta \Delta_x^{\beta,\lambda_2}\mu(x) \\ -\beta\lambda_2^{\beta-1}\Delta_x^{1,\lambda_2} D_{+,x}^{1,\lambda_2}\mu(x) + \beta\lambda_2^{\beta-1}(\dfrac{1}{6} - c_\beta)h^2 D_{+,x}^{2,\lambda_2}(D_{+,x}^{1,\lambda_2}\mu(x)). \end{cases} \tag{2.17}$$

They have the following effective fourth-order approximation:

$$\begin{cases} \Lambda_x^{\beta,\lambda_1}[D_{-,x}^{\beta,\lambda_1}\mu(x) - \lambda_1^\beta \mu(x) - \beta\lambda_1^{\beta-1}\dfrac{\partial\mu(x)}{\partial x}] \\ \qquad = B_h^{\beta,\lambda_1}\mu(x) + (\beta-1)\lambda_1^\beta C_h^{\beta,\lambda_1}\mu(x) - \beta\lambda_1^{\beta-1}B_h^{1,\lambda_1}\mu(x) \\ \qquad + \beta\lambda_1^{\beta-1}(\dfrac{1}{6} - c_\beta)h^2[\lambda_1^3\mu(x) + 3\lambda_1^2\delta_x\mu(x) + 3\lambda_1\delta_x^2\mu(x) + \delta_{x,1}^3\mu(x)] + O(h^4), \end{cases} \tag{2.18}$$

$$\begin{cases}\Lambda_x^{\beta,\lambda_1}[D_{+,x}^{\beta,\lambda_2}\mu(x)-\lambda_2^{\beta}\mu(x)+\beta\lambda_2^{\beta-1}\dfrac{\partial\mu(x)}{\partial x}]\\ \qquad =\hat{B}_h^{\beta,\lambda_2}\mu(x)+(\beta-1)\lambda_2^{\beta}C_h^{\beta,\lambda_1}\mu(x)-\beta\lambda_2^{\beta-1}\hat{B}_h^{1,\lambda_2}\mu(x)\\ \qquad +\beta\lambda_2^{\beta-1}(\dfrac{1}{6}-c_\beta)h^2[\lambda_2^3\mu(x)-3\lambda_2^2\delta_x\mu(x)+3\lambda_2\delta_x^2\mu(x)-\delta_{x,2}^3\mu(x)]\\ \qquad +c_\beta h^2(\lambda_1^2-\lambda_2^2)[\hat{B}_h^{\beta,\lambda_2}\mu(x)-\beta\lambda_2^{\beta-1}\hat{B}_h^{1,\lambda_2}\mu(x)]\\ \qquad +2c_\beta h^2(\lambda_1+\lambda_2)[\lambda_2\hat{B}_h^{\beta,\lambda_2}\mu(x)-\hat{B}_h^{\beta+1,\lambda_2}\mu(x)\\ \qquad -\beta\lambda_2^{\beta-1}(\lambda_2\hat{B}_h^{1,\lambda_2}\mu(x)-\hat{B}_h^{2,\lambda_2}\mu(x))]+O(h^4),\end{cases}\tag{2.19}$$

$$\begin{cases}\Delta_x^{\beta,\lambda_2}[D_{-,x}^{\beta,\lambda_1}\mu(x)-\lambda_1^{\beta}\mu(x)-\beta\lambda_1^{\beta-1}\dfrac{\partial\mu(x)}{\partial x}]\\ =B_h^{\beta,\lambda_1}\mu(x)+(\beta-1)\lambda_1^{\beta}\hat{C}_h^{\beta,\lambda_2}\mu(x)-\beta\lambda_1^{\beta-1}B_h^{1,\lambda_1}\mu(x)\\ +\beta\lambda_1^{\beta-1}(\dfrac{1}{6}-c_\beta)h^2[\lambda_1^3\mu(x)+3\lambda_1^2\delta_x\mu(x)+3\lambda_1\delta_x^2\mu(x)+\delta_{x,1}^3\mu(x)]\\ +c_\beta h^2(\lambda_2^2-\lambda_1^2)[B_h^{\beta,\lambda_1}\mu(x)-\beta\lambda_1^{\beta-1}B_h^{1,\lambda_1}\mu(x)]\\ +2c_\beta h^2(\lambda_1+\lambda_2)[\lambda_1 B_h^{\beta,\lambda_1}\mu(x)-B_h^{\beta+1,\lambda_1}\mu(x)\\ -\beta\lambda_1^{\beta-1}(\lambda_1 B_h^{1,\lambda_1}\mu(x)-B_h^{2,\lambda_1}\mu(x))]+O(h^4),\end{cases}\tag{2.20}$$

and

$$\begin{cases}\Delta_x^{\beta,\lambda_2}[D_{+,x}^{\beta,\lambda_2}\mu(x)-\lambda_2^{\beta}\mu(x)+\beta\lambda_2^{\beta-1}\dfrac{\partial\mu(x)}{\partial x}]\\ =\hat{B}_h^{\beta,\lambda_2}\mu(x)+(\beta-1)\lambda_2^{\beta}\hat{C}_h^{\beta,\lambda_2}\mu(x)-\beta\lambda_2^{\beta-1}\hat{B}_h^{1,\lambda_2}\mu(x)\\ +\beta\lambda_2^{\beta-1}(\dfrac{1}{6}-c_\beta)h^2[\lambda_2^3\mu(x)-3\lambda_2^2\delta_x\mu(x)+3\lambda_2\delta_x^2\mu(x)-\delta_{x,2}^3\mu(x)]+O(h^4),\end{cases}\tag{2.21}$$

where

$$\begin{cases}B_h^{\xi,\lambda}\mu(x)=\displaystyle\sum_{p=-1}^{1}\gamma_p^{(\xi)}A_{h,p}^{\xi,\lambda}\mu(x)=\frac{1}{h^\xi}\sum_{k=0}^{[\frac{x-a}{h}]+1}w_k^{(\xi,\lambda)}\mu(x-(k-1)h),\\ \hat{B}_h^{\xi,\lambda}\mu(x)=\displaystyle\sum_{p=-1}^{1}\gamma_p^{(\xi)}\hat{A}_{h,p}^{\xi,\lambda}\mu(x)=\frac{1}{h^\xi}\sum_{k=0}^{[\frac{b-x}{h}]+1}w_k^{(\xi,\lambda)}\mu(x+(k-1)h),\\ w_k^{(\xi,\lambda)}=(\gamma_1^{(\xi)}g_k^{(\xi)}+\gamma_0^{(\xi)}g_{k-1}^{(\xi)}+\gamma_{-1}^{(\xi)}g_{k-2}^{(\xi)})e^{-(k-1)\lambda h}\quad(k\geq 0,g_{-2}^{(\xi)}=g_{-1}^{(\xi)}=0),\\ \displaystyle\sum_{k=0}^{+\infty}w_k^{(\xi,\lambda)}=(\gamma_1^{(\xi)}e^{\lambda h}+\gamma_0^{(\xi)}+\gamma_{-1}^{(\xi)}e^{-\lambda h})(1-e^{-\lambda h})^{\xi},\\ \gamma_1^{(\xi)}=\dfrac{1}{12}(\xi^2+3\xi+2),\ \ \gamma_0^{(\xi)}=\dfrac{1}{6}(-\xi^2+4),\ \ \gamma_{-1}^{(\xi)}=\dfrac{1}{12}(\xi^2-3\xi+2),\end{cases}\tag{2.22}$$

$$\begin{cases} C_h^{\beta,\lambda}\mu(x) = \mu(x) + c_\beta h^2 e^{-\lambda x}\delta_x^2[e^{\lambda x}\mu(x)], \\ \hat{C}_h^{\beta,\lambda}\mu(x) = \mu(x) + c_\beta h^2 e^{\lambda x}\delta_x^2[e^{-\lambda x}\mu(x)], \\ \delta_x\mu(x) = \dfrac{1}{2h}[\mu(x+h) - \mu(x-h)], \\ \delta_x^2\mu(x) = \dfrac{1}{h^2}[\mu(x+h) - 2\mu(x) + \mu(x-h)], \\ \delta_{x,1}^3\mu(x) = \dfrac{1}{2h^3}[\mu(x-3h) - 6\mu(x-2h) + 12\mu(x-h) - 10\mu(x) + 3\mu(x+h)], \\ \delta_{x,2}^3\mu(x) = \dfrac{1}{2h^3}[-\mu(x+3h) + 6\mu(x+2h) - 12\mu(x+h) + 10\mu(x) - 3\mu(x-h)], \end{cases} \tag{2.23}$$

$\xi \in \{1, 2, \beta, \beta + 1\}$, $\lambda \in \{\lambda_1, \lambda_2\}$.

Proof. Next, we prove this lemma in two parts.

(i) (Fourth-order quasi-compact approximation of tempered fractional derivatives)

From the definition of the tempered derivatives, we can easily derive

$$\begin{cases} D_{-,x}^{1,\lambda_1} = \lambda_1 I + D, \\ D_{+,x}^{1,\lambda_2} = \lambda_2 I - D, \\ D_{-,x}^{2,\lambda_1} = \lambda_1^2 I + 2\lambda_1 D + D^2, \\ D_{+,x}^{2,\lambda_2} = \lambda_2^2 I - 2\lambda_2 D + D^2, \\ \Lambda_x^{\beta,\lambda_1} = \Delta_x^{\beta,\lambda_2} + c_\beta h^2[(\lambda_1^2 - \lambda_2^2)I + 2(\lambda_1 + \lambda_2)D], \end{cases} \tag{2.24}$$

with I, D, and D^2 being the identity operator, first-order differential operator, and second-order differential operator, respectively. Therefore, Eqs (2.18)–(2.21) hold immediately.

(ii) (The effectiveness of fourth-order approximation)

In fact, for the numerical scheme to be valid, we need to ensure that the matrix corresponding to the discretization of the fractional derivative is negative definite.

Let the matrices corresponding to the discrete operators in the approximation formulas (2.18) and (2.20) be $B^{(\xi,\lambda)}$, $C^{(\beta,\lambda)}$, E, $D^{(1)}$, $D^{(2)}$, and $D^{(3)}$, respectively. Here

$$B^{(\xi,\lambda)} = \frac{1}{h^\xi}\begin{pmatrix} w_1^{(\xi,\lambda)} & w_0^{(\xi,\lambda)} & & & & \\ w_2^{(\xi,\lambda)} & w_1^{(\xi,\lambda)} & w_0^{(\xi,\lambda)} & & & \\ \vdots & \vdots & \ddots & \ddots & & \\ \vdots & \vdots & \vdots & \ddots & \ddots & \\ w_{M-2}^{(\xi,\lambda)} & w_{M-3}^{(\xi,\lambda)} & w_{M-4}^{(\xi,\lambda)} & \dots & w_1^{(\xi,\lambda)} & w_0^{(\xi,\lambda)} \\ w_{M-1}^{(\xi,\lambda)} & w_{M-2}^{(\xi,\lambda)} & w_{M-3}^{(\xi,\lambda)} & \dots & w_2^{(\xi,\lambda)} & w_1^{(\xi,\lambda)} \end{pmatrix}, \tag{2.25}$$

$$D^{(3)} = \frac{1}{2h^3}\begin{pmatrix} -10 & 3 & & & & \\ 12 & -10 & 3 & & & \\ -6 & 12 & -10 & 3 & & \\ 1 & -6 & 12 & -10 & 3 & \\ & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & 1 & -6 & 12 & -10 \end{pmatrix}, \tag{2.26}$$

$C^{(\beta,\lambda)} = tridiag\{c_\beta e^{-\lambda h}, 1-2c_\beta, c_\beta e^{\lambda h}\}$, $D^{(1)} = \frac{1}{2h} tridiag\{-1, 0, 1\}$, and $D^{(2)} = \frac{1}{h^2} tridiag\{1, -2, 1\}$ are tridiagonal matrices, and E is the identity matrix.

(a) When $0 < \lambda_2 h \leq \lambda_1 h \leq 1/5$, denote the matrix corresponding to the discrete operator in the approximation formula (2.18) as $P = B^{(\beta,\lambda_1)} + (\beta-1)\lambda_1^\beta C^{(\beta,\lambda_1)} - \beta\lambda_1^{\beta-1} B^{(1,\lambda_1)} + (\frac{1}{6} - c_\beta)\beta\lambda_1^{\beta-1} h^2(\lambda_1^3 E + 3\lambda_1^2 D^{(1)} + 3\lambda_1 D^{(2)} + D^{(3)})$. Let $H = \frac{P+P^T}{2}$. According to Definition 2.1, the generating function of the matrix $h^\beta H$ is a real-valued function $h^\beta f_H(\beta, \lambda_1 h, \lambda_2 h, x)$ defined on $[-\pi, \pi]$ as follows:

$$\begin{aligned} h^\beta f_H(\beta, \lambda_1 h, \lambda_2 h, x) &= \frac{1}{2}\{\sum_{p=-1}^{1} \gamma_p^{(\beta)} e^{p\lambda_1 h}(1 + e^{-2\lambda_1 h} - 2e^{-\lambda_1 h}\cos x)^{\frac{\beta}{2}} 2\cos(\beta\theta - px) \\ &+ (\beta-1)(\lambda_1 h)^\beta[2c_\beta \cos x(e^{\lambda_1 h} + e^{-\lambda_1 h}) + 2(1-2c_\beta)] - \beta(\lambda_1 h)^{\beta-1}\cos x(e^{\lambda_1 h} - e^{-\lambda_1 h}) \\ &+ (\frac{1}{6} - c_\beta)\beta(\lambda_1 h)^{\beta-1}[2(\lambda_1 h)^3 + 12\lambda_1 h(\cos x - 1) + \cos 3x - 6\cos 2x + 15\cos x - 10]\}, \end{aligned} \tag{2.27}$$

where

$$\theta = \arctan\frac{-e^{-\lambda_1 h}\sin x}{1 - e^{-\lambda_1 h}\cos x}. \tag{2.28}$$

It is easy to verify that the function $h^\beta H$ is an even function on the interval $[-\pi, \pi]$. Therefore, we consider the case of $[0, \pi]$. In fact, when $0 < \lambda_2 h \leq \lambda_1 h \leq 1/5$, the values of the function $h^\beta H$ in the region $(1, 2) \times [0, \pi]$ are negative. See Figure 1 for details. By Lemmas 2.2 and 2.3, we conclude that the matrix P is negative definite. That is, the approximation formula is valid.

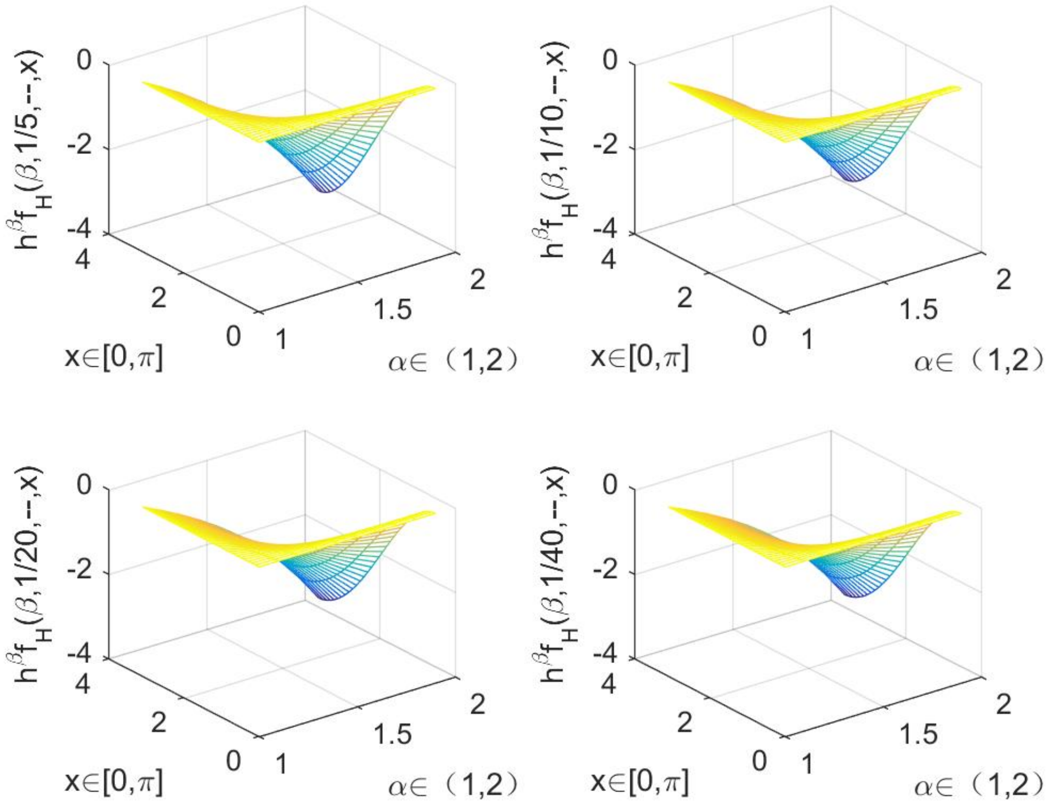

Figure 1. Images of the generating function $h^{\beta} f_H(\beta, \lambda_1 h, \lambda_2 h, x)$ in the finite domain $(1, 2) \times [0, \pi]$ when $\lambda_1 h = 1/5, 1/10, 1/20, 1/40$.

(b) When $0 < \lambda_2 h < \lambda_1 h \leq 1/5$, denote the matrix corresponding to the discrete operator in the approximation formula (2.19) as $Q = \{B^{(\beta,\lambda_2)} - \beta\lambda_2^{\beta-1}B^{(1,\lambda_2)} + (\frac{1}{6} - c_\beta)\beta\lambda_2^{\beta-1}h^2(\lambda_2^3 E + 3\lambda_2^2 D^{(1)} + 3\lambda_2 D^{(2)} + D^{(3)}) + c_\beta h^2(\lambda_1^2 - \lambda_2^2)(B^{(\beta,\lambda_2)} - \beta\lambda_2^{\beta-1}B^{(1,\lambda_2)}) + 2c_\beta h^2(\lambda_1 + \lambda_2)[\lambda_2 B^{(\beta,\lambda_2)} - B^{(\beta+1,\lambda_2)} - \beta\lambda_2^{\beta-1}(\lambda_2 B^{(1,\lambda_2)} - B^{(2,\lambda_2)})]\}^T + (\beta - 1)\lambda_2^{\beta} C^{(\beta,\lambda_1)}$. Let $H = \frac{Q+Q^T}{2}$. According to Definition 2.1, the generating function of the matrix $h^{\beta}H$ is a real-valued function $h^{\beta} f_H(\beta, \lambda_1 h, \lambda_2 h, x)$ defined on $[-\pi, \pi]$ as follows:

$$
\begin{aligned}
h^{\beta} f_H(\beta, \lambda_1 h, \lambda_2 h, x) &= \frac{1}{2}\{\sum_{p=-1}^{1} \gamma_p^{(\beta)} e^{p\lambda_2 h}(1 + e^{-2\lambda_2 h} - 2e^{-\lambda_2 h}\cos x)^{\frac{\beta}{2}} 2\cos(\beta\theta - px) \\
&+ (\beta - 1)(\lambda_2 h)^{\beta}[2c_\beta \cos x(e^{\lambda_1 h} + e^{-\lambda_1 h}) + 2(1 - 2c_\beta)] - \beta(\lambda_2 h)^{\beta-1}\cos x(e^{\lambda_2 h} - e^{-\lambda_2 h}) \\
&+ (\frac{1}{6} - c_\beta)\beta(\lambda_2 h)^{\beta-1}[2(\lambda_2 h)^3 + 12\lambda_2 h(\cos x - 1) + \cos 3x - 6\cos 2x + 15\cos x - 10] \\
&+ c_\beta[(\lambda_1 h)^2 - (\lambda_2 h)^2]\{\sum_{p=-1}^{1} \gamma_p^{(\beta)} e^{p\lambda_2 h}(1 + e^{-2\lambda_2 h} - 2e^{-\lambda_2 h}\cos x)^{\frac{\beta}{2}} 2\cos(\beta\theta - px) \\
&- \beta(\lambda_2 h)^{\beta-1}(e^{\lambda_2 h} - e^{-\lambda_2 h})\cos(x)\}
\end{aligned}
$$

$$+ 2c_\beta(\lambda_1 h + \lambda_2 h)[\lambda_2 h \sum_{p=-1}^{1} \gamma_p^{(\beta)} e^{p\lambda_2 h}(1 + e^{-2\lambda_2 h} - 2e^{-\lambda_2 h}\cos x)^{\frac{\beta}{2}} 2\cos(\beta\theta - px)$$

$$- \sum_{p=-1}^{1} \gamma_p^{(\beta+1)} e^{p\lambda_2 h}(1 + e^{-2\lambda_2 h} - 2e^{-\lambda_2 h}\cos x)^{\frac{\beta+1}{2}} 2\cos((\beta+1)\theta - px)]$$

$$- 2\beta c_\beta(\lambda_1 h + \lambda_2 h)(\lambda_2 h)^{\beta-1}[\lambda_2 h(e^{\lambda_2 h} - e^{-\lambda_2 h})2\cos x - 2\cos x(e^{\lambda_2 h} + e^{-\lambda_2 h}) + 4]\}, \tag{2.29}$$

where

$$\theta = \arctan\frac{-e^{-\lambda_2 h}\sin x}{1 - e^{-\lambda_2 h}\cos x}. \tag{2.30}$$

Similar to the proof process in (a), we find that the values of the function $h^\beta f_H(\beta, \lambda_1 h, \lambda_2 h, x)$ in the region $(1, 2) \times [0, \pi]$ are negative. See Figure 2 for details. The validity of the approximation scheme is also immediately established.

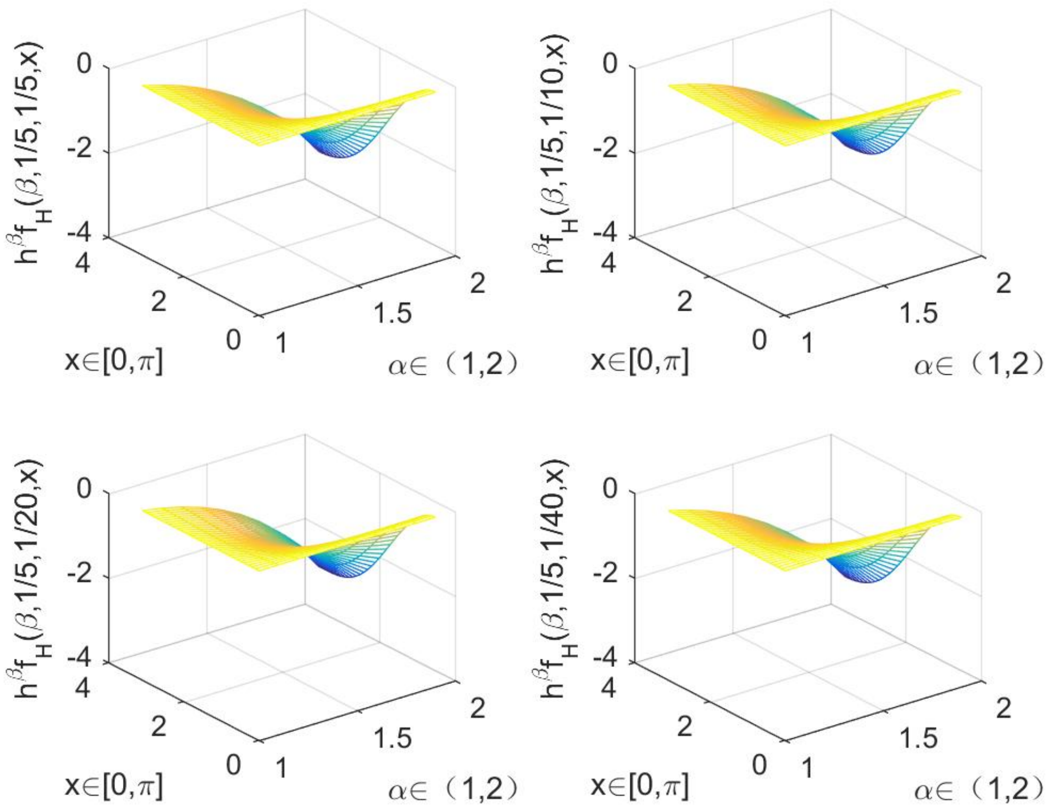

Figure 2. Images of the generating function $h^\beta f_H(\beta, \lambda_1 h, \lambda_2 h, x)$ in the finite domain $(1, 2) \times [0, \pi]$ when $\lambda_1 h = 1/5, \lambda_2 h = 1/5, 1/10, 1/20, 1/40$.

There exists a transpose relationship among the matrices corresponding to the approximation operators in each approximation formula. The methods for proving the validity of the approximation formulas (2.20) and (2.21) are exactly the same as the above-mentioned ones. We omit the process here and directly present the results:

(c) When $0 < \lambda_1 h < \lambda_2 h \leq 1/5$, the approximation formulas (2.20) and (2.21) are valid.

Thus, the lemma is proved. □

Remark 2.2. *For the left Riemann-Liouville time tempered fractional derivative* $D^{\alpha,\kappa}_{-,t}u(x,t)$ $\left(u(\cdot,t) \in \mathfrak{F}^{2+\alpha}_{\kappa}(\mathbb{R})\right)$ *in problem* (1.1)*, we adopt the method in [54] to approximate the time tempered fractional derivative, that is,*

$$D^{\alpha,\kappa}_{-,t}u(x,t) = \frac{1}{\tau^{\alpha}} \sum_{k=0}^{[\frac{t}{\tau}]+1} w^{(\alpha,\kappa)}_{k} u(x, t-(k-1)\tau) + O(\tau^2), \tag{2.31}$$

and the properties of the coefficient $w^{(\alpha,\kappa)}_{k}$ *can be found in Lemma 3.1 in Section 3.*

2.2. Derivation of the numerical scheme

We always assume that the solutions in this paper satisfy certain smoothness conditions, that is, $u(x,\cdot) \in \mathfrak{F}^{4+\beta}_{\lambda}(\mathbb{R})$ and $u(\cdot,t) \in \mathfrak{F}^{2+\alpha}_{\kappa}(\mathbb{R})$ after zero extension.

Let $u^n_i = u(x_i, t_n)$ and U^n_i denote the exact solution and the numerical solution at the point (x_i, t_n), respectively, and denote $f^n_i = f(x_i, t_n)$.

Considering the point (x_i, t_n), the tempered weighted and shifted Grünwald difference operator is used to discretize the Riemann-Liouville tempered fractional derivative in time at t_n. The fourth-order quasi-compact tempered difference operators are used to discretize the normalized left and right tempered fractional derivatives in space at x_i. Specifically, there are two cases:

(i) If $\lambda_1 \geq \lambda_2$, apply the fourth-order quasi-compact difference operator $\Lambda^{\beta,\lambda_1}_{x} = (I + c_{\beta}h^2 D^{2,\lambda_1}_{-,x})$ to discretize the normalized left and right tempered fractional derivatives, and then we can obtain

$$\begin{cases} \dfrac{1}{\tau^{\alpha}} \displaystyle\sum_{k=0}^{n+1} C^{(\beta,\lambda_1)}_{h} w^{(\alpha,\kappa)}_{k} u^{n-k+1}_{i} = l\{B^{\beta,\lambda_1}_{h} u^n_i + (\beta-1)\lambda_1^{\beta} C^{\beta,\lambda_1}_{h} u^n_i - \beta\lambda_1^{\beta-1} B^{1,\lambda_1}_{h} u^n_i \\ \qquad + \beta\lambda_1^{\beta-1}(\dfrac{1}{6} - c_{\beta})h^2[\lambda_1^3 u^n_i + 3\lambda_1^2 \delta_x u^n_i + 3\lambda_1 \delta_x^2 u^n_i + \delta^3_{x,1} u^n_i]\} \\ \qquad + r\{\hat{B}^{\beta,\lambda_2}_{h} u^n_i + (\beta-1)\lambda_2^{\beta} C^{\beta,\lambda_1}_{h} u^n_i - \beta\lambda_2^{\beta-1} \hat{B}^{1,\lambda_2}_{h} u^n_i \\ \qquad + \beta\lambda_2^{\beta-1}(\dfrac{1}{6} - c_{\beta})h^2[\lambda_2^3 u^n_i - 3\lambda_2^2 \delta_x u^n_i + 3\lambda_2 \delta_x^2 u^n_i - \delta^3_{x,2} u^n_i] \\ \qquad + c_{\beta}h^2(\lambda_1^2 - \lambda_2^2)[\hat{B}^{\beta,\lambda_2}_{h} u^n_i - \beta\lambda_2^{\beta-1} \hat{B}^{1,\lambda_2}_{h} u^n_i] \\ \qquad + 2c_{\beta}h^2(\lambda_1 + \lambda_2)[\lambda_2 \hat{B}^{\beta,\lambda_2}_{h} u^n_i - \hat{B}^{\beta+1,\lambda_2}_{h} u^n_i \\ \qquad - \beta\lambda_2^{\beta-1}(\lambda_2 \hat{B}^{1,\lambda_2}_{h} u^n_i - \hat{B}^{2,\lambda_2}_{h} u^n_i)]\} + C^{(\beta,\lambda_1)}_{h} f^n_i + O(\tau^2 + h^4), \\ \qquad 1 \leq n \leq N,\ 1 \leq i \leq M-1. \end{cases} \tag{2.32}$$

By discarding the local truncation error, the numerical scheme is obtained:

$$\begin{cases}\frac{1}{\tau^{\alpha}}\sum_{k=0}^{n+1}C_h^{(\beta,\lambda_1)}w_k^{(\alpha,\kappa)}U_i^{n-k+1}=l\{B_h^{\beta,\lambda_1}U_i^n+(\beta-1)\lambda_1^{\beta}C_h^{\beta,\lambda_1}U_i^n-\beta\lambda_1^{\beta-1}B_h^{1,\lambda_1}U_i^n\\ \qquad+\beta\lambda_1^{\beta-1}(\frac{1}{6}-c_\beta)h^2[\lambda_1^3U_i^n+3\lambda_1^2\delta_xU_i^n+3\lambda_1\delta_x^2U_i^n+\delta_{x,1}^3U_i^n]\}\\ \qquad+r\{\hat{B}_h^{\beta,\lambda_2}U_i^n+(\beta-1)\lambda_2^{\beta}C_h^{\beta,\lambda_1}U_i^n-\beta\lambda_2^{\beta-1}\hat{B}_h^{1,\lambda_2}U_i^n\\ \qquad+\beta\lambda_2^{\beta-1}(\frac{1}{6}-c_\beta)h^2[\lambda_2^3U_i^n-3\lambda_2^2\delta_xU_i^n+3\lambda_2\delta_x^2U_i^n-\delta_{x,2}^3U_i^n]\\ \qquad+c_\beta h^2(\lambda_1^2-\lambda_2^2)[\hat{B}_h^{\beta,\lambda_2}U_i^n-\beta\lambda_2^{\beta-1}\hat{B}_h^{1,\lambda_2}U_i^n]\\ \qquad+2c_\beta h^2(\lambda_1+\lambda_2)[\lambda_2\hat{B}_h^{\beta,\lambda_2}U_i^n-\hat{B}_h^{\beta+1,\lambda_2}U_i^n\\ \qquad-\beta\lambda_2^{\beta-1}(\lambda_2\hat{B}_h^{1,\lambda_2}U_i^n-\hat{B}_h^{2,\lambda_2}U_i^n)]\}+C_h^{(\beta,\lambda_1)}f_i^n.\end{cases}\tag{2.33}$$

The matrix form of the numerical format (2.33) is

$$\begin{aligned}&\frac{w_0^{(\alpha,\kappa)}}{\tau^{\alpha}}C^{(\beta,\lambda_1)}U^{n+1}+(\frac{w_1^{(\alpha,\kappa)}}{\tau^{\alpha}}C^{(\beta,\lambda_1)}-K)U^n\\ &\qquad=-\frac{1}{\tau^{\alpha}}\sum_{k=2}^{n+1}w_k^{(\alpha,\kappa)}C^{(\beta,\lambda_1)}U^{n-k+1}+C^{(\beta,\lambda_1)}f^n+F^n,\end{aligned}\tag{2.34}$$

where $U^n=(U_1^n,U_2^n,...,U_{M-2}^n,U_{M-1}^n)^T$, $f^n=(f_1^n,f_2^n,...,f_{M-2}^n,f_{M-1}^n)^T$, $K=lP+rQ$, P and Q are given in (a) and (b) of Lemma 2.4, respectively, and

$$F^n=\begin{pmatrix}c_\beta e^{-\lambda_1 h}f_0^n\\ \vdots\\ 0\\ \vdots\\ 0\\ c_\beta e^{\lambda_1 h}f_M^n\end{pmatrix}.\tag{2.35}$$

(ii) If $\lambda_2>\lambda_1$, apply the fourth-order quasi-compact difference operator $\Delta_x^{\beta,\lambda_2}=(I+c_\beta h^2D_{+,x}^{2,\lambda_2})$ to discretize the normalized left and right tempered fractional derivatives, and then we can obtain

$$\begin{cases}\frac{1}{\tau^{\alpha}}\sum_{k=0}^{n+1}\hat{C}_h^{(\beta,\lambda_2)}w_k^{(\alpha,\kappa)}u_i^{n-k+1}=l\{B_h^{\beta,\lambda_1}u_i^n+(\beta-1)\lambda_1^{\beta}\hat{C}_h^{\beta,\lambda_2}u_i^n-\beta\lambda_1^{\beta-1}B_h^{1,\lambda_1}u_i^n\\ \qquad+\beta\lambda_1^{\beta-1}(\frac{1}{6}-c_\beta)h^2[\lambda_1^3u_i^n+3\lambda_1^2\delta_xu_i^n+3\lambda_1\delta_x^2u_i^n+\delta_{x,1}^3u_i^n]\\ \qquad+c_\beta h^2(\lambda_2^2-\lambda_1^2)[B_h^{\beta,\lambda_1}u_i^n-\beta\lambda_1^{\beta-1}B_h^{1,\lambda_1}u_i^n]\\ \qquad+2c_\beta h^2(\lambda_1+\lambda_2)[\lambda_1B_h^{\beta,\lambda_1}u_i^n-B_h^{\beta+1,\lambda_1}u_i^n\\ \qquad-\beta\lambda_1^{\beta-1}(\lambda_1B_h^{1,\lambda_1}u_i^n-B_h^{2,\lambda_1}u_i^n)]\}\\ \qquad+r\{\hat{B}_h^{\beta,\lambda_2}u_i^n+(\beta-1)\lambda_2^{\beta}\hat{C}_h^{\beta,\lambda_2}u_i^n-\beta\lambda_2^{\beta-1}\hat{B}_h^{1,\lambda_2}u_i^n\\ \qquad+\beta\lambda_2^{\beta-1}(\frac{1}{6}-c_\beta)h^2[\lambda_2^3u_i^n-3\lambda_2^2\delta_xu_i^n+3\lambda_2\delta_x^2u_i^n-\delta_{x,2}^3u_i^n]\}\\ \qquad+\hat{C}_h^{(\beta,\lambda_2)}f_i^n+O(\tau^2+h^4),\quad 1\le n\le N,\ 1\le i\le M-1.\end{cases}\tag{2.36}$$

In the same way as the above steps, we obtain the numerical scheme:

$$\begin{cases} \frac{1}{\tau^\alpha}\sum_{k=0}^{n+1}\hat{C}_h^{(\beta,\lambda_2)}w_k^{(\alpha,\kappa)}U_i^{n-k+1} = l\{B_h^{\beta,\lambda_1}U_i^n + (\beta-1)\lambda_1^\beta\hat{C}_h^{\beta,\lambda_2}U_i^n - \beta\lambda_1^{\beta-1}B_h^{1,\lambda_1}U_i^n \\ \qquad + \beta\lambda_1^{\beta-1}(\frac{1}{6}-c_\beta)h^2[\lambda_1^3U_i^n + 3\lambda_1^2\delta_xU_i^n + 3\lambda_1\delta_x^2U_i^n + \delta_{x,1}^3U_i^n] \\ \qquad + c_\beta h^2(\lambda_2^2-\lambda_1^2)[B_h^{\beta,\lambda_1}U_i^n - \beta\lambda_1^{\beta-1}B_h^{1,\lambda_1}U_i^n] \\ \qquad + 2c_\beta h^2(\lambda_1+\lambda_2)[\lambda_1B_h^{\beta,\lambda_1}U_i^n - B_h^{\beta+1,\lambda_1}U_i^n \\ \qquad - \beta\lambda_1^{\beta-1}(\lambda_1B_h^{1,\lambda_1}U_i^n - B_h^{2,\lambda_1}U_i^n)]\} \\ \qquad + r\{\hat{B}_h^{\beta,\lambda_2}U_i^n + (\beta-1)\lambda_2^\beta\hat{C}_h^{\beta,\lambda_2}U_i^n - \beta\lambda_2^{\beta-1}\hat{B}_h^{1,\lambda_2}U_i^n \\ \qquad + \beta\lambda_2^{\beta-1}(\frac{1}{6}-c_\beta)h^2[\lambda_2^3U_i^n - 3\lambda_2^2\delta_xU_i^n + 3\lambda_2\delta_x^2U_i^n - \delta_{x,2}^3U_i^n]\} \\ \qquad + \hat{C}_h^{(\beta,\lambda_2)}f_i^n. \end{cases} \tag{2.37}$$

The matrix form of the numerical scheme (2.37) is

$$\begin{aligned} &\frac{w_0^{(\alpha,\kappa)}}{\tau^\alpha}\hat{C}^{(\beta,\lambda_2)}U^{n+1} + (\frac{w_1^{(\alpha,\kappa)}}{\tau^\alpha}\hat{C}^{(\beta,\lambda_2)} - K)U^n \\ &\qquad = -\frac{1}{\tau^\alpha}\sum_{k=2}^{n+1}w_k^{(\alpha,\kappa)}\hat{C}^{(\beta,\lambda_2)}U^{n-k+1} + \hat{C}^{(\beta,\lambda_2)}f^n + \hat{F}^n, \end{aligned} \tag{2.38}$$

where $K = lP + rQ$, $P = B^{(\beta,\lambda_1)} - \beta\lambda_1^{\beta-1}B^{(1,\lambda_1)} + (\frac{1}{6}-c_\beta)\beta\lambda_1^{\beta-1}h^2(\lambda_1^3E + 3\lambda_1^2D^{(1)} + 3\lambda_1D^{(2)} + D^{(3)}) + c_\beta h^2(\lambda_2^2 - \lambda_1^2)(B^{(\beta,\lambda_1)} - \beta\lambda_1^{\beta-1}B^{(1,\lambda_1)}) + 2c_\beta h^2(\lambda_1+\lambda_2)[\lambda_1B^{(\beta,\lambda_1)} - B^{(\beta+1,\lambda_1)} - \beta\lambda_1^{\beta-1}(\lambda_1B^{(1,\lambda_1)} - B^{(2,\lambda_1)})] + (\beta-1)\lambda_1^\beta(C^{(\beta,\lambda_2)})^T$, $Q = \{B^{(\beta,\lambda_2)} + (\beta-1)\lambda_2^\beta C^{(\beta,\lambda_2)} - \beta\lambda_2^{\beta-1}B^{(1,\lambda_2)} + (\frac{1}{6}-c_\beta)\beta\lambda_2^{\beta-1}h^2(\lambda_2^3E + 3\lambda_2^2D^{(1)} + 3\lambda_2D^{(2)} + D^{(3)})\}^T$, and

$$\hat{F}^n = \begin{pmatrix} c_\beta e^{\lambda_2h}f_0^n \\ \vdots \\ 0 \\ \vdots \\ 0 \\ c_\beta e^{-\lambda_2h}f_M^n \end{pmatrix}. \tag{2.39}$$

3. Numerical analysis

In this section, we rigorously and in detail prove the stability and convergence of the numerical scheme through the energy method. Now, define the discrete L_2-norm:

$$\|\mu\|_{L_2} = (h\sum_{i=1}^{M-1}\mu_i^2)^{1/2}.$$

Then some lemmas that are needed in the proof process are presented.

Lemma 3.1. *[54] Let $\kappa \geq 0, \tau > 0$.*
(i) For $0 < \alpha \leq \frac{\sqrt{17}-3}{2}$, if $\frac{3(\alpha^2+3\alpha-2)}{2(\alpha^2+3\alpha+2)} \leq \gamma_1^{(\alpha)} \leq 0$, then the following relationships are satisfied:

$$w_0^{(\alpha,\kappa)} + w_2^{(\alpha,\kappa)} \leq 0, \quad w_1^{(\alpha,\kappa)} \geq 0, \quad w_k^{(\alpha,\kappa)} \leq 0 \ (k \geq 3);$$

(ii) For $\frac{\sqrt{17}-3}{2} < \alpha < 1$, if $\frac{2(\alpha^2+3\alpha-4)}{\alpha^2+3\alpha+2} \leq \gamma_1^{(\alpha)} \leq 0$, then the following relationships are satisfied:

$$w_0^{(\alpha,\kappa)} + w_2^{(\alpha,\kappa)} \leq 0, \quad w_1^{(\alpha,\kappa)} \geq 0, \quad w_3^{(\alpha,\kappa)} \geq 0, \quad w_k^{(\alpha,\kappa)} \leq 0 \ (k \geq 4).$$

Lemma 3.2. *Let $0 < \lambda h \leq \frac{1}{5}$, and for any real column vector ε, the matrix $C^{(\beta,\lambda)}(\lambda \in \{\lambda_1, \lambda_2\})$ satisfies*

$$\frac{3}{10}\varepsilon^T\varepsilon \leq \varepsilon^T C^{(\beta,\lambda)}\varepsilon \leq \frac{6}{5}\varepsilon^T\varepsilon. \tag{3.1}$$

Proof. Let $H = \frac{C^{(\beta,\lambda)}+(C^{(\beta,\lambda)})^T}{2}$, and the generating function of the matrix H is

$$f(x) = 1 - 2c_\beta + c_\beta(e^{\lambda h} + e^{-\lambda h})\cos x, \quad x \in [-\pi, \pi]. \tag{3.2}$$

By Lemma 2.2, we obtain $1 - c_\beta(e^{\lambda h} + e^{-\lambda h} + 2) \leq \lambda(H) \leq 1 + c_\beta(e^{\lambda h} + e^{-\lambda h} - 2)$, and it is easy to check $\frac{3}{10} < \lambda(H) < \frac{6}{5}$, which means $\frac{3}{10}\epsilon^T\epsilon < \epsilon^T H\epsilon < \frac{6}{5}\epsilon^T\epsilon$, that is, $\frac{3}{10}\epsilon^T\epsilon < \epsilon^T C^{(\alpha)}\epsilon < \frac{6}{5}\epsilon^T\epsilon$.

Thus, the proof is completed. □

Lemma 3.3. *[3] Assume that $\{k_n\}$ and $\{p_n\}$ are nonnegative sequences, and the sequence $\{\phi_n\}$ satisfies*

$$\phi_0 \leq g_0, \ \phi_n \leq g_0 + \sum_{l=0}^{n-1} p_l + \sum_{l=0}^{n-1} k_l\phi_l, \ n \geq 1,$$

where $g_0 \geq 0$. Then the sequence $\{\phi_n\}$ satisfies

$$\phi_n \leq (g_0 + \sum_{l=0}^{n-1} p_l) exp(\sum_{l=0}^{n-1} k_l), \ n \geq 1.$$

3.1. Stability analysis

Theorem 3.1. *For $\alpha \in (0, 1), \beta \in (1, 2)$, let $\gamma_1^{(\alpha)} = 0$, and then if $0 < \lambda_2 h \leq \lambda_1 h \leq \frac{1}{5}$, the numerical scheme (2.33) is stable; otherwise, if $0 < \lambda_1 h < \lambda_2 h \leq \frac{1}{5}$, the numerical scheme (2.37) is stable.*

Proof. (1) If $0 < \lambda_2 h \leq \lambda_1 h \leq \frac{1}{5}$, considering a certain error in the initial condition, the numerical solution obtained by using the numerical scheme (2.33) is denoted as V_i^n. Let $\varepsilon^n = (\varepsilon_1^n, \varepsilon_2^n, ..., \varepsilon_{M-1}^n)^T$, $\varepsilon_i^n = U_i^n - V_i^n$. It can be obtained from Eq (2.34) that

$$\frac{w_0^{(\alpha,\kappa)}}{\tau^\alpha} C^{(\beta,\lambda_1)}\varepsilon^{n+1} + (\frac{w_1^{(\alpha,\kappa)}}{\tau^\alpha} C^{(\beta,\lambda_1)} - K)\varepsilon^n = -\frac{1}{\tau^\alpha}\sum_{k=2}^{n+1} w_k^{(\alpha,\kappa)} C^{(\beta,\lambda_1)}\varepsilon^{n-k+1}. \tag{3.3}$$

According to Lemma 3.1, we know that when $\gamma_1^{(\alpha)} = 0$, $w_0^{(\alpha,\kappa)} = 0$, and from Lemma 2.4, the matrix K is negative definite. If we left-multiply both sides of Eq (3.3) by $h(\varepsilon^n)^T$ simultaneously, we can obtain

$$w_1^{(\alpha,\kappa)}h(\varepsilon^n)^T C^{(\beta,\lambda_1)}\varepsilon^n \le -\sum_{k=2}^{n+1} w_k^{(\alpha,\kappa)}h(\varepsilon^n)^T C^{(\beta,\lambda_1)}\varepsilon^{n-k+1}. \tag{3.4}$$

(i) When $0 < \alpha \le \frac{\sqrt{17}-3}{2}$, since $w_1^{(\alpha,\kappa)} \ge 0, w_k^{(\alpha,\kappa)} \le 0(k \ge 2)$, according to the positive definiteness of the matrix $C^{(\beta,\lambda_1)}$, we can obtain from Eq (3.4) that

$$\begin{aligned} w_1^{(\alpha,\kappa)}h(\varepsilon^n)^T C^{(\beta,\lambda_1)}\varepsilon^n \le & -\frac{1}{2}[\sum_{k=2}^{n+1} w_k^{(\alpha,\kappa)}h(\varepsilon^n)^T C^{(\beta,\lambda_1)}\varepsilon^n \\ & + \sum_{k=2}^{n+1} w_k^{(\alpha,\kappa)}h(\varepsilon^{n-k+1})^T C^{(\beta,\lambda_1)}\varepsilon^{n-k+1}], \end{aligned} \tag{3.5}$$

and after further arrangement, we can get

$$\begin{aligned} w_1^{(\alpha,\kappa)}h(\varepsilon^n)^T C^{(\beta,\lambda_1)}\varepsilon^n & \le (\sum_{k=1}^{n+1} w_k^{(\alpha,\kappa)} + w_1^{(\alpha,\kappa)})h(\varepsilon^n)^T C^{(\beta,\lambda_1)}\varepsilon^n \\ & \le -\sum_{k=0}^{n-1} w_{n+1-k}^{(\alpha,\kappa)}h(\varepsilon^k)^T C^{(\beta,\lambda_1)}\varepsilon^k. \end{aligned} \tag{3.6}$$

According to Lemma 3.2, we can derive from Eq (3.6) that

$$\begin{aligned} \frac{3}{10}w_1^{(\alpha,\kappa)}\|\varepsilon^n\|_{L_2}^2 \le w_1^{(\alpha,\kappa)}h(\varepsilon^n)^T C^{(\beta,\lambda_1)}\varepsilon^n & \le (\sum_{k=1}^{n+1} w_k^{(\alpha,\kappa)} + w_1^{(\alpha,\kappa)})h(\varepsilon^n)^T C^{(\beta,\lambda_1)}\varepsilon^n \\ & \le -\sum_{k=0}^{n-1} w_{n+1-k}^{(\alpha,\kappa)}h(\varepsilon^k)^T C^{(\beta,\lambda_1)}\varepsilon^k \\ & \le -\frac{6}{5}\sum_{k=0}^{n-1} w_{n+1-k}^{(\alpha,\kappa)}\|\varepsilon^k\|_{L_2}^2, \end{aligned} \tag{3.7}$$

and by rearranging Eq (3.7) and applying Lemma 3.3, we can obtain

$$\|\varepsilon^n\|_{L_2}^2 \le e^{\sum_{k=0}^{n-1}\frac{-4w_{n+1-k}^{(\alpha,\kappa)}}{w_1^{(\alpha,\kappa)}}}\|\varepsilon^0\|_{L_2}^2 \le e^4\|\varepsilon^0\|_{L_2}^2. \tag{3.8}$$

(ii) When $\frac{\sqrt{17}-3}{2} < \alpha < 1$, since $w_1^{(\alpha,\kappa)} \ge 0, w_2^{(\alpha,\kappa)} \le 0, w_3^{(\alpha,\kappa)} \ge 0, w_k^{(\alpha,\kappa)} \le 0\,(k \ge 4)$, similar to the proof of (i), we have

$$\begin{aligned} w_1^{(\alpha,\kappa)}h(\varepsilon^n)^T C^{(\beta,\lambda_1)}\varepsilon^n \le & -\frac{1}{2}[\sum_{k=2}^{n+1} w_k^{(\alpha,\kappa)}h(\varepsilon^n)^T C^{(\beta,\lambda_1)}\varepsilon^n \\ & + \sum_{k=2}^{n+1} w_k^{(\alpha,\kappa)}h(\varepsilon^{n-k+1})^T C^{(\beta,\lambda_1)}\varepsilon^{n-k+1} \end{aligned}$$

$$-2w_3^{(\alpha,\kappa)}h(\varepsilon^n)^T C^{(\beta,\lambda_1)}\varepsilon^n - 2w_3^{(\alpha,\kappa)}h(\varepsilon^{n-2})^T C^{(\beta,\lambda_1)}\varepsilon^{n-2}], \tag{3.9}$$

and with Lemma 3.2, we obtain

$$\begin{aligned}
\frac{3}{10}(w_1^{(\alpha,\kappa)} - 2w_3^{(\alpha,\kappa)})\|\varepsilon^n\|_{L_2}^2 &\le (w_1^{(\alpha,\kappa)} - 2w_3^{(\alpha,\kappa)})h(\varepsilon^n)^T C^{(\beta,\lambda_1)}\varepsilon^n \\
&\le (\sum_{k=1}^{n+1} w_k^{(\alpha,\kappa)} + w_1^{(\alpha,\kappa)} - 2w_3^{(\alpha,\kappa)})h(\varepsilon^n)^T C^{(\beta,\lambda_1)}\varepsilon^n \\
&\le -\sum_{k=0}^{n-1} w_{n+1-k}^{(\alpha,\kappa)}h(\varepsilon^k)^T C^{(\beta,\lambda_1)}\varepsilon^k + 2w_3^{(\alpha,\kappa)}h(\varepsilon^{n-2})^T C^{(\beta,\lambda_1)}\varepsilon^{n-2} \\
&\le -\sum_{k=0}^{n-1}\frac{6}{5}w_{n+1-k}^{(\alpha,\kappa)}\|\varepsilon^k\|_{L_2}^2 + \frac{12}{5}w_3^{(\alpha,\kappa)}\|\varepsilon^{n-2}\|_{L_2}^2.
\end{aligned} \tag{3.10}$$

After arranging Eq (3.10) and by Lemma 3.3, we obtain

$$\begin{aligned}
\|\varepsilon^n\|_{L_2}^2 \le e^{\frac{\sum_{k=0}^{n-1} -4w_{n+1-k}^{(\alpha,\kappa)} + 8w_3^{(\alpha,\kappa)}}{w_1^{(\alpha,\kappa)} - 2w_3^{(\alpha,\kappa)}}}\|\varepsilon^0\|_{L_2}^2 &\le e^{\frac{4w_1^{(\alpha,\kappa)} + 8w_3^{(\alpha,\kappa)}}{w_1^{(\alpha,\kappa)} - 2w_3^{(\alpha,\kappa)}}}\|\varepsilon^0\|_{L_2}^2 \\
&\le e^{\frac{4(\alpha^3+3\alpha^2-\alpha+2)}{-\alpha^3-3\alpha^2+3\alpha+2}}\|\varepsilon^0\|_{L_2}^2.
\end{aligned} \tag{3.11}$$

(2) If $0 < \lambda_1 h < \lambda_2 h \le \frac{1}{5}$, the proof process of the stability of the numerical scheme (2.37) is exactly the same as that of the proof of (2.33), and we skip the proof process here.

In conclusion, we have completed the proof. □

3.2. Convergence analysis

Theorem 3.2. *For $\alpha \in (0,1), \beta \in (1,2)$, let $\gamma_1^{(\alpha)} = 0$, and then if $0 < \lambda_2 h \le \lambda_1 h \le \frac{1}{5}$, the numerical scheme (2.33) is convergent; otherwise, if $0 < \lambda_1 h < \lambda_2 h \le \frac{1}{5}$, the numerical scheme (2.37) is convergent.*

Proof. (1) If $0 < \lambda_2 h \le \lambda_1 h \le \frac{1}{5}$, subtract Eq (2.33) from Eq (2.32), and from the matrix form (2.34), we have

$$(\frac{w_1^{(\alpha,\kappa)}}{\tau^\alpha}C^{(\beta,\lambda_1)} - K)\eta^n = -\frac{1}{\tau^\alpha}\sum_{k=2}^{n+1} w_k^{(\alpha,\kappa)}C^{(\beta,\lambda_1)}\eta^{n-k+1} + R^n, \tag{3.12}$$

where $\eta^n = (\eta_1^n, \eta_2^n, ..., \eta_{M-1}^n)^T$, $\eta_i^n = u_i^n - U_i^n$, $R^n = (R_1^n, R_2^n, ..., R_{M-1}^n)^T$, and $R_i^n = O(\tau^2 + h^4)$ is the local truncation error.

Similar to the proof of Theorem 3.1, we rearrange Eq (3.12) to obtain

$$w_1^{(\alpha,\kappa)}h(\eta^n)^T C^{(\beta,\lambda_1)}\eta^n \le -\sum_{k=2}^{n+1} w_k^{(\alpha,\kappa)}h(\eta^n)^T C^{(\beta,\lambda_1)}\eta^{n-k+1} + \tau^\alpha h(\eta^n)^T R^n. \tag{3.13}$$

(i) When $0 < \alpha \le \frac{\sqrt{17}-3}{2}$, we can directly obtain from Eq (3.13) that

$$w_1^{(\alpha,\kappa)}h(\eta^n)^T C^{(\beta,\lambda_1)}\eta^n \le -\frac{1}{2}[\sum_{k=2}^{n+1} w_k^{(\alpha,\kappa)}h(\eta^n)^T C^{(\beta,\lambda_1)}\eta^n$$

$$+\sum_{k=2}^{n+1} w_k^{(\alpha,\kappa)} h(\eta^{n-k+1})^T C^{(\beta,\lambda_1)} \eta^{n-k+1}]$$
$$+\frac{\tau^\alpha}{2}(\frac{w_1^{(\alpha,\kappa)}}{10\tau^\alpha}\|\eta^n\|_{L_2}^2+\frac{10\tau^\alpha}{w_1^{(\alpha,\kappa)}}\|R^n\|_{L_2}^2), \tag{3.14}$$

and by arranging Eq (3.14), we obtain

$$\begin{aligned} w_1^{(\alpha,\kappa)} h(\eta^n)^T C^{(\beta,\lambda_1)} \eta^n &\leq (\sum_{k=1}^{n+1} w_k^{(\alpha,\kappa)} + w_1^{(\alpha,\kappa)}) h(\eta^n)^T C^{(\beta,\lambda_1)} \eta^n \\ &\leq -\sum_{k=0}^{n-1} w_{n+1-k}^{(\alpha,\kappa)} h(\eta^k)^T C^{(\beta,\lambda_1)} \eta^k \\ &+ \tau^\alpha(\frac{w_1^{(\alpha,\kappa)}}{10\tau^\alpha}\|\eta^n\|_{L_2}^2+\frac{10\tau^\alpha}{w_1^{(\alpha,\kappa)}}\|R^n\|_{L_2}^2). \end{aligned} \tag{3.15}$$

Combining Lemmas 3.2 and 3.3, we can infer that

$$\|\eta^n\|_{L_2}^2 \leq 50 e^{\sum_{k=0}^{n-1}\frac{-6w_{n+1-k}^{(\alpha,\kappa)}}{w_1^{(\alpha,\kappa)}}}\left(\frac{\tau^\alpha}{w_1^{(\alpha,\kappa)}}\right)^2 \max_{1\leq n\leq N}\|R^n\|_{L_2}^2 \leq 50\left(\frac{e^3\tau^\alpha}{w_1^{(\alpha,\kappa)}}\right)^2 \max_{1\leq n\leq N}\|R^n\|_{L_2}^2. \tag{3.16}$$

(ii) When $\frac{\sqrt{17}-3}{2} < \alpha < 1$, we can deduce from Eq (3.13) that

$$\begin{aligned} w_1^{(\alpha,\kappa)} h(\eta^n)^T C^{(\beta,\lambda_1)} \eta^n \leq &-\frac{1}{2}(\sum_{k=2}^{n+1} w_k^{(\alpha,\kappa)} h(\eta^n)^T C^{(\beta,\lambda_1)} \eta^n \\ &+\sum_{k=2}^{n+1} w_k^{(\alpha,\kappa)} h(\eta^{n-k+1})^T C^{(\beta,\lambda_1)} \eta^{n-k+1} \\ &- 2w_3^{(\alpha,\kappa)} h(\eta^n)^T C^{(\beta,\lambda_1)} \eta^n - 2w_3^{(\alpha,\kappa)} h(\eta^{n-2})^T C^{(\beta,\lambda_1)} \eta^{n-2}) \\ &+\frac{\tau^\alpha}{2}(\frac{w_1^{(\alpha,\kappa)}}{15\tau^\alpha}\|\eta^n\|_{L_2}^2+\frac{15\tau^\alpha}{w_1^{(\alpha,\kappa)}}\|R^n\|_{L_2}^2), \end{aligned} \tag{3.17}$$

and after further arrangement, we obtain

$$\begin{aligned} (w_1^{(\alpha,\kappa)} - 2w_3^{(\alpha,\kappa)}) h(\eta^n)^T C^{(\beta,\lambda_1)} \eta^n &\leq (\sum_{k=1}^{n+1} w_k^{(\alpha,\kappa)} + w_1^{(\alpha,\kappa)} - 2w_3^{(\alpha,\kappa)}) h(\eta^n)^T C^{(\beta,\lambda_1)} \eta^n \\ &\leq -\sum_{k=0}^{n-1} w_{n+1-k}^{(\alpha,\kappa)} h(\eta^k)^T C^{(\beta,\lambda_1)} \eta^k \\ &+ 2w_3^{(\alpha,\kappa)} h(\eta^{n-2})^T C^{(\beta,\lambda_1)} \eta^{n-2} \\ &+ \tau^\alpha(\frac{w_1^{(\alpha,\kappa)}}{15\tau^\alpha}\|\eta^n\|_{L_2}^2+\frac{15\tau^\alpha}{w_1^{(\alpha,\kappa)}}\|R^n\|_{L_2}^2). \end{aligned} \tag{3.18}$$

By integrating Lemmas 3.2 and 3.3, we derive that

$$\|\eta^n\|_{L_2}^2 \leq e^{\frac{\sum_{k=0}^{n-1} -36w_{n+1-k}^{(\alpha,\kappa)}+72w_3^{(\alpha,\kappa)}}{7w_1^{(\alpha,\kappa)}-18w_3^{(\alpha,\kappa)}}} \frac{450\tau^{2\alpha}}{(7w_1^{(\alpha,\kappa)} - 18w_3^{(\alpha,\kappa)})w_1^{(\alpha,\kappa)}} \max_{1\leq n\leq N} \|R^n\|_{L_2}^2. \tag{3.19}$$

By synthesizing these two cases of (i) and (ii), we obtain

$$\|\eta^n\|_{L_2} \leq C(\tau^2 + h^4),\ n = 1, 2, ..., N. \tag{3.20}$$

(2) For the case where $0 < \lambda_1 h < \lambda_2 h \leq \frac{1}{5}$, the proof of the convergence of the numerical scheme (2.37) is basically the same as that in case (1). To save space, we omit the proof process and directly obtain the conclusion.

Up to this point, we have rigorously proved the convergence of the numerical scheme. □

4. Numerical experiments

In this section, we carry out some numerical simulations to verify the accuracy and effectiveness of the proposed numerical scheme. First, we present the method for measuring the order of error in the L_2-norm:

$$\text{Order} = \log_m \left(\frac{\|e\|_{L_2,h}}{\|e\|_{L_2,h/m}} \right).$$

Example 4.1. *Consider the following initial and boundary value problem of the space-time tempered fractional differential equations:*

$$\begin{cases} {}^C D_{-,t}^{\alpha,\kappa} u(x,t) = l(D_{-,x}^{\beta,\lambda_1} u(x,t) - \lambda_1^\beta u(x,t) - \beta\lambda_1^{\beta-1} \frac{\partial u(x,t)}{\partial x}) \\ +r(D_{+,x}^{\beta,\lambda_2} u(x,t) - \lambda_2^\beta u(x,t) + \beta\lambda_2^{\beta-1} \frac{\partial u(x,t)}{\partial x}) + f(x,t), \\ & (x,t) \in (0,1) \times (0,1], \\ u(0,t) = 0, u(1,t) = 0, & t \in [0,1], \\ u(x,0) = 0, & x \in (0,1), \end{cases}$$

where $0 < \alpha < 1,\ 1 < \beta < 2,\ m, n \in \mathbb{N}$ $(m \geq 2, n \geq 4)$*, and*

$$\begin{aligned} f(x,t) = {} & x^n(1-x)^n e^{-\kappa t} \sum_{i=0}^{+\infty} \frac{(\kappa)^i}{i!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \frac{\Gamma(m+1+i+j)}{\Gamma(m+1+i+j-\alpha)} t^{m+i+j-\alpha} \\ & - t^m(1-t)^m [l e^{-\lambda_1 x} \sum_{i=0}^{+\infty} \frac{(\lambda_1)^i}{i!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{\Gamma(n+1+i+j)}{\Gamma(n+1+i+j-\beta)} x^{n+i+j-\beta} \\ & + r e^{\lambda_2 x} \sum_{i=0}^{+\infty} \frac{(-\lambda_2)^i}{i!} \sum_{j=0}^{n+i} (-1)^j \binom{n+i}{j} \frac{\Gamma(n+1+j)}{\Gamma(n+1+j-\beta)} (1-x)^{n+j-\beta} \\ & - (l\lambda_1^\beta + r\lambda_2^\beta) x^n (1-x)^n + (r\lambda_2^{\beta-1} - l\lambda_1^{\beta-1}) \beta n (x-x^2)^{n-1}(1-2x)]. \end{aligned}$$

The exact solution is $u(x,t) = t^m(1-t)^m x^n(1-x)^n$.

Consider $l = 1$, $r = 2$, $m = 2$, $n = 4$, $\kappa = 1$, $\lambda_1 = 2$, $\lambda_2 = 1$. We use the numerical scheme proposed in this paper to conduct the numerical simulations of Example 4.1. First, through the first numerical simulation, we measure that the convergence order of the numerical scheme in the time direction is 2, as shown in Table 1. Then, through the second numerical simulation, we measure that the convergence order of the numerical scheme in the spatial direction is 4, as shown in Table 2. Finally, the error surface graph of the numerical scheme, Figure 3, and the error curve graphs at different moments, Figure 4, are presented.

Table 1. Numerical results at time $t = 1$ for Example 4.1: $l = 1$, $r = 2$, $m = 2$, $n = 4$, $M = 500$.

	$\kappa = 1$	$\lambda_1 = 2$	$\lambda_2 = 1$	
α	β	$\tau(\frac{1}{N})$	$\|e\|_{L_2}$	Order
0.3	1.5	$\frac{1}{10}$	4.6447e-07	
		$\frac{1}{20}$	1.4193e-07	1.7104
		$\frac{1}{40}$	3.9081e-08	1.8606
		$\frac{1}{80}$	1.0245e-08	1.9315
0.5	1.8	$\frac{1}{10}$	5.9660e-07	
		$\frac{1}{20}$	1.8158e-07	1.7162
		$\frac{1}{40}$	4.9923e-08	1.8628
		$\frac{1}{80}$	1.3079e-08	1.9325
0.8	1.3	$\frac{1}{10}$	4.8099e-06	
		$\frac{1}{20}$	1.5188e-06	1.6631
		$\frac{1}{40}$	4.2633e-07	1.8329
		$\frac{1}{80}$	1.1292e-07	1.9167

Table 2. Numerical results at time $t = 1$ for Example 4.1: $l = 1$, $r = 2$, $m = 2$, $n = 4$, $N = M^2$.

	$\kappa = 1$	$\lambda_1 = 2$	$\lambda_2 = 1$	
α	β	$h(\frac{1}{M})$	$\|e\|_{L_2}$	Order
0.3	1.5	$\frac{1}{10}$	3.5929e-09	
		$\frac{1}{20}$	3.1905e-10	3.4933
		$\frac{1}{40}$	2.6934e-11	3.5663
		$\frac{1}{80}$	1.9034e-12	3.8228
0.5	1.8	$\frac{1}{10}$	4.2402e-09	
		$\frac{1}{20}$	2.7954e-10	3.9230
		$\frac{1}{40}$	2.0777e-11	3.7500
		$\frac{1}{80}$	1.4546e-12	3.8363
0.8	1.3	$\frac{1}{10}$	3.7577e-08	
		$\frac{1}{20}$	1.9448e-09	4.2722
		$\frac{1}{40}$	1.0785e-10	4.1725
		$\frac{1}{80}$	7.0233e-12	3.9407

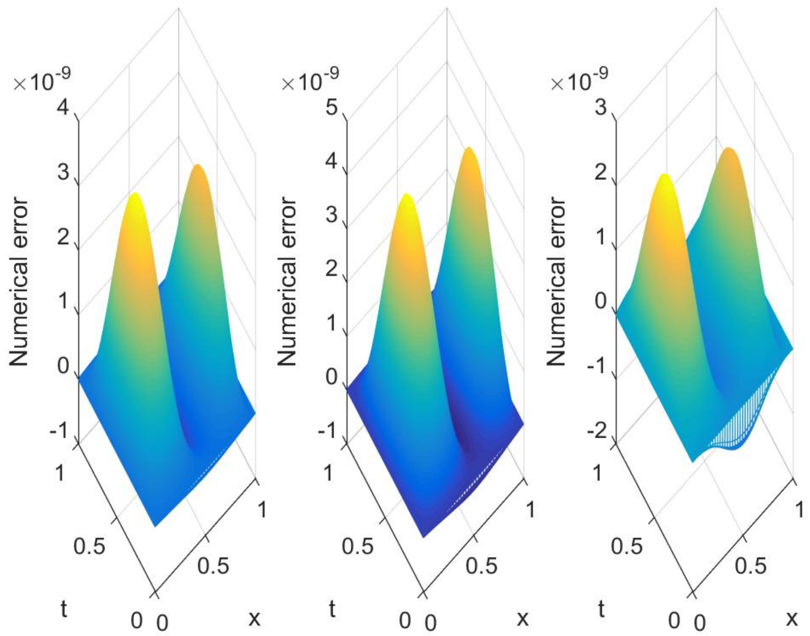

Figure 3. The numerical error graphs of Example 4.1 under different values of (α, β): $l = 1$, $r = 2$, $m = 2$, $n = 4$, $N = M^2$, $M = 40$.

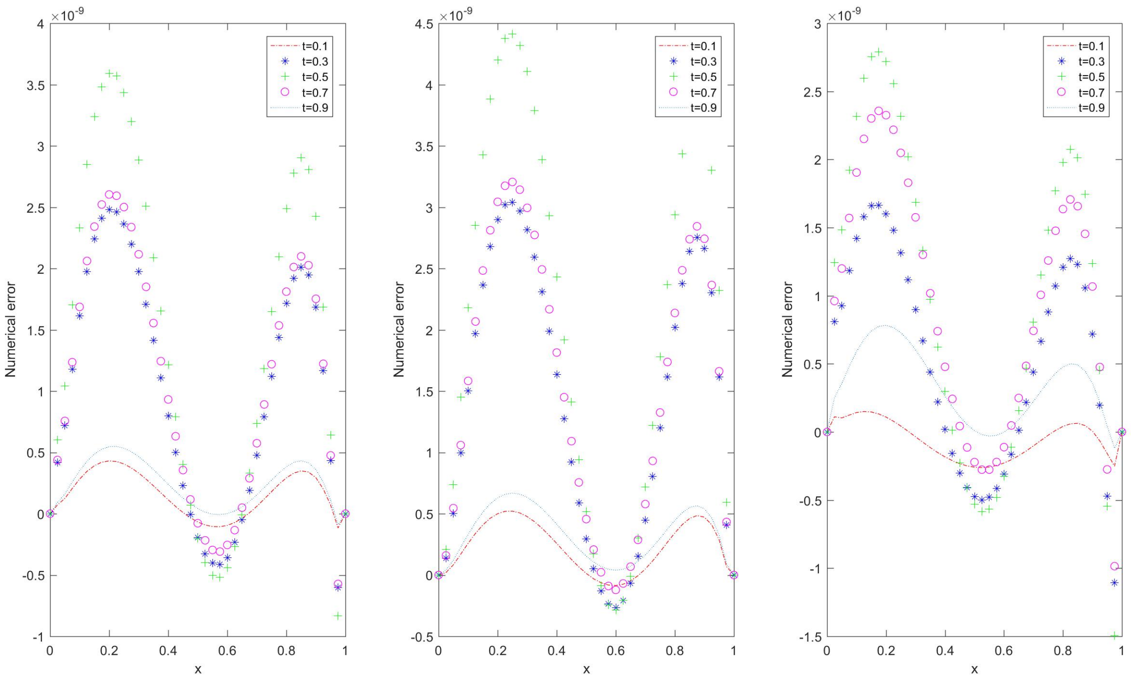

Figure 4. The error curve graphs at different moments corresponding to Figure 3.

Example 4.2. *Consider the following initial and boundary value problem of the space-time tempered fractional differential equations:*

$$\begin{cases} {}^{C}D_{-,t}^{\alpha,\kappa}u(x,t) = l(D_{-,x}^{\beta,\lambda_1}u(x,t) - \lambda_1^{\beta}u(x,t) - \beta\lambda_1^{\beta-1}\frac{\partial u(x,t)}{\partial x}) \\ +r(D_{+,x}^{\beta,\lambda_2}u(x,t) - \lambda_2^{\beta}u(x,t) + \beta\lambda_2^{\beta-1}\frac{\partial u(x,t)}{\partial x}) + f(x,t), \\ & (x,t) \in (0,1) \times (0,1], \\ u(0,t) = 0, u(1,t) = 0, & t \in [0,1], \\ u(x,0) = 0, & x \in (0,1), \end{cases}$$

where $0 < \alpha < 1$, $1 < \beta < 2$, $m, n \in \mathbb{N}$ $(m \geq 2, n \geq 4)$, *and*

$$\begin{aligned} f(x,t) = {} & x^n(1-x)^n e^{-\kappa t}\frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)}t^{m-\alpha} \\ & - e^{-\kappa t}t^m[le^{-\lambda_1 x}\sum_{i=0}^{+\infty}\frac{(\lambda_1)^i}{i!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{\Gamma(n+1+i+j)}{\Gamma(n+1+i+j-\beta)}x^{n+i+j-\beta} \\ & + re^{\lambda_2 x}\sum_{i=0}^{+\infty}\frac{(-\lambda_2)^i}{i!}\sum_{j=0}^{n+i}(-1)^j\binom{n+i}{j}\frac{\Gamma(n+1+j)}{\Gamma(n+1+j-\beta)}(1-x)^{n+j-\beta} \\ & - (l\lambda_1^{\beta} + r\lambda_2^{\beta})x^n(1-x)^n + (r\lambda_2^{\beta-1} - l\lambda_1^{\beta-1})\beta n(x-x^2)^{n-1}(1-2x)]. \end{aligned}$$

The exact solution is $u(x,t) = e^{-\kappa t}t^m x^n(1-x)^n$.

Let $l = 2$, $r = 1$, $m = 3$, $n = 5$, $\kappa = 1$, $\lambda_1 = 1$, $\lambda_2 = \frac{1}{5}$. The numerical scheme developed in this study is applied to perform numerical experiments for Example 4.2. In the first set of simulations, the temporal convergence order of the proposed numerical scheme is evaluated to be 2, as summarized in Table 3. Subsequently, the second simulation series reveals that the numerical scheme exhibits a spatial convergence order of 4, as detailed in Table 4. To further illustrate the performance, the error surface visualization of the numerical scheme is provided in Figure 5, while Figure 6 displays the error curves at various time instants.

Through a series of numerical simulations conducted on the above two numerical examples, we can find that these results are in complete agreement with our theoretical analysis, which verifies that the numerical scheme is efficient.

Table 3. Numerical results at time $t = 1$ for Example 4.2: $l = 2$, $r = 1$, $m = 3$, $n = 5$, $M = 500$.

	$\kappa = 1$	$\lambda_1 = 1$	$\lambda_2 = \frac{1}{5}$	
α	β	$\tau(\frac{1}{N})$	$\|\|e\|\|_{L_2}$	Order
0.3	1.8	$\frac{1}{10}$	4.0985e-08	
		$\frac{1}{20}$	1.0359e-08	1.9842
		$\frac{1}{40}$	2.5990e-09	1.9949
		$\frac{1}{80}$	6.5064e-10	1.9980
0.5	1.3	$\frac{1}{10}$	3.4051e-07	
		$\frac{1}{20}$	8.5590e-08	1.9922
		$\frac{1}{40}$	2.1414e-08	1.9989
		$\frac{1}{80}$	5.3533e-09	2.0000
0.8	1.5	$\frac{1}{10}$	3.4138e-07	
		$\frac{1}{20}$	8.3210e-08	2.0365
		$\frac{1}{40}$	2.0522e-08	2.0196
		$\frac{1}{80}$	5.0946e-09	2.0101

Table 4. Numerical results at time $t = 1$ for Example 4.2: $l = 2$, $r = 1$, $m = 3$, $n = 5$, $N = M^2$.

	$\kappa = 1$	$\lambda_1 = 1$	$\lambda_2 = \frac{1}{5}$	
α	β	$h(\frac{1}{M})$	$\|\|e\|\|_{L_2}$	Order
0.3	1.8	$\frac{1}{10}$	5.8511e-07	
		$\frac{1}{20}$	4.5388e-08	3.6883
		$\frac{1}{40}$	3.1406e-09	3.8532
		$\frac{1}{80}$	2.0609e-10	3.9297
0.5	1.3	$\frac{1}{10}$	5.7548e-07	
		$\frac{1}{20}$	4.5088e-08	3.6740
		$\frac{1}{40}$	3.2044e-09	3.8146
		$\frac{1}{80}$	2.1510e-10	3.8970
0.8	1.5	$\frac{1}{10}$	5.9423e-07	
		$\frac{1}{20}$	4.9114e-08	3.5968
		$\frac{1}{40}$	3.4959e-09	3.8124
		$\frac{1}{80}$	2.3319e-10	3.9061

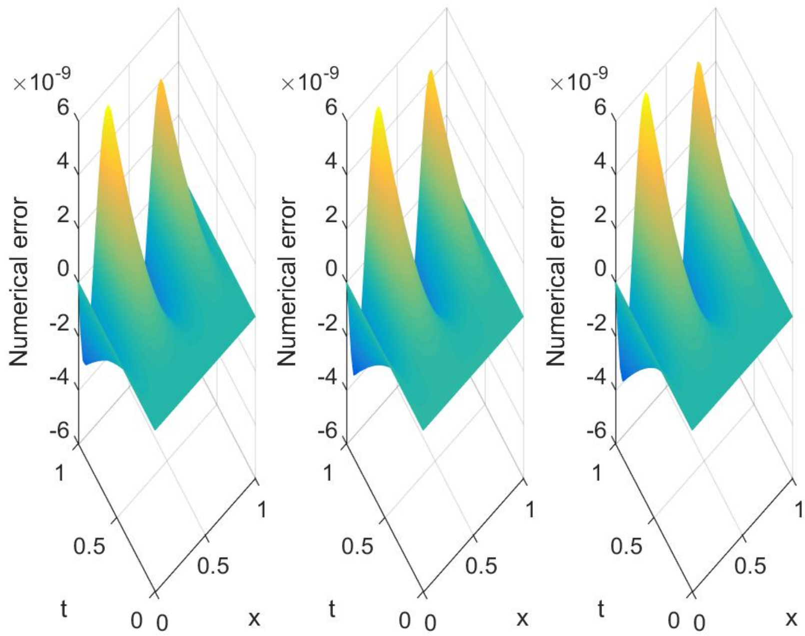

Figure 5. The numerical error graphs of Example 4.2 under different values of (α, β): $l = 2$, $r = 1$, $m = 3$, $n = 5$, $N = M^2$, $M = 40$.

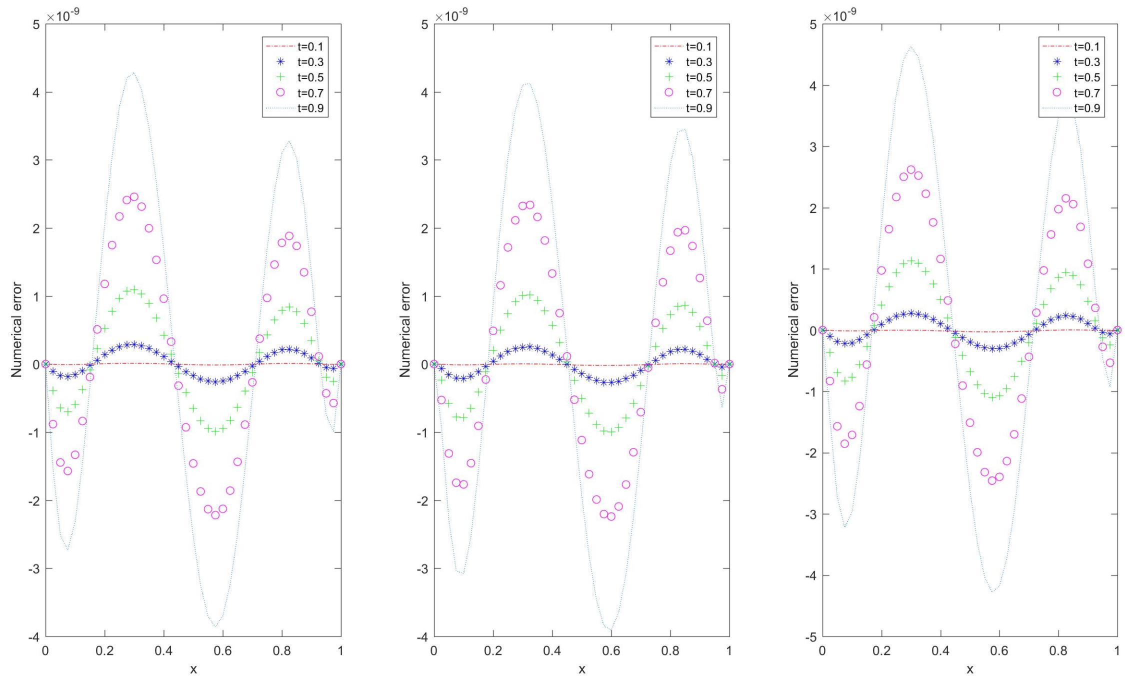

Figure 6. The error curve graphs at different moments corresponding to Figure 5.

5. Conclusions

This paper is dedicated to the investigation of an efficient numerical scheme for space-time tempered fractional differential equations. Inspired by the quasi-compact approximation of general diffusion equations, the concept of the fourth-order approximation of the normalized Riemann-Liouville tempered fractional derivative is proposed. The validity of the approximation formula is rigorously established via the generating function method. Moreover, a comprehensive theoretical analysis of the numerical scheme is presented, including stability and convergence analysis. The effectiveness and accuracy of the proposed numerical scheme are demonstrated through a series of numerical experiments on representative examples, which show excellent agreement with the theoretical predictions. In the follow-up work, we expect to consider the case of non-smooth solutions.

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Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

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