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Research article

Global stability of $\mathbb{S}\,\mathbb{E}\,\mathbb{I}_A\mathbb{I}_S\,\mathbb{V}$ epidemic models with two discrete time delays and control action

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Abstract: This article investigates two mathematical models describing the transmission dynamics of infectious diseases, with a particular focus on the impact of vaccination. Both models partition the total population into five compartments: susceptible individuals $\mathbb{S}(t)$, exposed individuals $\mathbb{E}(t)$, asymptomatic infected individuals $\mathbb{I}_A(t)$, symptomatic infected individuals $\mathbb{I}_S(t)$, and vaccinated individuals $\mathbb{V}(t)$. The first model incorporates discrete time delays to explore their effect on the stability of disease-free and endemic equilibrium. The second model introduces a control intervention to assess its influence on disease mitigation. For both frameworks, we establish the non-negativity and boundedness of solutions, ensuring biological feasibility. We then derive the basic reproduction number $\Re_0$ to characterize the local and global stability of the equilibrium. Global asymptotic stability is proven by constructing appropriate Lyapunov functions. Finally, numerical simulations are presented to illustrate and support the theoretical results, emphasizing the influence of time delays and vaccination strategies on the long-term behavior of the models.

Keywords: infectious disease; global stability; Lyapunov function; discrete time delay; control action
Mathematics Subject Classification: 34A12, 34K05, 34K20

1. Introduction

The study of mathematical models for infectious diseases is essential for understanding the dynamics, existence, stability, and control of epidemics [1]. Classical mathematical models, while foundational, often face limitations in capturing the full complexity of real-world phenomena. To overcome these challenges, various forms of differential equations have been introduced, offering improved accuracy and flexibility. These advanced modeling techniques have found applications not only in epidemiology but also in diverse disciplines such as production, optimization, artificial intelligence, medical diagnostics, robotics, and cosmology. In recent decades, mathematical modeling has been extensively employed to analyze biological systems and processes, for example see [2].

Real-world phenomena are frequently modeled using mathematical equations, which serve as robust tools for analyzing and interpreting complex systems. Mathematical epidemiology, a discipline that has evolved significantly over the past two centuries, employs such methodologies to study the spread and control of infectious diseases. Early foundational work includes D'Alembert's 1761 analysis of mortality risk and Bernoulli's 1766 modeling of endemic prevalence, life expectancy, and disease transmission dynamics [3]. While Bernoulli's contributions laid important groundwork, they received relatively limited recognition compared with later developments—most notably, the Kermack–McKendrick (Susceptible, Infected and Recovered) SIR model. This classical framework captures epidemic transitions via infection and removal processes, though it does not account for features such as endemic persistence or recurrent outbreaks [4]. Building on these foundations, subsequent researchers have extended the reach of mathematical modeling beyond epidemiology into diverse disciplines including the natural sciences, engineering, social sciences, and even domains such as music and philosophy. Mathematical models continue to be instrumental in analyzing systems' behavior, quantifying impacts, and forecasting outcomes in both theoretical and applied contexts [5].

In recent years, mathematical modeling has played an increasingly prominent role in the study of infectious diseases. Researchers have employed these models to capture a wide range of phenomena, including the spread of SARS-CoV-2 [6], dynamics within financial systems [7], predator–prey interactions [8], and the transmission patterns of COVID-19 [9]. For instance, the human respiratory syncytial virus has been investigated using optimal control theory and bifurcation analysis to better understand disease dynamics and intervention strategies [10]. These models serve as essential tools for predicting disease progression and formulating effective public health responses, enabling the development of strategies to contain or eradicate outbreaks before they escalate into pandemics.

The COVID-19 pandemic, which has profoundly affected global public health, economies, and societal structures over the past four years, has spurred significant research activity across multiple disciplines. In particular, mathematical models have served as essential tools for analyzing key aspects of the pandemic, including infection control, transmission mitigation, vaccine deployment, and treatment strategies involving immunization [11]. In recent developments, fractional calculus has been increasingly utilized to capture the complex memory and hereditary properties of COVID-19 dynamics [12]. These models offer enhanced flexibility in characterizing anomalous diffusion and nonlinear behavior that are often observed in epidemiological processes. Moreover, advanced mathematical techniques such as stability theory and numerical simulations involving stochastic and nonlocal fractional differential operators have further enriched the analytical framework, providing deeper insights into the system's behavior under uncertainty and spatial heterogeneity [13].

Mathematical modeling remains a vital tool for analyzing the dynamics of infectious diseases and designing effective control strategies. In the classical compartmental model, the population is divided into three classes: susceptible, infected, and recovered individuals [4]. To enhance the realism and applicability of this model, various extensions have been proposed that incorporate additional epidemiological factors and refine the population structure. These modifications aim to improve the accuracy and predictive power of the model, thereby enabling the development of more targeted and efficient intervention strategies. Common approaches include subdividing the population into further categories and incorporating mechanisms such as control actions that influence the transmission dynamics. In models that incorporate control measures, several intervention strategies are considered, each designed to reduce the transmission rate between susceptible and infectious individuals. The optimal choice of control strategy depends on factors such as the nature of the disease, host characteristics, and the severity of the outbreak. For further studies discussing epidemic models with control interventions, see [14]. In addition, time delays are often integrated into epidemic models to more accurately reflect the biological and behavioral processes involved in disease transmission. These delays represent incubation periods, delayed immune responses, or the time required for interventions to take effect. Mathematically, they are modeled using delay differential equations. For more detailed investigations on time-delay in epidemic models, see [15].

This study aims to develop two extended epidemic models that build upon the framework introduced in [16]. The proposed models incorporate transmission dynamics in five epidemiological compartments: susceptible individuals $\mathbb{S}$, exposed individuals $\mathbb{E}$, asymptomatic infectious individuals $\mathbb{I}_A$, symptomatic infectious individuals $\mathbb{I}_S$, and vaccinated individuals $\mathbb{V}$. Since recovered individuals are assumed to have no further impact on disease transmission, they are excluded from the current modeling framework. The first model refines the structure in [16] by introducing two discrete time delays: $\mathcal{J}$ and $\mathcal{N}$. The delay $\mathcal{J}$ accounts for the incubation period between exposure and infectiousness, while $\mathcal{N}$ represents the lag associated with vaccine deployment or the time required for individuals to develop immunity following vaccination. The second model incorporates a control variable ν to represent intervention strategies aimed at reducing transmission. This control-based framework enables the evaluation of policy-driven actions in mitigating the spread of the disease.

The structure of this paper is as follows. In Section 2, we present and analyze the discrete time-delay model. Section 3 introduces the control-based model and investigates its dynamic behavior. Numerical simulations illustrating the theoretical results and exploring the impact of key parameters are provided in Section 4. Finally, Section 5 summarizes the main findings and offers concluding remarks.

2. Discrete time-delay model $\mathbb{S}\mathbb{E}\mathbb{I}_A\mathbb{I}_S\mathbb{V}$

In this section, we extend the epidemic model originally proposed in [16] by formulating a system of delay differential equations. The modified model incorporates time-delay factors to investigate their influence on the transmission dynamics of infectious diseases. Specifically, we analyze how incubation and vaccination delays affect the progression and control of outbreaks. Let $\mathbb{S}(t)$, $\mathbb{E}(t)$, $\mathbb{I}_A(t)$, $\mathbb{I}_S(t)$, and $\mathbb{V}(t)$ denote the populations of susceptible individuals, exposed individuals, asymptomatic infected individuals, symptomatic infected individuals, and vaccinated individuals, respectively, at time t. These compartments reflect the epidemiological structure upon which the delay-based model is constructed.

2.1. System formulation

The developed model is described as

$$\begin{aligned}
\frac{d\,\mathbb{S}(t)}{dt} &= \Xi - \mathbb{S}(t)\left(\varrho_1 \mathbb{I}_A(t) + \varrho_2 \mathbb{I}_S(t)\right) - (\zeta + \iota)\mathbb{S}(t),\\
\frac{d\mathbb{E}(t)}{dt} &= e^{-\epsilon_1 \mathcal{J}}\mathbb{S}(t-\mathcal{J})\left(\varrho_1 \mathbb{I}_A(t-\mathcal{J}) + \varrho_2 \mathbb{I}_S(t-\mathcal{J})\right) + e^{-\epsilon_2 \mathcal{N}}\varepsilon\,\mathbb{V}(t-\mathcal{N})(\varrho_1 \mathbb{I}_A(t-\mathcal{N}) + \varrho_2 \mathbb{I}_S(t-\mathcal{N}))\\
&\quad - (\sigma + \iota)\mathbb{E}(t),\\
\frac{d\mathbb{I}_A(t)}{dt} &= \alpha\,\sigma\,\mathbb{E}(t) - (\varpi_1 + \iota)\mathbb{I}_A(t),\\
\frac{d\mathbb{I}_S(t)}{dt} &= (1-\alpha)\,\sigma\,\mathbb{E}(t) - (\varpi_2 + \iota)\mathbb{I}_S(t),\\
\frac{d\mathbb{V}(t)}{dt} &= \zeta\,\mathbb{S}(t) - \varepsilon\,\varrho_1 \mathbb{V}(t)\,\mathbb{I}_A(t) - \varepsilon\,\varrho_2 \mathbb{V}(t)\,\mathbb{I}_S(t) - (\varrho + \iota)\,\mathbb{V}(t),
\end{aligned} \tag{2.1}$$

Hence, $\mathcal{J}$ is assigned to delay in infection due to the incubation period of the disease and $\mathcal{N}$ is defined as the delay in vaccination, which reflects the time lag in the production of the vaccine or the delay in individuals receiving the vaccination. The probabilities of infected individuals surviving during the delay periods $[0, \mathcal{J}]$ and $[0, \mathcal{N}]$ are given by $e^{-\epsilon_1 \mathcal{J}}$ and $e^{-\epsilon_2 \mathcal{N}}$, respectively. The parameters of the model are illustrated in Table 1.The initial conditions for the proposed model (2.1) are

$$\begin{aligned}
&\mathbb{S}(\mathcal{X}) = \bar{\mathcal{Z}}_1(\mathcal{X}),\ \ \mathbb{E}(\mathcal{X}) = \bar{\mathcal{Z}}_2(\mathcal{X}),\ \ \mathbb{I}_A(\mathcal{X}) = \bar{\mathcal{Z}}_3(\mathcal{X}),\ \ \mathbb{I}_S(\mathcal{X}) = \bar{\mathcal{Z}}_4(\mathcal{X}),\ \ \mathbb{V}(\mathcal{X}) = \bar{\mathcal{Z}}_5(\mathcal{X}),\\
&\bar{\mathcal{Z}}_i(\mathcal{X}) \geq 0,\ \ \mathcal{X} \in [-\mathcal{T}, 0) > 0,\ i = 1, ..., 5,
\end{aligned} \tag{2.2}$$

where $\mathcal{T} = \max\{\mathcal{N}, \mathcal{J}\}$ and $(\bar{\mathcal{Z}}_1(\mathcal{X}), \bar{\mathcal{Z}}_2(\mathcal{X}), \bar{\mathcal{Z}}_3(\mathcal{X}), \bar{\mathcal{Z}}_4(\mathcal{X}), \bar{\mathcal{Z}}_5(\mathcal{X})) \in C([-\mathcal{T}, 0], \mathbf{R}^5_{\geq 0})$ such that C is the Banach space of continuous functions mapping the interval $[-\mathcal{T}, 0]$ into $\mathbb{R}^5_{\geq 0}$. From [17], the model (2.1) has a unique solution satisfying the initial states (2.2).

Table 1. Description of the parameters in the proposed model [16].

Parameters	Description
Ξ	The rate of addition to the susceptible population
ϱ_1	The transmission rate among asymptomatic susceptible individuals
ϱ_2	The rate of infection among susceptible individuals exhibiting symptoms
ζ	Rate of vaccination
ι	Rate of natural mortality
ε	Ineffectiveness of the vaccine , $0 \leq \varepsilon \leq 1$
$\alpha\,\sigma$	Rate of progression from $\mathbb{E}(t)$ into $\mathbb{I}_A(t)$, where $0 \leq \alpha \leq 1$
$(1-\alpha)\,\sigma$	Rate of progression from $\mathbb{E}(t)$ into $\mathbb{I}_S(t)$
ϱ	Vaccinated rate of individuals and that transition to the recovered group
ϖ_1	Infected recovery rate without symptoms
ϖ_2	Infected recovery rate with symptoms

2.2. Preliminaries

This section illustrates the important features of the system (2.1).

Theorem 1. *The solutions of the system (2.1) with the initial data, are non-negative and bounded, such that any solution with the initial conditions in* $\mathfrak{B}$ *remains within* $\mathfrak{B}$ *for all* $t \geq 0$*. Thus,* $\mathfrak{B}$ *is positive invariant, where* $\mathfrak{B}$ *is defined as*

$$\mathfrak{B} = \left\{ (\mathbb{S}, \mathbb{E}, \mathbb{I}_A, \mathbb{I}_S, \mathbb{V}) \in \mathbf{R}^5_{\geq 0} : 0 \leq \mathbb{S}, \mathbb{E}, \mathbb{I}_A, \mathbb{I}_S, \mathbb{V} \leq \frac{\Xi}{\iota} \right\}.$$

Proof. It is clear from the first equation of (2.1) that

$$\frac{d\mathbb{S}(t)}{dt}|_{\mathbb{S}=0} \; = \Xi \; \geq 0$$

for all $t > 0$.

Similarly, it follows from the second equation of the system (2.1) that

$$\begin{aligned} \frac{d\mathbb{E}(t)}{dt}|_{E=0} \quad &= e^{-\epsilon_1 \mathcal{J}} \mathbb{S}(t - \mathcal{J}) \left(\varrho_1 \mathbb{I}_A(t - \mathcal{J}) + \varrho_2 \mathbb{I}_S(t - \mathcal{J})\right) \\ &+ e^{-\epsilon_2 \mathcal{N}} \varepsilon \, \mathbb{V}(t - \mathcal{N})(\varrho_1 \mathbb{I}_A(t - \mathcal{N}) + \varrho_2 \mathbb{I}_S(t - \mathcal{N})) \geq 0 \end{aligned}$$

for all $t > 0$. Furthermore, from the third equation of system (2.1), it can be shown that

$$\frac{d\mathbb{I}_A(t)}{dt} \quad |_{\mathbb{I}_A=0} = \alpha \, \sigma \, \mathbb{E}(t) \; \geq 0$$

Using the same approach as that for $\mathbb{S}(t), \mathbb{E}(t)$, and $\mathbb{I}_A(t)$, it is easy to show that $\mathbb{I}_S(t) > 0, \mathbb{V}(t) > 0$.

Adding the first three equations and the last three equations in the system (2.1) gives,

$$\frac{dN_Y}{dt} = \Xi - \iota N_Y$$

such that

$$N_Y = \mathbb{S}(t) + \mathbb{E}(t) + \mathbb{I}_A(t) + \mathbb{I}_S(t) + \mathbb{V}(t).$$

It follows that

$$\Xi - (\iota + \Xi) N_Y \leq \frac{dN_Y}{dt} < \Xi - \iota N_Y$$

So that

$$\frac{\Xi}{\iota + \Xi} \leq \lim_{t \to \infty} inf N_Y(t) \leq \lim_{t \to \infty} sup N_Y(t) \leq \frac{\Xi}{\iota}$$

Hence

$$\lim_{t \to \infty} sup N_Y(t) \leq \frac{\Xi}{\iota}.$$

□

2.3. The basic reproduction number

One of the important tools that can be used to establish the behavior of equilibria is the basic reproduction number $\Re_0$. As a result, we will derive $\Re_0$ by constructing the next-generation matrix $\Im\Pi^{-1}$ as follows [18]:

$$\Im = \begin{pmatrix} 0 & \varrho_1\,(e^{-\epsilon_1 \mathcal{J}}\mathbb{S}_0 + e^{-\epsilon_2 N}\varepsilon\mathbb{V}_0) & \varrho_2\,(e^{-\epsilon_1 \mathcal{J}}\mathbb{S}_0 + e^{-\epsilon_2 N}\varepsilon\mathbb{V}_0) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where, $\Im$ is the Jacobian matrix of the new infection terms evaluated at the disease-free equilibrium $\mathcal{U}_0$. These terms correspond to the nonlinear components in the equations governing the $\mathbb{E}$, $\mathbb{I}_A$, and $\mathbb{I}_S$ compartments. We then determine the matrix Π as follows:

$$\Pi = \begin{pmatrix} \sigma + \iota & 0 & 0 \\ -\alpha\,\sigma & \varpi_1 + \iota & 0 \\ -(1-\alpha)\,\sigma & 0 & \varpi_2 + \iota \end{pmatrix},$$

where, Π represents the Jacobian of the transition terms evaluated at the disease-free equilibrium. These terms are typically linear in the infected state variables. Thus, we get

$$\Im\Pi^{-1} = \begin{pmatrix} \frac{\sigma(e^{-\epsilon_1 \mathcal{J}}\mathbb{S}_0 + e^{-\epsilon_2 N}\varepsilon\mathbb{V}_0)}{(\sigma+\iota)}\left(\frac{\varrho_1\,\alpha}{\varpi_1+\iota} + \frac{\varrho_2\,(1-\alpha)}{\varpi_2+\iota}\right) & \frac{\varrho_1\,(e^{-\epsilon_1 \mathcal{J}}\mathbb{S}_0 + e^{-\epsilon_2 N}\varepsilon\mathbb{V}_0)}{\varpi_1+\iota} & \frac{\varrho_2\,(e^{-\epsilon_1 \mathcal{J}}\mathbb{S}_0 + e^{-\epsilon_2 N}\varepsilon\mathbb{V}_0)}{\varpi_2+\iota} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, the basic reproduction number $\Re_0$ is the maximum eigenvalue of of the matrix $\Im\Pi^{-1}$, which is obtained as follows:

$$\begin{aligned} \Re_0 &= \frac{\sigma(e^{-\epsilon_1 \mathcal{J}}\mathbb{S}_0 + e^{-\epsilon_2 N}\varepsilon\mathbb{V}_0)}{(\sigma+\iota)}\left(\frac{\varrho_1\ \alpha}{\varpi_1+\iota} + \frac{\varrho_2\ (1-\alpha)}{\varpi_2+\iota}\right) \\ &= \underbrace{\frac{\sigma(e^{-\epsilon_1 \mathcal{J}}\mathbb{S}_0 + e^{-\epsilon_2 N}\varepsilon\mathbb{V}_0)}{(\sigma+\iota)}\frac{\varrho_1\,\alpha}{\varpi_1+\iota}}_{\mathcal{R}_{0,1}} + \underbrace{\frac{\sigma(e^{-\epsilon_1 \mathcal{J}}\mathbb{S}_0 + e^{-\epsilon_2 N}\varepsilon\mathbb{V}_0)}{(\sigma+\iota)}\frac{\varrho_2\ (1-\alpha)}{\varpi_2+\iota}}_{\mathcal{R}_{0,2}}. \end{aligned}$$

Hence, $\mathcal{R}_{0,1}$ and $\mathcal{R}_{0,2}$ describe the average number of secondary cases caused by contact with the infected individuals without symptoms and the average number of secondary cases related to infected individuals with symptoms, respectively.

2.4. Steady states

Lemma 2. System (2.1) has a positive basic reproduction number $\Re_0$ such that the following hold:

(i) If $\Re_0 \leq 1$, then (2.1) has only one fixed point $\mathcal{U}_0$.
(ii) If $\Re_0 > 1$, then (2.1) has two equilibrium points $\mathcal{U}_0$ and $\mathcal{U}^*$.

Proof. To obtain the equilibria of (2.1), Let $(\mathbb{S}, \mathbb{E}, \mathbb{I}_A, \mathbb{I}_S, \mathbb{V})$ be an equilibrium point that solves the following system:

$$0 = \Xi - \varrho_1 \mathbb{S}\,\mathbb{I}_A - \varrho_2 \mathbb{S}\,\mathbb{I}_S - (\zeta+\iota)\mathbb{S}, \tag{2.3}$$

$$
\begin{aligned}
0 &= e^{-\epsilon_1 \mathcal{J}}\mathbb{S}\,(\varrho_1\mathbb{I}_A+\varrho_2\mathbb{I}_S)+e^{-\epsilon_2 \mathcal{N}}\varepsilon\,\mathbb{V}(\varrho_1\mathbb{I}_A+\varrho_2\mathbb{I}_S)-(\sigma+\iota)\mathbb{E}, &(2.4)\\
0 &= \alpha\,\sigma\,\mathbb{E}-(\varpi_1+\iota)\mathbb{I}_A, &(2.5)\\
0 &= (1-\alpha)\,\sigma\,\mathbb{E}-(\varpi_2+\iota)\mathbb{I}_S, &(2.6)\\
0 &= \zeta\,\mathbb{S}-\varepsilon\varrho_1\mathbb{V}\,\mathbb{I}_A-\varepsilon\varrho_2\mathbb{V}\,\mathbb{I}_S-(\varrho+\iota)\,\mathbb{V}. &(2.7)
\end{aligned}
$$

Then, by some calculations, we get

$$
\begin{aligned}
\mathbb{S} &= \frac{\Xi}{\left(\frac{\varrho_1\,\alpha\,\sigma}{\varpi_1+\iota}+\frac{\varrho_2\,(1-\alpha)\,\sigma}{\varpi_2+\iota}\right)\mathbb{E}+(\zeta+\iota)}, &(2.8)\\
\mathbb{I}_A &= \frac{\alpha\,\sigma\,\mathbb{E}}{\varpi_1+\iota}, &(2.9)\\
\mathbb{I}_S &= \frac{(1-\alpha)\,\sigma\,\mathbb{E}}{\varpi_2+\iota}, &(2.10)\\
\mathbb{V} &= \left(\frac{\zeta\Xi}{\left(\frac{\varrho_1\,\alpha\,\sigma}{\varpi_1+\iota}+\frac{\varrho_2\,(1-\alpha)\,\sigma}{\varpi_2+\iota}\right)\mathbb{E}+(\zeta+\iota)}\right)\left(\frac{1}{\left(\frac{\varrho_1\,\alpha\,\sigma}{\varpi_1+\iota}+\frac{\varrho_2\,(1-\alpha)\,\sigma}{\varpi_2+\iota}\right)\varepsilon\mathbb{E}+(\varrho+\iota)}\right). &(2.11)
\end{aligned}
$$

By substituting Eqs (2.8)–(2.11) into (2.4), we have

$$
\mathcal{H}\,\mathbb{E}\,(\mathcal{B}_1\mathrm{E}^2+\mathcal{B}_2\mathbb{E}+\mathcal{B}_3)=0, \tag{2.12}
$$

where,

$$
\begin{aligned}
\mathcal{H} &= \frac{1}{-\varepsilon\,\mathcal{M}_1{}^2\,\mathbb{E}^2-\mathcal{M}_1\,\left((\iota+\varrho)\,+\varepsilon\,(\zeta+\iota)\right)\,\mathbb{E}-(\zeta+\iota)\,(\iota+\varrho)},\\
\mathcal{B}_1 &= \varepsilon\,(\iota+\sigma)\,\mathcal{M}_1^2,\\
\mathcal{B}_2 &= \mathcal{M}_1\left[(\varrho+\iota)\,(\varepsilon(\zeta+\iota)+(\sigma+\iota))-\Xi\,e^{-\epsilon_1\mathcal{J}}\,\varepsilon\mathcal{M}_1\right],\\
\mathcal{B}_3 &= (1-\Re_0).
\end{aligned}
$$

where

$$
\mathcal{M}_1 = \left(\frac{\varrho_1\,\sigma\,\alpha}{(\varpi_1+\iota)}+\frac{\varrho_2\,\sigma\,(1-\alpha)}{(\varpi_2+\iota)}\right),
$$

Hence, from (2.12) we see the following:

(i) If $\mathbb{E}=0$, then from (2.8)–(2.11), we see that the disease-free equilibrium point will be $\mathcal{U}_0 = (\mathbb{S}_0,0,0,0,\mathbb{V}_0)=\left(\frac{\Xi}{\zeta+\iota},0,0,0,\frac{\Xi\zeta}{(\varrho+\iota)(\zeta+\iota)}\right)$.

(ii) If $\mathbb{E}\neq 0$, then $\mathcal{B}_1\mathrm{E}^2+\mathcal{B}_2\mathbb{E}+\mathcal{B}_3=0$. Since $\mathcal{B}_2^2-4\,\mathcal{B}_1\,\mathcal{B}_3 > 0$ and $\mathcal{B}_3 < 0$ if and only if $\Re_0 > 1$, which implies that there is a positive real root $\mathbb{E}^*$ when $\Re_0 > 1$. By substituting $\mathbb{E}^*$ into (2.8)–(2.11), we get

$$
\mathbb{I}_A^* = \frac{\alpha\,\sigma\,\mathbb{E}^*}{\varpi_1+\iota}, \quad \text{and } \mathbb{I}_S^* = \frac{(1-\alpha)\,\sigma\,\mathbb{E}^*}{\varpi_2+\iota},
$$

$$\mathbb{S}^* = \frac{\Xi}{\left(\frac{\varrho_1\,\alpha\,\sigma}{\varpi_1+\iota} + \frac{\varrho_2\,(1-\alpha)\,\sigma}{\varpi_2+\iota}\right)\mathbb{E}^* + (\zeta+\iota)},$$

and

$$\mathbb{V}^* = \left(\frac{\zeta\Xi}{\left(\frac{\varrho_1\,\alpha\,\sigma}{\varpi_1+\iota} + \frac{\varrho_2\,(1-\alpha)\,\sigma}{\varpi_2+\iota}\right)\mathbb{E}^* + (\zeta+\iota)}\right)\left(\frac{1}{\left(\frac{\varrho_1\,\alpha\,\sigma}{\varpi_1+\iota} + \frac{\varrho_2\,(1-\alpha)\,\sigma}{\varpi_2+\iota}\right)\varepsilon\mathbb{E}^* + (\varrho+\iota)}\right).$$

It is observed that the endemic equilibrium point $\mathcal{U}^* = (\mathbb{S}^*, \mathbb{E}^*, \mathbb{I}_A^*, \mathbb{I}_S^*, \mathbb{V}^*)$ exists if $\Re_0 > 1$. □

2.5. *Global stability of the equilibrium points*

This section explores the global stability of the fixed points of the system (2.1) by establishing suitable Lyapunov functions. Suppose that the function $\mathcal{D} : \mathbb{R}_{>0} \longrightarrow \mathbb{R}_{\geq 0}$ is defined as $\mathcal{G}(\mathcal{Y}) = \mathcal{Y} - 1 - \ln \mathcal{Y}$. To simplify, we assume $(\mathbb{S}(t), \mathbb{E}(t), \mathbb{I}_A(t), \mathbb{I}_S(t), \mathbb{V}(t)) = (\mathbb{S}, \mathbb{E}, \mathbb{I}_A, \mathbb{I}_S, \mathbb{V})$.

Theorem 3. If $\Re_0 \leq 1$, then the disease-free equilibrium point $\mathcal{U}_0$ of the system (2.1) is globally asymptotically stable ($\mathcal{GAS}$).

Proof. See Appendix A, subsection "Proof of Theorem 3". □

Theorem 4. The endemic equilibrium point $\mathcal{U}^*$ of the system (2.1) is globally asymptotically stable ($\mathcal{GAS}$) if and only if $\Re_0 > 1$.

Proof. See Appendix A, subsection "Proof of Theorem 4".

□

3. Control of the $\mathbb{SEI}_A\mathbb{I}_S\mathbb{V}$ model

The present section deals with the analysis of the mathematical model under the effect of a control action $\nu(t)$, such that $\nu(t) \in [0, 1]$. The case $\nu(t) = 0$ corresponds to no control action, while the case where $\nu(t) = 1$ suggests that transmission is totally prevented.

3.1. *System formulation*

We develop the model in [16] to incorporate a control action to reduce the outbreak of infectious disease $\nu(t)$. The proposed model is follows:

$$\begin{aligned}
\frac{d\mathbb{S}(t)}{dt} &= \Xi - (1-\nu(t))\,\mathbb{S}(t)\,(\varrho_1\mathbb{I}_A(t) - \varrho_2\mathbb{I}_S(t)) - (\zeta+\iota)\mathbb{S}(t),\\
\frac{d\mathbb{E}(t)}{dt} &= (1-\nu(t))(\varrho_1\mathrm{S}\,\mathbb{I}_A + \varrho_2\mathrm{S}\,\mathbb{I}_S + \varepsilon\,\varrho_1\mathbb{V}\,\mathbb{I}_A + \varepsilon\,\varrho_2\mathbb{V}\,\mathbb{I}_S) - (\sigma+\iota)\mathbb{E},\\
\frac{d\mathbb{I}_A(t)}{dt} &= \alpha\,\sigma\,\mathbb{E}(t) - (\varpi_1+\iota)\mathbb{I}_A(t),\\
\frac{d\mathbb{I}_S(t)}{dt} &= (1-\alpha)\,\sigma\,\mathbb{E}(t) - (d_2+\iota)\mathbb{I}_S(t),\\
\frac{d\mathbb{V}(t)}{dt} &= \zeta\,\mathbb{S}(t) - (1-\nu(t))\,(\varepsilon\,\varrho_1\mathbb{V}(t)\,\mathbb{I}_A(t) - \varepsilon\,\varrho_2\mathbb{V}(t)\,\mathbb{I}_S(t)) - (\varrho+\iota)\,\mathbb{V}(t),
\end{aligned} \tag{3.1}$$

3.2. Preliminaries

Theorem 5. *All solutions of the model (3.1) are positive and bounded, such that any solution with the initial conditions in $\bar{\mathfrak{B}}$ remains within $\bar{\mathfrak{B}}$ for all $t \geq 0$. Thus, $\bar{\mathfrak{B}}$ is positive invariant, where $\bar{\mathfrak{B}}$ is defined as*

$$\bar{\mathfrak{B}} = \left\{ (\mathbb{S}, \mathbb{E}, \mathbb{I}_A, \mathbb{I}_S, \mathbb{V}) \in \mathbf{R}^5_{\geq 0} : 0 \leq \mathbb{S}, \mathbb{E}, \mathbb{I}_A, \mathbb{I}_S, \mathbb{V} \leq \frac{\Xi}{\iota} \right\}.$$

Proof. The first equation of (3.1) clearly shows that

$$\frac{d\mathbb{S}(t)}{dt}|_{\mathbb{S}=0} \;= \Xi \;\geq 0$$

for all $t > 0$.

In the same way, the second equation of the system (3.1) implies that

$$\frac{d\mathbb{E}(t)}{dt}|_{\mathbb{E}=0} \;= (1 - \nu(t))(\varrho_1 \mathrm{S}\, \mathbb{I}_A + \varrho_2 \mathrm{S}\, \mathbb{I}_S \;+ \varepsilon \varrho_1 \mathbb{V}\, \mathbb{I}_A + \varepsilon \varrho_2 \mathbb{V}\, \mathbb{I}_S) \;\geq 0$$

for all $t > 0$.

Additionally, it can be demonstrated from the third equation of the system (3.1), that

$$\frac{d\mathbb{I}_A(t)}{dt} \;|_{\mathbb{I}_A=0} = \alpha\, \sigma\, \mathbb{E}(t) \;\geq 0$$

By applying the same approach used for $\mathbb{S}(t), \mathbb{E}(t)$, and $\mathbb{I}_A(t)$, it is straight forward to demonstrate that $\mathbb{I}_S(t) > 0$, and $\mathbb{V}(t) > 0$.

Adding the first three equations and the last three equations in the system (3.1) results in:

$$\frac{dN}{dt} = \Xi - \iota N.$$

In this way

$$N = \mathbb{S}(t) + \mathbb{E}(t) + \mathbb{I}_A(t) + \mathbb{I}_S(t) + \mathbb{V}(t).$$

It can be concluded that

$$\Xi - (\iota + \Xi)N \leq \frac{dN}{dt} < \Xi - \iota N.$$

In order, this leads to

$$\frac{\Xi}{\iota + \Xi} \leq \lim_{t\to\infty} inf N(t) \leq \lim_{t\to\infty} sup N(t) \leq \frac{\Xi}{\iota}.$$

Therefore

$$\lim_{t\to\infty} sup N(t) \leq \frac{\Xi}{\iota}.$$

□

3.3. The basic reproduction number

In order to establish the basic reproduction number $\Re_0(\nu)$, we assume that $\nu(t) = \nu$ is constant. We then determine the next-generation matrix $\Im\Pi^{-1}$ following a procedure similar to that in Section 2.3. Thus, we have:

$$\Im = \begin{pmatrix} 0 & (1-\nu)\varrho_1(\mathbb{S}_0+\varepsilon\mathbb{V}_0) & \varrho_2(1-\nu)(\mathbb{S}_0+\varepsilon\mathbb{V}_0), \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\Pi = \begin{pmatrix} \sigma+\iota & 0 & 0 \\ -\alpha\sigma & \varpi_1+\iota & 0 \\ -(1-\alpha)\sigma & 0 & \varpi_2+\iota \end{pmatrix}.$$

Thus, we get

$$\Im\Pi^{-1} = \begin{pmatrix} \frac{(1-\nu)\sigma(\mathbb{S}_0+\varepsilon\mathbb{V}_0)}{(\sigma+\iota)}\left(\frac{\varrho_1\,\alpha}{\varpi_1+\iota}+\frac{\varrho_2\,(1-\alpha)}{\varpi_2+\iota}\right) & \frac{(1-\nu)\varrho_1\,(\mathbb{S}_0+\varepsilon\mathbb{V}_0)}{\varpi_1+\iota} & \frac{(1-\nu)\varrho_2\,(\mathbb{S}_0+\varepsilon\mathbb{V}_0)}{\varpi_2+\iota} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence, the basic reproduction number $\Re_0(\nu)$ is as

$$\begin{aligned} \Re_0(\nu) &= \frac{(1-\nu)\sigma(\mathbb{S}_0+\varepsilon\mathbb{V}_0)}{(\sigma+\iota)}\left(\frac{\varrho_1\,\alpha}{\varpi_1+\iota}+\frac{\varrho_2\,(1-\alpha)}{\varpi_2+\iota}\right) \\ &= \underbrace{\frac{(1-\nu)\sigma(\mathbb{S}_0+\varepsilon\mathbb{V}_0)}{(\sigma+\iota)}\frac{\varrho_1\,\alpha}{\varpi_1+\iota}}_{\mathcal{R}_{0,1}} + \underbrace{\frac{(1-\nu)\sigma(\mathbb{S}_0+\varepsilon\mathbb{V}_0)}{(\sigma+\iota)}\frac{\varrho_2\,(1-\alpha)}{\varpi_2+\iota}}_{\mathcal{R}_{0,2}}. \end{aligned}$$

Hence, $\mathcal{R}_{0,1}(\nu)$ and $\mathcal{R}_{0,2}(\nu)$ describe the average number of secondary cases caused by contact with the infected individuals without symptoms and the average number of secondary cases related to infected individuals with symptoms, respectively.

3.4. Steady states

Lemma 6. The system (3.1) has a positive basic reproduction number $\Re_0(\nu)$ such that the following hold:

(i) When $\Re_0(\nu) \le 1$, then only one equilibrium point $\mathcal{U}_0$ exists.
(ii) When $\Re_0(\nu) > 1$, then two equilibria $\mathcal{U}_0$ and $\mathcal{U}^*$ exist.

Proof. To find the equilibria of (3.1), we assume that the point $(\mathbb{S},\mathbb{E},\mathbb{I}_A,\mathbb{I}_S,\mathbb{V})$ is an equilibrium that solving the following equations:

$$0 = \Xi - (1-\nu(t))(\varrho_1\mathbb{S}\,\mathbb{I}_A + (\varrho_2\mathbb{S}\,\mathbb{I}_S) - (\zeta+\iota)\mathbb{S}, \tag{3.2}$$

$$0 = (1-\nu(t))(\varrho_1\mathrm{S}\,\mathbb{I}_A + \varrho_2\mathrm{S}\,\mathbb{I}_S + \varepsilon\varrho_1\mathbb{V}\,\mathbb{I}_A + \varepsilon\varrho_2\mathbb{V}\,\mathbb{I}_S) - (\sigma+\iota)\mathbb{E} \tag{3.3}$$

$$0 = \alpha\sigma\mathbb{E} - (\varpi_1+\iota)\mathbb{I}_A, \tag{3.4}$$

$$0 = (1-\alpha)\sigma\mathbb{E} - (\varpi_2+\iota)\mathbb{I}_S, \tag{3.5}$$

$$0 = \zeta\,\mathbb{S} - (1-\nu(t))(\varepsilon\varrho_1\mathbb{V}\,\mathbb{I}_A - \varepsilon\varrho_2\mathbb{V}\,\mathbb{I}_S) - (\varrho+\iota)\mathbb{V}. \tag{3.6}$$

To simplify, consider $\delta_1 = (1-\nu)\varrho_1$ and $\delta_2 = (1-\nu)\varrho_2$. After some calculations, we have

$$\mathbb{S} = \frac{\Xi}{\left(\frac{\delta_1\,\alpha\,\sigma}{\varpi_1+\iota} + \frac{\delta_2\,(1-\alpha)\,\sigma}{\varpi_2+\iota}\right)\mathbb{E} + (\zeta+\iota)}, \tag{3.7}$$

$$\mathbb{I}_A = \frac{\alpha\,\sigma\,\mathbb{E}}{\varpi_1+\iota}, \tag{3.8}$$

$$\mathbb{I}_S = \frac{(1-\alpha)\,\sigma\,\mathbb{E}}{\varpi_2+\iota}, \tag{3.9}$$

$$\mathbb{V} = \left(\frac{\zeta\Xi}{\left(\frac{\delta_1\,\alpha\,\sigma}{\varpi_1+\iota} + \frac{\delta_2\,(1-\alpha)\,\sigma}{\varpi_2+\iota}\right)\mathbb{E} + (\zeta+\iota)}\right)\left(\frac{1}{\left(\frac{\delta_1\,\alpha\,\sigma}{\varpi_1+\iota} + \frac{\delta_2\,(1-\alpha)\,\sigma}{\varpi_2+\iota}\right)\varepsilon\mathbb{E} + (\varrho+\iota)}\right). \tag{3.10}$$

Now, by substituting Eqs (3.7)–(3.10) into (3.3), we get

$$\mathcal{H}\,\mathbb{E}\,(\mathcal{B}_1\mathrm{E}^2 + \mathcal{B}_2\mathbb{E} + \mathcal{B}_3) = 0, \tag{3.11}$$

where,

$$\begin{aligned}
\mathcal{H} &= \frac{1}{-\varepsilon\,\mathcal{M}_1{}^2\,\mathbb{E}^2 - \mathcal{M}_1\,\left((\iota+\varrho) + \varepsilon\,(\zeta+\iota)\right)\mathbb{E} - (\zeta+\iota)\,(\iota+\varrho)},\\
\mathcal{B}_1 &= \varepsilon\,(\iota+\sigma)\,\mathcal{M}_1^2,\\
\mathcal{B}_2 &= \mathcal{M}_1\left[(\varrho+\iota)\,(\varepsilon(\zeta+\iota) + (\sigma+\iota)) - \Xi\,\varepsilon\mathcal{M}_1\right],\\
\mathcal{B}_3 &= (1-\Re_0(\nu)).
\end{aligned}$$

where

$$\mathcal{M}_1 = \sigma\left(\frac{\delta_1\,\alpha}{(\varpi_1+\iota)} + \frac{\delta_2\,(1-\alpha)}{(\varpi_2+\iota)}\right),$$

Thus, from (3.11), we conclude the following:

(i) When $\mathbb{E} = 0$, then, from (3.7)–(3.10), we obtain the disease-free equilibrium point $\mathcal{U}_0 = (\mathbb{S}_0, 0, 0, 0, \mathbb{V}_0) = \left(\frac{\Xi}{\zeta+\iota}, 0, 0, 0, \frac{\Xi\zeta}{(\varrho+\iota)(\zeta+\iota)}\right)$.

(ii) When $\mathbb{E} \neq 0$, then we find $\mathcal{B}_1\mathrm{E}^2 + \mathcal{B}_2\mathbb{E} + \mathcal{B}_3 = 0$. Since $\mathcal{B}_2^2 - 4\,\mathcal{B}_1\,\mathcal{B}_3 > 0$ and $\mathcal{B}_3 < 0$ if and only if $\Re_0(\nu) > 1$, implies a positive real root $\mathbb{E}^*$ if $\Re_0(\nu) > 1$ exists. By substituting $\mathbb{E}^*$ into (3.7)–(3.10), we get

$$\begin{aligned}
\mathbb{I}_A^* &= \frac{\alpha\,\sigma\,\mathbb{E}^*}{\varpi_1+\iota}, \quad \text{and } \mathbb{I}_S^* = \frac{(1-\alpha)\,\sigma\,\mathbb{E}^*}{\varpi_2+\iota},\\
\mathbb{S}^* &= \frac{\Xi}{\left(\frac{\delta_1\,\alpha\,\sigma}{\varpi_1+\iota} + \frac{\delta_2\,(1-\alpha)\,\sigma}{\varpi_2+\iota}\right)\mathbb{E}^* + (\zeta+\iota)},
\end{aligned}$$

and

$$\mathbb{V}^* = \left(\frac{\zeta\Xi}{\left(\frac{\delta_1\,\alpha\,\sigma}{\varpi_1+\iota} + \frac{\delta_2\,(1-\alpha)\,\sigma}{\varpi_2+\iota}\right)\mathbb{E}^* + (\zeta+\iota)}\right)\left(\frac{1}{\left(\frac{\delta_1\,\alpha\,\sigma}{\varpi_1+\iota} + \frac{\delta_2\,(1-\alpha)\,\sigma}{\varpi_2+\iota}\right)\varepsilon\mathbb{E}^* + (\varrho+\iota)}\right).$$

It is clear that the endemic equilibrium point $U^* = (\mathbb{S}^*, \mathbb{E}^*, \mathbb{I}_A^*, \mathrm{I}_S^*, \mathbb{V}^*)$ exists if $\Re_0(\nu) > 1$. □

3.5. Asymptotic controllability

In order to simplify the calculations, we set $\varrho_1(1-\nu) = \Xi_1$ and $\varrho_2(1-\nu) = \Xi_2$. Hence, we examine the global stability of the disease-free equilibrium point $\mathcal{U}_0$ and the endemic equilibrium point $\mathcal{U}^*$ by constructing the Lyapunov function. Model (3.1) can be formulated as a nonlinear control system

$$\dot{\mathbb{X}} = \mathbb{Z}(\mathbb{X}(t), \nu(t)), \ \ \mathbb{X}(0) = \mathbb{X}_0,$$

where $\mathbb{X} = (\mathbb{S}, \mathbb{E}, \mathbb{I}_A, \mathbb{I}_S, \mathbb{V})^T \in \Upsilon$, $\nu(t) = \nu$, and $\nu \in [0,1]$. Next, we prove that for any initial $\mathbb{Y} \in \Upsilon$, $\nu \in [0,1]$ exists such that $\lim_{t\longrightarrow\infty} \mathbb{X}(0) = \mathcal{U}_0$.

Theorem 7. The system (3.1) is asymptotically controllable from Υ to $\mathcal{U}_0$ if $\nu \geq \nu^m$, where $\nu^m = \max\left\{0, 1 - \left(\frac{(\sigma+\iota)}{\sigma(\mathbb{S}_0+\varepsilon\mathbb{V}_0)}\right)\left(\frac{1}{\frac{\varrho_1\,\alpha}{\varpi_1+\iota}+\frac{\varrho_2\,(1-\alpha)}{\varpi_2+\iota}}\right)\right\}$.

Proof. See Appendix A, subsection "Proof of Theorem 7". □

Theorem 8. The endemic equilibrium point $\mathcal{U}^*$ is $\mathcal{GAS}$ if and only if $\Re_0(\nu) > 1$.

Proof. See Appendix A, subsection "Proof of Theorem 8". □

4. Numerical simulations

4.1. Sensitivity analysis

Sensitivity analysis of the model with respect to the basic reproduction number $\Re_0$ provides valuable insight into the extent to which changes in the model's parameters affect disease transmission dynamics. To perform this analysis, we compute the partial derivatives of $\Re_0$ with respect to each parameter, thereby quantifying the sensitivity of $\Re_0$ to perturbations in those parameters. This approach helps identify which factors most critically influence the potential for an outbreak and informs targeted intervention strategies.

$$\Re_0 = \frac{\sigma((\varrho+\iota)e^{-\epsilon_1\mathcal{J}} + e^{-\epsilon_2 N}\varepsilon)}{(\sigma+\iota)(\zeta+\iota)}\left(\frac{\varrho_1\ \alpha}{\varpi_1+\iota} + \frac{\varrho_2\ (1-\alpha)}{\varpi_2+\iota}\right)\frac{\Xi}{\zeta+\iota}$$

We notice that $\frac{\partial\Re_0}{\partial\varrho_1} > 0$, $\frac{\partial\Re_0}{\partial\varrho_2} > 0$, $\frac{\partial\Re_0}{\partial\varpi_1} < 0$, $\frac{\partial\Re_0}{\partial\varpi_2} < 0$, $\frac{\partial\Re_0}{\partial\zeta} < 0$, $\frac{\partial\Re_0}{\partial\Xi} > 0$,$\frac{\partial\Re_0}{\partial\sigma} > 0$, and

$$\frac{\partial\Re_0}{\partial\iota} = \frac{(\varrho_1\ \alpha(\varpi_2+\iota) + \varrho_2\ (1-\alpha)(\varpi_1+\iota))\sigma\Xi(1-((\varrho+\iota)e^{-\epsilon_1\mathcal{J}} + e^{-\epsilon_2 N}\varepsilon)\Gamma)}{((\sigma+\iota)(\zeta+\iota)(\varpi_1+\iota)(\varpi_2+\iota)(\zeta+\iota))^2},$$

where $\Gamma = (\varpi_1+\iota)(\varpi_2+\iota)(\zeta+\iota)(\sigma+\iota)(\zeta+\iota) + (\varrho+\iota)e^{-\epsilon_1\mathcal{J}}(\varpi_2+\iota)(\zeta+\iota)(\sigma+\iota) + (\varpi_1+\iota)(\varrho+\iota)e^{-\epsilon_1\mathcal{J}}(\zeta+\iota)(\sigma+\iota) + (\varpi_1+\iota)(\varpi_2+\iota)(\zeta+\iota)(\varrho+\iota)e^{-\epsilon_1\mathcal{J}}$. Consequently, an increase in the parameters ϱ_1, ϱ_2, Ξ, and σ results in an elevated basic reproduction number $\Re_0$, indicating a heightened potential for disease transmission. In contrast, variations in the parameter ι exert no influence on $\Re_0$. The remaining model parameters exhibit a negative sensitivity, such that increases in their values contribute to a reduction in $\Re_0$. These findings highlight the parameters that are most critical in shaping epidemic outcomes and can thus inform targeted public health strategies.

To support our analytical findings, we present numerical simulations for System (2.1) and System (3.1). These simulations confirm the validity of our theoretical results and offer further insight into the systems' dynamics. In particular, we illustrate the influence of time delays and the control parameter on the asymptotic behavior of the system's fixed points. Additionally, we examine the impact of the vaccination rate on mitigating the spread of infectious diseases, demonstrating its effectiveness in reducing an outbreak's intensity.

As our study is performed to discuss a general infectious disease model, the parameter values will be assumed and are given in Table 2.

Table 2. Parameters values.

Ξ	ϱ_1	ϱ_1	ζ	ν	σ	α	ϖ_1	ϖ_2	ρ	ι	ϵ_1	ϵ_2	ε
10	varied	varied	vaied	varied	0.3	0.5	0.3	0.3	0.2	0.2	0.4	0.4	0.3

4.2. Numerical examples of the system (2.1)

To solve the system (2.1), we apply the parameter values in Table 2. We also submit the following three different initial conditions:

- $\mathbf{IC}_1$ $\mathbb{S}(0) = 60, \mathbb{E}(0) = 20, \mathbb{I}_{\mathbb{A}}(0) = 15, \mathbb{I}_S(0) = 10, \mathbb{V}(0) = 25.$
- $\mathbf{IC}_2$ $\mathbb{S}(0) = 40, \mathbb{E}(0) = 30, \mathbb{I}_{\mathbb{A}}(0) = 20, \mathbb{I}_S(0) = 15, \mathbb{V}(0) = 23.$
- $\mathbf{IC}_3$ $\mathbb{S}(0) = 10, \mathbb{E}(0) = 5, \mathbb{I}_{\mathbb{A}}(0) = 4, \mathbb{I}_S(0) = 4, \mathbb{V}(0) = 3.$

Case A: The effect of ϱ_1 and ϱ_2 on the stability of $\mathcal{U}_0$ and $\mathcal{U}^*$

In this case, we set $\mathcal{N} = \mathcal{J} = \tau = 0.3$. We then examine two scenarios.

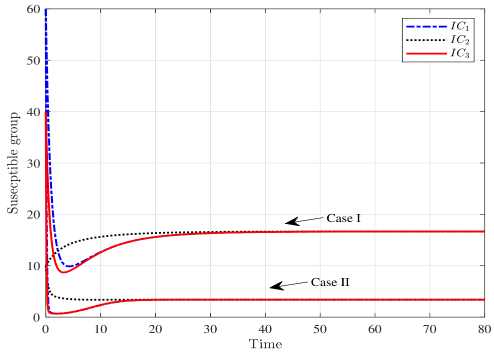
Scenario 1: We choose $\varrho_1 = 0.02$ and $\varrho_2 = 0.03$. We then compute $\Re_0 = 0.57 \leq 1$. From Figure 1, it is observed that the solutions approach $\mathcal{U}_0 \simeq (17, 0, 0, 0, 17)$. According to Theorem 3, the disease- free equilibrium point $\mathcal{U}_0$ is $\mathcal{GAS}$. This result corresponds to Theorem 3.

Scenario 2: We choose $\varrho_1 = 0.3$ and $\varrho_2 = 0.2$. We then compute $\Re_0 = 5.76 > 1$. From Figure 1, it is clear that the solutions tend to $\mathcal{U}^* \simeq (5, 16, 5, 5, 2)$. According to Theorem 3, the endemic equilibrium point $\mathcal{U}^*$ is $\mathcal{GAS}$. This outcome is in accordance with Theorem 4.

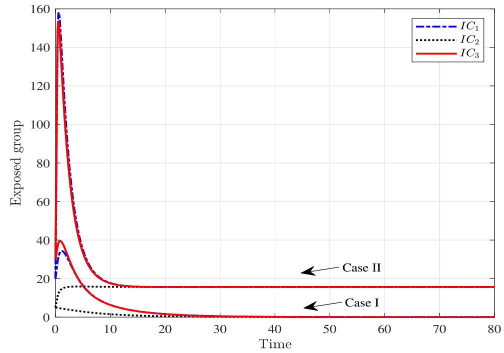
Case B: The effect of time delays on the behavior of the system (2.1)

To study the effect of time delays in decreasing the spread of infectious disease, we assume $\mathcal{N} = \mathcal{J} = \tau$ and we select the values of $\varrho_1 = 0.3$ and $\varrho_2 = 0.2$ in the case where $\Re_0 > 1$. From Table 3 and Figure 2, we conclude that the solution simulations reach to the equilibrium $\mathcal{U}_0$, when $\mathcal{N} = \mathcal{J}$ is increasing. Additionally, in Figure 3(a), the vaccination delay $\mathcal{N}$ is held constant while the infection delay $\mathcal{J}$ is varied. The results indicate a clear decline in the basic reproduction number $\Re_0$ as $\mathcal{J}$ increases. Similarly, Figure 3(b) shows that varying $\mathcal{N}$ while keeping $\mathcal{J}$ fixed also leads to a reduction in $\Re_0$. A comparative analysis of Figures 3(a) and (b) suggests that the impact of $\mathcal{J}$ on $\Re_0$ is more pronounced than that of $\mathcal{N}$. In Figure 3(c), where both delays $\mathcal{N}$ and $\mathcal{J}$ are set to equal values, $\Re_0$ declines significantly and drops below unity, indicating that synchronized delays in both transmission pathways can effectively suppress disease spread. Biologically, this behavior can be interpreted as follows: The inclusion of both vaccination and infection delays yields distinct mechanisms for reducing

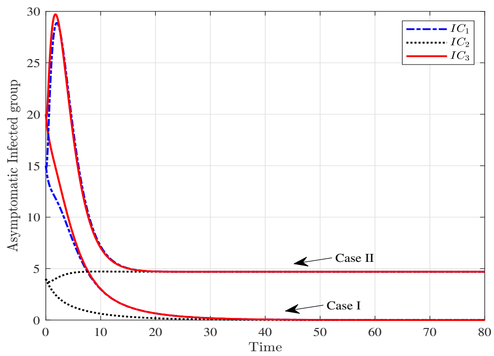
$\Re_0$. The vaccination delay $\mathcal{N}$ indirectly lowers transmission by limiting the contact and mobility of individuals awaiting vaccine efficacy, thereby reducing their contribution to the infection process. In contrast, the infection delay $\mathcal{J}$ directly interrupts transmission chains, decreasing the average number of secondary cases produced by an infectious individual. Mathematically, these effects are captured by exponential decay terms such as $e^{-\epsilon_1 \mathcal{J}}$ and $e^{-\epsilon_2 \mathcal{N}}$, which reflect the attenuation of transmission potential over time. Collectively, these delays exert a synergistic influence, leading to substantial suppression of the epidemic's propagation.



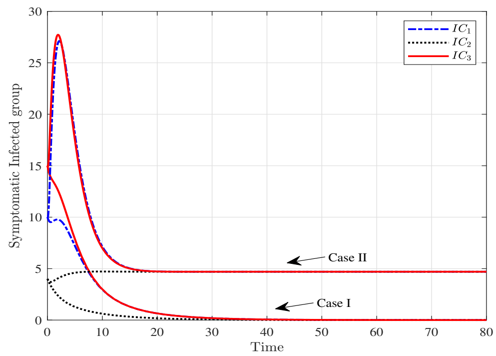
(a) Susceptible individuals.



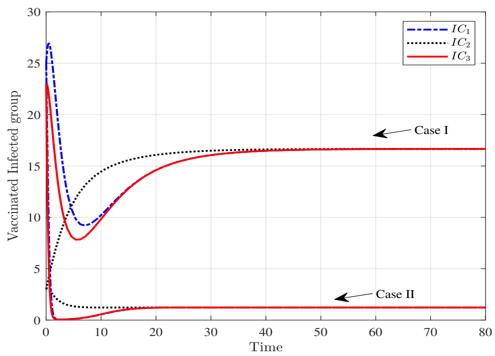
(b) Exposed individuals.



(c) Asymptotic infected individuals.

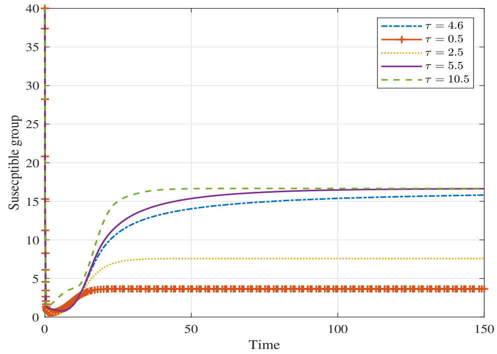


(d) Symptomatic infected individuals.

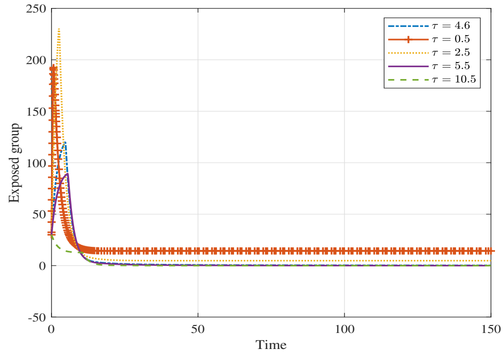


(e) Vaccinated individuals.

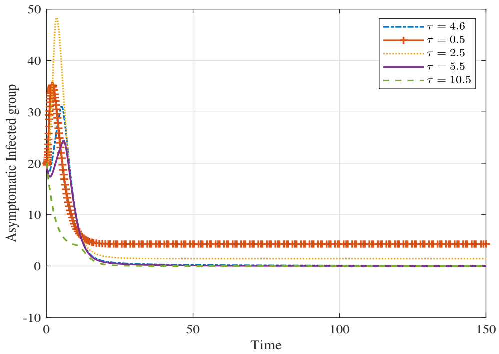
Figure 1. Solution of the system (2.1) with different initial values. Case I: $\Re_0 \leq 1$; Case II: $\Re_0 > 1$.



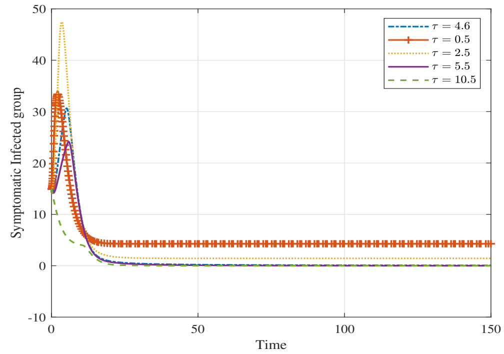
(a) Susceptible individuals.



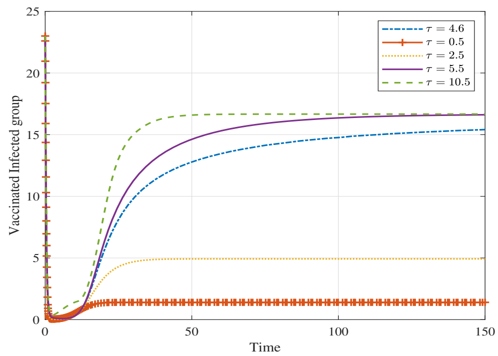
(b) Exposed individuals.



(c) Asymptotic infected individuals.



(d) Symptomatic infected individuals.

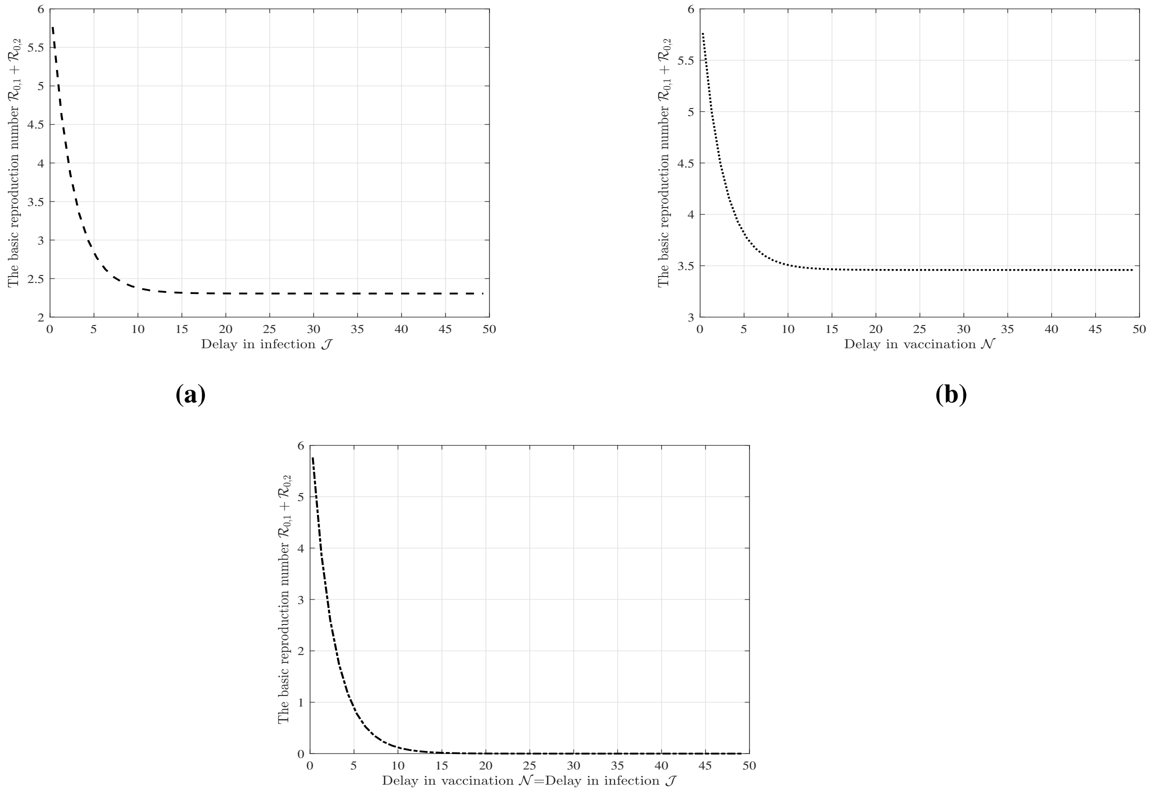


(e) Vaccinated individuals.

Figure 2. Solution of the system (2.1) with different values of time delays and $\Re_0 > 1$.

Table 3. The values of $\Re_0$ with different values of $\mathcal{N} = \mathcal{J}$.

$\mathcal{N} = \mathcal{J}$	Equilibria	$\Re_0$
0.5	$\mathcal{U}^* \simeq (4, 14, 4, 4, 17)$	5.32
2.5	$\mathcal{U}^* \simeq (8, 5, 2, 2, 5)$	2.39
4.6	$\mathcal{U}_0 \simeq (16, 0, 0, 0, 16)$	1.03
5.5	$\mathcal{U}_0 \simeq (16, 0, 0, 0, 16)$	0.72



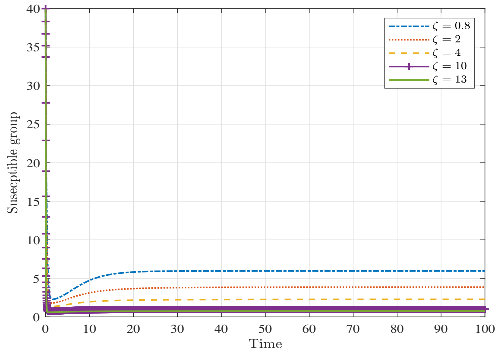
(a) **(b)**

(c)

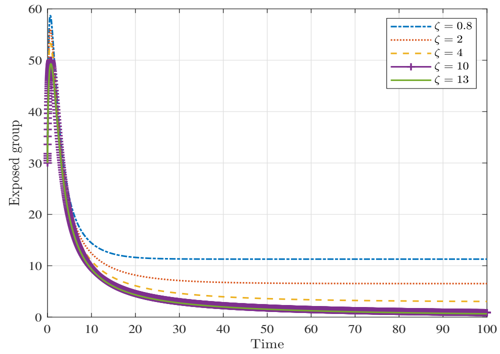
Figure 3. Effect of varying the delay times $\mathcal{N}$ and $\mathcal{J}$ on the basic reproduction number $\Re_0$.

Case C: The effect of the vaccination rate ζ on the solution trajectories of the model (2.1)

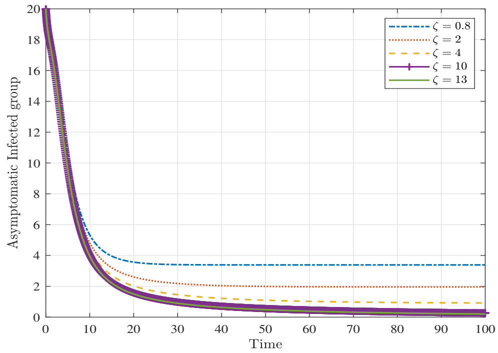
In this case, we numerically solve the system (2.1) for varying values of the vaccination rate parameter ζ to investigate its impact on the transmission dynamics of the infectious disease. The parameters are fixed at $\mathcal{N} = \mathcal{J} = 0.3$, $\varrho_1 = 0.3$, and $\varrho_2 = 0.2$. As illustrated in Figure 4, increasing the value of ζ causes the trajectories of the system (2.1) to converge toward the disease-free equilibrium $\mathcal{U}_0$. This outcome indicates that higher vaccination rates are effective in steering the system toward disease eradication. Consequently, we conclude that vaccination plays a fundamental role in controlling and preventing epidemic outbreaks.



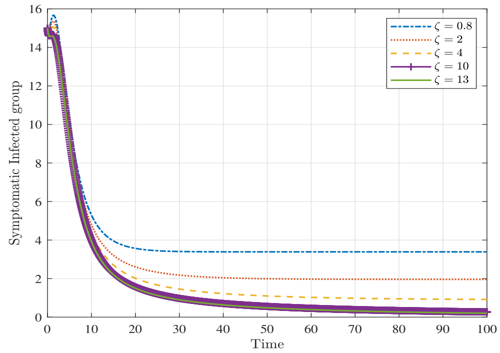
(a) Susceptible individuals.



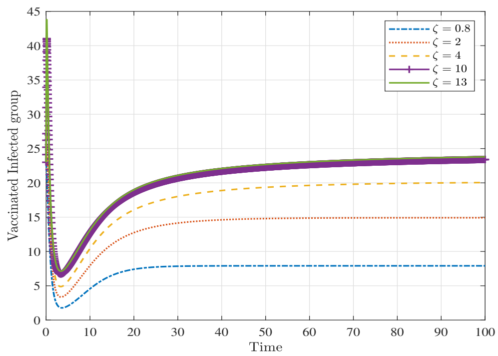
(b) Exposed individuals.



(c) Asymptotic infected individuals.



(d) Symptomatic infected individuals.



(e) Vaccinated individuals.

Figure 4. Solution of the system (2.1) with different values of the vaccination rate ζ.

4.3. Numerical examples of the system (3.1)

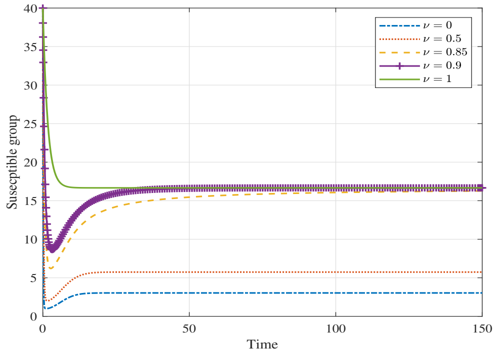
In this subsection, we present the solution of system (3.1).

Case A: Effect of the control parameter ν on the stability of the equilibria for the model (3.1)

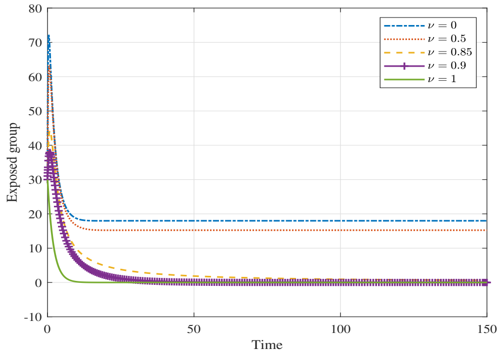
We choose $\varrho_1 = 0.3$ and $\varrho_2 = 0.2$. According to Table 4 and Figure 5, the value of $\Re_0(\nu)$ is decreased as ν increased. We observe that the control parameter helps to prevent the outbreak of infectious diseases.

Table 4. The values of $\Re_0(\nu)$ with different values of ν.

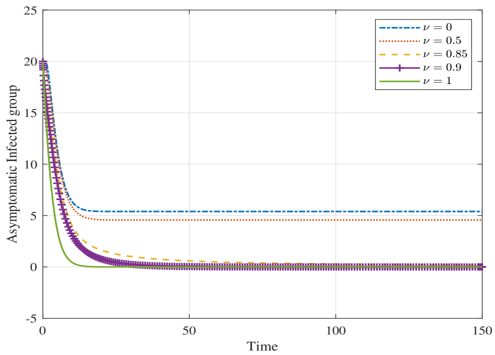
ν	Equilibria	$\Re_0(\nu)$
0	$\mathcal{U}^* \simeq (3, 18, 5, 5, 1)$	6.5
0.5	$\mathcal{U}^* \simeq (6, 15, 5, 5, 3)$	3.25
0.85	$\mathcal{U}_0 \simeq (16, 0, 0, 0, 16)$	0.97
0.9	$\mathcal{U}_0 \simeq (16, 0, 0, 0, 16)$	0.65



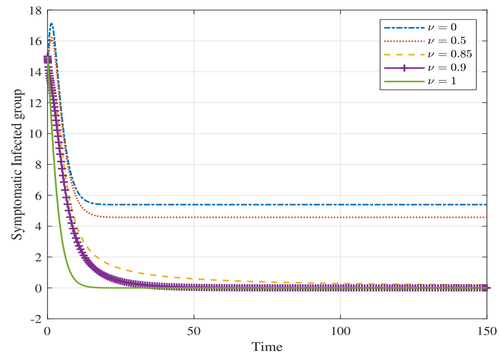
(a) Susceptible individuals.



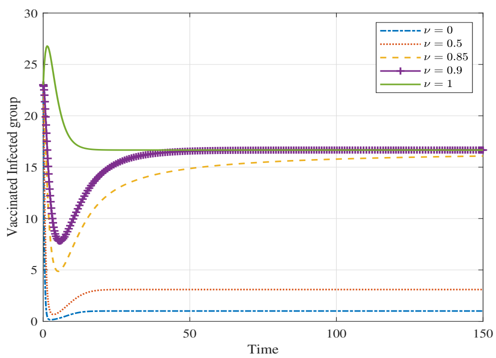
(b) Exposed individuals.



(c) Asymptotic infected individuals.



(d) Symptomatic infected individuals.

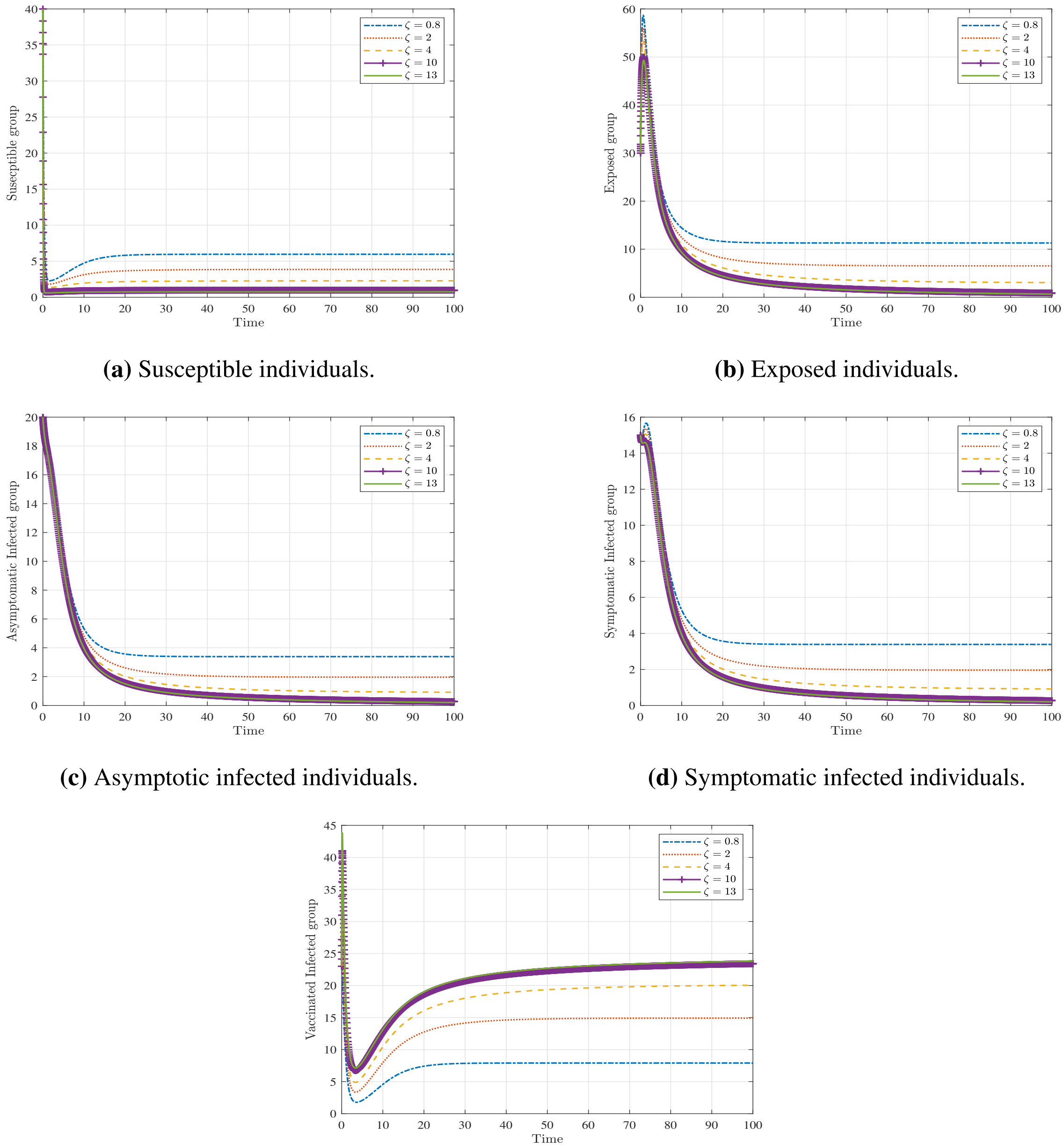


(e) Vaccinated individuals.

Figure 5. Solution of the system (3.1) with different values of the control parameter ν.

Case B: Effect of the vaccination rate ζ on the stability of the model (3.1)

In this case, we set $\nu = 0.6$. Figure 6 displays the solution trajectories of the system (3.1) with different values of ζ. Furthermore, we see that the solution approaches $\mathcal{U}_0$ as ζ reaches 1. This finding agrees with Theorem 7 and Theorem 8.



(a) Susceptible individuals.

(b) Exposed individuals.

(c) Asymptotic infected individuals.

(d) Symptomatic infected individuals.

(e) Vaccinated individuals.

Figure 6. Solution of the system (3.1) with different values of the vaccination rate ζ.

5. Conclusions

In this study, we developed two mathematical models to examine the effects of key epidemiological factors—including vaccination, time delays, and control interventions—on the transmission dynamics of infectious diseases. The proposed models stratify the population into five compartments: susceptible individuals, exposed individuals, asymptomatic infected individuals, symptomatic infected individuals, and vaccinated individuals. The first model incorporates two discrete time delays: $\mathcal{J}$, representing the incubation period between exposure and infectiousness, and $\mathcal{N}$, accounting for delays in vaccination availability or administration. The second model introduces a control parameter ν that reflects intervention efforts aimed at reducing disease transmission. For both models, we established the non-negative and boundedness of the solutions, ensuring their biological validity. We derived the basic reproduction number $\Re_0$ and used it as a threshold parameter to analyze the system's equilibria. Specifically, we identified the disease-free equilibrium $\mathcal{U}_0$ and the endemic equilibrium $\mathcal{U}^*$, and examined their global stability using Lyapunov function techniques in conjunction with La Salle's invariance principle. Two key findings emerged from the analysis: (i) $\mathcal{U}_0$ is globally asymptotically stable ($\mathcal{GAS}$) when $\Re_0 \leq 1$, implying that the disease will die out under these conditions; and (ii) the endemic equilibrium $\mathcal{U}^*$ is $\mathcal{GAS}$ if and only if $\Re_0 \geq 1$, indicating sustained disease presence. Numerical simulations were conducted to validate and illustrate the theoretical results, confirming consistency between analytical insights and the dynamic behavior of the models.

Author contributions

A. Alshareef: Conceptualization, Formal analysis, Methodology, Writing–review and editing, Investigation, Software; F. M. Alotaibi: Conceptualization, Formal analysis, Methodology, Writing–review and editing, Investigation, funding acquisition. All authors contributed equally to this paper. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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A. Appendix A: Proofs of the main results

A.1. Proof of Theorem 3

Proof. We construct the Lyapunov function $\bar{Q}$ as follows:

$$
\begin{aligned}
\bar{Q}(t) \;=\;& \frac{e^{-\epsilon_1 \mathcal{J}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\,\mathbb{S}_0+\varepsilon\, e^{-\epsilon_2 \mathcal{N}}\,\mathbb{V}_0)}\,\mathcal{G}(\mathbb{S})+\frac{e^{-\epsilon_2 \mathcal{N}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\,\mathbb{S}_0+\varepsilon\, e^{-\epsilon_2 \mathcal{N}}\,\mathbb{V}_0)}\,\mathcal{G}(\mathbb{V})+\mathbb{E}+(\varrho_1\,(\iota+\varpi_2))\,\mathbb{I}_A \\
&+(\varrho_2\,(\iota+\varpi_1))\,\mathbb{I}_S+\frac{e^{-\epsilon_1 \mathcal{J}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\,\mathbb{S}_0+\varepsilon\, e^{-\epsilon_2 \mathcal{N}}\,\mathbb{V}_0)}\,\varrho_1 \int_0^{\mathcal{J}} \mathbb{S}(t-\lambda)\,\mathbb{I}_A(t-\lambda)d\lambda \\
&+\frac{e^{-\epsilon_2 \mathcal{N}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\,\mathbb{S}_0+\varepsilon\, e^{-\epsilon_2 \mathcal{N}}\,\mathbb{V}_0)}\,\varepsilon\,\varrho_1 \int_0^{\mathcal{N}} \mathbb{V}(t-\lambda)\,\mathbb{I}_A(t-\lambda)d\lambda \\
&+\frac{e^{-\epsilon_1 \mathcal{J}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\,\mathbb{S}_0+\varepsilon\, e^{-\epsilon_2 \mathcal{N}}\,\mathbb{V}_0)}\,\varrho_2 \int_0^{\mathcal{J}} \mathbb{S}(t-\lambda)\,\mathbb{I}_S(t-\lambda)d\lambda \\
&+\frac{e^{-\epsilon_2 \mathcal{N}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\,\mathbb{S}_0+\varepsilon\, e^{-\epsilon_2 \mathcal{N}}\,\mathbb{V}_0)}\,\varepsilon\,\varrho_2 \int_0^{\mathcal{N}} \mathbb{V}(t-\lambda)\,\mathbb{I}_S(t-\lambda)d\lambda.
\end{aligned}
$$

Clearly, $\bar{Q}$ is positive definite. The time derivative of $\bar{Q}$ along the solutions of (2.1) is introduced as

$$
\begin{aligned}
\frac{d}{dt}\bar{Q} \;\leq\;& e^{-\epsilon_1 \mathcal{J}}\frac{(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2 \mathcal{N}}\varepsilon\,\mathbb{V}_0)}\left(1-\frac{\mathbb{S}_0}{\mathbb{S}}\right)\frac{d\mathbb{S}}{dt}+\frac{(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2 \mathcal{N}}\varepsilon\,\mathbb{V}_0)}\frac{d\mathbb{E}}{d} \\
&+e^{-\epsilon_2 \mathcal{N}}\frac{(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2 \mathcal{N}}\varepsilon\,\mathbb{V}_0)}\left(1-\frac{\mathbb{V}_0}{\mathbb{V}}\right)\frac{d\mathbb{V}}{dt}+(\varrho_1\,(\iota+\varpi_2))\,\frac{d\mathbb{I}_A}{dt}+(\varrho_2\,(\iota+\varpi_1))\frac{d\mathbb{I}_S}{dt} \\
&+\frac{e^{-\epsilon_1 \mathcal{J}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\,\mathbb{S}_0+\varepsilon\, e^{-\epsilon_2 \mathcal{N}}\,\mathbb{V}_0)}\,\varrho_1\,\mathbb{S}\,\mathbb{I}_A-\frac{e^{-\epsilon_1 \mathcal{J}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\,\mathbb{S}_0+\varepsilon\, e^{-\epsilon_2 \mathcal{N}}\,\mathbb{V}_0)}\,\varrho_1\,\mathbb{S}(t-\mathcal{J})\,\mathbb{I}_A(t-\mathcal{J}) \\
&+\frac{e^{-\epsilon_1 \mathcal{J}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\,\mathbb{S}_0+\varepsilon\, e^{-\epsilon_2 \mathcal{N}}\,\mathbb{V}_0)}\,\varrho_2\,\mathbb{S}\,\mathbb{I}_S-\frac{e^{-\epsilon_1 \mathcal{J}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\,\mathbb{S}_0+\varepsilon\, e^{-\epsilon_2 \mathcal{N}}\,\mathbb{V}_0)}\,\varrho_2\,\mathbb{S}(t-\mathcal{J})\,\mathbb{I}_S(t-\mathcal{J}) \\
&+\frac{e^{-\epsilon_2 \mathcal{N}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\,\mathbb{S}_0+\varepsilon\, e^{-\epsilon_2 \mathcal{N}}\,\mathbb{V}_0)}\,\varepsilon\,\varrho_1\,\mathbb{V}\,\mathbb{I}_A-\frac{e^{-\epsilon_2 \mathcal{N}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\,\mathbb{S}_0+\varepsilon\, e^{-\epsilon_2 \mathcal{N}}\,\mathbb{V}_0)}\,\varepsilon\varrho_1\,\mathbb{V}(t-\mathcal{N})\,\mathbb{I}_A(t-\mathcal{N})
\end{aligned}
$$

$$+\frac{e^{-\epsilon_2 \mathcal{N}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\,\mathbb{S}_0+\varepsilon\,e^{-\epsilon_2 \mathcal{N}}\,\mathbb{V}_0)}\,\varepsilon\,\varrho_2\,\mathbb{V}\,\mathbb{I}_S-\frac{e^{-\epsilon_2 \mathcal{N}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\,\mathbb{S}_0+\varepsilon\,e^{-\epsilon_2 \mathcal{N}}\,\mathbb{V}_0)}\,\varepsilon\varrho_2\,\mathbb{V}(t-\mathcal{N})\,\mathbb{I}_S(t-\mathcal{N}),$$

From the model (2.1), we have

$$\begin{aligned}
\leq\;& \frac{e^{-\epsilon_1 \mathcal{J}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2 \mathcal{N}}\varepsilon\,\mathbb{V}_0)}\left(1-\frac{\mathbb{S}_0}{\mathbb{S}}\right)(\Xi-\varrho_1\mathbb{S}\,\mathbb{I}_A-\varrho_2\mathbb{S}\,\mathbb{I}_S-(\zeta+\iota)\,\mathbb{S})\\
&+\frac{(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\mathbb{S}_0+\varepsilon\,e^{-\epsilon_2 \mathcal{N}}\,\mathbb{V}_0)}\,(\varrho_1\,\mathbb{S}(t-\mathcal{N})\,\mathbb{I}_A(t-\mathcal{J})+\varepsilon\,\varrho_1\mathbb{V}(t-\mathcal{J})\mathbb{I}_A(t-\mathcal{J})-(\sigma+\iota)\mathbb{E}\\
&+\varrho_2\,\mathbb{S}(t-\mathcal{N})\,\mathbb{I}_S(t-\mathcal{J})+\varepsilon\,\varrho_2\mathbb{V}(t-\mathcal{J})\mathbb{I}_2(t-\mathcal{J}))\\
&+e^{-\epsilon_2 \mathcal{N}}\frac{(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2 \mathcal{N}}\varepsilon\,\mathbb{V}_0)}\left(1-\frac{\mathbb{V}_0}{\mathbb{V}}\right)(\zeta\,\mathbb{S}-\varepsilon\,\varrho_1\mathbb{V}\,\mathbb{I}_A-\varepsilon\,\varrho_2\mathbb{V}\,\mathbb{I}_S-(\varrho+\iota)\,\mathbb{V})\\
&+(\varrho_1\,(\iota+\varpi_2))(\alpha\,\gamma\,\mathbb{E}-(\varpi_1+\iota)\,\mathbb{I}_A)+(\varrho_2\,(\iota+\varpi_1))((1-\alpha)\,\gamma\,\mathbb{E}-(\varpi_2+\iota)\mathbb{I}_S)\\
&+\frac{e^{-\epsilon_1 \mathcal{J}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\,\mathbb{S}_0+\varepsilon\,e^{-\epsilon_2 \mathcal{N}}\,\mathbb{V}_0)}\,\varrho_1\,\mathbb{S}\,\mathbb{I}_A-\frac{e^{-\epsilon_1 \mathcal{J}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\,\mathbb{S}_0+\varepsilon\,e^{-\epsilon_2 \mathcal{N}}\,\mathbb{V}_0)}\,\varrho_1\,\mathbb{S}(t-\mathcal{J})\,\mathbb{I}_A(t-\mathcal{J})\\
&+\frac{e^{-\epsilon_1 \mathcal{J}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\,\mathbb{S}_0+\varepsilon\,e^{-\epsilon_2 \mathcal{N}}\,\mathbb{V}_0)}\,\varrho_2\,\mathbb{S}\,\mathbb{I}_S-\frac{e^{-\epsilon_1 \mathcal{J}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\,\mathbb{S}_0+\varepsilon\,e^{-\epsilon_2 \mathcal{N}}\,\mathbb{V}_0)}\,\varrho_2\,\mathbb{S}(t-\mathcal{J})\,\mathbb{I}_S(t-\mathcal{J})\\
&+\frac{e^{-\epsilon_2 \mathcal{N}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\,\mathbb{S}_0+\varepsilon\,e^{-\epsilon_2 \mathcal{N}}\,\mathbb{V}_0)}\,\varepsilon\,\varrho_1\,\mathbb{V}\,\mathbb{I}_A-\frac{e^{-\epsilon_2 \mathcal{N}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\,\mathbb{S}_0+\varepsilon\,e^{-\epsilon_2 \mathcal{N}}\,\mathbb{V}_0)}\,\varepsilon\varrho_1\,\mathbb{V}(t-\mathcal{N})\,\mathbb{I}_A(t-\mathcal{N})\\
&+\frac{e^{-\epsilon_2 \mathcal{N}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\,\mathbb{S}_0+\varepsilon\,e^{-\epsilon_2 \mathcal{N}}\,\mathbb{V}_0)}\,\varepsilon\,\varrho_2\,\mathbb{V}\,\mathbb{I}_S-\frac{e^{-\epsilon_2 \mathcal{N}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\,\mathbb{S}_0+\varepsilon\,e^{-\epsilon_2 \mathcal{N}}\,\mathbb{V}_0)}\,\varepsilon\varrho_2\,\mathbb{V}(t-\mathcal{N})\,\mathbb{I}_S(t-\mathcal{N}),
\end{aligned}$$

By using $\Xi=(\zeta+\iota)\,\mathbb{S}_0$ and $\zeta\,\mathbb{S}_0=(\varrho+\iota)\,\mathbb{V}_0$, we obtain

$$\begin{aligned}
\frac{d}{dt}\bar{\mathcal{Q}}\;\leq\;& -(\zeta+\iota)\frac{e^{-\epsilon_1 \mathcal{J}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{\epsilon_1 \mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2 \mathcal{N}}\varepsilon\,\mathbb{V}_0)}\left(\frac{(\mathbb{S}_0-\mathbb{S})^2}{\mathbb{S}}\right)+\frac{e^{-\epsilon_1 \mathcal{J}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2 \mathcal{N}}\varepsilon\,\mathbb{V}_0)}\,\varrho_1\,\mathbb{S}_0\,\mathbb{I}_A\\
&+\frac{e^{-\epsilon_1 \mathcal{J}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2 \mathcal{N}}\varepsilon\,\mathbb{V}_0)}\,\varrho_2\,\mathbb{S}_0\,\mathbb{I}_S-\frac{(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2 \mathcal{N}}\varepsilon\,\mathbb{V}_0)}\,(\sigma+\iota)\mathbb{E}+\varrho_1\,(\iota+\varpi_2)\,\sigma\,\alpha\,\mathbb{E}\\
&-\varrho_1\,(\iota+\varpi_1)\,(\iota+\varpi_2)\,\mathbb{I}_A+\varrho_2\,(\iota+\varpi_1)\,\sigma\,(1-\alpha)\,\mathbb{E}-\varrho_2\,(\iota+\varpi_1)\,\mathbb{I}_S+\varrho_2\,(\iota+\varpi_1)\,\sigma\,(1-\alpha)\,\mathbb{E}\\
&+\frac{e^{-\epsilon_2 \mathcal{N}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\mathbb{S}_0+e^{\epsilon_2 \mathcal{N}}\varepsilon\,\mathbb{V}_0)}\,\varrho_1\,\varepsilon\,\mathbb{V}_0\,\mathbb{I}_A+\frac{e^{-\epsilon_2 \mathcal{N}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2 \mathcal{N}}\varepsilon\,\mathbb{V}_0)}\,\varrho_2\,\varepsilon\,\mathbb{V}_0\,\mathbb{I}_S\\
&-\frac{e^{-\epsilon_2 \mathcal{N}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2 \mathcal{N}}\varepsilon\,\mathbb{V}_0)}\,(\varrho+\iota)\,\mathbb{V}+\frac{e^{-\epsilon_2 \mathcal{N}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2 \mathcal{N}}\varepsilon\,\mathbb{V}_0)}\,(\varrho+\iota)\,\mathbb{V}_0\\
&+\frac{e^{-\epsilon_2 \mathcal{N}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2 \mathcal{N}}\varepsilon\,\mathbb{V}_0)}\left(1-\frac{\mathbb{V}_0}{\mathbb{V}}\right)\zeta\,\mathbb{S},
\end{aligned}$$

After some calculations, we get

$$\begin{aligned}
\frac{d}{dt}\bar{\mathcal{Q}}\;\leq\;& -(\zeta+\iota)\frac{e^{-\epsilon_1 \mathcal{J}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2 \mathcal{N}}\varepsilon\,\mathbb{V}_0)}\left(\frac{(\mathbb{S}_0-\mathbb{S})^2}{\mathbb{S}}\right)+\left(\frac{e^{-\epsilon_1 \mathcal{J}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2 \mathcal{N}}\varepsilon\,\mathbb{V}_0)}\,\varrho_1\,\mathbb{S}_0\right.\\
&\left.+\;\frac{e^{-\epsilon_2 \mathcal{N}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2 \mathcal{N}}\varepsilon\,\mathbb{V}_0)}\,\varrho_1\,\varepsilon\,\mathbb{V}_0-\varrho_1\,(\iota+\varpi_1)\,(\iota+\varpi_2)\right)\mathbb{I}_A\\
&+\left(\frac{e^{-\epsilon_1 \mathcal{J}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2 \mathcal{N}}\varepsilon\,\mathbb{V}_0)}\,\varrho_2\,\mathbb{S}_0+\frac{e^{-\epsilon_2 \mathcal{N}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2 \mathcal{N}}\varepsilon\,\mathbb{V}_0)}\,\varrho_2\,\varepsilon\,\mathbb{V}_0-\varrho_1\,(\iota+\varpi_1)\,(\iota+\varpi_2)\right)\mathbb{I}_S
\end{aligned}$$

$$\left(-\frac{(\sigma+\iota)(\iota+\varpi_1)(\iota+\varpi_2)}{(e^{-\epsilon_1\mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2\mathcal{N}}\varepsilon\,\mathbb{V}_0)}+\varrho_1(\iota+\varpi_2)\,\alpha\,\sigma+\varrho_2(\iota+\varpi_1)(1-\alpha)\,\sigma\right)\mathbb{E}$$
$$+\frac{e^{-\epsilon_2\mathcal{N}}(\iota+\varpi_1)(\iota+\varpi_2)}{(e^{-\epsilon_1\mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2\mathcal{N}}\varepsilon\,\mathbb{V}_0)}\zeta\,\mathbb{S}_0\left(1-\frac{\mathbb{V}}{\mathbb{V}_0}-\frac{\mathbb{S}\,\mathbb{V}_0}{\mathbb{S}_0\mathbb{V}}+\frac{\mathbb{S}}{\mathbb{S}_0}\right),$$

By collecting the terms, we get

$$\begin{aligned}\frac{d}{dt}\bar{Q} \;\le\;& -(\zeta+\iota)\frac{e^{-\epsilon_1\mathcal{J}}(\iota+\varpi_1)(\iota+\varpi_2)}{(e^{-\epsilon_1\mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2\mathcal{N}}\varepsilon\,\mathbb{V}_0)}\left(\frac{(\mathbb{S}_0-\mathbb{S})^2}{\mathbb{S}}\right)+(\Re_0-1)\,\mathbb{E}\\
&+\frac{e^{-\epsilon_2\mathcal{N}}(\iota+\varpi_1)(\iota+\varpi_2)}{(e^{-\epsilon_1\mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2\mathcal{N}}\varepsilon\,\mathbb{V}_0)}\zeta\,\mathbb{S}_0\left(1-\frac{\mathbb{V}}{\mathbb{V}_0}-\frac{\mathbb{S}\,\mathbb{V}_0}{\mathbb{S}_0\mathbb{V}}+\frac{\mathbb{S}}{\mathbb{S}_0}\right)+\frac{e^{-\epsilon_2\mathcal{N}}(\iota+\varpi_1)(\iota+\varpi_2)}{(e^{-\epsilon_1\mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2\mathcal{N}}\varepsilon\mathbb{V}_0)}\left(1-\frac{\mathbb{S}}{\mathbb{S}_0}\right)\zeta\,\mathbb{S}_0\\
&-\frac{e^{-\epsilon_2\mathcal{N}}(\iota+\varpi_1)(\iota+\varpi_2)}{(e^{-\epsilon_1\mathcal{J}}\mathbb{S}_0+e^{\epsilon_2\mathcal{N}}\varepsilon\mathbb{V}_0)}\left(1-\frac{\mathbb{S}}{\mathbb{S}_0}\right)\zeta\,\mathbb{S}_0+\frac{e^{-\epsilon_2\mathcal{N}}(\iota+\varpi_1)(\iota+\varpi_2)}{(e^{-\epsilon_1\mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2\mathcal{N}}\varepsilon\mathbb{V}_0)}\left(1-\frac{\mathbb{S}_0}{\mathbb{S}}\right)\zeta\,\mathbb{S}_0\\
&-\frac{e^{-\epsilon_2\mathcal{N}}(\iota+\varpi_1)(\iota+\varpi_2)}{(e^{-\epsilon_1\mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2\mathcal{N}}\varepsilon\mathbb{V}_0)}\left(1-\frac{\mathbb{S}_0}{\mathbb{S}}\right)\zeta\,\mathbb{S}_0,\end{aligned}$$

Thus

$$\begin{aligned}\frac{d}{dt}\bar{Q} \;\le\;& -(\zeta+\iota)\frac{e^{-\epsilon_1\mathcal{J}}(\iota+\varpi_1)(\iota+\varpi_2)}{(e^{-\epsilon_1\mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2\mathcal{N}}\varepsilon\,\mathbb{V}_0)}\left(\frac{(\mathbb{S}_0-\mathbb{S})^2}{\mathbb{S}}\right)+(\Re_0-1)\,\mathbb{E}\\
&+\frac{e^{-\epsilon_2\mathcal{N}}(\iota+\varpi_1)(\iota+\varpi_2)}{(e^{-\epsilon_1\mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2\mathcal{N}}\varepsilon\,\mathbb{V}_0)}\zeta\,\mathbb{S}_0\left(3-\frac{\mathbb{V}}{\mathbb{V}_0}-\frac{\mathbb{S}\,\mathbb{V}_0}{\mathbb{S}_0\mathbb{V}}-\frac{\mathbb{S}_0}{\mathbb{S}}\right)\\
&+\frac{e^{-\epsilon_2\mathcal{N}}(\iota+\varpi_1)(\iota+\varpi_2)}{(e^{-\epsilon_1\mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2\mathcal{N}}\varepsilon\,\mathbb{V}_0)}\zeta\,\mathbb{S}_0\left(\frac{\mathbb{S}_0}{\mathbb{S}}+\frac{\mathbb{S}}{\mathbb{S}_0}-2\right)+\frac{e^{-\epsilon_2\mathcal{N}}(\iota+\varpi_1)(\iota+\varpi_2)}{(e^{-\epsilon_1\mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2\mathcal{N}}\varepsilon\,\mathbb{V}_0)}\zeta\,\mathbb{S}_0\,\ln\left(\frac{\mathbb{S}}{S_0}\right)\\
&-\frac{e^{-\epsilon_2\mathcal{N}}(\iota+\varpi_1)(\iota+\varpi_2)}{(e^{-\epsilon_1\mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2\mathcal{N}}\varepsilon\,\mathbb{V}_0)}\zeta\,\mathbb{S}_0\,\ln\left(\frac{\mathbb{S}}{S_0}\right)+\frac{e^{-\epsilon_2\mathcal{N}}(\iota+\varpi_1)(\iota+\varpi_2)}{(e^{-\epsilon_1\mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2\mathcal{N}}\varepsilon\,\mathbb{V}_0)}\zeta\,\mathbb{S}_0\,\ln\left(\frac{\mathbb{S}_0}{S}\right)\\
&-\frac{e^{-\epsilon_2\mathcal{N}}(\iota+\varpi_1)(\iota+\varpi_2)}{(e^{-\epsilon_1\mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2\mathcal{N}}\varepsilon\,\mathbb{V}_0)}\zeta\,\mathbb{S}_0\,\ln\left(\frac{\mathbb{S}_0}{S}\right),\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\bar{Q} \;\le\;& -(\zeta+\iota)\frac{e^{-\epsilon_1\mathcal{J}}(\iota+\varpi_1)(\iota+\varpi_2)}{(e^{-\epsilon_1\mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2\mathcal{N}}\varepsilon\,\mathbb{V}_0)}\left(\frac{(\mathbb{S}_0-\mathbb{S})^2}{\mathbb{S}}\right)+(\Re_0-1)\,\mathbb{E}\\
&+\frac{e^{-\epsilon_2\mathcal{N}}(\iota+\varpi_1)(\iota+\varpi_2)}{(e^{-\epsilon_1\mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2\mathcal{N}}\varepsilon\,\mathbb{V}_0)}\zeta\,\mathbb{S}_0\left(3-\frac{\mathbb{V}}{\mathbb{V}_0}-\frac{\mathbb{S}\,\mathbb{V}_0}{\mathbb{S}_0\mathbb{V}}-\frac{\mathbb{S}_0}{\mathbb{S}}\right)\\
&+\frac{e^{-\epsilon_2\mathcal{N}}(\iota+\varpi_1)(\iota+\varpi_2)}{(e^{-\epsilon_1\mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2\mathcal{N}}\varepsilon\,\mathbb{V}_0)}\zeta\,\mathbb{S}_0\left[\left(-1+\frac{\mathbb{S}}{\mathbb{S}_0}-\ln\left(\frac{\mathbb{S}}{\mathbb{S}_0}\right)\right)\right.\\
&\left.+\left(-1+\frac{\mathbb{S}_0}{\mathbb{S}}-\ln\left(\frac{\mathbb{S}_0}{\mathbb{S}}\right)\right)\right]+\frac{e^{-\epsilon_2\mathcal{N}}(\iota+\varpi_1)(\iota+\varpi_2)}{(e^{-\epsilon_1\mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2\mathcal{N}}\varepsilon\,\mathbb{V}_0)}\zeta\,\mathbb{S}_0\left[\ln\left(\frac{\mathbb{S}_0}{\mathbb{S}}\,\frac{\mathbb{S}}{\mathbb{S}_0}\right)\right].\end{aligned}$$

The derivative of $\bar{Q}$ will then be

$$\frac{d}{dt}\bar{Q} \;\le\; -(\zeta+\iota)\frac{e^{-\epsilon_1\mathcal{J}}(\iota+\varpi_1)(\iota+\varpi_2)}{(e^{-\epsilon_1\mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2\mathcal{N}}\varepsilon\,\mathbb{V}_0)}\left(\frac{(\mathbb{S}_0-\mathbb{S})^2}{\mathbb{S}}\right)+(\Re_0-1)\,\mathbb{E}$$

$$+\frac{e^{-\epsilon_2 \mathcal{N}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2 \mathcal{N}}\varepsilon\,\mathbb{V}_0)}\,\zeta\,\mathbb{S}_0\,\left(3-\frac{\mathbb{V}}{\mathbb{V}_0}-\frac{\mathbb{S}\,\mathbb{V}_0}{\mathbb{S}_0\mathbb{V}}-\frac{\mathbb{S}_0}{\mathbb{S}}\right)$$
$$-\frac{e^{-\epsilon_2 \mathcal{N}}\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{(e^{-\epsilon_1 \mathcal{J}}\mathbb{S}_0+e^{-\epsilon_2 \mathcal{N}}\varepsilon\,\mathbb{V}_0)}\,\zeta\,\mathbb{S}_0\,\left[\mathcal{G}\left(\frac{\mathbb{S}_0}{\mathbb{S}}\right)+\mathcal{G}\left(\frac{\mathbb{S}}{\mathbb{S}_0}\right)\right].$$

Thus, since the geometric mean is less than or equal arithmetical mean, we observe

$$3\ \le\ \frac{\mathbb{S}_0}{\mathbb{S}}+\frac{\mathbb{V}_0\,\mathbb{S}}{\mathbb{V}\,\mathbb{S}_0}+\frac{\mathbb{V}}{\mathbb{V}_0}.$$

Furthermore, when $\Re_0\ \le\ 1$, we have $\frac{d\bar{Q}}{dt}\ \le\ 0$ for all $\mathbb{S},\mathbb{E},\mathbb{I}_A,\mathbb{I}_S,\mathbb{V}\ >\ 0$. Note that $\frac{d\bar{Q}}{dt}\ =\ 0$ if and only if $\mathbb{S}\ =\ \mathbb{S}_0$, $\mathbb{E}\ =\ 0$, and $\mathbb{V}\ =\ \mathbb{V}_0$. Thus, the subset $\mathcal{F}'$ is the largest invariant set of $\mathcal{F}\ =\ \left\{(\mathbb{S},\mathbb{E},\mathbb{I}_A,\mathbb{I}_S,\mathbb{V})\ \in\ \alpha:\frac{d\bar{Q}}{dt}=0\right\}$. In addition, for each elements of $\mathcal{F}'$, we have $\mathbb{E}=0$, then $\frac{d\mathbb{E}}{dt}=0$, and we find

$$0=e^{-\epsilon_1 \mathcal{J}}\mathbb{S}(t-\mathcal{J})\,(\varrho_1\mathbb{I}_A(t-\mathcal{J})+\varrho_2\mathbb{I}_S(t-\mathcal{J}))+e^{-\epsilon_2 \mathcal{N}}\varepsilon\,\mathbb{V}(t-\mathcal{N})(\varrho_1\mathbb{I}_A(t-\mathcal{N})+\varrho_2\mathbb{I}_S(t-\mathcal{N})).\quad (A.1)$$

From (A.1), we conclude that $\mathbb{I}_A\ =0$ and $\mathbb{I}_S\ =0$. By applying La Salle's invariant principle [19], we obtain all solutions approaches to $\mathcal{F}'$. In $\mathcal{F}'$, the elements are equal to $\mathbb{S}\ =\ \mathbb{S}_0$, $\mathbb{V}\ =\ \mathbb{V}_0$, and $\mathbb{E}\ =\ 0$. Thus, $\mathcal{F}'\ =\ \{(\mathbb{S},\mathbb{E},\mathbb{I}_A,\mathbb{I}_S,\mathbb{V})\ \in\ \mathcal{F}:\mathbb{S}=\mathbb{S}_0,\mathbb{V}=\mathbb{V}_0,\mathbb{E}=\mathbb{I}_A=\mathbb{I}_S=0\}\ =\ \{\mathcal{U}_0\}$. Therefore, $\mathcal{U}_0$ is $\mathcal{GAS}$ when $\Re_0\ \le 1$. □

A.2. Proof of Theorem 4

Proof. Suppose

$$\begin{aligned}
Q\ &=\ e^{-\epsilon_1 \mathcal{J}}\,\mathcal{G}\left(\frac{\mathbb{S}}{\mathbb{S}^*}\right)+\mathcal{G}\left(\frac{\mathbb{E}}{\mathbb{E}^*}\right)+e^{-\epsilon_2 \mathcal{N}}\,\mathcal{G}\left(\frac{\mathbb{V}}{\mathbb{V}^*}\right)+\frac{\varrho_1\,\mathbb{I}_A^*(e^{-\epsilon_1 \mathcal{J}}\,\mathbb{S}^*+e^{-\epsilon_2 \mathcal{N}}\,\varepsilon\,\mathbb{V}^*)}{\alpha\,\sigma\,\mathbb{E}^*}\,\mathcal{G}\left(\frac{\mathbb{I}_A}{\mathbb{I}_A^*}\right)\\
&\quad+\frac{\varrho_2\,\mathbb{I}_S^*(e^{-\epsilon_1 \mathcal{J}}\,\mathbb{S}^*+e^{-\epsilon_2 \mathcal{N}}\,\varepsilon\,\mathbb{V}^*)}{\alpha\,(1-\sigma)\,\mathbb{E}^*}\,\mathcal{G}\left(\frac{\mathbb{I}_S}{\mathbb{I}_S^*}\right)+e^{-\epsilon_1 \mathcal{J}}\,\varrho_1\,\mathbb{I}_A^*\,\mathbb{S}^*\int_0^{\mathcal{J}}\mathcal{G}\left(\frac{\mathbb{S}(t-\lambda)\,\mathbb{I}_A(t-\lambda)}{\mathbb{S}^*\,\mathbb{I}_A^*}\right)d\lambda\\
&\quad+e^{-\epsilon_1 \mathcal{J}}\varrho_2\mathbb{I}_S^*\,\mathbb{S}^*\int_0^{\mathcal{J}}\mathcal{G}\left(\frac{\mathbb{S}(t-\lambda)\mathbb{I}_S(t-\lambda)}{\mathbb{S}^*\mathbb{I}_S^*}\right)d\lambda+e^{-\epsilon_2 \mathcal{N}}\varepsilon\varrho_1\mathbb{I}_A^*\mathbb{V}^*\int_0^{\mathcal{N}}\mathcal{G}\left(\frac{\mathbb{V}(t-\lambda)\mathbb{I}_A(t-\lambda)}{\mathbb{V}^*\mathbb{I}_A^*}\right)d\lambda\\
&\quad+e^{-\epsilon_2 \mathcal{N}}\varepsilon\varrho_2\mathbb{I}_S^*\,\mathbb{V}^*\int_0^{\mathcal{N}}\mathcal{G}\left(\frac{\mathbb{V}(t-\lambda)\mathbb{I}_S(t-\lambda)}{\mathbb{V}^*\mathbb{I}_S^*}\right)d\lambda.
\end{aligned}$$

The $\frac{dQ}{dt}$ is expressed as

$$\begin{aligned}
\frac{dQ}{dt}\ &=\ e^{-\epsilon_1 \mathcal{J}}\left(1-\frac{\mathbb{S}^*}{\mathbb{S}}\right)(\Xi-\mathbb{S}(t)\,(\varrho_1\mathbb{I}_A(t)+\varrho_2\mathbb{I}_S(t))-(\zeta+\iota)\mathbb{S}(t))+\left(1-\frac{\mathbb{E}^*}{\mathbb{E}}\right)(-(\sigma+\iota)\mathbb{E}(t)\\
&\quad+e^{-\epsilon_1 \mathcal{J}}\mathbb{S}(t-\mathcal{J})\,(\varrho_1\mathbb{I}_A(t-\mathcal{J})+\varrho_2\mathbb{I}_S(t-\mathcal{J}))+e^{-\epsilon_2 \mathcal{N}}\varepsilon\,\mathbb{V}(t-\mathcal{N})(\varrho_1\mathbb{I}_A(t-\mathcal{N})+\varrho_2\mathbb{I}_S(t-\mathcal{N})))\\
&\quad+e^{-\epsilon_2 \mathcal{N}}\left(1-\frac{\mathbb{V}^*}{\mathbb{V}}\right)(\zeta\,\mathbb{S}(t)-\varepsilon\,\varrho_1\mathbb{V}(t)\,\mathbb{I}_A(t)-\varepsilon\,\varrho_2\mathbb{V}(t)\,\mathbb{I}_S(t)-(\varrho+\iota)\,\mathbb{V}(t))\\
&\quad+\frac{\varrho_1\,\mathbb{I}_A^*(e^{-\epsilon_1 \mathcal{J}}\,\mathbb{S}^*+e^{-\epsilon_2 \mathcal{N}}\,\varepsilon\,\mathbb{V}^*)}{\alpha\,\sigma\,\mathbb{E}^*}\left(1-\frac{\mathbb{I}_A^*}{\mathbb{I}_A}\right)(\alpha\,\sigma\,\mathbb{E}(t)-(\varpi_1+\iota)\mathbb{I}_A(t))
\end{aligned}$$

$$+\frac{\varrho_2\,\mathbb{I}_S^*(e^{-\epsilon_1\mathcal{J}}\,\mathbb{S}^*+e^{-\epsilon_2\mathcal{N}}\,\varepsilon\,\mathbb{V}^*)}{\alpha\,(1-\sigma)\,\mathbb{E}^*}\left(1-\frac{\mathbb{I}_S^*}{\mathbb{I}_S}\right)((1-\alpha)\,\sigma\,\mathbb{E}(t)-(\varpi_2+\iota)\mathbb{I}_S(t))$$
$$+e^{-\epsilon_1\mathcal{J}}\,\varrho_1\,\mathbb{I}_A^*\,\mathbb{S}^*\left(\frac{\mathbb{S}\,\mathbb{I}_A}{\mathbb{S}^*\,\mathbb{I}_A^*}-\frac{\mathbb{S}(t-\mathcal{J})\,\mathbb{I}_A(t-\mathcal{J})}{\mathbb{S}^*\mathbb{I}_A^*}-\ln\left(\frac{\mathbb{S}(t-\mathcal{J})\,\mathbb{I}_A(t-\mathcal{J})}{\mathbb{S}\,\mathbb{I}_A}\right)\right)$$
$$+e^{-\epsilon_1\mathcal{J}}\,\varrho_2\,\mathbb{I}_S^*\,\mathbb{S}^*\left(\frac{\mathbb{S}\,\mathbb{I}_S}{\mathbb{S}^*\,\mathbb{I}_A^*}-\frac{\mathbb{S}(t-\mathcal{J})\,\mathbb{I}_S(t-\mathcal{J})}{\mathbb{S}^*\mathbb{I}_S^*}-\ln\left(\frac{\mathbb{S}(t-\mathcal{J})\,\mathbb{I}_S(t-\mathcal{J})}{\mathbb{S}\,\mathbb{I}_S}\right)\right)$$
$$+\varepsilon\,e^{-\epsilon_2\mathcal{N}}\,\varrho_1\,\mathbb{I}_A^*\,\mathbb{V}^*\left(\frac{\mathbb{V}\,\mathbb{I}_A}{\mathbb{V}^*\,\mathbb{I}_A^*}-\frac{\mathbb{V}(t-\mathcal{N})\,\mathbb{I}_A(t-\mathcal{N})}{\mathbb{V}^*\mathbb{I}_A^*}-\ln\left(\frac{\mathbb{V}(t-\mathcal{N})\,\mathbb{I}_A(t-\mathcal{N})}{\mathbb{V}\,\mathbb{I}_A}\right)\right)$$
$$+\varepsilon\,e^{-\epsilon_2\mathcal{N}}\,\varrho_2\,\mathbb{I}_S^*\,\mathbb{V}^*\left(\frac{\mathbb{V}\,\mathbb{I}_S}{\mathbb{V}^*\,\mathbb{I}_S^*}-\frac{\mathbb{V}(t-\mathcal{N})\,\mathbb{I}_S(t-\mathcal{N})}{\mathbb{V}^*\mathbb{I}_S^*}-\ln\left(\frac{\mathbb{V}(t-\mathcal{N})\,\mathbb{I}_S(t-\mathcal{N})}{\mathbb{V}\,\mathbb{I}_S}\right)\right),$$

$$\begin{aligned}
\frac{dQ}{dt} \;=\;& e^{-\epsilon_1\mathcal{J}}\left(1-\frac{\mathbb{S}^*}{\mathbb{S}}\right)(\Xi-(\zeta+\iota)\,\mathbb{S}(t))-e^{-\epsilon_1\mathcal{J}}\,\varrho_1\,\mathbb{S}\,\mathbb{I}_A-e^{-\epsilon_1\mathcal{J}}\,\varrho_2\,\mathbb{S}\,\mathbb{I}_S+e^{-\epsilon_1\mathcal{J}}\,\varrho_1\,\mathbb{S}^*\,\mathbb{I}_A\\
&+e^{-\epsilon_1\mathcal{J}}\,\varrho_2\,\mathbb{S}^*\,\mathbb{I}_S+e^{-\epsilon_1\mathcal{J}}\,\varrho_1\mathbb{S}(t-\mathcal{J})\mathbb{I}_A(t-\mathcal{J})+e^{-\epsilon_1\mathcal{J}}\,\varrho_2\mathbb{S}(t-\mathcal{J})\mathbb{I}_S(t-\mathcal{J})\\
&+e^{-\epsilon_2\mathcal{N}}\varepsilon\varrho_1\mathbb{V}(t-\mathcal{N})\mathbb{I}_A(t-\mathcal{N})+e^{-\epsilon_2\mathcal{N}}\varepsilon\varrho_2\mathbb{V}(t-\mathcal{N})\mathbb{I}_S(t-\mathcal{N})-\left(\frac{\mathbb{E}^*}{\mathbb{E}}\right)e^{-\epsilon_1\mathcal{J}}\varrho_1\mathbb{S}(t-\mathcal{J})\mathbb{I}_A(t-\mathcal{J})\\
&-\left(\frac{\mathbb{E}^*}{\mathbb{E}}\right)e^{-\epsilon_1\mathcal{J}}\,\varrho_2\mathbb{S}(t-\mathcal{J})\mathbb{I}_S(t-\mathcal{J})-\left(\frac{\mathbb{E}^*}{\mathbb{E}}\right)e^{-\epsilon_2\mathcal{N}}\,\varrho_1\varepsilon\mathbb{V}(t-\mathcal{N})\mathbb{I}_A(t-\mathcal{N})-(\sigma+\iota)\mathbb{E}\\
&-\left(\frac{\mathbb{E}^*}{\mathbb{E}}\right)e^{-\epsilon_2\mathcal{N}}\,\varrho_2\varepsilon\mathbb{V}(t-\mathcal{N})\mathbb{I}_S(t-\mathcal{N})+(\sigma+\iota)\,\mathbb{E}^*+\frac{e^{-\epsilon_1\mathcal{J}}\,\varrho_1\,\mathbb{S}^*\mathbb{I}_A^*\,\mathbb{I}_A\,\mathbb{E}}{\mathbb{E}^*}+\frac{e^{-\epsilon_2\mathcal{N}}\,\varepsilon\varrho_1\,\mathbb{V}^*\mathbb{I}_A^*\,\mathbb{I}_A\,\mathbb{E}}{\mathbb{E}^*}\\
&-e^{-\epsilon_1\mathcal{J}}\,\varrho_1\,\mathbb{S}^*\,\mathbb{I}_A-e^{-\epsilon_2\mathcal{N}}\,\varepsilon\,\varrho_1\,\mathbb{V}^*\,\mathbb{I}_A-\frac{e^{-\epsilon_1\mathcal{J}}\,\varrho_1\mathbb{S}^*\,\mathbb{I}_A^*\,\mathbb{I}_A^*\mathbb{E}}{\mathbb{E}^*}-\frac{e^{-\epsilon_2\mathcal{N}}\,\varrho_1\varepsilon\mathbb{V}^*\,\mathbb{I}_A^*\,\mathbb{I}_A^*\mathbb{E}}{\mathbb{E}^*}+e^{-\epsilon_1\mathcal{J}}\,\varrho_1\mathbb{S}^*\,\mathbb{I}_A^*\\
&+e^{-\epsilon_2\mathcal{N}}\,\varepsilon\,\varrho_1\mathbb{V}^*\,\mathbb{I}_A^*+\frac{e^{-\epsilon_1\mathcal{J}}\,\varrho_2\,\mathbb{S}^*\mathbb{I}_S^*\,\mathbb{I}_S\,\mathbb{E}}{\mathbb{E}^*}+\frac{e^{-\epsilon_2\mathcal{N}}\,\varepsilon\varrho_2\,\mathbb{V}^*\mathbb{I}_S^*\,\mathbb{I}_S\,\mathbb{E}}{\mathbb{E}^*}-e^{-\epsilon_1\mathcal{J}}\,\varrho_2\,\mathbb{S}^*\,\mathbb{I}_S-e^{-\epsilon_2\mathcal{N}}\,\varepsilon\,\varrho_2\,\mathbb{V}^*\,\mathbb{I}_S\\
&-\frac{e^{-\epsilon_1\mathcal{J}}\,\varrho_2\mathbb{S}^*\,\mathbb{I}_S^*\,\mathbb{I}_S^*\mathbb{E}}{\mathbb{E}^*}-\frac{e^{-\epsilon_2\mathcal{N}}\,\varrho_2\varepsilon\mathbb{V}^*\,\mathbb{I}_S^*\,\mathbb{I}_S^*\mathbb{E}}{\mathbb{E}^*}+e^{-\epsilon_1\mathcal{J}}\,\varrho_2\mathbb{S}^*\,\mathbb{I}_S^*+e^{-\epsilon_2\mathcal{N}}\,\varepsilon\,\varrho_2\mathbb{V}^*\,\mathbb{I}_S^*+e^{-\epsilon_2\mathcal{N}}\left(1-\frac{\mathbb{V}^*}{\mathbb{V}}\right)\zeta\mathbb{S}\\
&-e^{-\epsilon_2\mathcal{N}}\varrho_1\varepsilon\mathbb{V}\mathbb{I}_A-e^{-\epsilon_2\mathcal{N}}\,\varrho_2\varepsilon\mathbb{V}\,\mathbb{I}_S+e^{-\epsilon_2\mathcal{N}}\varrho_1\varepsilon\mathbb{V}^*\,\mathbb{I}_A+e^{-\epsilon_2\mathcal{N}}\,\varrho_2\varepsilon\mathbb{V}^*\mathbb{I}_S-e^{-\epsilon_2\mathcal{N}}\left(1-\frac{\mathbb{V}^*}{\mathbb{V}}\right)(\varrho+\iota)\mathbb{V}
\end{aligned}$$

$$+e^{-\epsilon_1\mathcal{J}}\,\varrho_1\,\mathbb{I}_A^*\,\mathbb{S}^*\left(\frac{\mathbb{S}\,\mathbb{I}_A}{\mathbb{S}^*\,\mathbb{I}_A^*}-\frac{\mathbb{S}(t-\mathcal{J})\,\mathbb{I}_A(t-\mathcal{J})}{\mathbb{S}^*\mathbb{I}_A^*}-\ln\left(\frac{\mathbb{S}(t-\mathcal{J})\,\mathbb{I}_A(t-\mathcal{J})}{\mathbb{S}\,\mathbb{I}_A}\right)\right)$$
$$+e^{-\epsilon_1\mathcal{J}}\,\varrho_2\,\mathbb{I}_S^*\,\mathbb{S}^*\left(\frac{\mathbb{S}\,\mathbb{I}_S}{\mathbb{S}^*\,\mathbb{I}_A^*}-\frac{\mathbb{S}(t-\mathcal{J})\,\mathbb{I}_S(t-\mathcal{J})}{\mathbb{S}^*\mathbb{I}_S^*}-\ln\left(\frac{\mathbb{S}(t-\mathcal{J})\,\mathbb{I}_S(t-\mathcal{J})}{\mathbb{S}\,\mathbb{I}_S}\right)\right)$$
$$+\varepsilon\,e^{-\epsilon_2\mathcal{N}}\,\varrho_1\,\mathbb{I}_A^*\,\mathbb{V}^*\left(\frac{\mathbb{V}\,\mathbb{I}_A}{\mathbb{V}^*\,\mathbb{I}_A^*}-\frac{\mathbb{V}(t-\mathcal{N})\,\mathbb{I}_A(t-\mathcal{N})}{\mathbb{V}^*\mathbb{I}_A^*}-\ln\left(\frac{\mathbb{V}(t-\mathcal{N})\,\mathbb{I}_A(t-\mathcal{N})}{\mathbb{V}\,\mathbb{I}_A}\right)\right)$$
$$+\varepsilon\,e^{-\epsilon_2\mathcal{N}}\,\varrho_2\,\mathbb{I}_S^*\,\mathbb{V}^*\left(\frac{\mathbb{V}\,\mathbb{I}_S}{\mathbb{V}^*\,\mathbb{I}_S^*}-\frac{\mathbb{V}(t-\mathcal{N})\,\mathbb{I}_S(t-\mathcal{N})}{\mathbb{V}^*\mathbb{I}_S^*}-\ln\left(\frac{\mathbb{V}(t-\mathcal{N})\,\mathbb{I}_S(t-\mathcal{N})}{\mathbb{V}\,\mathbb{I}_S}\right)\right).$$

From the equilibria, we set $\Xi = \varrho_1\,\mathbb{S}^*\,\mathbb{I}_A^* + \varrho_2\,\mathbb{S}^*\,\mathbb{I}_S^* + (\zeta+\iota)\mathbb{S}^*$. After deleting some terms, we get

$$\begin{aligned}
\frac{dQ}{dt} &\le -e^{-\epsilon_1\mathcal{J}}\frac{(\zeta+\iota)(\mathbb{S}^*-\mathbb{S})^2}{\mathbb{S}} + \left(2-\frac{\mathbb{S}^*}{\mathbb{S}}-\frac{\mathbb{I}_A^*\,\mathbb{E}}{\mathbb{I}_A\,\mathbb{E}^*}\right)e^{-\epsilon_1\mathcal{J}}\varrho_2\mathbb{S}^*\mathbb{I}_A^* + \left(2-\frac{\mathbb{S}^*}{\mathbb{S}}-\frac{\mathbb{I}_S^*\,\mathbb{E}}{\mathbb{I}_S\,\mathbb{E}^*}\right)e^{-\epsilon_1\mathcal{J}}\varrho_1\mathbb{S}^*\mathbb{I}_S^* \\
&+ \left(e^{-\epsilon_1\mathcal{J}}\varrho_1\mathbb{S}^*\,\mathbb{I}_A^* + e^{-\epsilon_1\mathcal{J}}\varrho_2\mathbb{S}^*\,\mathbb{I}_S^* + e^{-\epsilon_2\mathcal{N}}\varrho_1\varepsilon\mathbb{V}^*\,\mathbb{I}_A^* + e^{-\epsilon_2\mathcal{N}}\varrho_2\varepsilon\mathbb{V}^*\,\mathbb{I}_A^*\right)\left(\frac{\mathbb{E}}{\mathbb{E}^*}\right) - (\varrho+\iota)\mathbb{E} \\
&+ (\varrho+\iota)\mathbb{E}^* + e^{-\epsilon_2\mathcal{N}}\,\varepsilon\,\varrho_1\mathbb{V}^*\mathbb{I}_A^* - \frac{e^{-\epsilon_2\mathcal{N}}\,\varepsilon\,\varrho_1\mathbb{V}^*\mathbb{I}_A^*\,\mathbb{I}_A^*\mathbb{E}}{\mathbb{I}_A\,\mathbb{E}^*} + e^{-\epsilon_2\mathcal{N}}\,\varepsilon\,\varrho_2\mathbb{V}^*\mathbb{I}_S^* - \frac{e^{-\epsilon_2\mathcal{N}}\,\varepsilon\,\varrho_2\mathbb{V}^*\mathbb{I}_S^*\,\mathbb{I}_S^*\mathbb{E}}{\mathbb{I}_S\,\mathbb{E}^*} \\
&+ e^{-\epsilon_2\mathcal{N}}\left(1-\frac{\mathbb{V}^*}{\mathbb{V}}\right)\zeta\,\mathbb{S} - e^{-\epsilon_2\mathcal{N}}\left(1-\frac{\mathbb{V}^*}{\mathbb{V}}\right)(\varrho+\iota)\,\mathbb{V} - \left(\frac{\mathbb{E}^*}{\mathbb{E}}\right)e^{-\epsilon_1\mathcal{J}}\varrho_1\mathbb{S}(t-\mathcal{J})\mathbb{I}_A(t-\mathcal{J}) \\
&- \left(\frac{\mathbb{E}^*}{\mathbb{E}}\right)e^{-\epsilon_1\mathcal{J}}\,\varrho_2\mathbb{S}(t-\mathcal{J})\mathbb{I}_S(t-\mathcal{J}) - \left(\frac{\mathbb{E}^*}{\mathbb{E}}\right)e^{-\epsilon_2\mathcal{N}}\,\varrho_1\varepsilon\mathbb{V}(t-\mathcal{N})\mathbb{I}_A(t-\mathcal{N}) - (\sigma+\iota)\mathbb{E} \\
&- \left(\frac{\mathbb{E}^*}{\mathbb{E}}\right)e^{-\epsilon_2\mathcal{N}}\,\varrho_2\varepsilon\mathbb{V}(t-\mathcal{N})\mathbb{I}_S(t-\mathcal{N}) + e^{-\epsilon_1\mathcal{J}}\varrho_1\,\mathbb{S}^*\mathbb{I}_A^*\,\ln\left(\frac{\mathbb{S}(t-\mathcal{J})\mathbb{I}_A(t-\mathcal{J})}{\mathbb{S}\,\mathbb{I}_A}\right) \\
&+ e^{-\epsilon_1\mathcal{J}}\varrho_2\,\mathbb{S}^*\mathbb{I}_S^*\,\ln\left(\frac{\mathbb{S}(t-\mathcal{J})\mathbb{I}_S(t-\mathcal{J})}{\mathbb{S}\,\mathbb{I}_S}\right) + e^{-\epsilon_2\mathcal{N}}\varrho_1\,\varepsilon\mathbb{V}^*\mathbb{I}_A^*\,\ln\left(\frac{\mathbb{V}(t-\mathcal{N})\mathbb{I}_A(t-\mathcal{N})}{\mathbb{V}\,\mathbb{I}_A}\right) \\
&+ e^{-\epsilon_2\mathcal{N}}\varrho_2\,\varepsilon\mathbb{V}^*\mathbb{I}_S^*\,\ln\left(\frac{\mathbb{V}(t-\mathcal{N})\mathbb{I}_S(t-\mathcal{N})}{\mathbb{V}\,\mathbb{I}_S}\right).
\end{aligned} \tag{A.2}$$

We then have the following equalities:

$$\begin{aligned}
\ln\left(\frac{\mathbb{S}(t-\mathcal{J})\,\mathbb{I}_A(t-\mathcal{J})}{\mathbb{S}\,\mathbb{I}_A}\right) &= \ln\left(\frac{\mathbb{S}(t-\mathcal{J})\mathbb{I}_A(t-\mathcal{J})\,\mathbb{E}^*}{\mathbb{S}^*\,\mathbb{I}_A^*\,\mathbb{E}}\right) + \ln\left(\frac{\mathbb{I}_A^*\,\mathbb{E}}{\mathbb{I}_A\,\mathbb{E}^*}\right) + \ln\left(\frac{\mathbb{S}^*}{\mathbb{S}}\right), \\
\ln\left(\frac{\mathbb{S}(t-\mathcal{J})\,\mathbb{I}_S(t-\mathcal{J})}{\mathbb{S}\,\mathbb{I}_S}\right) &= \ln\left(\frac{\mathbb{S}(t-\mathcal{J})\mathbb{I}_S(t-\mathcal{J})\,\mathbb{E}^*}{\mathbb{S}^*\,\mathbb{I}_S^*\,\mathbb{E}}\right) + \ln\left(\frac{\mathbb{I}_S^*\,\mathbb{E}}{\mathbb{I}_S\,\mathbb{E}^*}\right) + \ln\left(\frac{\mathbb{S}^*}{\mathbb{S}}\right), \\
\ln\left(\frac{\mathbb{V}(t-\mathcal{N})\,\mathbb{I}_A(t-\mathcal{N})}{\mathbb{V}\,\mathbb{I}_A}\right) &= \ln\left(\frac{\mathbb{V}(t-\mathcal{N})\mathbb{I}_A(t-\mathcal{N})\,\mathbb{E}^*}{\mathbb{V}^*\,\mathbb{I}_A^*\,\mathbb{E}}\right) + \ln\left(\frac{\mathbb{I}_A^*\,\mathbb{E}}{\mathbb{I}_A\,\mathbb{E}^*}\right) + \ln\left(\frac{\mathbb{V}^*}{\mathbb{V}}\right), \\
\ln\left(\frac{\mathbb{V}(t-\mathcal{N})\,\mathbb{I}_S(t-\mathcal{N})}{\mathbb{V}\,\mathbb{I}_S}\right) &= \ln\left(\frac{\mathbb{V}(t-\mathcal{N})\mathbb{I}_S(t-\mathcal{N})\,\mathbb{E}^*}{\mathbb{V}^*\,\mathbb{I}_S^*\,\mathbb{E}}\right) + \ln\left(\frac{\mathbb{I}_S^*\,\mathbb{E}}{\mathbb{I}_S\,\mathbb{E}^*}\right) + \ln\left(\frac{\mathbb{V}^*}{\mathbb{V}}\right),
\end{aligned} \tag{A.3}$$

By using $e^{-\epsilon_1\mathcal{J}}\varrho_1\mathbb{S}^*\,\mathbb{I}_A^* + e^{-\epsilon_1\mathcal{J}}\varrho_2\mathbb{S}^*\,\mathbb{I}_S^* + e^{-\epsilon_2\mathcal{N}}\varrho_1\varepsilon\mathbb{V}^*\,\mathbb{I}_A^* + e^{-\epsilon_2\mathcal{N}}\varrho_2\varepsilon\mathbb{V}^*\,\mathbb{I}_S^* = (\varrho+\iota)\,\mathbb{E}^*$ and (A.3) in (A.2), we get

$$\begin{aligned}
\frac{dQ}{dt} \;\le\; & -e^{-\epsilon_1\mathcal{J}}\frac{(\zeta+\iota)(\mathbb{S}^*-\mathbb{S})^2}{\mathbb{S}} + e^{-\epsilon_2\mathcal{N}}\left(1-\frac{\mathbb{V}^*}{\mathbb{V}}\right)\zeta\,\mathbb{S} - e^{-\epsilon_2\mathcal{N}}\left(1-\frac{\mathbb{V}^*}{\mathbb{V}}\right)(\varrho+\iota)\,\mathbb{V} \\
& + e^{-\epsilon_1\mathcal{J}}\,\varrho_1\,\mathbb{S}^*\mathbb{I}_A\left(1-\frac{\mathbb{E}^*\,\mathbb{S}(t-\mathcal{J})\,\mathbb{I}_A(t-\mathcal{J})}{\mathbb{E}\,\mathbb{S}^*\mathbb{I}_A^*} + \ln\left(\frac{\mathbb{E}^*\,\mathbb{S}(t-\mathcal{J})\,\mathbb{I}_A(t-\mathcal{J})}{\mathbb{E}\,\mathbb{S}^*\mathbb{I}_A^*}\right)\right) \\
& + e^{-\epsilon_1\mathcal{J}}\,\varrho_1\,\mathbb{S}^*\mathbb{I}_A\left(1-\frac{\mathbb{I}_A^*\,\mathbb{E}}{\mathbb{I}_A\,\mathbb{E}^*} + \ln\left(\frac{\mathbb{I}_A^*\,\mathbb{E}}{\mathbb{I}_A\,\mathbb{E}^*}\right)\right) + e^{-\epsilon_1\mathcal{J}}\,\varrho_1\,\mathbb{S}^*\mathbb{I}_A\left(1-\frac{\mathbb{S}^*}{\mathbb{S}} + \ln\left(\frac{\mathbb{S}^*}{\mathbb{S}}\right)\right) \\
& + e^{-\epsilon_1\mathcal{J}}\,\varrho_2\,\mathbb{S}^*\mathbb{I}_S\left(1-\frac{\mathbb{E}^*\,\mathbb{S}(t-\mathcal{J})\,\mathbb{I}_S(t-\mathcal{J})}{\mathbb{E}\,\mathbb{S}^*\mathbb{I}_S^*} + \ln\left(\frac{\mathbb{E}^*\,\mathbb{S}(t-\mathcal{J})\,\mathbb{I}_S(t-\mathcal{J})}{\mathbb{E}\,\mathbb{S}^*\mathbb{I}_S^*}\right)\right) \\
& + e^{-\epsilon_1\mathcal{J}}\,\varrho_2\,\mathbb{S}^*\mathbb{I}_S\left(1-\frac{\mathbb{I}_A^*\,\mathbb{E}}{\mathbb{I}_S\,\mathbb{E}^*} + \ln\left(\frac{\mathbb{I}_S^*\,\mathbb{E}}{\mathbb{I}_S\,\mathbb{E}^*}\right)\right) + e^{-\epsilon_1\mathcal{J}}\,\varrho_2\,\mathbb{S}^*\mathbb{I}_S\left(1-\frac{\mathbb{S}^*}{\mathbb{S}} + \ln\left(\frac{\mathbb{S}^*}{\mathbb{S}}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&+e^{-\epsilon_2\mathcal{N}}\varrho_1\,\varepsilon\mathbb{V}^*\mathbb{I}_A\left(1-\frac{\mathbb{E}^*\,\mathbb{V}(t-\mathcal{N})\,\mathbb{I}_A(t-\mathcal{N})}{\mathbb{E}\,\mathbb{V}^*\mathbb{I}_A^*}+\ln\left(\frac{\mathbb{E}^*\,\mathbb{V}(t-\mathcal{N})\,\mathbb{I}_A(t-\mathcal{N})}{\mathbb{E}\,\mathbb{V}^*\mathbb{I}_A^*}\right)\right)\\
&+e^{-\epsilon_2\mathcal{N}}\varepsilon\varrho_1\,\mathbb{V}^*\mathbb{I}_A\left(1-\frac{\mathbb{I}_A^*\,\mathbb{E}}{\mathbb{I}_A\,\mathbb{E}^*}+\ln\left(\frac{\mathbb{I}_A^*\,\mathbb{E}}{\mathbb{I}_A\,\mathbb{E}^*}\right)\right)+e^{-\epsilon_2\mathcal{N}}\varepsilon\varrho_2\,\mathbb{V}^*\mathbb{I}_S\left(1-\frac{\mathbb{I}_S^*\,\mathbb{E}}{\mathbb{I}_S\,\mathbb{E}^*}+\ln\left(\frac{\mathbb{I}_S^*\,\mathbb{E}}{\mathbb{I}_S\,\mathbb{E}^*}\right)\right)\\
&+e^{-\epsilon_2\mathcal{N}}\varrho_2\,\varepsilon\mathbb{V}^*\mathbb{I}_S\left(1-\frac{\mathbb{E}^*\,\mathbb{V}(t-\mathcal{N})\,\mathbb{I}_S(t-\mathcal{N})}{\mathbb{E}\,\mathbb{V}^*\mathbb{I}_S^*}+\ln\left(\frac{\mathbb{E}^*\,\mathbb{V}(t-\mathcal{N})\,\mathbb{I}_S(t-\mathcal{N})}{\mathbb{E}\,\mathbb{V}^*\mathbb{I}_S^*}\right)\right)\\
&+e^{-\epsilon_2\mathcal{N}}\varrho_1\,\varepsilon\mathbb{V}^*\mathbb{I}_A\ln\left(\frac{\mathbb{V}^*}{\mathbb{V}}\right)+e^{-\epsilon_2\mathcal{N}}\varrho_2\,\varepsilon\mathbb{V}^*\mathbb{I}_S\ln\left(\frac{\mathbb{V}^*}{\mathbb{V}}\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\frac{dQ}{dt}\ \leq\ & -e^{-\epsilon_1\mathcal{J}}\frac{(\zeta+\iota)(\mathbb{S}^*-\mathbb{S})^2}{\mathbb{S}}+e^{-\epsilon_2\mathcal{N}}\left(1-\frac{\mathbb{V}^*}{\mathbb{V}}\right)\zeta\,\mathbb{S}-e^{-\epsilon_2\mathcal{N}}\left(1-\frac{\mathbb{V}^*}{\mathbb{V}}\right)(\varrho+\iota)\,\mathbb{V}\\
&-e^{\epsilon_1\mathcal{J}}\varrho_1\mathbb{S}^*\mathbb{I}_A^*\left(\mathcal{G}\left(\frac{\mathbb{E}^*\mathbb{S}(t-\mathcal{J})\mathbb{I}_A(t-\mathcal{J})}{\mathbb{E}\,\mathbb{S}^*\mathbb{I}_A^*}\right)+\mathcal{G}\left(\frac{\mathbb{E}\,\mathbb{I}_A^*}{\mathbb{E}^*\mathbb{I}_A}\right)+\mathcal{G}\left(\frac{\mathbb{S}^*}{\mathbb{S}}\right)\right)\\
&-e^{\epsilon_1\mathcal{J}}\varrho_2\mathbb{S}^*\mathbb{I}_S^*\left(\mathcal{G}\left(\frac{\mathbb{E}^*\mathbb{S}(t-\mathcal{J})\mathbb{I}_S(t-\mathcal{J})}{\mathbb{E}\,\mathbb{S}^*\mathbb{I}_S^*}\right)+\mathcal{G}\left(\frac{\mathbb{E}\,\mathbb{I}_S^*}{\mathbb{E}^*\mathbb{I}_S}\right)+\mathcal{G}\left(\frac{\mathbb{S}^*}{\mathbb{S}}\right)\right)\\
&-e^{\epsilon_2\mathcal{N}}\varrho_1\varepsilon\,\mathbb{V}^*\mathbb{I}_A^*\left(\mathcal{G}\left(\frac{\mathbb{E}^*\mathbb{V}(t-\mathcal{N})\mathbb{I}_A(t-\mathcal{N})}{\mathbb{E}\,\mathbb{V}^*\mathbb{I}_A^*}\right)+\mathcal{G}\left(\frac{\mathbb{E}\,\mathbb{I}_A^*}{\mathbb{E}^*\mathbb{I}_A}\right)\right)+e^{-\epsilon_2\mathcal{N}}\varrho_1\,\varepsilon\mathbb{V}^*\mathbb{I}_A\ln\left(\frac{\mathbb{V}^*}{\mathbb{V}}\right)\\
&-e^{\epsilon_2\mathcal{N}}\varrho_2\varepsilon\,\mathbb{V}^*\mathbb{I}_S^*\left(\mathcal{G}\left(\frac{\mathbb{E}^*\mathbb{V}(t-\mathcal{N})\mathbb{I}_S(t-\mathcal{N})}{\mathbb{E}\,\mathbb{V}^*\mathbb{I}_S^*}\right)+\mathcal{G}\left(\frac{\mathbb{E}\,\mathbb{I}_S^*}{\mathbb{E}^*\mathbb{I}_S}\right)\right)+e^{-\epsilon_2\mathcal{N}}\varrho_2\,\varepsilon\mathbb{V}^*\mathbb{I}_S\ln\left(\frac{\mathbb{V}^*}{\mathbb{V}}\right)\\
&+e^{-\epsilon_2\mathcal{N}}\left(1-\frac{\mathbb{V}^*}{\mathbb{V}}\right)(\varepsilon\varrho_1\,\mathbb{V}^*\,\mathbb{I}_A^*+\varepsilon\varrho_2\,\mathbb{V}^*\,\mathbb{I}_S^*)-e^{-\epsilon_2\mathcal{N}}\left(1-\frac{\mathbb{V}^*}{\mathbb{V}}\right)(\varepsilon\varrho_1\,\mathbb{V}^*\,\mathbb{I}_A^*+\varepsilon\varrho_2\,\mathbb{V}^*\,\mathbb{I}_S^*).
\end{aligned}
$$

From the equilibria, $\varepsilon\varrho_1\,\mathbb{V}^*\,\mathbb{I}_A^*+\varepsilon\varrho_2\,\mathbb{V}^*\,\mathbb{I}_S^*=\zeta\,\mathbb{S}^*-(\varrho+\iota)\mathbb{V}^*$

$$
\begin{aligned}
\frac{dQ}{dt}\ \leq\ & -e^{-\epsilon_1\mathcal{J}}\frac{(\zeta+\iota)(\mathbb{S}^*-\mathbb{S})^2}{\mathbb{S}}+e^{-\epsilon_2\mathcal{N}}\left(1-\frac{\mathbb{V}^*}{\mathbb{V}}\right)(\zeta\,\mathbb{S}-\zeta\,\mathbb{S}^*)-e^{-\epsilon_2\mathcal{N}}(\varrho+\iota)\left(\frac{(\mathbb{V}^*-\mathbb{V})^2}{\mathbb{V}}\right)\\
&-e^{\epsilon_1\mathcal{J}}\varrho_1\mathbb{S}^*\mathbb{I}_A^*\left(\mathcal{G}\left(\frac{\mathbb{E}^*\mathbb{S}(t-\mathcal{J})\mathbb{I}_A(t-\mathcal{J})}{\mathbb{E}\,\mathbb{S}^*\mathbb{I}_A^*}\right)+\mathcal{G}\left(\frac{\mathbb{E}\,\mathbb{I}_A^*}{\mathbb{E}^*\mathbb{I}_A}\right)+\mathcal{G}\left(\frac{\mathbb{S}^*}{\mathbb{S}}\right)\right)\\
&-e^{\epsilon_1\mathcal{J}}\varrho_2\mathbb{S}^*\mathbb{I}_S^*\left(\mathcal{G}\left(\frac{\mathbb{E}^*\mathbb{S}(t-\mathcal{J})\mathbb{I}_S(t-\mathcal{J})}{\mathbb{E}\,\mathbb{S}^*\mathbb{I}_S^*}\right)+\mathcal{G}\left(\frac{\mathbb{E}\,\mathbb{I}_S^*}{\mathbb{E}^*\mathbb{I}_S}\right)+\mathcal{G}\left(\frac{\mathbb{S}^*}{\mathbb{S}}\right)\right)\\
&-e^{\epsilon_2\mathcal{N}}\varrho_1\varepsilon\,\mathbb{V}^*\mathbb{I}_A^*\left(\mathcal{G}\left(\frac{\mathbb{E}^*\mathbb{V}(t-\mathcal{N})\mathbb{I}_A(t-\mathcal{N})}{\mathbb{E}\,\mathbb{V}^*\mathbb{I}_A^*}\right)+\mathcal{G}\left(\frac{\mathbb{E}\,\mathbb{I}_A^*}{\mathbb{E}^*\mathbb{I}_A}\right)\right)\\
&+e^{-\epsilon_2\mathcal{N}}\varrho_1\,\varepsilon\mathbb{V}^*\mathbb{I}_A\left(1-\frac{\mathbb{V}^*}{\mathbb{V}}+\ln\left(\frac{\mathbb{V}^*}{\mathbb{V}}\right)\right)++e^{-\epsilon_2\mathcal{N}}\varrho_2\,\varepsilon\mathbb{V}^*\mathbb{I}_S\left(1-\frac{\mathbb{V}^*}{\mathbb{V}}+\ln\left(\frac{\mathbb{V}^*}{\mathbb{V}}\right)\right)\\
&-e^{\epsilon_2\mathcal{N}}\varrho_2\varepsilon\,\mathbb{V}^*\mathbb{I}_S^*\left(\mathcal{G}\left(\frac{\mathbb{E}^*\mathbb{V}(t-\mathcal{N})\mathbb{I}_S(t-\mathcal{N})}{\mathbb{E}\,\mathbb{V}^*\mathbb{I}_S^*}\right)+\mathcal{G}\left(\frac{\mathbb{E}\,\mathbb{I}_S^*}{\mathbb{E}^*\mathbb{I}_S}\right)\right).
\end{aligned}
$$

This implies

$$
\frac{dQ}{dt}\ \leq\ -e^{-\epsilon_1\mathcal{J}}\frac{(\zeta+\iota)(\mathbb{S}^*-\mathbb{S})^2}{\mathbb{S}}+e^{-\epsilon_2\mathcal{N}}\left(1-\frac{\mathbb{V}^*}{\mathbb{V}}\right)(\zeta\,\mathbb{S}-\zeta\,\mathbb{S}^*)-e^{-\epsilon_2\mathcal{N}}(\varrho+\iota)\left(\frac{(\mathbb{V}^*-\mathbb{V})^2}{\mathbb{V}}\right)
$$

$$
\begin{aligned}
&-e^{\epsilon_1\mathcal{J}}\varrho_1\mathbb{S}^*\mathbb{I}_A^*\left(\mathcal{G}\left(\frac{\mathbb{E}^*\mathbb{S}(t-\mathcal{J})\mathbb{I}_A(t-\mathcal{J})}{\mathbb{E}\,\mathbb{S}^*\mathbb{I}_A^*}\right)+\mathcal{G}\left(\frac{\mathbb{E}\,\mathbb{I}_A^*}{\mathbb{E}^*\mathbb{I}_A}\right)+\mathcal{G}\left(\frac{\mathbb{S}^*}{\mathbb{S}}\right)\right)\\
&-e^{\epsilon_1\mathcal{J}}\varrho_2\mathbb{S}^*\mathbb{I}_S^*\left(\mathcal{G}\left(\frac{\mathbb{E}^*\mathbb{S}(t-\mathcal{J})\mathbb{I}_S(t-\mathcal{J})}{\mathbb{E}\,\mathbb{S}^*\mathbb{I}_S^*}\right)+\mathcal{G}\left(\frac{\mathbb{E}\,\mathbb{I}_S^*}{\mathbb{E}^*\mathbb{I}_S}\right)+\mathcal{G}\left(\frac{\mathbb{S}^*}{\mathbb{S}}\right)\right)\\
&-e^{\epsilon_2\mathcal{N}}\varrho_1\varepsilon\,\mathbb{V}^*\mathbb{I}_A^*\left(\mathcal{G}\left(\frac{\mathbb{E}^*\mathbb{V}(t-\mathcal{N})\mathbb{I}_A(t-\mathcal{N})}{\mathbb{E}\,\mathbb{V}^*\mathbb{I}_A^*}\right)+\mathcal{G}\left(\frac{\mathbb{E}\,\mathbb{I}_A^*}{\mathbb{E}^*\mathbb{I}_A}\right)+\mathcal{G}\left(\frac{\mathbb{V}^*}{\mathbb{V}}\right)\right)\\
&-e^{\epsilon_2\mathcal{N}}\varrho_2\varepsilon\,\mathbb{V}^*\mathbb{I}_S^*\left(\mathcal{G}\left(\frac{\mathbb{E}^*\mathbb{V}(t-\mathcal{N})\mathbb{I}_S(t-\mathcal{N})}{\mathbb{E}\,\mathbb{V}^*\mathbb{I}_S^*}\right)+\mathcal{G}\left(\frac{\mathbb{E}\,\mathbb{I}_S^*}{\mathbb{E}^*\mathbb{I}_S}\right)+\mathcal{G}\left(\frac{\mathbb{V}^*}{\mathbb{V}}\right)\right).
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{dQ}{dt} \leq\;& -e^{-\epsilon_1\mathcal{J}}\frac{(\zeta+\iota)(\mathbb{S}^*-\mathbb{S})^2}{\mathbb{S}}+e^{-\epsilon_2\mathcal{N}}\left(1-\frac{\mathbb{V}^*}{\mathbb{V}}\right)(\zeta\,\mathbb{S}-\zeta\,\mathbb{S}^*)-e^{-\epsilon_2\mathcal{N}}(\varrho+\iota)\left(\frac{(\mathbb{V}^*-\mathbb{V})^2}{\mathbb{V}}\right)\\
&-e^{\epsilon_1\mathcal{J}}\varrho_1\mathbb{S}^*\mathbb{I}_A^*\left(\mathcal{G}\left(\frac{\mathbb{E}^*\mathbb{S}(t-\mathcal{J})\mathbb{I}_A(t-\mathcal{J})}{\mathbb{E}\,\mathbb{S}^*\mathbb{I}_A^*}\right)+\mathcal{G}\left(\frac{\mathbb{E}\,\mathbb{I}_A^*}{\mathbb{E}^*\mathbb{I}_A}\right)+\mathcal{G}\left(\frac{\mathbb{S}^*}{\mathbb{S}}\right)\right)\\
&-e^{\epsilon_1\mathcal{J}}\varrho_2\mathbb{S}^*\mathbb{I}_S^*\left(\mathcal{G}\left(\frac{\mathbb{E}^*\mathbb{S}(t-\mathcal{J})\mathbb{I}_S(t-\mathcal{J})}{\mathbb{E}\,\mathbb{S}^*\mathbb{I}_S^*}\right)+\mathcal{G}\left(\frac{\mathbb{E}\,\mathbb{I}_S^*}{\mathbb{E}^*\mathbb{I}_S}\right)+\mathcal{G}\left(\frac{\mathbb{S}^*}{\mathbb{S}}\right)\right)\\
&-e^{\epsilon_2\mathcal{N}}\varrho_1\varepsilon\,\mathbb{V}^*\mathbb{I}_A^*\left(\mathcal{G}\left(\frac{\mathbb{E}^*\mathbb{V}(t-\mathcal{N})\mathbb{I}_A(t-\mathcal{N})}{\mathbb{E}\,\mathbb{V}^*\mathbb{I}_A^*}\right)+\mathcal{G}\left(\frac{\mathbb{E}\,\mathbb{I}_A^*}{\mathbb{E}^*\mathbb{I}_A}\right)+\mathcal{G}\left(\frac{\mathbb{V}^*}{\mathbb{V}}\right)\right)\\
&-e^{\epsilon_2\mathcal{N}}\varrho_2\varepsilon\,\mathbb{V}^*\mathbb{I}_S^*\left(\mathcal{G}\left(\frac{\mathbb{E}^*\mathbb{V}(t-\mathcal{N})\mathbb{I}_S(t-\mathcal{N})}{\mathbb{E}\,\mathbb{V}^*\mathbb{I}_S^*}\right)+\mathcal{G}\left(\frac{\mathbb{E}\,\mathbb{I}_S^*}{\mathbb{E}^*\mathbb{I}_S}\right)+\mathcal{G}\left(\frac{\mathbb{V}^*}{\mathbb{V}}\right)\right)\\
&+e^{-\epsilon_2\mathcal{N}}\zeta\,\mathbb{S}\left(\frac{\mathbb{V}^*\,\mathbb{S}^*}{\mathbb{V}\mathbb{S}}\right)-e^{-\epsilon_2\mathcal{N}}\zeta\,\mathbb{S}\left(\frac{\mathbb{V}^*\,\mathbb{S}^*}{\mathbb{V}\mathbb{S}}\right)+e^{-\epsilon_2\mathcal{N}}\zeta\,\mathbb{S}\left(\frac{\mathbb{V}\,\mathbb{S}}{\mathbb{V}^*\mathbb{S}^*}\right)-e^{-\epsilon_2\mathcal{N}}\zeta\,\mathbb{S}\left(\frac{\mathbb{V}\,\mathbb{S}}{\mathbb{V}^*\mathbb{S}^*}\right).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{dQ}{dt} \leq\;& -e^{-\epsilon_1\mathcal{J}}\frac{(\zeta+\iota)(\mathbb{S}^*-\mathbb{S})^2}{\mathbb{S}}-e^{-\epsilon_2\mathcal{N}}(\varrho+\iota)\left(\frac{(\mathbb{V}^*-\mathbb{V})^2}{\mathbb{V}}\right)+e^{-\epsilon_2\mathcal{N}}\zeta\,\mathbb{S}\left(3-\frac{\mathbb{V}^*}{\mathbb{V}}-\frac{\mathbb{S}^*}{\mathbb{S}}-\frac{\mathbb{V}\mathbb{S}}{\mathbb{V}^*\mathbb{S}^*}\right)\\
&-e^{\epsilon_1\mathcal{J}}\varrho_1\mathbb{S}^*\mathbb{I}_A^*\left(\mathcal{G}\left(\frac{\mathbb{E}^*\mathbb{S}(t-\mathcal{J})\mathbb{I}_A(t-\mathcal{J})}{\mathbb{E}\,\mathbb{S}^*\mathbb{I}_A^*}\right)+\mathcal{G}\left(\frac{\mathbb{E}\,\mathbb{I}_A^*}{\mathbb{E}^*\mathbb{I}_A}\right)+\mathcal{G}\left(\frac{\mathbb{S}^*}{\mathbb{S}}\right)\right)\\
&-e^{\epsilon_1\mathcal{J}}\varrho_2\mathbb{S}^*\mathbb{I}_S^*\left(\mathcal{G}\left(\frac{\mathbb{E}^*\mathbb{S}(t-\mathcal{J})\mathbb{I}_S(t-\mathcal{J})}{\mathbb{E}\,\mathbb{S}^*\mathbb{I}_S^*}\right)+\mathcal{G}\left(\frac{\mathbb{E}\,\mathbb{I}_S^*}{\mathbb{E}^*\mathbb{I}_S}\right)+\mathcal{G}\left(\frac{\mathbb{S}^*}{\mathbb{S}}\right)\right)\\
&-e^{\epsilon_2\mathcal{N}}\varrho_1\varepsilon\,\mathbb{V}^*\mathbb{I}_A^*\left(\mathcal{G}\left(\frac{\mathbb{E}^*\mathbb{V}(t-\mathcal{N})\mathbb{I}_A(t-\mathcal{N})}{\mathbb{E}\,\mathbb{V}^*\mathbb{I}_A^*}\right)+\mathcal{G}\left(\frac{\mathbb{E}\,\mathbb{I}_A^*}{\mathbb{E}^*\mathbb{I}_A}\right)+\mathcal{G}\left(\frac{\mathbb{V}^*}{\mathbb{V}}\right)\right)\\
&-e^{\epsilon_2\mathcal{N}}\varrho_2\varepsilon\,\mathbb{V}^*\mathbb{I}_S^*\left(\mathcal{G}\left(\frac{\mathbb{E}^*\mathbb{V}(t-\mathcal{N})\mathbb{I}_S(t-\mathcal{N})}{\mathbb{E}\,\mathbb{V}^*\mathbb{I}_S^*}\right)+\mathcal{G}\left(\frac{\mathbb{E}\,\mathbb{I}_S^*}{\mathbb{E}^*\mathbb{I}_S}\right)+\mathcal{G}\left(\frac{\mathbb{V}^*}{\mathbb{V}}\right)\right)\\
&+e^{-\epsilon_2\mathcal{N}}\zeta\,\mathbb{S}\left(-1+\frac{\mathbb{V}^*\,\mathbb{S}^*}{\mathbb{V}\mathbb{S}}-\ln\left(\frac{\mathbb{V}^*\mathbb{S}^*}{\mathbb{V}\mathbb{S}}\right)\right)+e^{-\epsilon_2\mathcal{N}}\zeta\,\mathbb{S}\left(-1+\frac{\mathbb{V}\,\mathbb{S}}{\mathbb{V}^*\mathbb{S}^*}-\ln\left(\frac{\mathbb{V}\mathbb{S}}{\mathbb{V}^*\mathbb{S}^*}\right)\right)\\
&+e^{-\epsilon_2\mathcal{N}}\zeta\,\mathbb{S}\ln\left(\frac{\mathbb{V}^*\,\mathbb{S}^*}{\mathbb{V}\mathbb{S}}\right)+e^{-\epsilon_2\mathcal{N}}\zeta\,\mathbb{S}\ln\left(\frac{\mathbb{V}\,\mathbb{S}}{\mathbb{V}^*\mathbb{S}^*}\right),\\
\frac{dQ}{dt} \leq\;& -e^{-\epsilon_1\mathcal{J}}\frac{(\zeta+\iota)(\mathbb{S}^*-\mathbb{S})^2}{\mathbb{S}}-e^{-\epsilon_2\mathcal{N}}(\varrho+\iota)\left(\frac{(\mathbb{V}^*-\mathbb{V})^2}{\mathbb{V}}\right)+e^{-\epsilon_2\mathcal{N}}\zeta\,\mathbb{S}\left(3-\frac{\mathbb{V}^*}{\mathbb{V}}-\frac{\mathbb{S}^*}{\mathbb{S}}-\frac{\mathbb{V}\mathbb{S}}{\mathbb{V}^*\mathbb{S}^*}\right)
\end{aligned}
$$

$$
\begin{aligned}
&-e^{\epsilon_1 \mathcal{J}}\varrho_1 \mathbb{S}^*\mathbb{I}_A^* \left(\mathcal{G}\left(\frac{\mathbb{E}^*\mathbb{S}(t-\mathcal{J})\mathbb{I}_A(t-\mathcal{J})}{\mathbb{E}\,\mathbb{S}^*\mathbb{I}_A^*}\right)+\mathcal{G}\left(\frac{\mathbb{E}\,\mathbb{I}_A^*}{\mathbb{E}^*\mathbb{I}_A}\right)+\mathcal{G}\left(\frac{\mathbb{S}^*}{\mathbb{S}}\right)\right)\\
&-e^{\epsilon_1 \mathcal{J}}\varrho_2 \mathbb{S}^*\mathbb{I}_S^* \left(\mathcal{G}\left(\frac{\mathbb{E}^*\mathbb{S}(t-\mathcal{J})\mathbb{I}_S(t-\mathcal{J})}{\mathbb{E}\,\mathbb{S}^*\mathbb{I}_S^*}\right)+\mathcal{G}\left(\frac{\mathbb{E}\,\mathbb{I}_S^*}{\mathbb{E}^*\mathbb{I}_S}\right)+\mathcal{G}\left(\frac{\mathbb{S}^*}{\mathbb{S}}\right)\right)\\
&-e^{\epsilon_2 \mathcal{N}}\varrho_1 \varepsilon\,\mathbb{V}^*\mathbb{I}_A^* \left(\mathcal{G}\left(\frac{\mathbb{E}^*\mathbb{V}(t-\mathcal{N})\mathbb{I}_A(t-\mathcal{N})}{\mathbb{E}\,\mathbb{V}^*\mathbb{I}_A^*}\right)+\mathcal{G}\left(\frac{\mathbb{E}\,\mathbb{I}_A^*}{\mathbb{E}^*\mathbb{I}_A}\right)+\mathcal{G}\left(\frac{\mathbb{V}^*}{\mathbb{V}}\right)\right)\\
&-e^{\epsilon_2 \mathcal{N}}\varrho_2 \varepsilon\,\mathbb{V}^*\mathbb{I}_S^* \left(\mathcal{G}\left(\frac{\mathbb{E}^*\mathbb{V}(t-\mathcal{N})\mathbb{I}_S(t-\mathcal{N})}{\mathbb{E}\,\mathbb{V}^*\mathbb{I}_S^*}\right)+\mathcal{G}\left(\frac{\mathbb{E}\,\mathbb{I}_S^*}{\mathbb{E}^*\mathbb{I}_S}\right)+\mathcal{G}\left(\frac{\mathbb{V}^*}{\mathbb{V}}\right)\right)\\
&-e^{-\epsilon_2 \mathcal{N}}\zeta\,\mathbb{S}\left(\mathcal{G}\left(\frac{\mathbb{V}^*\,\mathbb{S}^*}{\mathbb{V}\mathbb{S}}\right)+\mathcal{G}\left(\frac{\mathbb{V}\,\mathbb{S}}{\mathbb{V}^*\mathbb{S}^*}\right)\right)+e^{-\epsilon_2 \mathcal{N}}\zeta\,\mathbb{S}\ln\left(\frac{\mathbb{V}^*\,\mathbb{S}^*}{\mathbb{V}\mathbb{S}}\frac{\mathbb{V}\,\mathbb{S}}{\mathbb{V}^*\mathbb{S}^*}\right).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{dQ}{dt} \;\le\; & -e^{-\epsilon_1 \mathcal{J}}\frac{(\zeta+\iota)(\mathbb{S}^*-\mathbb{S})^2}{\mathbb{S}}-e^{-\epsilon_2 \mathcal{N}}(\varrho+\iota)\left(\frac{(\mathbb{V}^*-\mathbb{V})^2}{\mathbb{V}}\right)+e^{-\epsilon_2 \mathcal{N}}\zeta\,\mathbb{S}\left(3-\frac{\mathbb{V}^*}{\mathbb{V}}-\frac{\mathbb{S}^*}{\mathbb{S}}-\frac{\mathbb{V}\mathbb{S}}{\mathbb{V}^*\mathbb{S}^*}\right)\\
&-e^{\epsilon_1 \mathcal{J}}\varrho_1 \mathbb{S}^*\mathbb{I}_A^* \left(\mathcal{G}\left(\frac{\mathbb{E}^*\mathbb{S}(t-\mathcal{J})\mathbb{I}_A(t-\mathcal{J})}{\mathbb{E}\,\mathbb{S}^*\mathbb{I}_A^*}\right)+\mathcal{G}\left(\frac{\mathbb{E}\,\mathbb{I}_A^*}{\mathbb{E}^*\mathbb{I}_A}\right)+\mathcal{G}\left(\frac{\mathbb{S}^*}{\mathbb{S}}\right)\right)\\
&-e^{\epsilon_1 \mathcal{J}}\varrho_2 \mathbb{S}^*\mathbb{I}_S^* \left(\mathcal{G}\left(\frac{\mathbb{E}^*\mathbb{S}(t-\mathcal{J})\mathbb{I}_S(t-\mathcal{J})}{\mathbb{E}\,\mathbb{S}^*\mathbb{I}_S^*}\right)+\mathcal{G}\left(\frac{\mathbb{E}\,\mathbb{I}_S^*}{\mathbb{E}^*\mathbb{I}_S}\right)+\mathcal{G}\left(\frac{\mathbb{S}^*}{\mathbb{S}}\right)\right)\\
&-e^{\epsilon_2 \mathcal{N}}\varrho_1 \varepsilon\,\mathbb{V}^*\mathbb{I}_A^* \left(\mathcal{G}\left(\frac{\mathbb{E}^*\mathbb{V}(t-\mathcal{N})\mathbb{I}_A(t-\mathcal{N})}{\mathbb{E}\,\mathbb{V}^*\mathbb{I}_A^*}\right)+\mathcal{G}\left(\frac{\mathbb{E}\,\mathbb{I}_A^*}{\mathbb{E}^*\mathbb{I}_A}\right)+\mathcal{G}\left(\frac{\mathbb{V}^*}{\mathbb{V}}\right)\right)\\
&-e^{\epsilon_2 \mathcal{N}}\varrho_2 \varepsilon\,\mathbb{V}^*\mathbb{I}_S^* \left(\mathcal{G}\left(\frac{\mathbb{E}^*\mathbb{V}(t-\mathcal{N})\mathbb{I}_S(t-\mathcal{N})}{\mathbb{E}\,\mathbb{V}^*\mathbb{I}_S^*}\right)+\mathcal{G}\left(\frac{\mathbb{E}\,\mathbb{I}_S^*}{\mathbb{E}^*\mathbb{I}_S}\right)+\mathcal{G}\left(\frac{\mathbb{V}^*}{\mathbb{V}}\right)\right)\\
&-e^{-\epsilon_2 \mathcal{N}}\zeta\,\mathbb{S}\left(\mathcal{G}\left(\frac{\mathbb{V}^*\,\mathbb{S}^*}{\mathbb{V}\mathbb{S}}\right)+\mathcal{G}\left(\frac{\mathbb{V}\,\mathbb{S}}{\mathbb{V}^*\mathbb{S}^*}\right)\right).
\end{aligned}
$$

Hence, from the geometrical and arithmetical means relationship, we see that

$$
3 \le \frac{\mathbb{S}^*}{\mathbb{S}}+\frac{\mathbb{V}^*}{\mathbb{V}}+\frac{\mathbb{V}\,\mathbb{S}}{\mathbb{S}^*\,\mathbb{V}^*},
$$

Accordingly, $\frac{dQ}{dt} \le 0$ if $\Re_0 > 1$ for all $\mathbb{S},\mathbb{E},\mathbb{I}_A,\mathbb{I}_S,\mathbb{V} > 0$. Moreover, $\frac{dQ}{dt} = 0$ if and only if $\mathbb{S}=\mathbb{S}^*$, $\mathbb{E}=\mathbb{E}^*$, $\mathbb{I}_A=\mathbb{I}_A^*$, $\mathbb{I}_S=\mathbb{I}_S^*$, and $\mathbb{V}=\mathbb{V}^*$. Let $\bar{\mathcal{F}}'$ be the largest subset of $\bar{\mathcal{F}} = \{(\mathbb{S},\mathbb{E},\mathbb{I}_A,\mathbb{I}_S,\mathbb{V}) : \frac{d}{dt}\bar{C}=0\}$. Then, $\bar{\mathcal{H}}' = \{\mathcal{U}^*\}$. Thus, from La Salle's invariant principle [19], the endemic equilibrium point $\mathcal{U}^*$ is $\mathcal{GAS}$ when $\Re_0 > 1$. □

A.3. Proof of Theorem 7

Proof. We establish a Lyapunov function $\alpha \longrightarrow \mathbb{R}_+^5$ as follows:

$$
C(t) = \mathfrak{a}_1\left(\mathcal{G}(\mathbb{S})+\mathcal{G}(\mathbb{V})+\mathbb{E}\right)+\mathfrak{a}_2\,\mathbb{I}_A+\mathfrak{a}_3\,\mathbb{I}_S,
$$

where, $\mathfrak{a}_1 = \frac{(\iota+\varpi_1)(\iota+\varpi_2)}{(\mathbb{S}_0+\varepsilon\mathbb{V}_0)}$, $\mathfrak{a}_2 = \Xi_1(\iota+\varpi_2)$, and $\mathfrak{a}_3 = \Xi_2(\iota+\varpi_1)$. It is clear that C is positive definite. The time derivative of C along the solutions of (3.1) is defined as

$$
\frac{d}{dt}C \;\le\; \mathfrak{a}_1\left(1-\frac{\mathbb{S}_0}{\mathbb{S}}\right)\frac{d}{dt}\mathbb{S}+\mathfrak{a}_1\frac{d}{dt}\mathbb{E}+\mathfrak{a}_1\left(1-\frac{\mathbb{V}_0}{\mathbb{V}}\right)\frac{d}{dt}\mathbb{V}+\mathfrak{a}_2\frac{d}{dt}\mathbb{I}_A+\mathfrak{a}_3\frac{d}{dt}\mathbb{I}_S,
$$

$$
\begin{aligned}
&\leq \mathfrak{a}_1\left(1-\frac{\mathbb{S}_0}{\mathbb{S}}\right)\Big(\Xi-\Xi_1\mathbb{S}\,\mathrm{I_c}-\Xi_2\mathbb{S}\,\mathrm{I}_\eta-(\zeta+\iota)\,\mathbb{S}\Big)+\mathfrak{a}_1\left(\Xi_1\mathbb{S}\,\mathrm{I_c}+\Xi_2\mathbb{S}\,\mathrm{I}_\eta+\varepsilon\,\Xi_1\mathbb{V}\,\mathrm{I_c}\right.\\
&\left.+\varepsilon\,\Xi_2\mathbb{V}\,\mathrm{I}_\eta-(\sigma+\iota)\mathbb{E}\right)+\mathfrak{a}_1\left(1-\frac{\mathbb{V}_0}{\mathbb{V}}\right)(\zeta\,\mathbb{S}-\varepsilon\,\Xi_1\mathbb{V}\,\mathrm{I_c}-\varepsilon\,\Xi_2\mathbb{V}\,\mathrm{I}_\eta-(\varrho+\iota)\,\mathbb{V})\\
&+\mathfrak{a}_2(\alpha\,\sigma\,\mathbb{E}-(\varpi_1+\iota)\,\mathrm{I_c})+\mathfrak{a}_3((1-\alpha)\,\sigma\,\mathbb{E}-(\varpi_2+\iota)\mathrm{I}_\eta).
\end{aligned}
$$

By using $\Xi = (\zeta + \iota)\,\mathbb{S}_0$ and $\zeta\,\mathbb{S}_0 = (\varrho + \iota)\,\mathbb{V}_0$, we obtain

$$
\begin{aligned}
\frac{d}{dt}C &\leq \mathfrak{a}_1\left(1-\frac{\mathbb{S}_0}{\mathbb{S}}\right)(\iota\,\mathbb{S}_0-\iota\,\mathbb{S})+\mathfrak{a}_1\,\Xi_1\,\mathbb{S}_0\,\mathbb{I}_A+\mathfrak{a}_1\,\Xi_2\,\mathbb{S}_0\,\mathbb{I}_S+\mathfrak{a}_1\,\zeta\,\mathbb{S}_0-\mathfrak{a}_1\,(\sigma+\iota)\,\mathbb{E}\\
&+\mathfrak{a}_2\,\alpha\,\sigma\,\mathbb{E}-\mathfrak{a}_2\,(\varpi_1+\iota)\,\mathbb{I}_A+\mathfrak{a}_3\,\sigma\,(1-\alpha)\,\mathbb{E}-\mathfrak{a}_3\,(\varpi_2+\iota)\mathbb{I}_S-\mathfrak{a}_1\,(\varrho+\iota)\,\mathbb{V}\\
&-\mathfrak{a}_1\,\zeta\,\mathbb{S}\left(\frac{\mathbb{V}_0}{\mathbb{V}}\right)+\mathfrak{a}_1\,\varepsilon\,\Xi_1\,\mathbb{V}_0\,\mathbb{I}_A+\mathfrak{a}_1\,\varepsilon\,\Xi_2\,\mathbb{V}_0\,\mathbb{I}_S+\mathfrak{a}_1\,(\varrho+\iota)\,\mathbb{V}_0+\mathfrak{a}_1\left(1-\frac{\mathbb{S}_0}{\mathbb{S}}\right)\zeta\,\mathbb{S}_0,
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}C &\leq \mathfrak{a}_1\left(1-\frac{\mathbb{S}_0}{\mathbb{S}}\right)(\iota\,\mathbb{S}_0-\iota\,\mathbb{S})+\mathfrak{a}_1\,\zeta\,\mathbb{S}_0\left(2-\frac{\mathbb{S}_0}{\mathbb{S}}-\frac{\mathbb{V}_0\,\mathbb{S}}{\mathbb{S}_0\,\mathbb{V}}\right)+\mathfrak{a}_1\,(\varrho+\iota)\,\mathbb{V}_0\left(1-\frac{\mathbb{V}}{\mathbb{V}_0}\right)\\
&+\left(\frac{(\iota+\varpi_1)\,(\iota+\varpi_2)}{\mathbb{S}_0+\varepsilon\mathbb{V}_0}\,(\mathbb{S}_0+\varepsilon\mathbb{V}_0)\,\Xi_1-(\iota+\varpi_1)\,(\iota+\varpi_2)\,\Xi_1)\right)\mathbb{I}_A\\
&+\left(\frac{(\iota+\varpi_1)\,(\iota+\varpi_2)}{\mathbb{S}_0+\varepsilon\mathbb{V}_0}\,(\mathbb{S}_0+\varepsilon\mathbb{V}_0)\,\Xi_2-(\iota+\varpi_1)\,(\iota+\varpi_2)\,\Xi_2)\right)\mathbb{I}_S\\
&+\left(\sigma\,(\iota+\varpi_2)\,\Xi_1\,\alpha\;+\sigma\,(\iota+\varpi_1)\,\Xi_2\,(1-\alpha)-\frac{(\iota+\varpi_1)\,(\iota+\varpi_2)}{\mathbb{S}_0+\varepsilon\mathbb{V}_0}\,(\iota+\sigma)\right)\mathbb{E}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{d}{dt}C &\leq \mathfrak{a}_1\left(1-\frac{\mathbb{S}_0}{\mathbb{S}}\right)(\iota\,\mathbb{S}_0-\iota\,\mathbb{S})+\mathfrak{a}_1\,\zeta\,\mathbb{S}_0\left(2-\frac{\mathbb{S}_0}{\mathbb{S}}-\frac{\mathbb{V}_0\,\mathbb{S}}{\mathbb{S}_0\,\mathbb{V}}\right)+\mathfrak{a}_1\,\zeta\,\mathbb{S}_0\left(1-\frac{\mathbb{V}}{\mathbb{V}_0}\right)\\
&+\left((\iota+\varpi_2)\,\Xi_1\,\alpha\,\sigma+(\iota+\varpi_1)\,\Xi_2\,(1-\alpha)\sigma-\frac{(\iota+\varpi_1)\,(\iota+\varpi_2)}{\mathbb{S}_0+\varepsilon\mathbb{V}_0}\,(\iota+\sigma)\right)\mathbb{E}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{d}{dt}C &\leq \mathfrak{a}_1\left(1-\frac{\mathbb{S}_0}{\mathbb{S}}\right)(\iota\,\mathbb{S}_0-\iota\,\mathbb{S})+\mathfrak{a}_1\,\zeta\,\mathbb{S}_0\left(3-\frac{\mathbb{S}_0}{\mathbb{S}}-\frac{\mathbb{V}_0\,\mathbb{S}}{\mathbb{S}_0\,\mathbb{V}}-\frac{\mathbb{V}}{\mathbb{V}_0}\right)\\
&+\left(\frac{(\mathbb{S}_0+\varepsilon\mathbb{V}_0)\,(\iota+\varpi_2)\,\Xi_1\,\alpha\,\sigma}{(\iota+\varpi_1)\,(\iota+\varpi_2)\,(\iota+\sigma)}+\frac{(\mathbb{S}_0+\varepsilon\mathbb{V}_0)\,(\iota+\varpi_1)\,\Xi_2\,(1-\alpha)\sigma}{(\iota+\varpi_1)\,(\iota+\varpi_2)\,(\iota+\sigma)}-1\right)\mathbb{E}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{d}{dt}C &\leq -\frac{(\iota+\varpi_1)\,(\iota+\varpi_2)}{\iota\,\mathbb{S}\,(\mathbb{S}_0+\varepsilon\,\mathbb{V}_0)}\,(\mathbb{S}-\mathbb{S}_0)^2+\frac{\zeta\,\mathbb{S}_0\,(\iota+\varpi_1)\,(\iota+\varpi_2)}{\mathbb{S}_0+\varepsilon\,\mathbb{V}_0}\left(3-\frac{\mathbb{S}_0}{\mathbb{S}}-\frac{\mathbb{V}_0\,\mathbb{S}}{\mathbb{S}_0\,\mathbb{V}}-\frac{\mathbb{V}}{\mathbb{V}_0}\right)\\
&+(\Re_0(\nu)-1)\,\mathbb{E}.
\end{aligned}
$$

Since the geometric mean is less than or equal to the arithmetical mean, we have

$$
3 \leq \frac{\mathbb{S}_0}{\mathbb{S}}+\frac{\mathbb{V}_0\,\mathbb{S}}{\mathbb{V}\,\mathbb{S}_0}+\frac{\mathbb{V}}{\mathbb{V}_0}.
$$

Moreover, in cases where $(1-\nu) \le \left(\frac{(\sigma+\iota)}{\sigma(\mathbb{S}_0+\varepsilon\mathbb{V}_0)}\right)\left(\frac{1}{\frac{\varrho_1\,\alpha}{\varpi_1+\iota}+\frac{\varrho_2\,(1-\alpha)}{\varpi_2+\iota}}\right)$ implies $\Re_0(\nu) \le 1$, $\frac{d}{dt}C \le 0$ for all $\mathbb{S},\mathbb{E},\mathbb{I}_A,\mathbb{I}_S,\mathbb{V} > 0$. Clearly, $\frac{d}{dt}C = 0$ if and only if $\mathbb{S}=\mathbb{S}_0$, $\mathbb{E}=0$, and $\mathbb{V}=\mathbb{V}_0$. Therefore, the subset $\mathcal{D}'$ is the largest invariant set of $\mathcal{D}=\left\{(\mathbb{S},\mathbb{E},\mathbb{I}_A,\mathbb{I}_S,\mathbb{V})\in\alpha : \frac{d}{dt}C=0\right\}$. By applying La Salle's invariant principle [19], we see that all solutions converge to $\mathcal{D}'$. In $\mathcal{D}'$, the elements are equal to $\mathbb{S}=\mathbb{S}_0$, $\mathbb{V}=\mathbb{V}_0$, and $\mathbb{E}=0$. Moreover, from the system (3.1), we get $\mathbb{I}_A=0$ and $\mathbb{I}_S=0$ when $\mathbb{E}=0$. Thus, $\mathcal{D}'=\{(\mathbb{S},\mathbb{E},\mathbb{I}_A,\mathbb{I}_S,\mathbb{V})\in\mathcal{D}:\mathbb{S}=\mathbb{S}_0,\mathbb{V}=\mathbb{V}_0,\mathbb{E}=\mathbb{I}_A=\mathbb{I}_S=0\}=\{\mathcal{U}_0\}$. Hence, $\mathcal{U}_0$ is $\mathcal{GAS}$ when $\Re_0(\nu)\le 1$. □

A.4. Proof of Theorem 8

Proof. We construct a Lyapunov function as follows:

$$\bar{C} = \mathcal{G}(\mathbb{S})+\mathcal{G}(\mathbb{E})+\frac{\Xi_1\,\mathbb{I}_A^*(\mathbb{S}^*+\varepsilon\,\mathbb{V}^*)}{\alpha\,\sigma\,\mathbb{E}^*}\,\mathcal{G}(\mathbb{I}_A)+\frac{\Xi_2\,\mathbb{I}_S^*\,(\mathbb{S}^*+\varepsilon\,\mathbb{V}^*)}{(1-\alpha)\,\sigma\,\mathbb{E}^*}\,\mathcal{G}(\mathbb{I}_S)+\mathcal{G}(\mathbb{V}).$$

Clearly, $\bar{C}$ is positive definite. By obtaining the time derivative of $\bar{C}$ along the solutions of (3.1), we find

$$\begin{aligned}\frac{d}{dt}\bar{C} \;&\le\; \left(1-\frac{\mathbb{S}^*}{\mathbb{S}}\right)\frac{d}{dt}\mathbb{S}+\left(1-\frac{\mathbb{E}^*}{\mathbb{E}}\right)\frac{d}{dt}\mathbb{E}+\frac{\Xi_1\,\mathbb{I}_A^*\,(\mathbb{S}^*+\varepsilon\,\mathbb{V}^*)}{\alpha\,\sigma\,\mathbb{E}^*}\left(1-\frac{\mathbb{I}_A^*}{\mathbb{I}_A}\right)\frac{d}{dt}\mathbb{I}_A\\ &+\frac{\Xi_2\,\mathbb{I}_S^*(\mathbb{S}^*+\varepsilon\,\mathbb{V}^*)}{(1-\alpha)\,\sigma\,\mathbb{E}^*}\left(1-\frac{\mathbb{I}_S^*}{\mathbb{I}_S}\right)\frac{d}{dt}\mathbb{I}_S+\left(1-\frac{\mathbb{V}^*}{\mathbb{V}}\right)\frac{d}{dt}\mathbb{V},\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\bar{C} \;&\le\; \left(1-\frac{\mathbb{S}^*}{\mathbb{S}}\right)(\Xi-\Xi_1\mathbb{S}\,\mathrm{I_c}-\Xi_2\mathbb{S}\,\mathrm{I}_\eta-(\zeta+\iota)\mathbb{S})+\left(1-\frac{\mathbb{E}^*}{\mathbb{E}}\right)(\Xi_1\mathbb{S}\,\mathrm{I_c}+\Xi_2\mathbb{S}\,\mathrm{I}_\eta+\varepsilon\,\Xi_1\mathbb{V}\,\mathrm{I_c}\\ &+\varepsilon\,\Xi_2\mathbb{V}\,\mathrm{I}_\eta-(\sigma+\iota)\mathbb{E})+\frac{\Xi_1\,\mathbb{I}_A^*(\mathbb{S}^*+\varepsilon\,\mathbb{V}^*)}{\alpha\,\sigma\,\mathbb{E}^*}\left(1-\frac{\mathbb{I}_A^*}{\mathbb{I}_A}\right)(\alpha\,\sigma\,\mathbb{E}-(\varpi_1+\iota)\mathrm{I_c})\\ &+\frac{\Xi_2\,\mathbb{I}_S^*\,(\mathbb{S}^*+\varepsilon\,\mathbb{V}^*)}{(1-\alpha)\,\sigma\,\mathbb{E}^*}\left(1-\frac{\mathbb{I}_S^*}{\mathbb{I}_S}\right)((1-\alpha)\,\sigma\,\mathbb{E}-(\varpi_2+\iota)\mathrm{I}_\eta)\\ &+\left(1-\frac{\mathbb{V}^*}{\mathbb{V}}\right)(\zeta\,\mathbb{S}-\varepsilon\,\Xi_1\mathbb{V}\,\mathrm{I_c}-\varepsilon\,\Xi_2\mathbb{V}\,\mathrm{I}_\eta-(\varrho+\iota)\,\mathbb{V}).\end{aligned}$$

From the equilibria, we have the following relationships:

$$\begin{cases}\Xi \;=\; \Xi_1\,\mathbb{S}^*\,\mathrm{I}_c^*+\Xi_2\,\mathbb{S}^*\,\mathrm{I}_\eta^*+(\zeta+\iota)\,\mathbb{S}^*,\\ \alpha\,\sigma\,\mathbb{E}^* \;=\; (\varpi_1+\iota)\,\mathbb{I}_A^*,\\ (1-\alpha)\,\sigma\,\mathbb{E}^* \;=\; (\varpi_2+\iota)\,\mathbb{I}_S^*\\ (\varrho+\iota)\,\mathbb{V}^* \;=\; \zeta\,\mathbb{S}^*-\varepsilon\,\Xi_1\,\mathbb{I}_A^*\mathbb{V}^*-\varepsilon\,\Xi_2\,\mathbb{I}_S^*\,\mathbb{V}^*\\ \zeta\,\mathbb{S}^*=(\varrho+\iota)\,\mathbb{V}^*\;+\varepsilon\,\Xi_1\,\mathbb{V}^*\,\mathbb{I}_A^*+\varepsilon\,\Xi_2\,\mathbb{V}^*\,\mathbb{I}_S^*,\\ (\sigma+\iota)\,\mathbb{E}^* \;=\; \Xi_1\,\mathbb{S}^*\,\mathrm{I}_c^*+\Xi_2\,\mathbb{S}^*\,\mathrm{I}_\eta^*+\varepsilon\,\Xi_1\,\mathbb{V}^*\,\mathbb{I}_A^*+\varepsilon\,\Xi_2\,\mathbb{V}^*\,\mathbb{I}_S^*.\end{cases}$$

By substituting the previous relationships and then eliminating some terms, we get

$$\frac{d}{dt}\bar{C} \;\le\; \left(1-\frac{\mathbb{S}^*}{\mathbb{S}}\right)((\zeta+\iota)\,\mathbb{S}^*-(\zeta+\iota)\,\mathbb{S})+\left(1-\frac{\mathbb{S}^*}{\mathbb{S}}\right)\Xi_1\,\mathbb{S}^*\,\mathbb{I}_A^*+\left(1-\frac{\mathbb{S}^*}{\mathbb{S}}\right)\Xi_2\,\mathbb{S}^*\,\mathbb{I}_S^*$$

$$
\begin{aligned}
&+\Xi_1\,\mathbb{S}^*\,\mathbb{I}_A+\Xi_2\,\mathbb{S}^*\,\mathbb{I}_S-(\sigma+\iota)\,\mathbb{E}-\left(\frac{\mathbb{E}^*}{\mathbb{E}}\right)\Xi_1\,\mathbb{S}\,\mathbb{I}_A-\left(\frac{\mathbb{E}^*}{\mathbb{E}}\right)\Xi_2\,\mathbb{S}\,\mathbb{I}_S-\left(\frac{\mathbb{E}^*}{\mathbb{E}}\right)\varepsilon\,\Xi_1\,\mathbb{V}\,\mathbb{I}_A\\
&-\left(\frac{\mathbb{E}^*}{\mathbb{E}}\right)\varepsilon\,\Xi_2\,\mathbb{V}\,\mathbb{I}_S+(\Xi_1\,\mathbb{S}^*\,\mathbb{I}_A^*+\Xi_2\,\mathbb{S}^*\,\mathbb{I}_S^*+\varepsilon\,\Xi_1\,\mathbb{V}^*\,\mathbb{I}_A^*+\varepsilon\,\Xi_2\,\mathbb{V}^*\,\mathbb{I}_S^*)+\left(\frac{\Xi_1\,\mathbb{I}_A^*(\mathbb{S}^*+\varepsilon\,\mathbb{V}^*)}{\alpha\,\sigma\,\mathbb{E}^*}\right)\alpha\,\sigma\,\mathbb{E}\\
&-\left(\frac{\Xi_1\,\mathbb{I}_A^*(\mathbb{S}^*+\varepsilon\,\mathbb{V}^*)}{\alpha\,\sigma\,\mathbb{E}^*}\right)(\varpi_1+\iota)\,I_c-\left(\frac{\Xi_1\,\mathbb{I}_A^*(\mathbb{S}^*+\varepsilon\,\mathbb{V}^*)}{\alpha\,\sigma\,\mathbb{E}^*}\right)\left(\frac{\mathrm{I}_c^*}{\mathrm{I}_c}\right)\alpha\,\sigma\,\mathbb{E}\\
&+\left(\frac{\Xi_1\,\mathbb{I}_A^*(\mathbb{S}^*+\varepsilon\,\mathbb{V}^*)}{\alpha\,\sigma\,\mathbb{E}^*}\right)(\varpi_1+\iota)\,\mathbb{I}_A^*+\left(\frac{\Xi_2\,\mathbb{I}_S^*(\mathbb{S}^*+\varepsilon\,\mathbb{V}^*)}{(1-\alpha)\,\sigma\,\mathbb{E}^*}\right)(1-\alpha)\,\sigma\,\mathbb{E}\\
&-\left(\frac{\Xi_2\,\mathbb{I}_S^*(\mathbb{S}^*+\varepsilon\,\mathbb{V}^*)}{(1-\alpha)\,\sigma\,\mathbb{E}^*}\right)(\varpi_2+\iota)\,\mathbb{I}_S-\left(\frac{\mathbb{I}_S^*}{\mathbb{I}_S}\right)\left(\frac{\Xi_2\,\mathbb{I}_S^*(\mathbb{S}^*+\varepsilon\,\mathbb{V}^*)}{(1-\alpha)\,\sigma\,\mathbb{E}^*}\right)(1-\alpha)\,\sigma\,\mathbb{E}\\
&+\left(\frac{\Xi_2\,\mathbb{I}_S^*(\mathbb{S}^*+\varepsilon\,\mathbb{V}^*)}{(1-\alpha)\,\sigma\,\mathbb{E}^*}\right)(\varpi_2+\iota)\,\mathbb{I}_S^*-\left(1-\frac{\mathbb{V}^*}{\mathbb{V}}\right)(\iota+\varrho)\,\mathbb{V}+\left(1-\frac{\mathbb{V}^*}{\mathbb{V}}\right)\zeta\,\mathbb{S}\\
&+\varepsilon\,\Xi_1\,\mathbb{V}^*\,\mathbb{I}_A+\varepsilon\,\Xi_2\,\mathbb{V}^*\,\mathbb{I}_S.
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\bar{\mathcal{C}} \;\leq\;& -(\zeta+\iota)\,\frac{(\mathbb{S}^*-\mathbb{S})^2}{\mathbb{S}}+\left(1-\frac{\mathbb{S}^*}{\mathbb{S}}\right)(\Xi_1\,\mathbb{S}^*\,\mathbb{I}_A^*+\Xi_2\,\mathbb{S}^*\,\mathbb{I}_S^*)+\Xi_1\,\mathbb{S}^*\,\mathbb{I}_A+\Xi_2\,\mathbb{S}^*\,\mathbb{I}_S\\
&-\frac{\mathbb{E}^*}{\mathbb{E}}\,\Xi_1\,\mathbb{S}\,\mathbb{I}_A-\frac{\mathbb{E}^*}{\mathbb{E}}\,\Xi_2\,\mathbb{S}\,\mathbb{I}_S-\frac{\mathbb{E}^*}{\mathbb{E}}\,\varepsilon\,\Xi_1\,\mathbb{V}\,\mathbb{I}_A-\frac{\mathbb{E}^*}{\mathbb{E}}\,\varepsilon\,\Xi_2\,\mathbb{V}\,\mathbb{I}_S-(\sigma+\iota)\,\mathbb{E}\\
&+\frac{\mathbb{E}}{\mathbb{E}^*}\,(\Xi_1\,\mathbb{S}^*\,\mathbb{I}_A^*+\varepsilon\,\Xi_1\,\mathbb{V}^*\,\mathbb{I}_A^*+\Xi_2\,\mathbb{S}^*\,\mathbb{I}_S^*+\varepsilon\,\Xi_2\,\mathbb{V}^*\,\mathbb{I}_S^*)-\Xi_1\,\mathbb{S}^*\,\mathbb{I}_A-\Xi_2\,\mathbb{S}^*\,\mathbb{I}_S\\
&-\left(\frac{\mathbb{I}_A^*\,\mathbb{E}}{\mathbb{I}_A\,\mathbb{E}^*}\right)\Xi_1\,\mathbb{S}^*\,\mathbb{I}_A^*-\left(\frac{\mathbb{I}_A^*\,\mathbb{E}}{\mathbb{I}_A\,\mathbb{E}^*}\right)\varepsilon\,\Xi_1\,\mathbb{V}^*\,\mathbb{I}_A^*-\left(\frac{\mathbb{I}_S^*\,\mathbb{E}}{\mathbb{I}_S\,\mathbb{E}^*}\right)\Xi_2\,\mathbb{S}^*\,\mathbb{I}_S^*-\left(\frac{\mathbb{I}_S^*\,\mathbb{E}}{\mathbb{I}_S\,\mathbb{E}^*}\right)\varepsilon\,\Xi_2\,\mathbb{V}^*\,\mathbb{I}_S^*\\
&+2\,\Xi_1\,\mathbb{S}^*\,\mathbb{I}_A^*+2\,\varepsilon\,\Xi_1\,\mathbb{V}^*\,\mathbb{I}_A^*+2\,\Xi_2\,\mathbb{S}^*\,\mathbb{I}_S^*+2\,\varepsilon\,\Xi_2\,\mathbb{V}^*\,\mathbb{I}_S^*-\varepsilon\,\Xi_1\,\mathbb{V}^*\,\mathbb{I}_A-\varepsilon\,\Xi_2\,\mathbb{V}^*\,\mathbb{I}_S\\
&+\varepsilon\,\Xi_1\,\mathbb{V}^*\,\mathbb{I}_A+\varepsilon\,\Xi_2\,\mathbb{V}^*\,\mathbb{I}_S-\left(1-\frac{\mathbb{V}^*}{\mathbb{V}}\right)(\iota+\varrho)\,\mathbb{V}+\left(1-\frac{\mathbb{V}^*}{\mathbb{V}}\right)\zeta\,\mathbb{S}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{d}{dt}\bar{\mathcal{C}} \;\leq\;& -(\zeta+\iota)\,\frac{(\mathbb{S}^*-\mathbb{S})^2}{\mathbb{S}}+\left(1-\frac{\mathbb{S}^*}{\mathbb{S}}\right)(\Xi_1\,\mathbb{S}^*\,\mathbb{I}_A^*+\Xi_2\,\mathbb{S}^*\,\mathbb{I}_S^*)-(\sigma+\iota)\,\mathbb{E}\\
&-\frac{\mathbb{E}^*}{\mathbb{E}}\,\Xi_1\,\mathbb{S}\,\mathbb{I}_A-\frac{\mathbb{E}^*}{\mathbb{E}}\,\Xi_2\,\mathbb{S}\,\mathbb{I}_S-\frac{\mathbb{E}^*}{\mathbb{E}}\,\varepsilon\,\Xi_1\,\mathbb{V}\,\mathbb{I}_A-\frac{\mathbb{E}^*}{\mathbb{E}}\,\varepsilon\,\Xi_2\,\mathbb{V}\,\mathbb{I}_S+\frac{\mathbb{E}}{\mathbb{E}^*}\,(\iota+\sigma)\,\mathbb{E}^*\\
&-\left(\frac{\mathbb{I}_A^*\,\mathbb{E}}{\mathbb{I}_A\,\mathbb{E}^*}\right)\Xi_1\,\mathbb{S}^*\,\mathbb{I}_A^*-\left(\frac{\mathbb{I}_A^*\,\mathbb{E}}{\mathbb{I}_A\,\mathbb{E}^*}\right)\varepsilon\,\Xi_1\,\mathbb{V}^*\,\mathbb{I}_A^*-\left(\frac{\mathbb{I}_S^*\,\mathbb{E}}{\mathbb{I}_S\,\mathbb{E}^*}\right)\Xi_2\,\mathbb{S}^*\,\mathbb{I}_S^*-\left(\frac{\mathbb{I}_S^*\,\mathbb{E}}{\mathbb{I}_S\,\mathbb{E}^*}\right)\varepsilon\,\Xi_2\,\mathbb{V}^*\,\mathbb{I}_S^*\\
&+2\,\Xi_1\,\mathbb{S}^*\,\mathbb{I}_A^*+2\,\varepsilon\,\Xi_1\,\mathbb{V}^*\,\mathbb{I}_A^*+2\,\Xi_2\,\mathbb{S}^*\,\mathbb{I}_S^*+2\,\varepsilon\,\Xi_2\,\mathbb{V}^*\,\mathbb{I}_S^*-\left(1-\frac{\mathbb{V}^*}{\mathbb{V}}\right)(\iota+\varrho)\,\mathbb{V}+\left(1-\frac{\mathbb{V}^*}{\mathbb{V}}\right)\zeta\,\mathbb{S}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{d}{dt}\bar{\mathcal{C}} \;\leq\;& -(\zeta+\iota)\,\frac{(\mathbb{S}^*-\mathbb{S})^2}{\mathbb{S}}+\Xi_1\,\mathbb{S}^*\mathbb{I}_A^*\left(3-\frac{\mathbb{S}^*}{\mathbb{S}}-\frac{\mathbb{S}\,\mathbb{I}_A\,\mathbb{E}^*}{\mathbb{S}^*\,\mathbb{I}_A^*\,\mathbb{E}}-\frac{\mathbb{E}\,\mathbb{I}_A^*}{\mathbb{E}^*\,\mathbb{I}_A}\right)-(\sigma+\iota)\,\mathbb{E}\\
&+\Xi_2\,\mathbb{S}^*\mathbb{I}_S^*\left(3-\frac{\mathbb{S}^*}{\mathbb{S}}-\frac{\mathbb{S}\,\mathbb{I}_S\,\mathbb{E}^*}{\mathbb{S}^*\,\mathbb{I}_S^*\,\mathbb{E}}-\frac{\mathbb{E}\,\mathbb{I}_S^*}{\mathbb{E}^*\,\mathbb{I}_S}\right)+(\sigma+\iota)\,\mathbb{E}+\varepsilon\,\Xi_1\,\mathbb{V}^*\mathbb{I}_A^*\left(2-\frac{\mathbb{V}\,\mathbb{I}_A\,\mathbb{E}^*}{\mathbb{V}^*\,\mathbb{I}_A^*\,\mathbb{E}}-\frac{\mathbb{E}\,\mathbb{I}_A^*}{\mathbb{E}^*\,\mathbb{I}_A}\right)
\end{aligned}
$$

$$+\varepsilon\,\Xi_2\,\mathbb{V}^*\mathbb{I}_S^*\left(2-\frac{\mathbb{V}\,\mathbb{I}_S\,\mathbb{E}^*}{\mathbb{V}^*\,\mathbb{I}_S^*\,\mathbb{E}}-\frac{\mathbb{E}\,\mathbb{I}_S^*}{\mathbb{E}^*\,\mathbb{I}_S}\right)-\left(1-\frac{\mathbb{V}^*}{\mathbb{V}}\right)(\iota+\varrho)\,\mathbb{V}+\left(1-\frac{\mathbb{V}^*}{\mathbb{V}}\right)\zeta\,\mathbb{S}.$$

Then

$$\begin{aligned}\frac{d}{dt}\bar{C} \;\le\;& -(\zeta+\iota)\,\frac{(\mathbb{S}^*-\mathbb{S})^2}{\mathbb{S}}+\Xi_1\,\mathbb{S}^*\mathbb{I}_A^*\left(3-\frac{\mathbb{S}^*}{\mathbb{S}}-\frac{\mathbb{S}\,\mathbb{I}_A\,\mathbb{E}^*}{\mathbb{S}^*\,\mathbb{I}_A^*\,\mathbb{E}}-\frac{\mathbb{E}\,\mathbb{I}_A^*}{\mathbb{E}^*\,\mathbb{I}_A}\right)-(\sigma+\iota)\,\mathbb{E}\\ &+\Xi_2\,\mathbb{S}^*\mathbb{I}_S^*\left(3-\frac{\mathbb{S}^*}{\mathbb{S}}-\frac{\mathbb{S}\,\mathbb{I}_S\,\mathbb{E}^*}{\mathbb{S}^*\,\mathbb{I}_S^*\,\mathbb{E}}-\frac{\mathbb{E}\,\mathbb{I}_S^*}{\mathbb{E}^*\,\mathbb{I}_S}\right)+(\sigma+\iota)\,\mathbb{E}+\varepsilon\,\Xi_1\,\mathbb{V}^*\mathbb{I}_A^*\left(2-\frac{\mathbb{V}\,\mathbb{I}_A\,\mathbb{E}^*}{\mathbb{V}^*\,\mathbb{I}_A^*\,\mathbb{E}}-\frac{\mathbb{E}\,\mathbb{I}_A^*}{\mathbb{E}^*\,\mathbb{I}_A}\right)\\ &+\varepsilon\,\Xi_2\,\mathbb{V}^*\mathbb{I}_S^*\left(2-\frac{\mathbb{V}\,\mathbb{I}_S\,\mathbb{E}^*}{\mathbb{V}^*\,\mathbb{I}_S^*\,\mathbb{E}}-\frac{\mathbb{E}\,\mathbb{I}_S^*}{\mathbb{E}^*\,\mathbb{I}_S}\right)-\left(1-\frac{\mathbb{V}^*}{\mathbb{V}}\right)(\iota+\varrho)\,\mathbb{V}+\left(1-\frac{\mathbb{V}^*}{\mathbb{V}}\right)\zeta\,\mathbb{S}\\ &+\left(1-\frac{\mathbb{V}^*}{\mathbb{V}}\right)(\varepsilon\,\Xi_1\,\mathbb{V}^*\,\mathbb{I}_A^*+\varepsilon\,\Xi_2\,\mathbb{V}^*\,\mathbb{I}_S^*)-\left(1-\frac{\mathbb{V}^*}{\mathbb{V}}\right)(\varepsilon\,\Xi_1\,\mathbb{V}^*\,\mathbb{I}_A^*+\varepsilon\,\Xi_2\,\mathbb{V}^*\,\mathbb{I}_S^*).\end{aligned}$$

We obtain

$$\begin{aligned}\frac{d}{dt}\bar{C} \;\le\;& -(\zeta+\iota)\,\frac{(\mathbb{S}^*-\mathbb{S})^2}{\mathbb{S}}+\Xi_1\,\mathbb{S}^*\mathbb{I}_A^*\left(3-\frac{\mathbb{S}^*}{\mathbb{S}}-\frac{\mathbb{S}\,\mathbb{I}_A\,\mathbb{E}^*}{\mathbb{S}^*\,\mathbb{I}_A^*\,\mathbb{E}}-\frac{\mathbb{E}\,\mathbb{I}_A^*}{\mathbb{E}^*\,\mathbb{I}_A}\right)-(\sigma+\iota)\,\mathbb{E}\\ &+\Xi_2\,\mathbb{S}^*\mathbb{I}_S^*\left(3-\frac{\mathbb{S}^*}{\mathbb{S}}-\frac{\mathbb{S}\,\mathbb{I}_S\,\mathbb{E}^*}{\mathbb{S}^*\,\mathbb{I}_S^*\,\mathbb{E}}-\frac{\mathbb{E}\,\mathbb{I}_S^*}{\mathbb{E}^*\,\mathbb{I}_S}\right)+(\sigma+\iota)\,\mathbb{E}+\varepsilon\,\Xi_1\,\mathbb{V}^*\mathbb{I}_A^*\left(3-\frac{\mathbb{V}^*}{\mathbb{V}}-\frac{\mathbb{V}\,\mathbb{I}_A\,\mathbb{E}^*}{\mathbb{V}^*\,\mathbb{I}_A^*\,\mathbb{E}}-\frac{\mathbb{E}\,\mathbb{I}_A^*}{\mathbb{E}^*\,\mathbb{I}_A}\right)\\ &+\varepsilon\,\Xi_2\,\mathbb{V}^*\mathbb{I}_S^*\left(3-\frac{\mathbb{V}^*}{\mathbb{V}}-\frac{\mathbb{V}\,\mathbb{I}_S\,\mathbb{E}^*}{\mathbb{V}^*\,\mathbb{I}_S^*\,\mathbb{E}}-\frac{\mathbb{E}\,\mathbb{I}_S^*}{\mathbb{E}^*\,\mathbb{I}_S}\right)-\left(1-\frac{\mathbb{V}^*}{\mathbb{V}}\right)(\iota+\varrho)\,\mathbb{V}+\left(1-\frac{\mathbb{V}^*}{\mathbb{V}}\right)\zeta\,\mathbb{S}\\ &-\left(1-\frac{\mathbb{V}^*}{\mathbb{V}}\right)(\zeta\,\mathbb{S}^*-(\iota+\varrho)\,\mathbb{V}^*).\end{aligned}$$

Thus

$$\begin{aligned}\frac{d}{dt}\bar{C} \;\le\;& -\iota\,\frac{(\mathbb{S}^*-\mathbb{S})^2}{\mathbb{S}}+\Xi_1\,\mathbb{S}^*\mathbb{I}_A^*\left(3-\frac{\mathbb{S}^*}{\mathbb{S}}-\frac{\mathbb{S}\,\mathbb{I}_A\,\mathbb{E}^*}{\mathbb{S}^*\,\mathbb{I}_A^*\,\mathbb{E}}-\frac{\mathbb{E}\,\mathbb{I}_A^*}{\mathbb{E}^*\,\mathbb{I}_A}\right)-(\sigma+\iota)\,\mathbb{E}\\ &+\Xi_2\,\mathbb{S}^*\mathbb{I}_S^*\left(3-\frac{\mathbb{S}^*}{\mathbb{S}}-\frac{\mathbb{S}\,\mathbb{I}_S\,\mathbb{E}^*}{\mathbb{S}^*\,\mathbb{I}_S^*\,\mathbb{E}}-\frac{\mathbb{E}\,\mathbb{I}_S^*}{\mathbb{E}^*\,\mathbb{I}_S}\right)+(\sigma+\iota)\,\mathbb{E}+\varepsilon\,\Xi_1\,\mathbb{V}^*\mathbb{I}_A^*\left(3-\frac{\mathbb{V}^*}{\mathbb{V}}-\frac{\mathbb{V}\,\mathbb{I}_A\,\mathbb{E}^*}{\mathbb{V}^*\,\mathbb{I}_A^*\,\mathbb{E}}-\frac{\mathbb{E}\,\mathbb{I}_A^*}{\mathbb{E}^*\,\mathbb{I}_A}\right)\\ &+\varepsilon\,\Xi_2\,\mathbb{V}^*\mathbb{I}_S^*\left(3-\frac{\mathbb{V}^*}{\mathbb{V}}-\frac{\mathbb{V}\,\mathbb{I}_S\,\mathbb{E}^*}{\mathbb{V}^*\,\mathbb{I}_S^*\,\mathbb{E}}-\frac{\mathbb{E}\,\mathbb{I}_S^*}{\mathbb{E}^*\,\mathbb{I}_S}\right)-(\iota+\varrho)\frac{(\mathbb{V}^*-\mathbb{V})^2}{\mathbb{V}}+\zeta\,\mathbb{S}^*\left(1-\frac{\mathbb{S}^*}{\mathbb{S}}-\frac{\mathbb{S}\,\mathbb{V}^*}{\mathbb{S}^*\mathbb{V}}+\frac{\mathbb{V}^*}{\mathbb{V}}\right).\end{aligned}$$

Therefore

$$\begin{aligned}\frac{d}{dt}\bar{C} \;\le\;& -\iota\,\frac{(\mathbb{S}^*-\mathbb{S})^2}{\mathbb{S}}+\Xi_1\,\mathbb{S}^*\mathbb{I}_A^*\left(3-\frac{\mathbb{S}^*}{\mathbb{S}}-\frac{\mathbb{S}\,\mathbb{I}_A\,\mathbb{E}^*}{\mathbb{S}^*\,\mathbb{I}_A^*\,\mathbb{E}}-\frac{\mathbb{E}\,\mathbb{I}_A^*}{\mathbb{E}^*\,\mathbb{I}_A}\right)-(\sigma+\iota)\,\mathbb{E}\\ &+\Xi_2\,\mathbb{S}^*\mathbb{I}_S^*\left(3-\frac{\mathbb{S}^*}{\mathbb{S}}-\frac{\mathbb{S}\,\mathbb{I}_S\,\mathbb{E}^*}{\mathbb{S}^*\,\mathbb{I}_S^*\,\mathbb{E}}-\frac{\mathbb{E}\,\mathbb{I}_S^*}{\mathbb{E}^*\,\mathbb{I}_S}\right)+(\sigma+\iota)\,\mathbb{E}+\varepsilon\,\Xi_1\,\mathbb{V}^*\mathbb{I}_A^*\left(3-\frac{\mathbb{V}^*}{\mathbb{V}}-\frac{\mathbb{V}\,\mathbb{I}_A\,\mathbb{E}^*}{\mathbb{V}^*\,\mathbb{I}_A^*\,\mathbb{E}}-\frac{\mathbb{E}\,\mathbb{I}_A^*}{\mathbb{E}^*\,\mathbb{I}_A}\right)\\ &+\varepsilon\,\Xi_2\,\mathbb{V}^*\mathbb{I}_S^*\left(3-\frac{\mathbb{V}^*}{\mathbb{V}}-\frac{\mathbb{V}\,\mathbb{I}_S\,\mathbb{E}^*}{\mathbb{V}^*\,\mathbb{I}_S^*\,\mathbb{E}}-\frac{\mathbb{E}\,\mathbb{I}_S^*}{\mathbb{E}^*\,\mathbb{I}_S}\right)-(\iota+\varrho)\frac{(\mathbb{V}^*-\mathbb{V})^2}{\mathbb{V}}\\ &+(\varepsilon\,\Xi_1\,\mathbb{V}^*\,\mathbb{I}_A^*+\varepsilon\,\Xi_2\,\mathbb{V}^*\,\mathbb{I}_S^*+(\iota+\varrho)\,\mathbb{V}^*)\left(1-\frac{\mathbb{S}^*}{\mathbb{S}}-\frac{\mathbb{S}\,\mathbb{V}^*}{\mathbb{S}^*\mathbb{V}}+\frac{\mathbb{V}^*}{\mathbb{V}}\right).\end{aligned}$$

Finally, we have

$$\begin{aligned}\frac{d}{dt}\bar{C} \;\leq\; & -\iota\,\frac{(\mathbb{S}^*-\mathbb{S})^2}{\mathbb{S}}+\Xi_1\,\mathbb{S}^*\mathbb{I}_A^*\left(3-\frac{\mathbb{S}^*}{\mathbb{S}}-\frac{\mathbb{S}\,\mathbb{I}_A\,\mathbb{E}^*}{\mathbb{S}^*\,\mathbb{I}_A^*\,\mathbb{E}}-\frac{\mathbb{E}\,\mathbb{I}_A^*}{\mathbb{E}^*\,\mathbb{I}_A}\right)+\Xi_2\,\mathbb{S}^*\mathbb{I}_S^*\left(3-\frac{\mathbb{S}^*}{\mathbb{S}}-\frac{\mathbb{S}\,\mathbb{I}_S\,\mathbb{E}^*}{\mathbb{S}^*\,\mathbb{I}_S^*\,\mathbb{E}}-\frac{\mathbb{E}\,\mathbb{I}_S^*}{\mathbb{E}^*\,\mathbb{I}_S}\right)\\ & +\varepsilon\,\Xi_1\,\mathbb{V}^*\mathbb{I}_A^*\left(4-\frac{\mathbb{S}^*}{\mathbb{S}}-\frac{\mathbb{S}\,\mathbb{V}^*}{\mathbb{S}^*\,\mathbb{V}}-\frac{\mathbb{V}\,\mathbb{I}_A\,\mathbb{E}^*}{\mathbb{V}^*\,\mathbb{I}_A^*\,\mathbb{E}}-\frac{\mathbb{E}\,\mathbb{I}_A^*}{\mathbb{E}^*\,\mathbb{I}_A}\right)+(\iota+\varrho)\mathbb{V}^*\left(3-\frac{\mathbb{S}^*}{\mathbb{S}}-\frac{\mathbb{S}\,\mathbb{V}^*}{\mathbb{S}^*\mathbb{V}}-\frac{\mathbb{V}}{\mathbb{V}^*}\right)\\ & +\varepsilon\,\Xi_2\,\mathbb{V}^*\mathbb{I}_S^*\left(4-\frac{\mathbb{S}^*}{\mathbb{S}}-\frac{\mathbb{S}\,\mathbb{V}^*}{\mathbb{S}^*\,\mathbb{V}}-\frac{\mathbb{V}\,\mathbb{I}_S\,\mathbb{E}^*}{\mathbb{V}^*\,\mathbb{I}_S^*\,\mathbb{E}}-\frac{\mathbb{E}\,\mathbb{I}_S^*}{\mathbb{E}^*\,\mathbb{I}_S}\right).\end{aligned}$$

Hence, from the geometrical and arithmetical means relationship, we find

$$\begin{aligned}3 &\leq \frac{\mathbb{S}^*}{\mathbb{S}}+\frac{\mathbb{S}\,\mathbb{I}_A\,\mathbb{E}^*}{\mathbb{S}^*\,\mathbb{I}_A^*\,\mathbb{E}}+\frac{\mathbb{E}\,\mathbb{I}_A^*}{\mathbb{E}^*\,\mathbb{I}_A},\\ 3 &\leq \frac{\mathbb{S}^*}{\mathbb{S}}+\frac{\mathbb{S}\,\mathbb{I}_S\,\mathbb{E}^*}{\mathbb{S}^*\,\mathbb{I}_S^*\,\mathbb{E}}+\frac{\mathbb{E}\,\mathbb{I}_S^*}{\mathbb{E}^*\,\mathbb{I}_S},\\ 4 &\leq \frac{\mathbb{S}^*}{\mathbb{S}}+\frac{\mathbb{S}\,\mathbb{V}^*}{\mathbb{S}^*\,\mathbb{V}}+\frac{\mathbb{V}\,\mathbb{I}_A\,\mathbb{E}^*}{\mathbb{V}^*\,\mathbb{I}_A^*\,\mathbb{E}}+\frac{\mathbb{E}\,\mathbb{I}_A^*}{\mathbb{E}^*\,\mathbb{I}_A},\\ 4 &\leq \frac{\mathbb{S}^*}{\mathbb{S}}+\frac{\mathbb{S}\,\mathbb{V}^*}{\mathbb{S}^*\,\mathbb{V}}+\frac{\mathbb{V}\,\mathbb{I}_S\,\mathbb{E}^*}{\mathbb{V}^*\,\mathbb{I}_S^*\,\mathbb{E}}+\frac{\mathbb{E}\,\mathbb{I}_S^*}{\mathbb{E}^*\,\mathbb{I}_S}.\end{aligned}$$

Accordingly, $\frac{d}{dt}\bar{C} \leq 0$ if $\Re_0 > 1$ for all $\mathbb{S},\mathbb{E},\mathbb{I}_A,\mathbb{I}_S,\mathbb{V} > 0$. Moreover, $\frac{d}{dt}\bar{C} = 0$ if and only if $\mathbb{S} = \mathbb{S}^*$, $\mathbb{E} = \mathbb{E}^*$, $\mathbb{I}_A = \mathbb{I}_A^*$, $\mathbb{I}_S = \mathbb{I}_S^*$, and $\mathbb{V} = \mathbb{V}^*$. Assume that $\bar{\mathcal{D}}'$ is the largest subset of $\bar{\mathcal{D}} = \{(\mathbb{S},\mathbb{E},\mathbb{I}_A,\mathbb{I}_S,\mathbb{V}) : \frac{d}{dt}\bar{C} = 0\}$. Then, $\bar{\mathcal{D}}' = \{\mathcal{U}^*\}$. Hence, from La Salle's invariant principle [19], the endemic equilibrium point $\mathcal{U}^*$ is $\mathcal{GAS}$ when $\Re_0(\nu) > 1$. □