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Research article

Modeling bounded data with a new unit distribution: regression analysis and applications

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Abstract: This study proposes a new flexible unit distribution derived from the generalized exponential geometric model, which is intended to model data confined to the unit interval $(0, 1)$. The introduced distribution density shapes are extremely versatile, accommodating right-skewed, left-skewed, approximately symmetric, decreasing, U-shaped, increasing, and J-shaped. Furthermore, its hazard rate function is adaptable to bathtub, U-shaped, increasing, and J-shaped shapes, making it applicable to a wide range of real-world datasets. Essential statistical properties, such as moments, quantiles, and entropy measures, are closely derived and analyzed. Based on this distribution, a new regression model is developed to link bounded response variables to linear predictors, increasing its practical applicability. The maximum likelihood approach is used to estimate the parameters of the regression model and the suggested distribution. The performance of the maximum likelihood approach based on the suggested distribution and regression model is examined using a Monte Carlo simulation. The applicability of the regression model and the new distribution is demonstrated through real data analysis. The proposed distribution exhibits strong modeling capabilities for bounded data in the unit interval, making it highly applicable in fields such as reliability analysis, survival studies, and modeling of proportions or rates. Its superior performance over existing models, as demonstrated through simulation studies and real data applications, highlights its potential as a practical and flexible tool for applied statisticians.

Keywords: generalized exponential geometric; entropy; exponential transformation; simulation; quantile regression
Mathematics Subject Classification: 60E05, 60E99, 62E15

1. Introduction

In order to describe the uncertainty of a bounded phenomenon, such as the proportion of a certain characteristic, a continuous distribution with a bounded domain is required. The statistics literature has extremely few distributions specified on $(0, 1)$. The beta distribution is the well-known statistical distribution for modeling data sets on the interval $(0, 1)$. Despite this, its ability to model data may not be sufficient to model leptokurtic and strongly left-skewed data sets. So, in literature, several distributions have been proposed to improve the modeling accuracy of the data sets on $(0, 1)$. Amongst these distributions are the following: Kumaraswamy [1], unit inverse Gaussian [2], cosine-sine [3], power logarithmic [4], unit-Rayleigh [5], unit-Gompertz [6], log-weighted exponential [7], and the unit-Weibull [8]. As was previously established, data on $(0, 1)$ could arise from data scaling/transformation. In a comparable manner, unit distributions can be created via random variable transformation. By way of illustration, if the random variable Y is defined in $(0, \infty)$, then a unit distribution may be produced by the random variable $X = e^{-Y}$. The beta regression model is a well-known regression model that may be applied when the response variable is between 0 and 1. Many regression models have been proposed as an alternative to the beta regression model in recent years. Several of these distributions involve log-weighted exponential [9], unit-improved second-degree Lindley [10], unit inverse Gaussian [11], log-exponential-power [12], unit log-log [13], unit Weibull [14], unit folded normal [15], and one-parameter unit-Lindley distributions [16]. Although numerous constrained distributions exist in the literature, no single distribution is appropriate for describing diverse datasets. Therefore, we develop a new bounded distribution known as the unit generalized exponential geometric (UGEG) distribution and quantile regression modeling for it, aiming to model data on the unit intervals. Although the UGEG distribution is defined on the unit interval $(0, 1)$, this bounded support does not inherently limit its practical utility. In contrast, many important types of real-world data are naturally restricted to this interval. Examples include probabilities, proportions, normalized scores, biomedical rates (e.g., infection rate, body fat percentage), and bounded performance indices. These data types are frequently encountered in fields such as reliability engineering, survival analysis, biostatistics, economics, and quality control. Therefore, developing flexible models defined on $(0, 1)$, such as the UGEG distribution, addresses a common and important class of modeling problems. Moreover, the UGEG distribution can be extended to model data defined over any bounded interval (a, b) through a simple transformation:

$$X_{(a,b)} = a + (b - a)X, \quad \text{where } X \sim \text{UGEG distribution.}$$

This transformation preserves the statistical structure while extending its applicability to other bound domains.

The following are the motivations for establishing new models:

- Compared to the standard generalized exponential distribution, our proposed unit generalized exponential distribution will have more varied hazard rates and probability density shapes in terms of data modeling.
- A closed form for the UGEG distribution's quantile function may be found. As a result, it is simple to re-parameterize its probability density function (PDF) and cumulative distribution function (CDF) with regard to any quantiles.

- Our model could potentially be used to generate well-known distributions with a range of parameters and supports, including those that are power function, uniform, Kumaraswamy, a four-parameter upper bound variant of the generalized exponential, and generalized exponential distributions.
- Develop a bounded distribution that is capable of modeling data with right-skewed, approximately symmetric, left-skewed, decreasing, U, increasing, and J-shapes, and bathtub, U, J, and increasing hazard rate functions.

Recent contributions have focused on flexible modeling approaches for complex data structures. For example, Ozcalci and Kaya [17] proposed a hybrid model for forecasting web-scraped microdata, highlighting the importance of adaptable models in practical applications. In line with such advancements, the proposed UGEG distribution offers enhanced flexibility for modeling bounded data with varying hazard shapes.

The rest of the paper is organized as follows: The new distribution is proposed in Section 2. In Section 3, statistical properties of the new model are examined. Some entropies for the new distribution are investigated in Section 4. In Section 5, several estimation methods are studied for unknown parameters of the distribution, and a Monte Carlo simulation is conducted in Section 6. A real data analysis is performed to observe the capability of the new model in Section 7. In Section 8, a new quantile regression model is proposed, and some numerical and practical analyses are achieved. The article ends with a conclusion in Section 9.

2. Unit generalized exponential geometric distribution

In this part, we provide a novel bounded model generated by exponential transformation from the generalized exponential geometric (GEG) distribution proposed by Silva et al. [18]. Using the GEG distribution whose CDF is given by

$$F_{GEG}(y) = \left(\frac{1 - e^{-\beta y}}{1 - pe^{-\beta y}}\right)^{\alpha}, \ y > 0, \beta > 0, p \in (0, 1). \tag{2.1}$$

Then, we can look at our distribution as a result of the GEG distribution by using the following transformation: $X = e^{-Y}$.

Hence, the CDF of the unit generalized exponential geometric (UGEG) distribution is given by

$$F(x) = 1 - \left(\frac{x^{\beta} - 1}{px^{\beta} - 1}\right)^{\alpha}, \ 0 < x < 1, \beta > 0, \alpha > 0, 0 < p < 1, \tag{2.2}$$

and its corresponding PDF is given by

$$f(x) = \alpha\beta(1 - p)x^{\beta-1}\left(1 - x^{\beta}\right)^{\alpha-1}\left(1 - px^{\beta}\right)^{-(\alpha+1)}. \tag{2.3}$$

Its hazard rate function (HRF) is defined as follows:

$$h(x) = \frac{\alpha\beta(1 - p)x^{\beta-1}}{(1 - x^{\beta})(1 - px^{\beta})}. \tag{2.4}$$

Its quantile function is defined as follows:

$$Q(u) = \left[\frac{(1-u)^{1/\alpha} - 1}{p(1-u)^{1/\alpha} - 1}\right]^{1/\beta}, \quad 0 < u < 1, \tag{2.5}$$

Different plots of the PDF of the UGEG model are displayed in Figure 1 for several parameter values, which showed that our proposed model PDF can be right-skewed, approximately symmetric, left-skewed, decreasing, U, J-shaped, and increasing.

Our proposed model HRF can be a bathtub, U, J-shaped or an increasing function by different parameter values as shown in Figure 2. The flexibility of the HRF of the UGEG distribution, which includes increasing, decreasing, bathtub, and notably U-shaped forms, broadens its applicability in reliability and survival analysis. In particular, the U-shaped hazard rate makes the model especially suitable for scenarios involving early-life failures followed by a stable period and eventual wear-out failures commonly encountered in biological systems and mechanical components.

- It is noteworthy to note that the power function distribution matches the special case from Eq (2.3) for $\alpha = 1$ and $p = 0$, indicating the uniform distribution for $\beta = 1$.
- Further, under X having the UGEG model, and $p = 0$, Eq (2.3) reduces to the Kumaraswamy distribution.
- If we take $Y = \lambda X, \lambda > 0$ in Eq (2.3), we obtain a four-parameter upper bound variant of the unit generalized exponential geometric (4-UGEG) distribution with PDF:

$$f(y) = \frac{\alpha\beta(1-p)}{\lambda}\left(\frac{y}{\lambda}\right)^{\beta-1}\left[1-\left(\frac{y}{\lambda}\right)^{\beta}\right]^{\alpha-1}\left[1-p\left(\frac{y}{\lambda}\right)^{\beta}\right]^{-(\alpha+1)}, \; 0 < y < \lambda, \;\; \alpha,\beta,\lambda > 0, \;\; 0 < p < 1.$$

- Actually, the generalized exponential distribution proposed by Gupta and Kundu [19] is a particular case of Eq (2.3), for $p = 0$ and the transformation $Y = -\log(X)$.

Throughout the rest of the paper, the definitions and results stated below will be mentioned.

i. Regularized hypergeometric function is defined by

$${}_p\tilde{\mathrm{F}}_q(a_1,\ldots,a_p;b_1,\ldots,b_p;z) = \frac{{}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_p;z)}{\Gamma(b_1)\ldots\Gamma(b_q)},$$

where ${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_p;z)$ is a generalized hypergeometric function.

ii. The Appell hypergeometric function of two variables is defined as follows:

$$F_1(a,b_1,b_2;c;x,y) = \sum_{k,m=0}^{\infty}\frac{(a)_{k+m}(b_1)_k(b_2)_m}{(c)_{k+m}k!m!}x^k y^m.$$

iii. For $z > 0$, $n = 1,2,\ldots$, we define

$$\zeta(z;n,\alpha,\beta,p) = \int_0^z x^n f(x;\alpha,\beta,p)dx = \frac{\alpha\beta(1-p)}{n+\beta}z^{n+\beta}F_1\left(\frac{n+\beta}{\beta}, 1-\alpha, 1+\alpha; 2+\frac{n}{\beta}; z^{\beta}; pz^{\beta}\right). \tag{2.6}$$

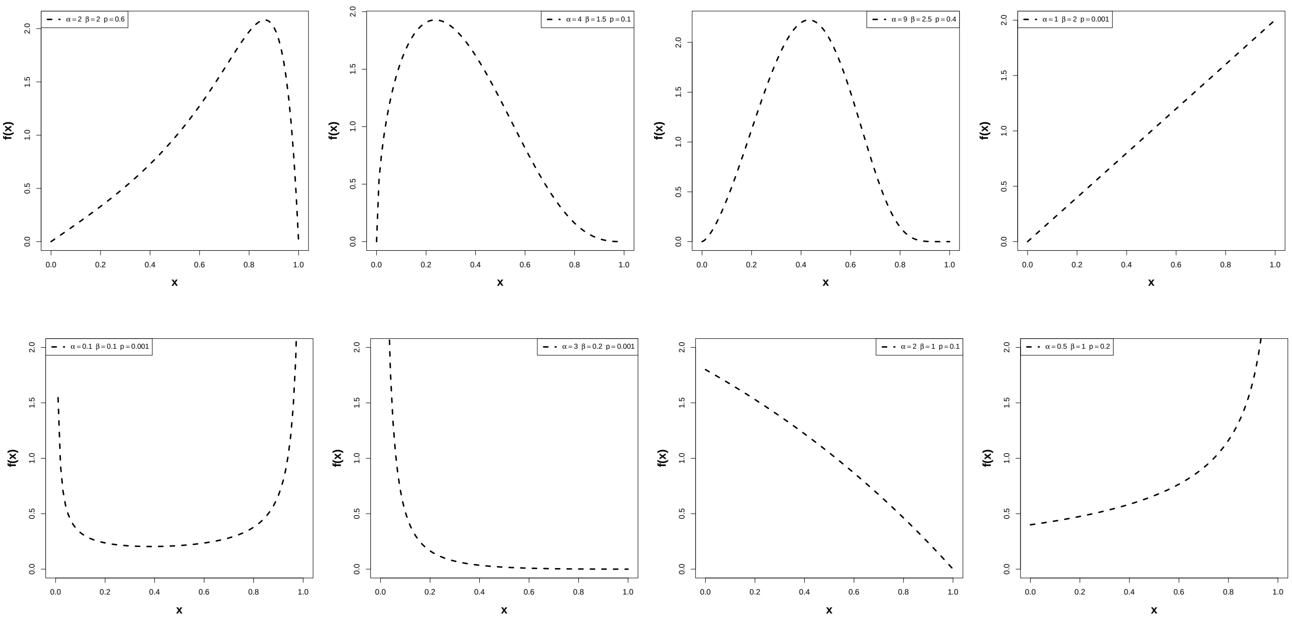

Figure 1. Plots of PDF of UGEG distribution.

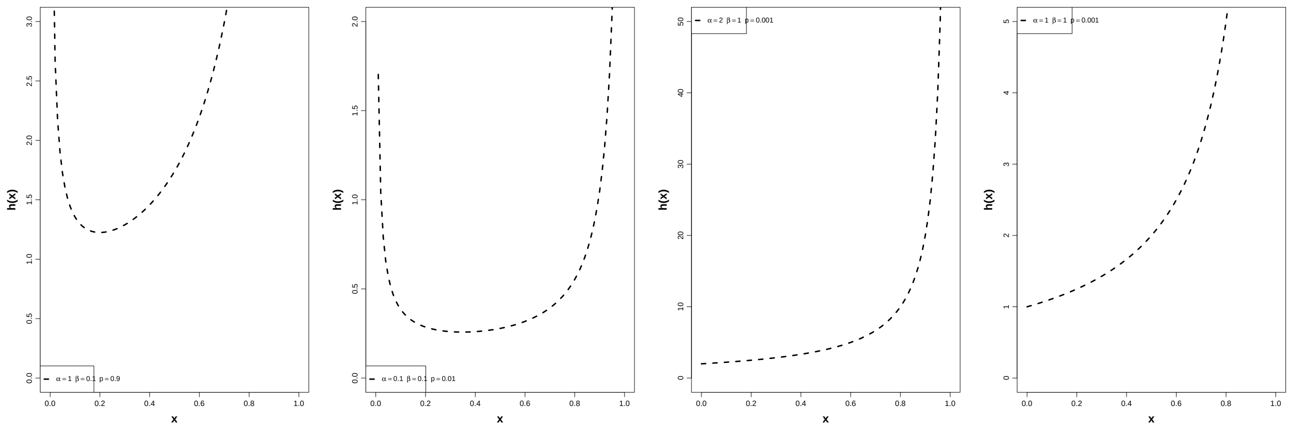

Figure 2. Plots of HRF of UGEG distribution.

3. Statistical properties

This section discusses some statistical features of the UGEG distribution, which include moments, incomplete moments, mean deviation, Bonferroni, and Lorenz curves.

3.1. Moments

The rth moment about the original of X with UGEG distribution is given, for $r = 1, 2, \ldots,$ by

$$\begin{aligned}\mu_r = E(x^r) &= \int_0^1 \alpha\beta(1-p)x^{\beta+r-1}\left(1-x^{\beta}\right)^{\alpha-1}\left(1-px^{\beta}\right)^{-(\alpha+1)}dx \\ &= \alpha(1-p)\Gamma(\alpha)\Gamma\left(\frac{r+\beta}{\beta}\right) {}_2\tilde{F}_1\left(\alpha+1, \frac{r+\beta}{\beta}; \frac{r}{\beta}+\alpha+1; p\right). \end{aligned} \tag{3.1}$$

Remark 1. *Based on Eq* (3.1)*, the mean of X can be expressed as*

$$E(x) = \frac{\alpha}{\beta}(1-p)\Gamma(\alpha)\Gamma\left(\frac{1}{\beta}\right) {}_2\tilde{F}_1\left(\alpha+1, 1+\frac{1}{\beta}; \alpha+\frac{1}{\beta}+1; p\right),$$

and the moment of order 2 of the UGEG distribution can be obtained as

$$E(x^2) = \alpha(1-p)\Gamma(\alpha)\Gamma\left(\frac{\beta+2}{\beta}\right) {}_2\tilde{F}_1\left(\alpha+1, \frac{\beta+2}{\beta}; \alpha+\frac{2}{\beta}+1; p\right).$$

From the above moments, we can obtain the variance of the UGEG distribution.

3.2. Incomplete moments

The n^{th} incomplete moment of a random variable X with the UGEG model is given, for $n = 1, 2, \ldots$, and $y > 0$, by

$$m_n(y) = \int_0^y x^n f(x; \alpha, \beta, p)dx = \alpha\beta(1-p)\int_0^y x^{\beta+n-1}\left(1-x^{\beta}\right)^{\alpha-1}\left(1-px^{\beta}\right)^{-(\alpha+1)} dx. \tag{3.2}$$

Based on Eq (2.6), the n^{th} incomplete moments can be reduced to

$$m_n(y) = \frac{\alpha\beta(1-p)}{n+\beta} y^{n+\beta} F_1\left(\frac{n+\beta}{\beta}, 1-\alpha, 1+\alpha; 2+\frac{n}{\beta}; y^{\beta}; py^{\beta}\right). \tag{3.3}$$

An essential application of the incomplete moments refers to the mean deviations about the mean and the median. Additionally, it can be used to represent the Bonferroni and Lorenz curves.

3.3. Mean deviations

The mean deviation of X about the mean μ and about the median M can be represented as

$$\eta_1 = 2\mu F(\mu) - 2m_1(\mu), \; \eta_2 = \mu - 2m_1(M),$$

respectively. For the UGEG model, $F(.)$ is easily obtained by Eq (2.2), μ can be obtained from (3.1), and the median (M) of X follows (2.5) as $M = Q(1/2)$ and $m_1(.)$ given by Eq (3.3) with $n = 1$.

3.4. Bonferroni and Lorenz curves

The Bonferroni and Lorenz curves are widely used in applied fields to analyze inequality or concentration phenomena. For instance, in economics, they are fundamental tools to measure income or wealth inequality across populations. In insurance and risk management, these curves can be used to evaluate risk concentration, helping actuaries assess whether a few large claims dominate total losses. Moreover, in marketing, Lorenz curves can illustrate market share distributions and customer segmentation, guiding strategies for targeting high-value segments. The Bonferroni and Lorenz curves are defined by

$$B(\rho) = \frac{1}{\rho\mu}\int_0^q xf(x)dx, \; L(\rho) = \frac{1}{\mu}\int_0^q xf(x)dx,$$

respectively, where $0 \leq \rho \leq 1$ and $q = Q(\rho; \alpha, \beta, p)$.

For the UGEG model, based on Eq (2.6), the Bonferroni and Lorenz curves are written as follows:

$$B(\rho) = \frac{1}{\rho\mu} \frac{\alpha\beta(1-p)}{1+\beta} q^{1+\beta} F_1 \left(\frac{1+\beta}{\beta}, 1-\alpha, 1+\alpha; 2+\frac{1}{\beta}; q^{\beta}; pq^{\beta} \right),$$

and

$$L(\rho) = \frac{1}{\mu} \frac{\alpha\beta(1-p)}{1+\beta} q^{1+\beta} F_1 \left(\frac{1+\beta}{\beta}, 1-\alpha, 1+\alpha; 2+\frac{1}{\beta}; q^{\beta}; pq^{\beta} \right),$$

respectively, where $q = Q(\rho; \alpha, \beta, p)$ is easily obtained by Eq (2.5) and μ can be obtained from (3.1).

4. Entropies

Entropy measures have diverse practical applications beyond pure probability and statistics. In information theory, entropy quantifies the uncertainty or information content of communication signals, playing a crucial role in data compression and transmission efficiency. In finance, entropy-based approaches are used to design diversified portfolios by minimizing concentration risk. In ecological studies, entropy metrics help measure biodiversity by assessing the evenness and richness of species distributions within ecosystems. In the distribution, there are several types of entropy. This section explores the Rényi entropy [20], the Mathi-Haubold entropy [21], and Shannon entropy [22], three of the relevant entropy measurements. In the setting of UGEG distribution, the Rényi entropy, the Mathi-Haubold entropy, and the Shannon entropy are defined as

$$R_\gamma = \frac{1}{1-\gamma} \log \left(\int_R f^{\gamma}(x; \alpha, \beta, p) dx \right), \ \gamma > 0, \gamma \neq 1, \tag{4.1}$$

$$J_{MH} = \frac{1}{\delta - 1} \left(\int_R f(x; \alpha, \beta, p)^{2-\delta} dx - 1 \right), \delta \neq 1, \delta < 2, \tag{4.2}$$

and

$$SH = -\int_R f(x; \alpha, \beta, p) \log(f(x; \alpha, \beta, p)) dx, \tag{4.3}$$

respectively. The next three propositions provide formal expressions of these entropies for the UGEG distribution.

Proposition 1. *The Rényi entropy of the UGEG model can be written*

$$\begin{aligned} R_\gamma &= \frac{\gamma}{1-\gamma} \log(\alpha\beta(1-p)) - \frac{1}{1-\gamma} \log(\beta) \\ &+ \frac{1}{1-\gamma} \log \left(\Gamma((\alpha-1)\gamma+1) \Gamma\left(\frac{(\beta-1)\gamma+1}{\beta} \right) {}_2\tilde{F}_1 \left((\alpha+1)\gamma, \frac{(\beta-1)\gamma+1}{\beta}; \frac{\alpha\gamma\beta+\beta-\gamma+1}{\beta}; p \right) \right). \end{aligned}$$

Proof. Using Eq (2.3), we have

$$\int_0^1 (f(x;\alpha,\beta,p))^\gamma dx = (\alpha\beta(1-p))^\gamma \int_0^1 x^{\gamma(\beta-1)}\left(1-x^\beta\right)^{\gamma(\alpha-1)}\left(1-px^\beta\right)^{-\gamma(\alpha+1)} dx = \frac{(1-p)^\gamma(\alpha\beta)^\gamma}{\beta}$$
$$\times\Gamma((\alpha-1)\gamma+1)\Gamma\left(\frac{(\beta-1)\gamma+1}{\beta}\right) {}_2\tilde{F}_1\left((\alpha+1)\gamma, \frac{(\beta-1)\gamma+1}{\beta}; \frac{\alpha\gamma\beta+\beta-\gamma+1}{\beta}; p\right).$$

By getting the logarithmic function of this expression and multiplying it by $1/(1-\gamma)$, we prove this proposition. □

Proposition 2. *The Mathai-Haubold entropy of the UGEG model can be expressed as*

$$J_{MH} = \frac{1}{\delta-1}\left[\frac{(\alpha\beta(1-p))^{2-\delta}}{\beta}\Gamma((\alpha-1)(2-\delta)+1)\Gamma\left(\frac{(\beta-1)(2-\delta)+1}{\beta}\right)\right.$$
$$\left.\times\ {}_2\tilde{F}_1\left((\alpha+1)(2-\delta), \frac{(\beta-1)(2-\delta)+1}{\beta}; \frac{\alpha(2-\delta)\beta+\beta-(2-\delta)+1}{\beta}; p\right)-1\right].$$

Proof. Proceeding as for R_γ with $(2-\delta)$ instead of γ, we have

$$\int_0^1 (f(x;\alpha,\beta,p))^{2-\delta} dx = \left[\frac{(\alpha\beta(1-p))^{2-\delta}}{\beta}\Gamma((\alpha-1)(2-\delta)+1)\Gamma\left(\frac{(\beta-1)(2-\delta)+1}{\beta}\right)\right.$$
$$\left.\times\ {}_2\tilde{F}_1\left((\alpha+1)(2-\delta), \frac{(\beta-1)(2-\delta)+1}{\beta}; \frac{\alpha(2-\delta)\beta+\beta-(2-\delta)+1}{\beta}; p\right)\right].$$

The proof is complete. □

Proposition 3. *The Shannon entropy of the UGEG model can be expressed as*

$$\begin{aligned}SH = &-\frac{(p-1)}{\beta}\left[{}_2F_1^{(0,0,1,0)}(1,\alpha+1,\alpha+1,p) + {}_2F_1^{(1,0,0,0)}(1,\alpha+1,\alpha+1,p)\right]\\ &+(p-1)\left[{}_2F_1^{(0,0,1,0)}(1,\alpha+1,\alpha+1,p) + {}_2F_1^{(1,0,0,0)}(1,\alpha+1,\alpha+1,p)\right]\\ &+H_\alpha\left(1+\frac{1}{\beta}\right) + \frac{(\alpha-1)(1-p)^{-\alpha}\,{}_2F_1\left(\alpha,\alpha;\alpha+1;\frac{p}{p-1}\right)}{\alpha} - p\,{}_2F_1(1,\alpha+1;\alpha+2;p) - \log(\alpha\beta(1-p)),\end{aligned}$$

where $H_\alpha = \sum_{i=1}^{\alpha}\frac{1}{i}$ *is a harmonic number,* ${}_2F_1^{(1,0,0,0)}(a,b;c;z) = \frac{\partial}{\partial a}{}_2F_1(a,b;c;z)$*, and* ${}_2F_1^{(0,0,1,0)}(a,b;c;z) = \frac{\partial}{\partial c}{}_2F_1(a,b;c;z)$*.*

Proof. By using Eqs (2.3) and (4.3), we have

$$SH = -\int_R \alpha\beta(1-p)x^{\beta-1}\left(1-x^\beta\right)^{\alpha-1}\left(1-px^\beta\right)^{-(\alpha+1)}\log(\alpha\beta(1-p)x^{\beta-1}\left(1-x^\beta\right)^{\alpha-1}\left(1-px^\beta\right)^{-(\alpha+1)})dx.$$

So,

$$I_1 = -\alpha\beta(1-p)\log(\alpha\beta(1-p))\int_0^1 \left(1-x^\beta\right)^{\alpha-1} x^{\beta-1}\left(1-px^\beta\right)^{-(\alpha+1)} dx = -\log(\alpha\beta(1-p)).$$

To simplify our calculations, we define the following transformation: $u = x^{\beta}$:

$$I_2 = -\alpha(1-p)\int_0^1 (1-u)^{\alpha-1}\log(u)(1-pu)^{-(\alpha+1)}du$$
$$= (p-1)\left({}_2F_1^{(0,0,1,0)}(1,\alpha+1,\alpha+1,p) + {}_2F_1^{(1,0,0,0)}(1,\alpha+1,\alpha+1,p)\right) + H_{\alpha},$$

and

$$I_3 = \frac{\alpha}{\beta}(1-p)\int_0^1 (1-u)^{\alpha-1}\log(u)(1-pu)^{-(\alpha+1)}du$$
$$= -\frac{(p-1)\left({}_2F_1^{(0,0,1,0)}(1,\alpha+1,\alpha+1,p) + {}_2F_1^{(1,0,0,0)}(1,\alpha+1,\alpha+1,p)\right) + H_{\alpha}}{\beta}.$$

Also,

$$I_4 = -\alpha(\alpha-1)(1-p)\int_0^1 (1-u)^{\alpha-1}\log(1-u)(1-pu)^{-(\alpha+1)}du$$
$$= -\frac{(\alpha-1)(p-1)(1-p)^{-\alpha-1}\,{}_2F_1\left(\alpha,\alpha;\alpha+1;\frac{p}{p-1}\right)}{\alpha}.$$

Finally,

$$I_5 = \alpha(\alpha+1)(1-p)\int_0^1 (1-u)^{\alpha-1}(1-pu)^{-(\alpha+1)}\log(1-pu)du = -p\,{}_2F_1(1,\alpha+1;\alpha+2;p).$$

After simplification and rearranging terms, the proof is complete. □

Remark 2. *It can be easily concluded that Shannon entropy is obtained by taking the limit of Rényi entropy when γ tends to 1, and negative Shannon entropy is obtained by taking the limit of Mathai-Haubold when δ tends to 1.*

5. Estimation methods

It will be discussed in this section how to use many traditional estimation methods to determine the estimators of the proposed model parameters α, β, and p that are produced by maximizing or minimizing an objective function.

The estimated parameters of our proposed model by the maximum likelihood estimation method (E1) are obtained by maximizing the log-likelihood function, which is our objective function in this case, and it is defined in the following equation:

$$l(\alpha,\beta,p) = n\log(\alpha\beta(1-p)) + (\beta-1)\sum_{i=1}^{n}\left(1-x_i^{\beta}\right) - (\alpha+1)\sum_{i=1}^{n}\left(1-px_i^{\beta}\right).$$

In the case of estimating our proposed model parameters by the Anderson-Darling estimation method (E2), our objective function is minimized, and it is defined as follows: $(x_{(1)} \le x_{(2)} \le \ldots \le x_{(n)})$

$$A(\alpha,\beta,p) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)[\log F(x_i) + \log S(x_i)]$$

$$= -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log\left(1-\left(\frac{x_i^{\beta}-1}{px_i^{\beta}-1}\right)^{\alpha}\right)+\log\left(\left(\frac{x_i^{\beta}-1}{px_i^{\beta}-1}\right)^{\alpha}\right)\right].$$

In the case of estimating our proposed model parameters by the Cramér-von Mises estimation method (E3), our objective function is minimized, and it is defined as follows: $(x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)})$

$$C(\alpha,\beta,p) = -\frac{1}{12n}+\sum_{i=1}^{n}\left[F(x_i)-\frac{2i-1}{2n}\right]^2 = -\frac{1}{12n}+\sum_{i=1}^{n}\left[1-\left(\frac{x_i^{\beta}-1}{px_i^{\beta}-1}\right)^{\alpha}-\frac{2i-1}{2n}\right]^2.$$

In the case of estimating our proposed model parameters by the maximum product of the spacings estimation method (E4), our objective function is maximized, and it is defined as follows: $(x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)})$

$$M(\alpha,\beta,p) = \frac{1}{n+1}\sum_{i=1}^{n+1}\log T_i,$$

where

$$T_i = F\left(x_{(i)}\right) - F\left(x_{(i-1)}\right) = \left(\frac{x_{i-1}^{\beta}-1}{px_{i-1}^{\beta}-1}\right)^{\alpha} - \left(\frac{x_i^{\beta}-1}{px_i^{\beta}-1}\right)^{\alpha}.$$

In the case of estimating our proposed model parameters by the least-squares estimation method (E5), our objective function is minimized, and it is defined as follows: $(x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)})$

$$V(\alpha,\beta,p) = \sum_{i=1}^{n}\left[F(x_i)-\frac{i}{n+1}\right]^2 = \sum_{i=1}^{n}\left[1-\left(\frac{x_i^{\beta}-1}{px_i^{\beta}-1}\right)^{\alpha}-\frac{i}{n+1}\right]^2.$$

In the case of estimating our proposed model parameters by the percentile estimation method (E6), our objective function is minimized, and it is defined as follows: $(x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)})$

$$PC = \sum_{i=1}^{n}[x_i - Q(e_i)]^2 = \sum_{i=1}^{n}\left[x_i - \left(\frac{(1-e_i)^{1/\alpha}-1}{p(1-e_i)^{1/\alpha}-1}\right)^{1/\beta}\right]^2,\ e_i = \frac{i}{n+1}.$$

In the case of estimating our proposed model parameters by the right-tail Anderson-Darling estimation method (E7), our objective function is minimized, and it is defined as follows: $(x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)})$

$$\begin{aligned} R(\alpha,\beta,p) &= \frac{n}{2} - 2\sum_{i=1}^{n}F\left(x_i\right) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log S\left(x_{n+1-i}\right) \\ &= \frac{n}{2} - 2\sum_{i=1}^{n}\left(1-\left(\frac{x_i^{\beta}-1}{px_i^{\beta}-1}\right)^{\alpha}\right) - \frac{\alpha}{n}\sum_{i=1}^{n}(2i-1)\log\left(\frac{x_{n+1-i}^{\beta}-1}{px_{n+1-i}^{\beta}-1}\right). \end{aligned}$$

In the case of estimating our proposed model parameters by the weighted least-squares estimation method (E8), our objective function is minimized, and it is defined as follows: $(x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)})$

$$W(\alpha,\beta,p) = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_i)-\frac{i}{n+1}\right]^2 = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[1-\left(\frac{x_i^{\beta}-1}{px_i^{\beta}-1}\right)^{\alpha}-\frac{i}{n+1}\right]^2.$$

6. Numerical simulation

All estimation methods presented in Section 5 will be used in this section for estimating our proposed model parameters by using randomly generated data sets. Our aim in this section is to study both the performance of these estimation methods and the behavior of our proposed model estimators. Also, we will compare all methods by some measures such as average of bias (BIAS) $|Bias(\widehat{\boldsymbol{\Psi}})| = \frac{1}{M}\sum_{i=1}^{M}|\widehat{\boldsymbol{\Psi}} - \boldsymbol{\Psi}|$, mean squared errors (MSE), $MSE = \frac{1}{M}\sum_{i=1}^{M}(\widehat{\boldsymbol{\Psi}} - \boldsymbol{\Psi})^2$, and mean relative errors (MRE) $MRE = \frac{1}{M}\sum_{i=1}^{M}|\widehat{\boldsymbol{\Psi}} - \boldsymbol{\Psi}|/\boldsymbol{\Psi}$, $\boldsymbol{\Psi} = (\alpha, \beta, p)$. One of the most important benefits of the simulation is to construct a guideline for the best estimating method for the proposed model parameters. A random sample of $M = 1000$ is generated using the R software for n = 25, 50, 75, 100, 125, and 150.

The numerical results of simulations are reported in Tables 1–5, and the graphical representation of these tables is presented in Figures 3–7, respectively. The power of any value relates to its rank in comparing all estimating methodologies. Our estimators' partial and overall rankings are displayed in Table 6.

From simulation and ranking tables, we conclude the following:

- The consistency characteristic is shown by all of the estimators.
- For all methods of estimating, the BIAS of all estimators reduces as n increases.
- For all methods of estimating, the MSE of all estimators reduces as n increases.
- For all methods of estimating, the MRE of all estimators reduces as n increases.
- The maximum product of spacing is the best method for estimating. So we advise researchers to use this method if they have data sets from our proposed model.

Table 1. Simulation values of BIAS, MSE and MRE for ($\alpha = 0.25,\ \beta = 0.5,\ p = 0.75$).

n	Est.	Est. Par.	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8
25	BIAS	$\hat{\alpha}$	$0.051591^{\{2\}}$	$0.052725^{\{4\}}$	$0.056642^{\{7\}}$	$0.049259^{\{1\}}$	$0.054353^{\{5\}}$	$0.091793^{\{8\}}$	$0.054763^{\{6\}}$	$0.052705^{\{3\}}$
		$\hat{\beta}$	$0.729677^{\{7\}}$	$0.620046^{\{3\}}$	$0.76638^{\{8\}}$	$0.530697^{\{1\}}$	$0.65465^{\{5\}}$	$0.547926^{\{2\}}$	$0.704129^{\{6\}}$	$0.621287^{\{4\}}$
		$\hat{p}$	$0.392441^{\{4\}}$	$0.367719^{\{2\}}$	$0.41821^{\{7\}}$	$0.361832^{\{1\}}$	$0.414717^{\{6\}}$	$0.436069^{\{8\}}$	$0.400713^{\{5\}}$	$0.381643^{\{3\}}$
	MSE	$\hat{\alpha}$	$0.004633^{\{2\}}$	$0.004783^{\{4\}}$	$0.005476^{\{7\}}$	$0.003802^{\{1\}}$	$0.004832^{\{5\}}$	$0.012473^{\{8\}}$	$0.005225^{\{6\}}$	$0.004652^{\{3\}}$
		$\hat{\beta}$	$1.417229^{\{7\}}$	$0.901267^{\{3\}}$	$1.445481^{\{8\}}$	$0.673892^{\{2\}}$	$1.011607^{\{5\}}$	$0.5839^{\{1\}}$	$1.408182^{\{6\}}$	$0.93405^{\{4\}}$
		$\hat{p}$	$0.238092^{\{5\}}$	$0.202639^{\{2\}}$	$0.244727^{\{7\}}$	$0.196464^{\{1\}}$	$0.239728^{\{6\}}$	$0.261606^{\{8\}}$	$0.234921^{\{4\}}$	$0.210419^{\{3\}}$
	MRE	$\hat{\alpha}$	$0.206364^{\{2\}}$	$0.2109^{\{4\}}$	$0.22657^{\{7\}}$	$0.197037^{\{1\}}$	$0.217414^{\{5\}}$	$0.367171^{\{8\}}$	$0.219053^{\{6\}}$	$0.210819^{\{3\}}$
		$\hat{\beta}$	$1.459354^{\{7\}}$	$1.240091^{\{3\}}$	$1.53276^{\{8\}}$	$1.061393^{\{1\}}$	$1.3093^{\{5\}}$	$1.095851^{\{2\}}$	$1.408257^{\{6\}}$	$1.242574^{\{4\}}$
		$\hat{p}$	$0.523255^{\{4\}}$	$0.490292^{\{2\}}$	$0.557613^{\{7\}}$	$0.482443^{\{1\}}$	$0.552956^{\{6\}}$	$0.581425^{\{8\}}$	$0.534285^{\{5\}}$	$0.508858^{\{3\}}$
	$\sum Ranks$		$40^{\{4\}}$	$27^{\{2\}}$	$66^{\{8\}}$	$10^{\{1\}}$	$48^{\{5\}}$	$53^{\{7\}}$	$50^{\{6\}}$	$30^{\{3\}}$
50	BIAS	$\hat{\alpha}$	$0.036992^{\{3\}}$	$0.037802^{\{4\}}$	$0.042282^{\{7\}}$	$0.035506^{\{1\}}$	$0.040109^{\{6\}}$	$0.074175^{\{8\}}$	$0.038145^{\{5\}}$	$0.036894^{\{2\}}$
		$\hat{\beta}$	$0.455474^{\{3\}}$	$0.474017^{\{4\}}$	$0.657027^{\{8\}}$	$0.394863^{\{1\}}$	$0.595202^{\{7\}}$	$0.436084^{\{2\}}$	$0.5833^{\{6\}}$	$0.526315^{\{5\}}$
		$\hat{p}$	$0.291789^{\{2\}}$	$0.308314^{\{3\}}$	$0.388133^{\{8\}}$	$0.274802^{\{1\}}$	$0.376931^{\{5\}}$	$0.37752^{\{6\}}$	$0.379313^{\{7\}}$	$0.334717^{\{4\}}$
	MSE	$\hat{\alpha}$	$0.002529^{\{5\}}$	$0.0024^{\{3\}}$	$0.003139^{\{7\}}$	$0.001996^{\{1\}}$	$0.002681^{\{6\}}$	$0.008343^{\{8\}}$	$0.002524^{\{4\}}$	$0.002339^{\{2\}}$
		$\hat{\beta}$	$0.460704^{\{4\}}$	$0.418422^{\{3\}}$	$0.827397^{\{8\}}$	$0.266079^{\{1\}}$	$0.640573^{\{6\}}$	$0.3174^{\{2\}}$	$0.653666^{\{7\}}$	$0.5165^{\{5\}}$
		$\hat{p}$	$0.154866^{\{3\}}$	$0.154373^{\{2\}}$	$0.220445^{\{8\}}$	$0.128013^{\{1\}}$	$0.206851^{\{5\}}$	$0.209449^{\{6\}}$	$0.217265^{\{7\}}$	$0.174514^{\{4\}}$
	MRE	$\hat{\alpha}$	$0.147969^{\{3\}}$	$0.151209^{\{4\}}$	$0.169129^{\{7\}}$	$0.142023^{\{1\}}$	$0.160435^{\{6\}}$	$0.2967^{\{8\}}$	$0.152578^{\{5\}}$	$0.147576^{\{2\}}$
		$\hat{\beta}$	$0.910949^{\{3\}}$	$0.948034^{\{4\}}$	$1.314054^{\{8\}}$	$0.789726^{\{1\}}$	$1.190404^{\{7\}}$	$0.872169^{\{2\}}$	$1.166599^{\{6\}}$	$1.05263^{\{5\}}$
		$\hat{p}$	$0.389052^{\{2\}}$	$0.411086^{\{3\}}$	$0.517511^{\{8\}}$	$0.366403^{\{1\}}$	$0.502575^{\{5\}}$	$0.50336^{\{6\}}$	$0.50575^{\{7\}}$	$0.446289^{\{4\}}$
	$\sum Ranks$		$28^{\{2\}}$	$30^{\{3\}}$	$69^{\{8\}}$	$9^{\{1\}}$	$53^{\{6\}}$	$48^{\{5\}}$	$54^{\{7\}}$	$33^{\{4\}}$
75	BIAS	$\hat{\alpha}$	$0.029929^{\{2\}}$	$0.030199^{\{3\}}$	$0.033998^{\{7\}}$	$0.028074^{\{1\}}$	$0.03277^{\{6\}}$	$0.066146^{\{8\}}$	$0.030995^{\{4\}}$	$0.031447^{\{5\}}$
		$\hat{\beta}$	$0.345732^{\{2\}}$	$0.41331^{\{4\}}$	$0.558164^{\{8\}}$	$0.335031^{\{1\}}$	$0.530064^{\{6\}}$	$0.394034^{\{3\}}$	$0.55219^{\{7\}}$	$0.451781^{\{5\}}$
		$\hat{p}$	$0.234296^{\{2\}}$	$0.271643^{\{3\}}$	$0.351568^{\{7\}}$	$0.223104^{\{1\}}$	$0.343758^{\{6\}}$	$0.328165^{\{5\}}$	$0.352375^{\{8\}}$	$0.292672^{\{4\}}$
	MSE	$\hat{\alpha}$	$0.001599^{\{3\}}$	$0.00158^{\{2\}}$	$0.00196^{\{7\}}$	$0.001265^{\{1\}}$	$0.001766^{\{6\}}$	$0.007024^{\{8\}}$	$0.001664^{\{4\}}$	$0.001681^{\{5\}}$
		$\hat{\beta}$	$0.262619^{\{3\}}$	$0.294135^{\{4\}}$	$0.520607^{\{7\}}$	$0.194272^{\{1\}}$	$0.466121^{\{6\}}$	$0.258269^{\{2\}}$	$0.541265^{\{8\}}$	$0.359719^{\{5\}}$
		$\hat{p}$	$0.10894^{\{2\}}$	$0.125033^{\{3\}}$	$0.188569^{\{7\}}$	$0.088574^{\{1\}}$	$0.180949^{\{6\}}$	$0.170754^{\{5\}}$	$0.194077^{\{8\}}$	$0.14185^{\{4\}}$
	MRE	$\hat{\alpha}$	$0.119718^{\{2\}}$	$0.120796^{\{3\}}$	$0.135991^{\{7\}}$	$0.112298^{\{1\}}$	$0.131079^{\{6\}}$	$0.264584^{\{8\}}$	$0.123981^{\{4\}}$	$0.125789^{\{5\}}$
		$\hat{\beta}$	$0.691464^{\{2\}}$	$0.826621^{\{4\}}$	$1.116328^{\{8\}}$	$0.670062^{\{1\}}$	$1.060128^{\{6\}}$	$0.788067^{\{3\}}$	$1.104381^{\{7\}}$	$0.903562^{\{5\}}$
		$\hat{p}$	$0.312394^{\{2\}}$	$0.36219^{\{3\}}$	$0.468757^{\{7\}}$	$0.297472^{\{1\}}$	$0.458344^{\{6\}}$	$0.437554^{\{5\}}$	$0.469834^{\{8\}}$	$0.390229^{\{4\}}$
	$\sum Ranks$		$20^{\{2\}}$	$29^{\{3\}}$	$65^{\{8\}}$	$9^{\{1\}}$	$54^{\{6\}}$	$47^{\{5\}}$	$58^{\{7\}}$	$42^{\{4\}}$
100	BIAS	$\hat{\alpha}$	$0.025731^{\{4\}}$	$0.025612^{\{2\}}$	$0.029349^{\{7\}}$	$0.025391^{\{1\}}$	$0.028974^{\{6\}}$	$0.060747^{\{8\}}$	$0.025666^{\{3\}}$	$0.026224^{\{5\}}$
		$\hat{\beta}$	$0.295691^{\{2\}}$	$0.358251^{\{3\}}$	$0.534747^{\{8\}}$	$0.282354^{\{1\}}$	$0.488634^{\{6\}}$	$0.360695^{\{4\}}$	$0.512707^{\{7\}}$	$0.408327^{\{5\}}$
		$\hat{p}$	$0.202535^{\{2\}}$	$0.238284^{\{3\}}$	$0.33572^{\{8\}}$	$0.186551^{\{1\}}$	$0.32742^{\{6\}}$	$0.306149^{\{5\}}$	$0.334309^{\{7\}}$	$0.263109^{\{4\}}$
	MSE	$\hat{\alpha}$	$0.001247^{\{5\}}$	$0.001077^{\{2\}}$	$0.001446^{\{7\}}$	$0.001048^{\{1\}}$	$0.001383^{\{6\}}$	$0.005847^{\{8\}}$	$0.001101^{\{3\}}$	$0.001144^{\{4\}}$
		$\hat{\beta}$	$0.192691^{\{2\}}$	$0.218961^{\{4\}}$	$0.455804^{\{8\}}$	$0.1317^{\{1\}}$	$0.363372^{\{6\}}$	$0.204047^{\{3\}}$	$0.448779^{\{7\}}$	$0.284322^{\{5\}}$
		$\hat{p}$	$0.087828^{\{2\}}$	$0.100459^{\{3\}}$	$0.174944^{\{7\}}$	$0.063794^{\{1\}}$	$0.166034^{\{6\}}$	$0.153375^{\{5\}}$	$0.180267^{\{8\}}$	$0.120239^{\{4\}}$
	MRE	$\hat{\alpha}$	$0.102923^{\{4\}}$	$0.102448^{\{2\}}$	$0.117397^{\{7\}}$	$0.101562^{\{1\}}$	$0.115895^{\{6\}}$	$0.242987^{\{8\}}$	$0.102664^{\{3\}}$	$0.104897^{\{5\}}$
		$\hat{\beta}$	$0.591382^{\{2\}}$	$0.716502^{\{3\}}$	$1.069494^{\{8\}}$	$0.564707^{\{1\}}$	$0.977268^{\{6\}}$	$0.72139^{\{4\}}$	$1.025413^{\{7\}}$	$0.816655^{\{5\}}$
		$\hat{p}$	$0.270047^{\{2\}}$	$0.317712^{\{3\}}$	$0.447626^{\{8\}}$	$0.248735^{\{1\}}$	$0.43656^{\{6\}}$	$0.408199^{\{5\}}$	$0.445746^{\{7\}}$	$0.350812^{\{4\}}$
	$\sum Ranks$		$25^{\{2.5\}}$	$25^{\{2.5\}}$	$68^{\{8\}}$	$9^{\{1\}}$	$54^{\{7\}}$	$50^{\{5\}}$	$52^{\{6\}}$	$41^{\{4\}}$
125	BIAS	$\hat{\alpha}$	$0.022809^{\{2\}}$	$0.02365^{\{4\}}$	$0.025511^{\{7\}}$	$0.021951^{\{1\}}$	$0.025128^{\{6\}}$	$0.056428^{\{8\}}$	$0.02354^{\{3\}}$	$0.024241^{\{5\}}$
		$\hat{\beta}$	$0.260277^{\{2\}}$	$0.334209^{\{3\}}$	$0.478685^{\{8\}}$	$0.252423^{\{1\}}$	$0.453397^{\{6\}}$	$0.335138^{\{4\}}$	$0.47487^{\{7\}}$	$0.374649^{\{5\}}$
		$\hat{p}$	$0.183759^{\{2\}}$	$0.218723^{\{3\}}$	$0.306083^{\{7\}}$	$0.170132^{\{1\}}$	$0.295728^{\{6\}}$	$0.281067^{\{5\}}$	$0.31502^{\{8\}}$	$0.244894^{\{4\}}$
	MSE	$\hat{\alpha}$	$0.000956^{\{4\}}$	$0.000914^{\{3\}}$	$0.001059^{\{7\}}$	$0.000759^{\{1\}}$	$0.001037^{\{6\}}$	$0.005186^{\{8\}}$	$0.000904^{\{2\}}$	$0.000977^{\{5\}}$
		$\hat{\beta}$	$0.147791^{\{2\}}$	$0.191032^{\{4\}}$	$0.361931^{\{7\}}$	$0.107823^{\{1\}}$	$0.315666^{\{6\}}$	$0.172298^{\{3\}}$	$0.378657^{\{8\}}$	$0.242323^{\{5\}}$
		$\hat{p}$	$0.07295^{\{2\}}$	$0.088244^{\{3\}}$	$0.150385^{\{7\}}$	$0.055235^{\{1\}}$	$0.141698^{\{6\}}$	$0.134041^{\{5\}}$	$0.165457^{\{8\}}$	$0.106311^{\{4\}}$
	MRE	$\hat{\alpha}$	$0.091237^{\{2\}}$	$0.094602^{\{4\}}$	$0.102044^{\{7\}}$	$0.087802^{\{1\}}$	$0.100512^{\{6\}}$	$0.225712^{\{8\}}$	$0.094161^{\{3\}}$	$0.096964^{\{5\}}$
		$\hat{\beta}$	$0.520554^{\{2\}}$	$0.668418^{\{3\}}$	$0.957369^{\{8\}}$	$0.504846^{\{1\}}$	$0.906795^{\{6\}}$	$0.670275^{\{4\}}$	$0.94974^{\{7\}}$	$0.749299^{\{5\}}$
		$\hat{p}$	$0.245012^{\{2\}}$	$0.29163^{\{3\}}$	$0.408111^{\{7\}}$	$0.226843^{\{1\}}$	$0.394304^{\{6\}}$	$0.374756^{\{5\}}$	$0.420027^{\{8\}}$	$0.326525^{\{4\}}$
	$\sum Ranks$		$20^{\{2\}}$	$30^{\{3\}}$	$65^{\{8\}}$	$9^{\{1\}}$	$54^{\{6.5\}}$	$50^{\{5\}}$	$54^{\{6.5\}}$	$42^{\{4\}}$
150	BIAS	$\hat{\alpha}$	$0.020671^{\{2\}}$	$0.021456^{\{4\}}$	$0.023372^{\{7\}}$	$0.019821^{\{1\}}$	$0.023177^{\{6\}}$	$0.055221^{\{8\}}$	$0.021642^{\{5\}}$	$0.021165^{\{3\}}$
		$\hat{\beta}$	$0.222641^{\{1\}}$	$0.297765^{\{3\}}$	$0.456608^{\{7\}}$	$0.227744^{\{2\}}$	$0.438811^{\{6\}}$	$0.31424^{\{4\}}$	$0.475774^{\{8\}}$	$0.328452^{\{5\}}$
		$\hat{p}$	$0.161433^{\{2\}}$	$0.197359^{\{3\}}$	$0.28375^{\{7\}}$	$0.148026^{\{1\}}$	$0.277267^{\{6\}}$	$0.260366^{\{5\}}$	$0.31378^{\{8\}}$	$0.216579^{\{4\}}$
	MSE	$\hat{\alpha}$	$0.000844^{\{5\}}$	$0.000753^{\{3\}}$	$0.00091^{\{7\}}$	$0.000625^{\{1\}}$	$0.000863^{\{6\}}$	$0.00494^{\{8\}}$	$0.000772^{\{4\}}$	$0.000725^{\{2\}}$
		$\hat{\beta}$	$0.105635^{\{2\}}$	$0.152554^{\{4\}}$	$0.332876^{\{7\}}$	$0.086648^{\{1\}}$	$0.297847^{\{6\}}$	$0.148357^{\{3\}}$	$0.37582^{\{8\}}$	$0.186554^{\{5\}}$
		$\hat{p}$	$0.058576^{\{2\}}$	$0.072592^{\{3\}}$	$0.131637^{\{7\}}$	$0.04064^{\{1\}}$	$0.125591^{\{6\}}$	$0.11713^{\{5\}}$	$0.163867^{\{8\}}$	$0.086676^{\{4\}}$
	MRE	$\hat{\alpha}$	$0.082682^{\{2\}}$	$0.085823^{\{4\}}$	$0.093489^{\{7\}}$	$0.079284^{\{1\}}$	$0.09271^{\{6\}}$	$0.220885^{\{8\}}$	$0.086567^{\{5\}}$	$0.084659^{\{3\}}$
		$\hat{\beta}$	$0.445282^{\{1\}}$	$0.59553^{\{3\}}$	$0.913216^{\{7\}}$	$0.455487^{\{2\}}$	$0.877622^{\{6\}}$	$0.628481^{\{4\}}$	$0.951549^{\{8\}}$	$0.656905^{\{5\}}$
		$\hat{p}$	$0.215244^{\{2\}}$	$0.263145^{\{3\}}$	$0.378334^{\{7\}}$	$0.197368^{\{1\}}$	$0.369689^{\{6\}}$	$0.347154^{\{5\}}$	$0.418373^{\{8\}}$	$0.288772^{\{4\}}$
	$\sum Ranks$		$19^{\{2\}}$	$30^{\{3\}}$	$63^{\{8\}}$	$11^{\{1\}}$	$54^{\{6\}}$	$50^{\{5\}}$	$62^{\{7\}}$	$35^{\{4\}}$

Table 2. Simulation values of BIAS, MSE and MRE for ($\alpha = 0.75,\ \beta = 1.5,\ p = 0.25$).

n	Est.	Est. Par.	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8
25	BIAS	$\hat{\alpha}$	0.199233$^{\{5\}}$	0.200079$^{\{6\}}$	0.201937$^{\{7\}}$	0.178752$^{\{1\}}$	0.196027$^{\{3\}}$	0.217508$^{\{8\}}$	0.198708$^{\{4\}}$	0.189079$^{\{2\}}$
		$\hat{\beta}$	0.491544$^{\{1\}}$	0.540826$^{\{2\}}$	0.582732$^{\{6\}}$	0.542269$^{\{3\}}$	0.605783$^{\{7\}}$	0.559959$^{\{4\}}$	0.609822$^{\{8\}}$	0.579491$^{\{5\}}$
		$\hat{p}$	0.269246$^{\{1\}}$	0.298309$^{\{2\}}$	0.299795$^{\{3\}}$	0.310599$^{\{5\}}$	0.334074$^{\{8\}}$	0.312526$^{\{6\}}$	0.323979$^{\{7\}}$	0.309123$^{\{4\}}$
	MSE	$\hat{\alpha}$	0.068967$^{\{7\}}$	0.066251$^{\{5\}}$	0.06889$^{\{6\}}$	0.052255$^{\{1\}}$	0.062344$^{\{3\}}$	0.076593$^{\{8\}}$	0.066226$^{\{4\}}$	0.058651$^{\{2\}}$
		$\hat{\beta}$	0.455951$^{\{3\}}$	0.448294$^{\{2\}}$	0.556649$^{\{6\}}$	0.430997$^{\{1\}}$	0.576642$^{\{7\}}$	0.491442$^{\{4\}}$	0.607324$^{\{8\}}$	0.541358$^{\{5\}}$
		$\hat{p}$	0.09189$^{\{1\}}$	0.116687$^{\{2\}}$	0.117005$^{\{3\}}$	0.12569$^{\{6\}}$	0.146045$^{\{8\}}$	0.12511$^{\{5\}}$	0.13941$^{\{7\}}$	0.124075$^{\{4\}}$
	MRE	$\hat{\alpha}$	0.265644$^{\{5\}}$	0.266772$^{\{6\}}$	0.26925$^{\{7\}}$	0.238336$^{\{1\}}$	0.261369$^{\{3\}}$	0.29001$^{\{8\}}$	0.264944$^{\{4\}}$	0.252105$^{\{2\}}$
		$\hat{\beta}$	0.327696$^{\{1\}}$	0.360551$^{\{2\}}$	0.388488$^{\{6\}}$	0.361513$^{\{3\}}$	0.403856$^{\{7\}}$	0.373306$^{\{4\}}$	0.406548$^{\{8\}}$	0.386328$^{\{5\}}$
		$\hat{p}$	1.076985$^{\{1\}}$	1.193234$^{\{2\}}$	1.19918$^{\{3\}}$	1.242397$^{\{5\}}$	1.336297$^{\{8\}}$	1.250104$^{\{6\}}$	1.295917$^{\{7\}}$	1.236492$^{\{4\}}$
	$\sum Ranks$		25$^{\{1\}}$	29$^{\{3\}}$	47$^{\{5\}}$	26$^{\{2\}}$	54$^{\{7\}}$	53$^{\{6\}}$	57$^{\{8\}}$	33$^{\{4\}}$
50	BIAS	$\hat{\alpha}$	0.147351$^{\{2\}}$	0.155187$^{\{4\}}$	0.161997$^{\{5\}}$	0.136246$^{\{1\}}$	0.165931$^{\{6\}}$	0.190405$^{\{8\}}$	0.16768$^{\{7\}}$	0.15425$^{\{3\}}$
		$\hat{\beta}$	0.349649$^{\{1\}}$	0.380335$^{\{2\}}$	0.435532$^{\{6\}}$	0.397148$^{\{4\}}$	0.467606$^{\{7\}}$	0.390263$^{\{3\}}$	0.502618$^{\{8\}}$	0.404016$^{\{5\}}$
		$\hat{p}$	0.252311$^{\{1\}}$	0.273101$^{\{2\}}$	0.285224$^{\{5\}}$	0.274737$^{\{3\}}$	0.297315$^{\{7\}}$	0.288902$^{\{6\}}$	0.312833$^{\{8\}}$	0.277021$^{\{4\}}$
	MSE	$\hat{\alpha}$	0.038689$^{\{2\}}$	0.043397$^{\{3\}}$	0.047489$^{\{5\}}$	0.031547$^{\{1\}}$	0.048216$^{\{6\}}$	0.061059$^{\{8\}}$	0.050459$^{\{7\}}$	0.044372$^{\{4\}}$
		$\hat{\beta}$	0.196123$^{\{1\}}$	0.238903$^{\{2\}}$	0.305884$^{\{6\}}$	0.247144$^{\{4\}}$	0.351975$^{\{7\}}$	0.244965$^{\{3\}}$	0.416972$^{\{8\}}$	0.256703$^{\{5\}}$
		$\hat{p}$	0.079849$^{\{1\}}$	0.095921$^{\{2\}}$	0.104819$^{\{5\}}$	0.097423$^{\{3\}}$	0.114973$^{\{7\}}$	0.106214$^{\{6\}}$	0.126998$^{\{8\}}$	0.100685$^{\{4\}}$
	MRE	$\hat{\alpha}$	0.196468$^{\{2\}}$	0.206916$^{\{4\}}$	0.215996$^{\{5\}}$	0.181662$^{\{1\}}$	0.221242$^{\{6\}}$	0.253873$^{\{8\}}$	0.223574$^{\{7\}}$	0.205666$^{\{3\}}$
		$\hat{\beta}$	0.233099$^{\{1\}}$	0.253556$^{\{2\}}$	0.290355$^{\{6\}}$	0.264765$^{\{4\}}$	0.311737$^{\{7\}}$	0.260175$^{\{3\}}$	0.335078$^{\{8\}}$	0.269344$^{\{5\}}$
		$\hat{p}$	1.009243$^{\{1\}}$	1.092404$^{\{2\}}$	1.140895$^{\{5\}}$	1.098948$^{\{3\}}$	1.189259$^{\{7\}}$	1.155608$^{\{6\}}$	1.251332$^{\{8\}}$	1.108085$^{\{4\}}$
	$\sum Ranks$		12$^{\{1\}}$	23$^{\{2\}}$	48$^{\{5\}}$	24$^{\{3\}}$	60$^{\{7\}}$	51$^{\{6\}}$	69$^{\{8\}}$	37$^{\{4\}}$
75	BIAS	$\hat{\alpha}$	0.127997$^{\{3\}}$	0.127553$^{\{2\}}$	0.153746$^{\{7\}}$	0.119979$^{\{1\}}$	0.140848$^{\{6\}}$	0.168106$^{\{8\}}$	0.13113$^{\{4\}}$	0.132409$^{\{5\}}$
		$\hat{\beta}$	0.301359$^{\{1\}}$	0.315425$^{\{2\}}$	0.376539$^{\{6\}}$	0.316915$^{\{3\}}$	0.380688$^{\{7\}}$	0.341773$^{\{5\}}$	0.397496$^{\{8\}}$	0.325934$^{\{4\}}$
		$\hat{p}$	0.241908$^{\{1\}}$	0.248791$^{\{3\}}$	0.277875$^{\{7\}}$	0.24865$^{\{2\}}$	0.272574$^{\{5\}}$	0.277067$^{\{6\}}$	0.282518$^{\{8\}}$	0.25495$^{\{4\}}$
	MSE	$\hat{\alpha}$	0.031611$^{\{3\}}$	0.030335$^{\{2\}}$	0.044765$^{\{7\}}$	0.026174$^{\{1\}}$	0.035734$^{\{6\}}$	0.050439$^{\{8\}}$	0.031909$^{\{4\}}$	0.032731$^{\{5\}}$
		$\hat{\beta}$	0.143223$^{\{1\}}$	0.157404$^{\{2\}}$	0.221151$^{\{6\}}$	0.157971$^{\{3\}}$	0.233783$^{\{7\}}$	0.185242$^{\{5\}}$	0.260175$^{\{8\}}$	0.170462$^{\{4\}}$
		$\hat{p}$	0.074523$^{\{1\}}$	0.0794$^{\{2\}}$	0.099301$^{\{7\}}$	0.081654$^{\{3\}}$	0.098249$^{\{5\}}$	0.098454$^{\{6\}}$	0.105546$^{\{8\}}$	0.084315$^{\{4\}}$
	MRE	$\hat{\alpha}$	0.170663$^{\{3\}}$	0.17007$^{\{2\}}$	0.204995$^{\{7\}}$	0.159972$^{\{1\}}$	0.187798$^{\{6\}}$	0.224142$^{\{8\}}$	0.17484$^{\{4\}}$	0.176545$^{\{5\}}$
		$\hat{\beta}$	0.200906$^{\{1\}}$	0.210283$^{\{2\}}$	0.251026$^{\{6\}}$	0.211277$^{\{3\}}$	0.253792$^{\{7\}}$	0.227849$^{\{5\}}$	0.264997$^{\{8\}}$	0.21729$^{\{4\}}$
		$\hat{p}$	0.967632$^{\{1\}}$	0.995164$^{\{3\}}$	1.111499$^{\{7\}}$	0.9946$^{\{2\}}$	1.090296$^{\{5\}}$	1.10827$^{\{6\}}$	1.130073$^{\{8\}}$	1.0198$^{\{4\}}$
	$\sum Ranks$		15$^{\{1\}}$	20$^{\{3\}}$	60$^{\{7.5\}}$	19$^{\{2\}}$	54$^{\{5\}}$	57$^{\{6\}}$	60$^{\{7.5\}}$	39$^{\{4\}}$
100	BIAS	$\hat{\alpha}$	0.109373$^{\{2\}}$	0.119326$^{\{4\}}$	0.132379$^{\{6\}}$	0.102004$^{\{1\}}$	0.133633$^{\{7\}}$	0.145378$^{\{8\}}$	0.120623$^{\{5\}}$	0.11665$^{\{3\}}$
		$\hat{\beta}$	0.261256$^{\{1\}}$	0.284333$^{\{2\}}$	0.33875$^{\{6\}}$	0.28547$^{\{3\}}$	0.343825$^{\{7\}}$	0.302514$^{\{5\}}$	0.352178$^{\{8\}}$	0.293869$^{\{4\}}$
		$\hat{p}$	0.221213$^{\{1\}}$	0.236717$^{\{3\}}$	0.259934$^{\{7\}}$	0.229795$^{\{2\}}$	0.258143$^{\{6\}}$	0.256117$^{\{5\}}$	0.269078$^{\{8\}}$	0.242987$^{\{4\}}$
	MSE	$\hat{\alpha}$	0.023327$^{\{2\}}$	0.026306$^{\{3\}}$	0.034228$^{\{7\}}$	0.01802$^{\{1\}}$	0.033303$^{\{6\}}$	0.03965$^{\{8\}}$	0.027235$^{\{5\}}$	0.026787$^{\{4\}}$
		$\hat{\beta}$	0.108518$^{\{1\}}$	0.12754$^{\{3\}}$	0.186014$^{\{6\}}$	0.126915$^{\{2\}}$	0.188393$^{\{7\}}$	0.142192$^{\{5\}}$	0.205567$^{\{8\}}$	0.139157$^{\{4\}}$
		$\hat{p}$	0.063063$^{\{1\}}$	0.073633$^{\{3\}}$	0.088122$^{\{7\}}$	0.067966$^{\{2\}}$	0.087758$^{\{6\}}$	0.085448$^{\{5\}}$	0.095961$^{\{8\}}$	0.076918$^{\{4\}}$
	MRE	$\hat{\alpha}$	0.145831$^{\{2\}}$	0.159102$^{\{4\}}$	0.176505$^{\{6\}}$	0.136005$^{\{1\}}$	0.178177$^{\{7\}}$	0.193837$^{\{8\}}$	0.160831$^{\{5\}}$	0.155534$^{\{3\}}$
		$\hat{\beta}$	0.174171$^{\{1\}}$	0.189555$^{\{2\}}$	0.225834$^{\{6\}}$	0.190314$^{\{3\}}$	0.229216$^{\{7\}}$	0.201676$^{\{5\}}$	0.234785$^{\{8\}}$	0.195913$^{\{4\}}$
		$\hat{p}$	0.884852$^{\{1\}}$	0.946868$^{\{3\}}$	1.039737$^{\{7\}}$	0.919181$^{\{2\}}$	1.032571$^{\{6\}}$	1.024469$^{\{5\}}$	1.076313$^{\{8\}}$	0.97195$^{\{4\}}$
	$\sum Ranks$		12$^{\{1\}}$	27$^{\{3\}}$	58$^{\{6\}}$	17$^{\{2\}}$	59$^{\{7\}}$	54$^{\{5\}}$	63$^{\{8\}}$	34$^{\{4\}}$
125	BIAS	$\hat{\alpha}$	0.095819$^{\{2\}}$	0.10287$^{\{3\}}$	0.116859$^{\{7\}}$	0.092213$^{\{1\}}$	0.115875$^{\{6\}}$	0.141544$^{\{8\}}$	0.104616$^{\{4\}}$	0.107491$^{\{5\}}$
		$\hat{\beta}$	0.233543$^{\{1\}}$	0.249439$^{\{2\}}$	0.303048$^{\{6\}}$	0.259756$^{\{3\}}$	0.305354$^{\{7\}}$	0.260569$^{\{4\}}$	0.332159$^{\{8\}}$	0.268521$^{\{5\}}$
		$\hat{p}$	0.209378$^{\{1\}}$	0.213157$^{\{2\}}$	0.249931$^{\{6\}}$	0.21809$^{\{3\}}$	0.253997$^{\{7\}}$	0.248482$^{\{5\}}$	0.25741$^{\{8\}}$	0.229738$^{\{4\}}$
	MSE	$\hat{\alpha}$	0.017716$^{\{2\}}$	0.018871$^{\{3\}}$	0.027101$^{\{7\}}$	0.01398$^{\{1\}}$	0.026531$^{\{6\}}$	0.038929$^{\{8\}}$	0.02028$^{\{4\}}$	0.021605$^{\{5\}}$
		$\hat{\beta}$	0.084135$^{\{1\}}$	0.100354$^{\{2\}}$	0.146043$^{\{6\}}$	0.10424$^{\{3\}}$	0.151363$^{\{7\}}$	0.106741$^{\{4\}}$	0.179886$^{\{8\}}$	0.112845$^{\{5\}}$
		$\hat{p}$	0.057572$^{\{1\}}$	0.059895$^{\{2\}}$	0.082578$^{\{6\}}$	0.062399$^{\{3\}}$	0.082992$^{\{7\}}$	0.07945$^{\{5\}}$	0.086629$^{\{8\}}$	0.069658$^{\{4\}}$
	MRE	$\hat{\alpha}$	0.127759$^{\{2\}}$	0.13716$^{\{3\}}$	0.155812$^{\{7\}}$	0.122951$^{\{1\}}$	0.154499$^{\{6\}}$	0.188726$^{\{8\}}$	0.139488$^{\{4\}}$	0.143322$^{\{5\}}$
		$\hat{\beta}$	0.155695$^{\{1\}}$	0.166293$^{\{2\}}$	0.202032$^{\{6\}}$	0.17317$^{\{3\}}$	0.203569$^{\{7\}}$	0.173713$^{\{4\}}$	0.22144$^{\{8\}}$	0.179014$^{\{5\}}$
		$\hat{p}$	0.837513$^{\{1\}}$	0.852629$^{\{2\}}$	0.999724$^{\{6\}}$	0.872361$^{\{3\}}$	1.015988$^{\{7\}}$	0.993927$^{\{5\}}$	1.029639$^{\{8\}}$	0.918951$^{\{4\}}$
	$\sum Ranks$		12$^{\{1\}}$	21$^{\{2.5\}}$	57$^{\{6\}}$	21$^{\{2.5\}}$	60$^{\{7.5\}}$	51$^{\{5\}}$	60$^{\{7.5\}}$	42$^{\{4\}}$
150	BIAS	$\hat{\alpha}$	0.090121$^{\{2\}}$	0.090951$^{\{3\}}$	0.108371$^{\{7\}}$	0.082727$^{\{1\}}$	0.099989$^{\{6\}}$	0.131326$^{\{8\}}$	0.094301$^{\{5\}}$	0.091773$^{\{4\}}$
		$\hat{\beta}$	0.219095$^{\{1\}}$	0.236805$^{\{3\}}$	0.282944$^{\{7\}}$	0.232666$^{\{2\}}$	0.279553$^{\{6\}}$	0.252834$^{\{5\}}$	0.298366$^{\{8\}}$	0.250229$^{\{4\}}$
		$\hat{p}$	0.203641$^{\{1\}}$	0.211166$^{\{3\}}$	0.235859$^{\{5\}}$	0.205182$^{\{2\}}$	0.237273$^{\{6\}}$	0.23802$^{\{7\}}$	0.240123$^{\{8\}}$	0.215468$^{\{4\}}$
	MSE	$\hat{\alpha}$	0.014768$^{\{2\}}$	0.015029$^{\{3\}}$	0.02327$^{\{7\}}$	0.011292$^{\{1\}}$	0.018628$^{\{6\}}$	0.034053$^{\{8\}}$	0.01645$^{\{5\}}$	0.015737$^{\{4\}}$
		$\hat{\beta}$	0.07517$^{\{1\}}$	0.088846$^{\{3\}}$	0.128175$^{\{7\}}$	0.082791$^{\{2\}}$	0.121755$^{\{6\}}$	0.099005$^{\{5\}}$	0.145997$^{\{8\}}$	0.097254$^{\{4\}}$
		$\hat{p}$	0.053707$^{\{1\}}$	0.05796$^{\{3\}}$	0.073004$^{\{6\}}$	0.05614$^{\{2\}}$	0.071759$^{\{5\}}$	0.074276$^{\{7\}}$	0.075184$^{\{8\}}$	0.060869$^{\{4\}}$
	MRE	$\hat{\alpha}$	0.120162$^{\{2\}}$	0.121268$^{\{3\}}$	0.144494$^{\{7\}}$	0.110302$^{\{1\}}$	0.133319$^{\{6\}}$	0.175101$^{\{8\}}$	0.125734$^{\{5\}}$	0.122365$^{\{4\}}$
		$\hat{\beta}$	0.146063$^{\{1\}}$	0.15787$^{\{3\}}$	0.188629$^{\{7\}}$	0.155111$^{\{2\}}$	0.186369$^{\{6\}}$	0.168556$^{\{5\}}$	0.198911$^{\{8\}}$	0.166819$^{\{4\}}$
		$\hat{p}$	0.814563$^{\{1\}}$	0.844666$^{\{3\}}$	0.943438$^{\{5\}}$	0.820728$^{\{2\}}$	0.949092$^{\{6\}}$	0.952081$^{\{7\}}$	0.96049$^{\{8\}}$	0.861872$^{\{4\}}$
	$\sum Ranks$		12$^{\{1\}}$	27$^{\{3\}}$	58$^{\{6\}}$	15$^{\{2\}}$	53$^{\{5\}}$	60$^{\{7\}}$	63$^{\{8\}}$	36$^{\{4\}}$

Table 3. Simulation values of BIAS, MSE and MRE for ($\alpha = 1.5,\ \beta = 0.75,\ p = 0.5$).

n	Est.	Est. Par.	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8
25	BIAS	$\hat{\alpha}$	0.412113$^{\{1\}}$	0.419768$^{\{3\}}$	0.433955$^{\{5\}}$	0.43559$^{\{6\}}$	0.4473$^{\{8\}}$	0.438541$^{\{7\}}$	0.415237$^{\{2\}}$	0.425532$^{\{4\}}$
		$\hat{\beta}$	0.251509$^{\{3\}}$	0.248097$^{\{2\}}$	0.291152$^{\{7\}}$	0.246944$^{\{1\}}$	0.27637$^{\{5\}}$	0.284622$^{\{6\}}$	0.300232$^{\{8\}}$	0.266728$^{\{4\}}$
		$\hat{p}$	0.324709$^{\{1\}}$	0.334691$^{\{3\}}$	0.367316$^{\{7\}}$	0.329872$^{\{2\}}$	0.365501$^{\{6\}}$	0.364488$^{\{5\}}$	0.369198$^{\{8\}}$	0.347776$^{\{4\}}$
	MSE	$\hat{\alpha}$	0.273157$^{\{4\}}$	0.273119$^{\{3\}}$	0.289704$^{\{7\}}$	0.274024$^{\{5\}}$	0.290754$^{\{8\}}$	0.286477$^{\{6\}}$	0.269432$^{\{1\}}$	0.269755$^{\{2\}}$
		$\hat{\beta}$	0.1114$^{\{3\}}$	0.105392$^{\{2\}}$	0.145218$^{\{7\}}$	0.093526$^{\{1\}}$	0.124659$^{\{5\}}$	0.130437$^{\{6\}}$	0.153405$^{\{8\}}$	0.117183$^{\{4\}}$
		$\hat{p}$	0.134881$^{\{1\}}$	0.14015$^{\{3\}}$	0.161611$^{\{7\}}$	0.135959$^{\{2\}}$	0.159807$^{\{6\}}$	0.159385$^{\{5\}}$	0.162373$^{\{8\}}$	0.147998$^{\{4\}}$
	MRE	$\hat{\alpha}$	0.274742$^{\{1\}}$	0.279846$^{\{3\}}$	0.289303$^{\{5\}}$	0.290394$^{\{6\}}$	0.2982$^{\{8\}}$	0.292361$^{\{7\}}$	0.276825$^{\{2\}}$	0.283688$^{\{4\}}$
		$\hat{\beta}$	0.335345$^{\{3\}}$	0.330796$^{\{2\}}$	0.388203$^{\{7\}}$	0.329259$^{\{1\}}$	0.368494$^{\{5\}}$	0.379496$^{\{6\}}$	0.400309$^{\{8\}}$	0.355638$^{\{4\}}$
		$\hat{p}$	0.649418$^{\{1\}}$	0.669382$^{\{3\}}$	0.734632$^{\{7\}}$	0.659744$^{\{2\}}$	0.731002$^{\{6\}}$	0.728976$^{\{5\}}$	0.738397$^{\{8\}}$	0.695553$^{\{4\}}$
	$\sum Ranks$		18$^{\{1\}}$	24$^{\{2\}}$	59$^{\{8\}}$	26$^{\{3\}}$	57$^{\{7\}}$	53$^{\{5.5\}}$	53$^{\{5.5\}}$	34$^{\{4\}}$
50	BIAS	$\hat{\alpha}$	0.374253$^{\{3\}}$	0.379171$^{\{4\}}$	0.393284$^{\{7\}}$	0.371934$^{\{2\}}$	0.397839$^{\{8\}}$	0.390429$^{\{6\}}$	0.37167$^{\{1\}}$	0.38277$^{\{5\}}$
		$\hat{\beta}$	0.180647$^{\{1\}}$	0.191245$^{\{3\}}$	0.221693$^{\{6\}}$	0.185111$^{\{2\}}$	0.214209$^{\{5\}}$	0.222554$^{\{7\}}$	0.235033$^{\{8\}}$	0.198316$^{\{4\}}$
		$\hat{p}$	0.271226$^{\{1\}}$	0.285305$^{\{3\}}$	0.321615$^{\{7\}}$	0.282073$^{\{2\}}$	0.320302$^{\{6\}}$	0.322252$^{\{8\}}$	0.318034$^{\{5\}}$	0.299318$^{\{4\}}$
	MSE	$\hat{\alpha}$	0.238065$^{\{4\}}$	0.23846$^{\{5\}}$	0.251165$^{\{8\}}$	0.214113$^{\{1\}}$	0.247208$^{\{7\}}$	0.243114$^{\{6\}}$	0.228906$^{\{2\}}$	0.233491$^{\{3\}}$
		$\hat{\beta}$	0.053204$^{\{2\}}$	0.058059$^{\{3\}}$	0.080683$^{\{7\}}$	0.052694$^{\{1\}}$	0.073562$^{\{5\}}$	0.079479$^{\{6\}}$	0.088944$^{\{8\}}$	0.062551$^{\{4\}}$
		$\hat{p}$	0.101786$^{\{1\}}$	0.109599$^{\{3\}}$	0.132364$^{\{8\}}$	0.106819$^{\{2\}}$	0.130712$^{\{6\}}$	0.131822$^{\{7\}}$	0.129793$^{\{5\}}$	0.118067$^{\{4\}}$
	MRE	$\hat{\alpha}$	0.249502$^{\{3\}}$	0.252781$^{\{4\}}$	0.262189$^{\{7\}}$	0.247956$^{\{2\}}$	0.265226$^{\{8\}}$	0.260286$^{\{6\}}$	0.24778$^{\{1\}}$	0.25518$^{\{5\}}$
		$\hat{\beta}$	0.240863$^{\{1\}}$	0.254993$^{\{3\}}$	0.295591$^{\{6\}}$	0.246815$^{\{2\}}$	0.285612$^{\{5\}}$	0.296739$^{\{7\}}$	0.313377$^{\{8\}}$	0.264422$^{\{4\}}$
		$\hat{p}$	0.542452$^{\{1\}}$	0.570611$^{\{3\}}$	0.64323$^{\{7\}}$	0.564147$^{\{2\}}$	0.640604$^{\{6\}}$	0.644503$^{\{8\}}$	0.636069$^{\{5\}}$	0.598636$^{\{4\}}$
	$\sum Ranks$		17$^{\{2\}}$	31$^{\{3\}}$	63$^{\{8\}}$	16$^{\{1\}}$	56$^{\{6\}}$	61$^{\{7\}}$	43$^{\{5\}}$	37$^{\{4\}}$
75	BIAS	$\hat{\alpha}$	0.345682$^{\{2\}}$	0.3508$^{\{5\}}$	0.376853$^{\{8\}}$	0.335996$^{\{1\}}$	0.376757$^{\{7\}}$	0.368559$^{\{6\}}$	0.349124$^{\{4\}}$	0.345925$^{\{3\}}$
		$\hat{\beta}$	0.153032$^{\{1\}}$	0.161111$^{\{3\}}$	0.186727$^{\{6\}}$	0.156121$^{\{2\}}$	0.183838$^{\{5\}}$	0.191516$^{\{7\}}$	0.203488$^{\{8\}}$	0.164548$^{\{4\}}$
		$\hat{p}$	0.24434$^{\{1\}}$	0.257623$^{\{3\}}$	0.292734$^{\{7\}}$	0.250332$^{\{2\}}$	0.28943$^{\{6\}}$	0.293429$^{\{8\}}$	0.289382$^{\{5\}}$	0.2676$^{\{4\}}$
	MSE	$\hat{\alpha}$	0.210311$^{\{5\}}$	0.209248$^{\{4\}}$	0.236532$^{\{8\}}$	0.179442$^{\{1\}}$	0.231219$^{\{7\}}$	0.222131$^{\{6\}}$	0.208492$^{\{3\}}$	0.200666$^{\{2\}}$
		$\hat{\beta}$	0.037472$^{\{2\}}$	0.040799$^{\{3\}}$	0.055329$^{\{6\}}$	0.036848$^{\{1\}}$	0.053235$^{\{5\}}$	0.058042$^{\{7\}}$	0.064896$^{\{8\}}$	0.042723$^{\{4\}}$
		$\hat{p}$	0.08571$^{\{1\}}$	0.093437$^{\{3\}}$	0.114327$^{\{8\}}$	0.088539$^{\{2\}}$	0.112148$^{\{6\}}$	0.114062$^{\{7\}}$	0.111913$^{\{5\}}$	0.099575$^{\{4\}}$
	MRE	$\hat{\alpha}$	0.230455$^{\{2\}}$	0.233867$^{\{5\}}$	0.251235$^{\{8\}}$	0.223998$^{\{1\}}$	0.251171$^{\{7\}}$	0.245706$^{\{6\}}$	0.232749$^{\{4\}}$	0.230616$^{\{3\}}$
		$\hat{\beta}$	0.204043$^{\{1\}}$	0.214815$^{\{3\}}$	0.248969$^{\{6\}}$	0.208161$^{\{2\}}$	0.245117$^{\{5\}}$	0.255355$^{\{7\}}$	0.271317$^{\{8\}}$	0.219398$^{\{4\}}$
		$\hat{p}$	0.488679$^{\{1\}}$	0.515246$^{\{3\}}$	0.585468$^{\{7\}}$	0.500664$^{\{2\}}$	0.57886$^{\{6\}}$	0.586858$^{\{8\}}$	0.578765$^{\{5\}}$	0.535201$^{\{4\}}$
	$\sum Ranks$		16$^{\{2\}}$	32$^{\{3.5\}}$	64$^{\{8\}}$	14$^{\{1\}}$	54$^{\{6\}}$	62$^{\{7\}}$	50$^{\{5\}}$	32$^{\{3.5\}}$
100	BIAS	$\hat{\alpha}$	0.31868$^{\{2\}}$	0.328026$^{\{4\}}$	0.366431$^{\{8\}}$	0.300767$^{\{1\}}$	0.359169$^{\{7\}}$	0.347175$^{\{6\}}$	0.321891$^{\{3\}}$	0.330038$^{\{5\}}$
		$\hat{\beta}$	0.135372$^{\{1\}}$	0.144278$^{\{3\}}$	0.1643$^{\{5\}}$	0.136063$^{\{2\}}$	0.165753$^{\{6\}}$	0.174007$^{\{7\}}$	0.183846$^{\{8\}}$	0.147204$^{\{4\}}$
		$\hat{p}$	0.22473$^{\{1\}}$	0.241062$^{\{3\}}$	0.27055$^{\{7\}}$	0.22667$^{\{2\}}$	0.270131$^{\{5\}}$	0.27497$^{\{8\}}$	0.270169$^{\{6\}}$	0.245435$^{\{4\}}$
	MSE	$\hat{\alpha}$	0.181413$^{\{3\}}$	0.184376$^{\{4\}}$	0.22786$^{\{8\}}$	0.147937$^{\{1\}}$	0.216615$^{\{7\}}$	0.201759$^{\{6\}}$	0.181072$^{\{2\}}$	0.186081$^{\{5\}}$
		$\hat{\beta}$	0.028765$^{\{2\}}$	0.032805$^{\{3\}}$	0.042297$^{\{5\}}$	0.028096$^{\{1\}}$	0.042371$^{\{6\}}$	0.047144$^{\{7\}}$	0.052772$^{\{8\}}$	0.033536$^{\{4\}}$
		$\hat{p}$	0.074849$^{\{1\}}$	0.083907$^{\{3\}}$	0.100866$^{\{7\}}$	0.075702$^{\{2\}}$	0.100235$^{\{5\}}$	0.103547$^{\{8\}}$	0.100473$^{\{6\}}$	0.085951$^{\{4\}}$
	MRE	$\hat{\alpha}$	0.212453$^{\{2\}}$	0.218684$^{\{4\}}$	0.244287$^{\{8\}}$	0.200511$^{\{1\}}$	0.239446$^{\{7\}}$	0.23145$^{\{6\}}$	0.214594$^{\{3\}}$	0.220025$^{\{5\}}$
		$\hat{\beta}$	0.180495$^{\{1\}}$	0.192371$^{\{3\}}$	0.219067$^{\{5\}}$	0.181417$^{\{2\}}$	0.221004$^{\{6\}}$	0.23201$^{\{7\}}$	0.245127$^{\{8\}}$	0.196273$^{\{4\}}$
		$\hat{p}$	0.44946$^{\{1\}}$	0.482125$^{\{3\}}$	0.541101$^{\{7\}}$	0.453339$^{\{2\}}$	0.540262$^{\{5\}}$	0.54994$^{\{8\}}$	0.540337$^{\{6\}}$	0.490871$^{\{4\}}$
	$\sum Ranks$		14$^{\{1.5\}}$	30$^{\{3\}}$	60$^{\{7\}}$	14$^{\{1.5\}}$	54$^{\{6\}}$	63$^{\{8\}}$	50$^{\{5\}}$	39$^{\{4\}}$
125	BIAS	$\hat{\alpha}$	0.297562$^{\{2\}}$	0.311158$^{\{5\}}$	0.344945$^{\{7\}}$	0.281066$^{\{1\}}$	0.345138$^{\{8\}}$	0.331865$^{\{6\}}$	0.306671$^{\{3\}}$	0.311092$^{\{4\}}$
		$\hat{\beta}$	0.122619$^{\{1\}}$	0.131994$^{\{3\}}$	0.153242$^{\{6\}}$	0.125458$^{\{2\}}$	0.151159$^{\{5\}}$	0.160571$^{\{7\}}$	0.172868$^{\{8\}}$	0.13422$^{\{4\}}$
		$\hat{p}$	0.207209$^{\{1\}}$	0.223833$^{\{3\}}$	0.25505$^{\{5\}}$	0.210164$^{\{2\}}$	0.257584$^{\{8\}}$	0.256014$^{\{7\}}$	0.255716$^{\{6\}}$	0.22813$^{\{4\}}$
	MSE	$\hat{\alpha}$	0.160348$^{\{2\}}$	0.169147$^{\{5\}}$	0.206423$^{\{8\}}$	0.131544$^{\{1\}}$	0.200806$^{\{7\}}$	0.185582$^{\{6\}}$	0.166706$^{\{3\}}$	0.168391$^{\{4\}}$
		$\hat{\beta}$	0.023683$^{\{1\}}$	0.02675$^{\{3\}}$	0.036381$^{\{6\}}$	0.023782$^{\{2\}}$	0.034875$^{\{5\}}$	0.039728$^{\{7\}}$	0.045308$^{\{8\}}$	0.027995$^{\{4\}}$
		$\hat{p}$	0.065063$^{\{1\}}$	0.074403$^{\{3\}}$	0.091714$^{\{6\}}$	0.066339$^{\{2\}}$	0.092854$^{\{8\}}$	0.091557$^{\{5\}}$	0.091829$^{\{7\}}$	0.075971$^{\{4\}}$
	MRE	$\hat{\alpha}$	0.198375$^{\{2\}}$	0.207439$^{\{5\}}$	0.229963$^{\{7\}}$	0.187377$^{\{1\}}$	0.230092$^{\{8\}}$	0.221243$^{\{6\}}$	0.204447$^{\{3\}}$	0.207394$^{\{4\}}$
		$\hat{\beta}$	0.163493$^{\{1\}}$	0.175992$^{\{3\}}$	0.204323$^{\{6\}}$	0.167277$^{\{2\}}$	0.201545$^{\{5\}}$	0.214094$^{\{7\}}$	0.230491$^{\{8\}}$	0.17896$^{\{4\}}$
		$\hat{p}$	0.414418$^{\{1\}}$	0.447667$^{\{3\}}$	0.5101$^{\{5\}}$	0.420329$^{\{2\}}$	0.515169$^{\{8\}}$	0.512027$^{\{7\}}$	0.511433$^{\{6\}}$	0.45626$^{\{4\}}$
	$\sum Ranks$		12$^{\{1\}}$	33$^{\{3\}}$	56$^{\{6\}}$	15$^{\{2\}}$	62$^{\{8\}}$	58$^{\{7\}}$	52$^{\{5\}}$	36$^{\{4\}}$
150	BIAS	$\hat{\alpha}$	0.278285$^{\{2\}}$	0.293718$^{\{4\}}$	0.332728$^{\{8\}}$	0.261735$^{\{1\}}$	0.329662$^{\{7\}}$	0.310057$^{\{6\}}$	0.290877$^{\{3\}}$	0.298799$^{\{5\}}$
		$\hat{\beta}$	0.113109$^{\{1\}}$	0.122159$^{\{3\}}$	0.142331$^{\{6\}}$	0.115722$^{\{2\}}$	0.141304$^{\{5\}}$	0.150581$^{\{7\}}$	0.157188$^{\{8\}}$	0.123659$^{\{4\}}$
		$\hat{p}$	0.194139$^{\{1\}}$	0.21211$^{\{3\}}$	0.244118$^{\{7\}}$	0.196232$^{\{2\}}$	0.241793$^{\{6\}}$	0.244466$^{\{8\}}$	0.238085$^{\{5\}}$	0.214122$^{\{4\}}$
	MSE	$\hat{\alpha}$	0.140774$^{\{2\}}$	0.152407$^{\{4\}}$	0.192542$^{\{8\}}$	0.114977$^{\{1\}}$	0.185759$^{\{7\}}$	0.165215$^{\{6\}}$	0.149815$^{\{3\}}$	0.156557$^{\{5\}}$
		$\hat{\beta}$	0.020292$^{\{2\}}$	0.023078$^{\{3\}}$	0.031646$^{\{6\}}$	0.020192$^{\{1\}}$	0.030719$^{\{5\}}$	0.034741$^{\{7\}}$	0.037679$^{\{8\}}$	0.023796$^{\{4\}}$
		$\hat{p}$	0.058034$^{\{1\}}$	0.06756$^{\{3\}}$	0.085778$^{\{8\}}$	0.059313$^{\{2\}}$	0.08455$^{\{6\}}$	0.085689$^{\{7\}}$	0.082187$^{\{5\}}$	0.068187$^{\{4\}}$
	MRE	$\hat{\alpha}$	0.185523$^{\{2\}}$	0.195812$^{\{4\}}$	0.221819$^{\{8\}}$	0.17449$^{\{1\}}$	0.219775$^{\{7\}}$	0.206705$^{\{6\}}$	0.193918$^{\{3\}}$	0.199199$^{\{5\}}$
		$\hat{\beta}$	0.150811$^{\{1\}}$	0.162879$^{\{3\}}$	0.189774$^{\{6\}}$	0.154296$^{\{2\}}$	0.188406$^{\{5\}}$	0.200775$^{\{7\}}$	0.209584$^{\{8\}}$	0.164879$^{\{4\}}$
		$\hat{p}$	0.388277$^{\{1\}}$	0.42422$^{\{3\}}$	0.488235$^{\{7\}}$	0.392465$^{\{2\}}$	0.483585$^{\{6\}}$	0.488932$^{\{8\}}$	0.47617$^{\{5\}}$	0.428244$^{\{4\}}$
	$\sum Ranks$		13$^{\{1\}}$	30$^{\{3\}}$	64$^{\{8\}}$	14$^{\{2\}}$	54$^{\{6\}}$	62$^{\{7\}}$	48$^{\{5\}}$	39$^{\{4\}}$

Table 4. Simulation values of BIAS, MSE and MRE for ($\alpha = 1.5,\ \beta = 1.5,\ p = 0.3$).

n	Est.	Est. Par.	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8
25	BIAS	$\hat{\alpha}$	0.403746{1}	0.407905{4}	0.425391{8}	0.407344{3}	0.420375{7}	0.419004{6}	0.403998{2}	0.409839{5}
		$\hat{\beta}$	0.392063{1}	0.405417{2}	0.445954{6}	0.418031{3}	0.451674{7}	0.424954{4}	0.459419{8}	0.425886{5}
		$\hat{p}$	0.278057{1}	0.286568{2}	0.293516{3}	0.297413{6}	0.301364{7}	0.295561{5}	0.301606{8}	0.294878{4}
	MSE	$\hat{\alpha}$	0.27773{7}	0.272792{5}	0.292725{8}	0.250539{1}	0.270047{4}	0.27515{6}	0.264287{2}	0.2663{3}
		$\hat{\beta}$	0.257319{1}	0.268341{3}	0.328133{7}	0.266225{2}	0.326578{6}	0.288465{4}	0.349379{8}	0.291237{5}
		$\hat{p}$	0.090347{1}	0.09706{2}	0.099797{3}	0.10626{6}	0.106958{8}	0.104182{5}	0.10628{7}	0.102084{4}
	MRE	$\hat{\alpha}$	0.269164{1}	0.271937{4}	0.283594{8}	0.271563{3}	0.28025{7}	0.279336{6}	0.269332{2}	0.273226{5}
		$\hat{\beta}$	0.261375{1}	0.270278{2}	0.297303{6}	0.278688{3}	0.301116{7}	0.283302{4}	0.306279{8}	0.283924{5}
		$\hat{p}$	0.926857{1}	0.955226{2}	0.978388{3}	0.991378{6}	1.004547{7}	0.985202{5}	1.005352{8}	0.982926{4}
	$\sum$ *Ranks*		15{1}	26{2}	52{6}	33{3}	60{8}	45{5}	53{7}	40{4}
50	BIAS	$\hat{\alpha}$	0.359996{5}	0.358306{4}	0.377875{8}	0.340348{1}	0.367862{6}	0.368344{7}	0.358104{3}	0.35295{2}
		$\hat{\beta}$	0.284975{1}	0.303214{2}	0.341287{7}	0.304188{3}	0.340215{6}	0.313058{4}	0.354671{8}	0.315107{5}
		$\hat{p}$	0.255822{1}	0.265886{3}	0.275822{5}	0.265838{2}	0.280121{7}	0.276182{6}	0.280344{8}	0.271023{4}
	MSE	$\hat{\alpha}$	0.234459{7}	0.226027{4}	0.24643{8}	0.19027{1}	0.226618{5}	0.23076{6}	0.222034{3}	0.214073{2}
		$\hat{\beta}$	0.131152{1}	0.146028{3}	0.189983{7}	0.143397{2}	0.184705{6}	0.156712{4}	0.205049{8}	0.157554{5}
		$\hat{p}$	0.080131{1}	0.085704{2}	0.090543{5}	0.087113{3}	0.093603{7}	0.091481{6}	0.093948{8}	0.088471{4}
	MRE	$\hat{\alpha}$	0.239997{5}	0.238871{4}	0.251917{8}	0.226899{1}	0.245241{6}	0.245562{7}	0.238736{3}	0.2353{2}
		$\hat{\beta}$	0.189983{1}	0.202143{2}	0.227525{7}	0.202792{3}	0.22681{6}	0.208705{4}	0.236447{8}	0.210072{5}
		$\hat{p}$	0.852739{1}	0.886285{3}	0.919406{5}	0.886127{2}	0.933736{7}	0.920606{6}	0.93448{8}	0.90341{4}
	$\sum$ *Ranks*		23{2}	27{3}	60{8}	18{1}	56{6}	50{5}	57{7}	33{4}
75	BIAS	$\hat{\alpha}$	0.329021{4}	0.329053{5}	0.348184{8}	0.306296{1}	0.347593{7}	0.342291{6}	0.329005{3}	0.322367{2}
		$\hat{\beta}$	0.241773{1}	0.254461{2}	0.289844{7}	0.256962{3}	0.286365{6}	0.265256{5}	0.306079{8}	0.263188{4}
		$\hat{p}$	0.242949{1}	0.25248{3}	0.264312{6}	0.249284{2}	0.269238{7}	0.260897{5}	0.269295{8}	0.253309{4}
	MSE	$\hat{\alpha}$	0.201993{5}	0.200528{4}	0.220709{8}	0.16211{1}	0.213232{7}	0.20772{6}	0.196193{3}	0.190043{2}
		$\hat{\beta}$	0.092198{1}	0.102254{3}	0.1351{7}	0.102248{2}	0.12988{6}	0.110961{5}	0.151632{8}	0.109181{4}
		$\hat{p}$	0.07384{1}	0.078396{3}	0.083939{6}	0.077991{2}	0.087204{7}	0.083295{5}	0.088128{8}	0.079273{4}
	MRE	$\hat{\alpha}$	0.219347{4}	0.219369{5}	0.232123{8}	0.204197{1}	0.231729{7}	0.228194{6}	0.219337{3}	0.214911{2}
		$\hat{\beta}$	0.161182{1}	0.16964{2}	0.193229{7}	0.171308{3}	0.19091{6}	0.176838{5}	0.204053{8}	0.175459{4}
		$\hat{p}$	0.809828{1}	0.841599{3}	0.881042{6}	0.830947{2}	0.89746{7}	0.869655{5}	0.89765{8}	0.844365{4}
	$\sum$ *Ranks*		19{2}	30{3.5}	63{8}	17{1}	60{7}	48{5}	57{6}	30{3.5}
100	BIAS	$\hat{\alpha}$	0.301891{2}	0.305291{4}	0.328256{8}	0.280379{1}	0.325124{6}	0.328078{7}	0.304184{3}	0.306966{5}
		$\hat{\beta}$	0.214136{1}	0.224368{2}	0.255908{6}	0.225041{3}	0.257388{7}	0.234428{5}	0.26916{8}	0.233571{4}
		$\hat{p}$	0.231789{1}	0.240802{3}	0.255153{6}	0.233778{2}	0.259896{8}	0.251249{5}	0.256525{7}	0.245569{4}
	MSE	$\hat{\alpha}$	0.174433{4}	0.174259{3}	0.198801{8}	0.136191{1}	0.192083{6}	0.198567{7}	0.172208{2}	0.176849{5}
		$\hat{\beta}$	0.071781{1}	0.079534{3}	0.104308{6}	0.078478{2}	0.104584{7}	0.086941{5}	0.116829{8}	0.086838{4}
		$\hat{p}$	0.068116{1}	0.072389{3}	0.079396{6}	0.06977{2}	0.081738{8}	0.078247{5}	0.080958{7}	0.074851{4}
	MRE	$\hat{\alpha}$	0.201261{2}	0.203527{4}	0.218837{8}	0.186919{1}	0.216749{6}	0.218718{7}	0.202789{3}	0.204644{5}
		$\hat{\beta}$	0.142757{1}	0.149578{2}	0.170605{6}	0.150028{3}	0.171592{7}	0.156285{5}	0.17944{8}	0.155714{4}
		$\hat{p}$	0.772629{1}	0.802675{3}	0.850509{6}	0.77926{2}	0.866321{8}	0.837495{5}	0.855084{7}	0.818563{4}
	$\sum$ *Ranks*		14{1}	27{3}	60{7}	17{2}	63{8}	51{5}	53{6}	39{4}
125	BIAS	$\hat{\alpha}$	0.286159{2}	0.292177{5}	0.319995{8}	0.264775{1}	0.312112{6}	0.313091{7}	0.286491{3}	0.290815{4}
		$\hat{\beta}$	0.194418{1}	0.207276{3}	0.23449{7}	0.205018{2}	0.23236{6}	0.21523{5}	0.249095{8}	0.213666{4}
		$\hat{p}$	0.221803{1}	0.231683{3}	0.24908{6}	0.222341{2}	0.249571{8}	0.243698{5}	0.249436{7}	0.236647{4}
	MSE	$\hat{\alpha}$	0.158345{3}	0.16351{5}	0.192521{8}	0.125964{1}	0.181012{6}	0.182864{7}	0.155432{2}	0.160329{4}
		$\hat{\beta}$	0.059425{1}	0.067471{3}	0.086347{7}	0.065106{2}	0.086148{6}	0.073824{5}	0.099496{8}	0.071341{4}
		$\hat{p}$	0.062992{1}	0.068004{3}	0.076491{6}	0.064627{2}	0.077169{7}	0.074341{5}	0.077172{8}	0.070338{4}
	MRE	$\hat{\alpha}$	0.190773{2}	0.194784{5}	0.21333{8}	0.176517{1}	0.208075{6}	0.208727{7}	0.190994{3}	0.193877{4}
		$\hat{\beta}$	0.129612{1}	0.138184{3}	0.156326{7}	0.136679{2}	0.154907{6}	0.143486{5}	0.166064{8}	0.142444{4}
		$\hat{p}$	0.739342{1}	0.772276{3}	0.830266{6}	0.741136{2}	0.831904{8}	0.812328{5}	0.831453{7}	0.788823{4}
	$\sum$ *Ranks*		13{1}	33{3}	63{8}	15{2}	59{7}	51{5}	54{6}	36{4}
150	BIAS	$\hat{\alpha}$	0.26308{2}	0.270109{3}	0.304752{8}	0.244961{1}	0.301563{7}	0.294663{6}	0.271687{4}	0.272606{5}
		$\hat{\beta}$	0.183261{1}	0.191297{3}	0.221729{7}	0.186114{2}	0.216654{6}	0.200184{5}	0.230175{8}	0.1978{4}
		$\hat{p}$	0.211407{1}	0.220585{3}	0.242269{7}	0.211605{2}	0.243099{8}	0.235471{5}	0.239003{6}	0.226618{4}
	MSE	$\hat{\alpha}$	0.133139{2}	0.140801{3}	0.175874{8}	0.107429{1}	0.17142{7}	0.162929{6}	0.142362{5}	0.141574{4}
		$\hat{\beta}$	0.052084{1}	0.057097{3}	0.077409{7}	0.054148{2}	0.074415{6}	0.062905{5}	0.084193{8}	0.060488{4}
		$\hat{p}$	0.058579{1}	0.062672{3}	0.073019{7}	0.059353{2}	0.073697{8}	0.069825{5}	0.071952{6}	0.065234{4}
	MRE	$\hat{\alpha}$	0.175387{2}	0.180073{3}	0.203168{8}	0.163307{1}	0.201042{7}	0.196442{6}	0.181125{4}	0.181737{5}
		$\hat{\beta}$	0.122174{1}	0.127531{3}	0.147819{7}	0.124076{2}	0.144436{6}	0.133456{5}	0.15345{8}	0.131867{4}
		$\hat{p}$	0.704689{1}	0.735283{3}	0.807564{7}	0.705351{2}	0.810329{8}	0.784904{5}	0.796676{6}	0.755392{4}
	$\sum$ *Ranks*		12{1}	27{3}	66{8}	15{2}	63{7}	48{5}	55{6}	38{4}

Table 5. Simulation values of BIAS, MSE and MRE for ($\alpha = 2.5,\ \beta = 2.5,\ p = 0.9$).

n	Est.	Est. Par.	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8
25	BIAS	$\hat{\alpha}$	0.858946[1]	0.918776[3]	0.985095[6]	0.965535[4]	1.03761[7]	1.281759[8]	0.896977[2]	0.972994[5]
		$\hat{\beta}$	1.852059[4]	1.849321[3]	2.408636[7]	1.549571[1]	2.193075[6]	1.696557[2]	2.852911[8]	2.043346[5]
		$\hat{p}$	0.227399[2]	0.237316[3]	0.340125[7]	0.201621[1]	0.316646[5]	0.383855[8]	0.324208[6]	0.274547[4]
	MSE	$\hat{\alpha}$	1.024469[1]	1.123682[3]	1.248262[6]	1.182519[4]	1.344654[7]	1.967689[8]	1.074019[2]	1.215117[5]
		$\hat{\beta}$	7.131575[4]	6.613852[3]	10.839655[7]	4.204113[1]	8.654809[6]	5.506888[2]	15.519889[8]	8.205978[5]
		$\hat{p}$	0.127676[2]	0.138405[3]	0.233363[7]	0.109775[1]	0.212767[5]	0.27599[8]	0.216356[6]	0.173098[4]
	MRE	$\hat{\alpha}$	0.343579[1]	0.36751[3]	0.394038[6]	0.386214[4]	0.415044[7]	0.512703[8]	0.358791[2]	0.389197[5]
		$\hat{\beta}$	0.740823[4]	0.739728[3]	0.963454[7]	0.619828[1]	0.87723[6]	0.678623[2]	1.141164[8]	0.817338[5]
		$\hat{p}$	0.252665[2]	0.263685[3]	0.377917[7]	0.224023[1]	0.351829[5]	0.426506[8]	0.360231[6]	0.305052[4]
	$\sum$ *Ranks*		23[2]	29[3]	54[6]	20[1]	56[7.5]	56[7.5]	42[4]	44[5]
50	BIAS	$\hat{\alpha}$	0.76617[1]	0.808795[4]	0.886604[6]	0.790624[2]	0.912196[7]	1.133855[8]	0.794417[3]	0.825447[5]
		$\hat{\beta}$	1.161002[3]	1.249331[4]	1.715644[7]	1.080913[1]	1.638662[6]	1.158025[2]	2.037234[8]	1.384381[5]
		$\hat{p}$	0.121476[2]	0.136092[3]	0.216937[7]	0.11226[1]	0.207987[6]	0.257767[8]	0.204169[5]	0.160239[4]
	MSE	$\hat{\alpha}$	0.872245[2]	0.933137[4]	1.05983[6]	0.862568[1]	1.099882[7]	1.598612[8]	0.901793[3]	0.939269[5]
		$\hat{\beta}$	2.595721[3]	2.883644[4]	5.44836[7]	1.953508[1]	4.781381[6]	2.431094[2]	7.683586[8]	3.535987[5]
		$\hat{p}$	0.042651[2]	0.052384[3]	0.117029[7]	0.036986[1]	0.10862[6]	0.155071[8]	0.104023[5]	0.071511[4]
	MRE	$\hat{\alpha}$	0.306468[1]	0.323518[4]	0.354641[6]	0.316249[2]	0.364879[7]	0.453542[8]	0.317767[3]	0.330179[5]
		$\hat{\beta}$	0.464401[3]	0.499733[4]	0.686258[7]	0.432365[1]	0.655465[6]	0.46321[2]	0.814893[8]	0.553752[5]
		$\hat{p}$	0.134973[2]	0.151213[3]	0.241041[7]	0.124733[1]	0.231097[6]	0.286408[8]	0.226854[5]	0.178043[4]
	$\sum$ *Ranks*		19[2]	33[3]	60[8]	11[1]	57[7]	54[6]	48[5]	42[4]
75	BIAS	$\hat{\alpha}$	0.698398[2]	0.740335[4]	0.813787[6]	0.692388[1]	0.837844[7]	1.048911[8]	0.735348[3]	0.746651[5]
		$\hat{\beta}$	0.884664[2]	0.989967[4]	1.376975[7]	0.874796[1]	1.322782[6]	0.904891[3]	1.601755[8]	1.084892[5]
		$\hat{p}$	0.082258[2]	0.097321[3]	0.156268[7]	0.081721[1]	0.154614[6]	0.19198[8]	0.147382[5]	0.111423[4]
	MSE	$\hat{\alpha}$	0.765459[2]	0.821595[5]	0.936457[6]	0.693943[1]	0.973528[7]	1.415093[8]	0.798324[3]	0.81151[4]
		$\hat{\beta}$	1.42356[3]	1.729049[4]	3.429264[7]	1.23332[1]	3.043453[6]	1.401448[2]	4.596074[8]	2.126172[5]
		$\hat{p}$	0.018505[1]	0.026497[3]	0.06838[7]	0.018939[2]	0.064714[6]	0.09548[8]	0.058772[5]	0.035536[4]
	MRE	$\hat{\alpha}$	0.279359[2]	0.296134[4]	0.325515[6]	0.276955[1]	0.335138[7]	0.419565[8]	0.294139[3]	0.298661[5]
		$\hat{\beta}$	0.353865[2]	0.395987[4]	0.55079[7]	0.349918[1]	0.529113[6]	0.361956[3]	0.640702[8]	0.433957[5]
		$\hat{p}$	0.091397[2]	0.108135[3]	0.173631[7]	0.090801[1]	0.171793[6]	0.213311[8]	0.163758[5]	0.123803[4]
	$\sum$ *Ranks*		18[2]	34[3]	60[8]	10[1]	57[7]	56[6]	48[5]	41[4]
100	BIAS	$\hat{\alpha}$	0.667656[2]	0.685517[3]	0.774683[7]	0.629503[1]	0.771076[6]	0.999678[8]	0.687054[4]	0.709363[5]
		$\hat{\beta}$	0.752113[2]	0.860077[4]	1.159436[7]	0.729878[1]	1.132593[6]	0.779635[3]	1.341641[8]	0.916105[5]
		$\hat{p}$	0.067215[2]	0.080779[3]	0.119445[6]	0.063324[1]	0.121582[7]	0.158819[8]	0.117345[5]	0.087948[4]
	MSE	$\hat{\alpha}$	0.72473[3]	0.732808[4]	0.88201[7]	0.59723[1]	0.851152[6]	1.310588[8]	0.717834[2]	0.757744[5]
		$\hat{\beta}$	1.029261[3]	1.280691[4]	2.34312[7]	0.843421[1]	2.259546[6]	1.013439[2]	3.212537[8]	1.489619[5]
		$\hat{p}$	0.011343[2]	0.016932[3]	0.039265[6]	0.010245[1]	0.040543[7]	0.069176[8]	0.038294[5]	0.01997[4]
	MRE	$\hat{\alpha}$	0.267062[2]	0.274207[3]	0.309873[7]	0.251801[1]	0.30843[6]	0.399871[8]	0.274822[4]	0.283745[5]
		$\hat{\beta}$	0.300845[2]	0.344031[4]	0.463774[7]	0.291951[1]	0.453037[6]	0.311854[3]	0.536656[8]	0.366442[5]
		$\hat{p}$	0.074683[2]	0.089755[3]	0.132717[6]	0.07036[1]	0.135091[7]	0.176465[8]	0.130383[5]	0.09772[4]
	$\sum$ *Ranks*		20[2]	31[3]	60[8]	9[1]	57[7]	56[6]	49[5]	42[4]
125	BIAS	$\hat{\alpha}$	0.62472[2]	0.652303[3]	0.735088[6]	0.5953[1]	0.738545[7]	0.927756[8]	0.661154[4]	0.665678[5]
		$\hat{\beta}$	0.647041[1]	0.74455[4]	1.022597[7]	0.647322[2]	1.015423[6]	0.682969[3]	1.178052[8]	0.792327[5]
		$\hat{p}$	0.056086[2]	0.06582[3]	0.101022[6]	0.053679[1]	0.104035[7]	0.125956[8]	0.095268[5]	0.072212[4]
	MSE	$\hat{\alpha}$	0.655818[2]	0.679237[3]	0.808795[7]	0.547405[1]	0.794708[6]	1.173006[8]	0.691347[4]	0.695482[5]
		$\hat{\beta}$	0.739885[2]	0.956316[4]	1.816763[7]	0.664472[1]	1.760827[6]	0.777984[3]	2.395581[8]	1.082904[5]
		$\hat{p}$	0.007245[2]	0.010434[3]	0.028545[6]	0.007008[1]	0.029895[7]	0.044752[8]	0.023237[5]	0.013447[4]
	MRE	$\hat{\alpha}$	0.249888[2]	0.260921[3]	0.294035[6]	0.23812[1]	0.295418[7]	0.371102[8]	0.264462[4]	0.266271[5]
		$\hat{\beta}$	0.258816[1]	0.29782[4]	0.409039[7]	0.258929[2]	0.406169[6]	0.273187[3]	0.471221[8]	0.316931[5]
		$\hat{p}$	0.062318[2]	0.073133[3]	0.112247[6]	0.059643[1]	0.115594[7]	0.139951[8]	0.105853[5]	0.080235[4]
	$\sum$ *Ranks*		16[2]	30[3]	58[7]	11[1]	59[8]	57[6]	51[5]	42[4]
150	BIAS	$\hat{\alpha}$	0.586758[2]	0.630992[4]	0.698849[6]	0.548019[1]	0.710583[7]	0.907536[8]	0.622597[3]	0.631003[5]
		$\hat{\beta}$	0.585925[1]	0.699227[5]	0.936644[7]	0.590736[2]	0.918488[6]	0.611663[3]	1.075592[8]	0.695614[4]
		$\hat{p}$	0.04922[2]	0.061926[4]	0.089823[7]	0.047982[1]	0.088862[6]	0.113046[8]	0.086796[5]	0.060798[3]
	MSE	$\hat{\alpha}$	0.58236[2]	0.649895[5]	0.738343[6]	0.47904[1]	0.757883[7]	1.114696[8]	0.619677[3]	0.639251[4]
		$\hat{\beta}$	0.589443[2]	0.836799[5]	1.509636[7]	0.55255[1]	1.443394[6]	0.628927[3]	1.993289[8]	0.814652[4]
		$\hat{p}$	0.005295[1]	0.008959[4]	0.020939[7]	0.005445[2]	0.020562[6]	0.035698[8]	0.018934[5]	0.00822[3]
	MRE	$\hat{\alpha}$	0.234703[2]	0.252397[4]	0.27954[6]	0.219208[1]	0.284233[7]	0.363014[8]	0.249039[3]	0.252401[5]
		$\hat{\beta}$	0.23437[1]	0.279691[5]	0.374658[7]	0.236294[2]	0.367395[6]	0.244665[3]	0.430237[8]	0.278246[4]
		$\hat{p}$	0.054689[2]	0.068807[4]	0.099803[7]	0.053314[1]	0.098736[6]	0.125606[8]	0.09644[5]	0.067553[3]
	$\sum$ *Ranks*		15[2]	40[4]	60[8]	12[1]	57[6.5]	57[6.5]	48[5]	35[3]

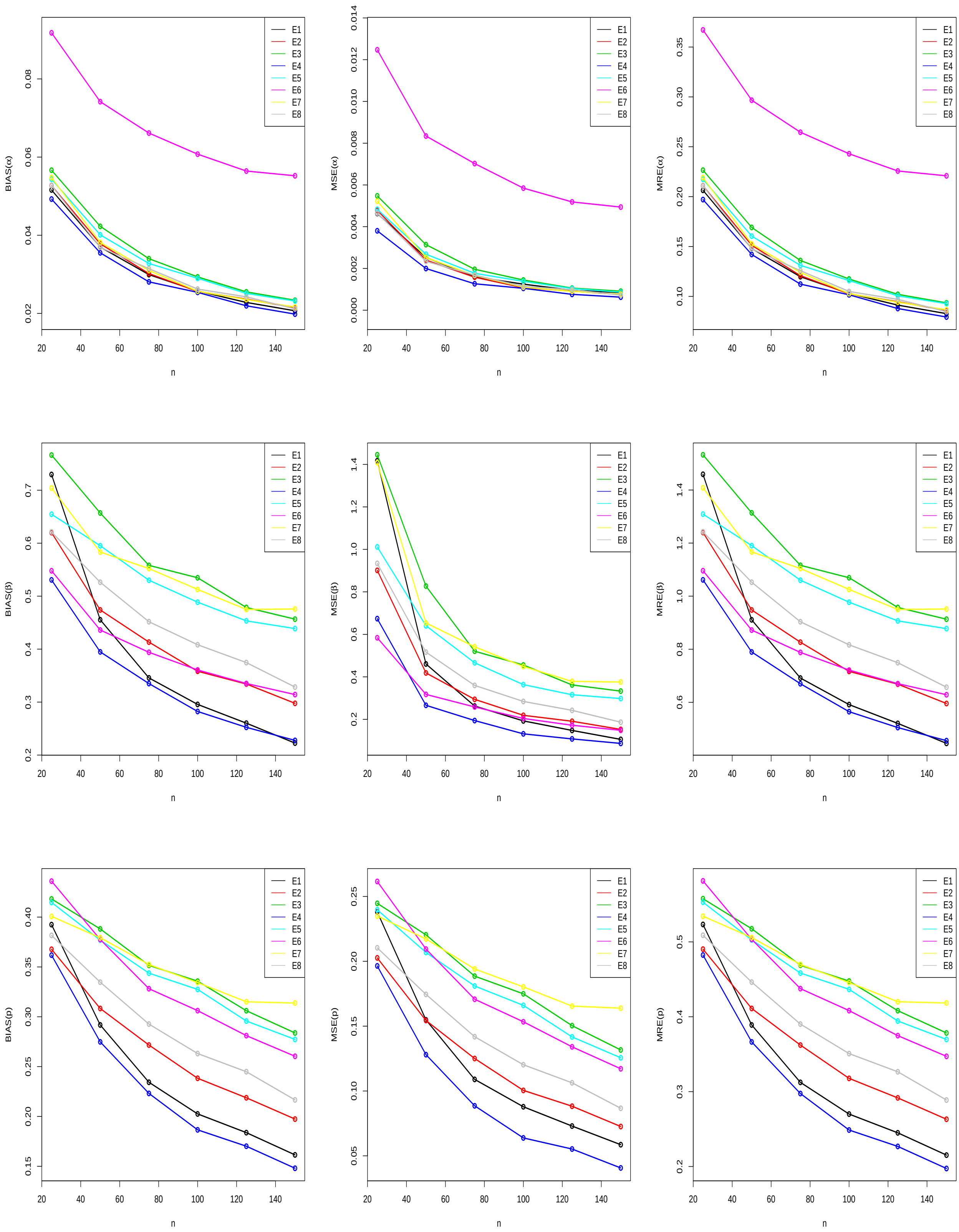

Figure 3. Graphical representation of BIAS, MSE, and MRE values in Table 1.

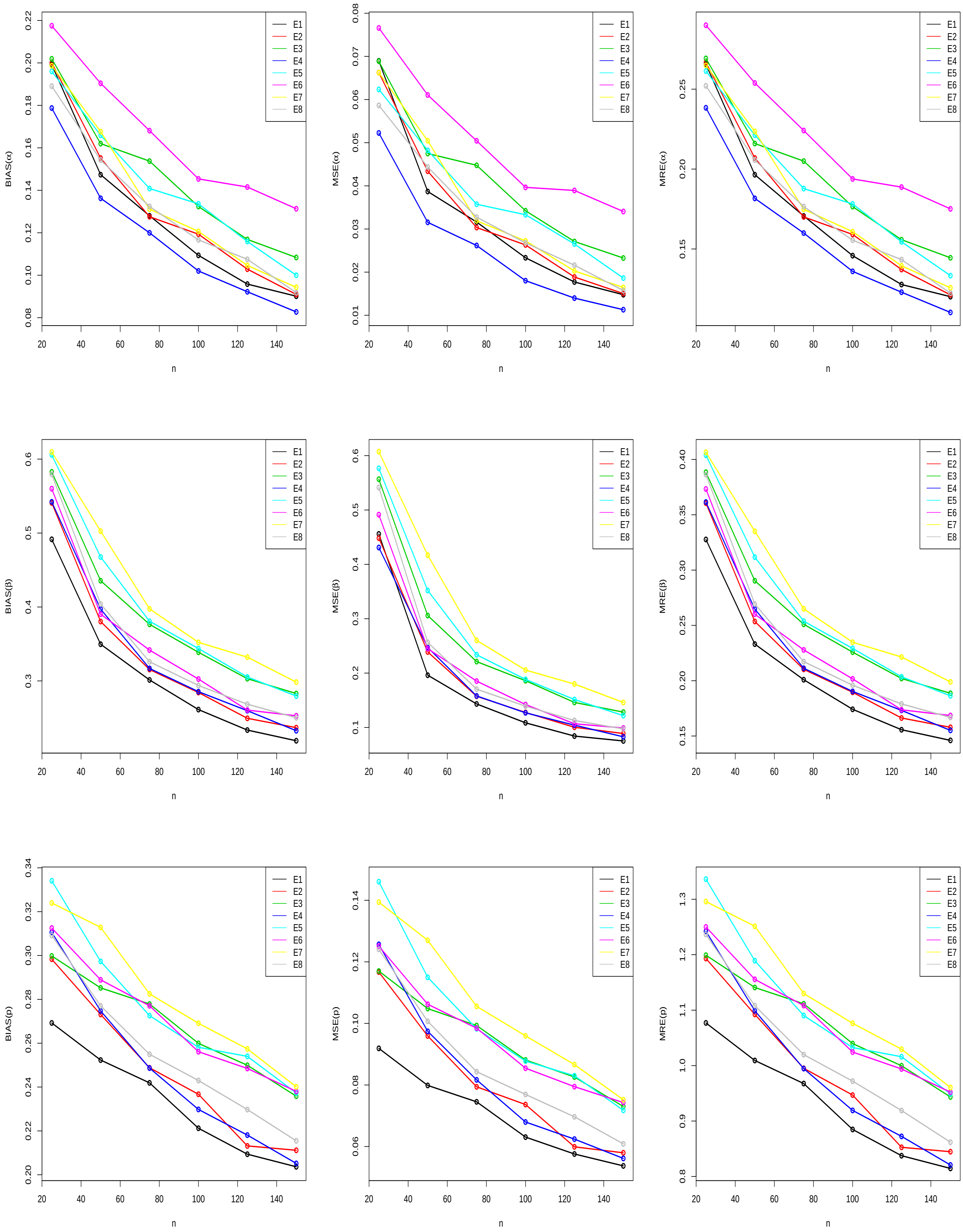

Figure 4. Graphical representation of BIAS, MSE, and MRE values in Table 2.

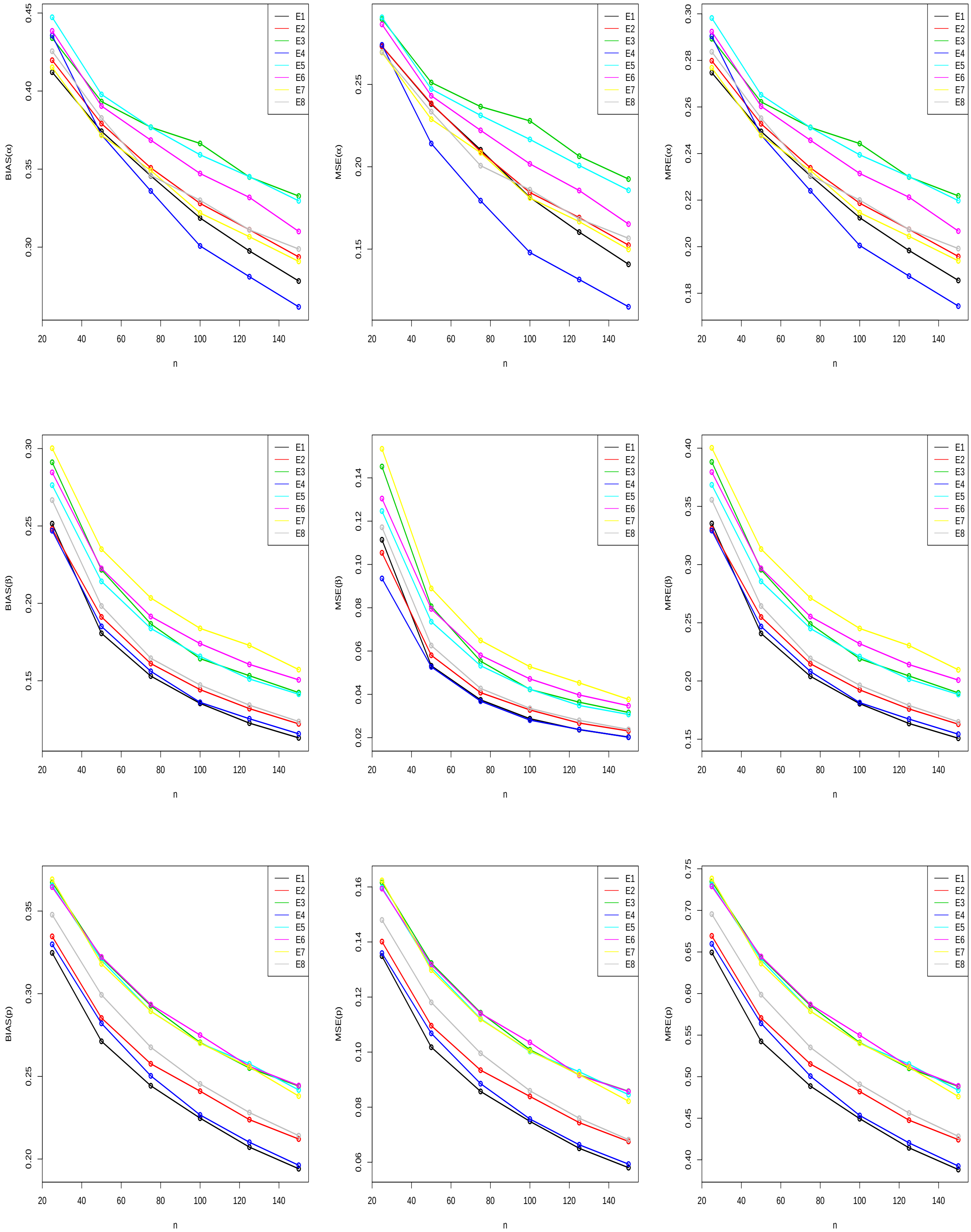

Figure 5. Graphical representation of BIAS, MS,E and MRE values in Table 3.

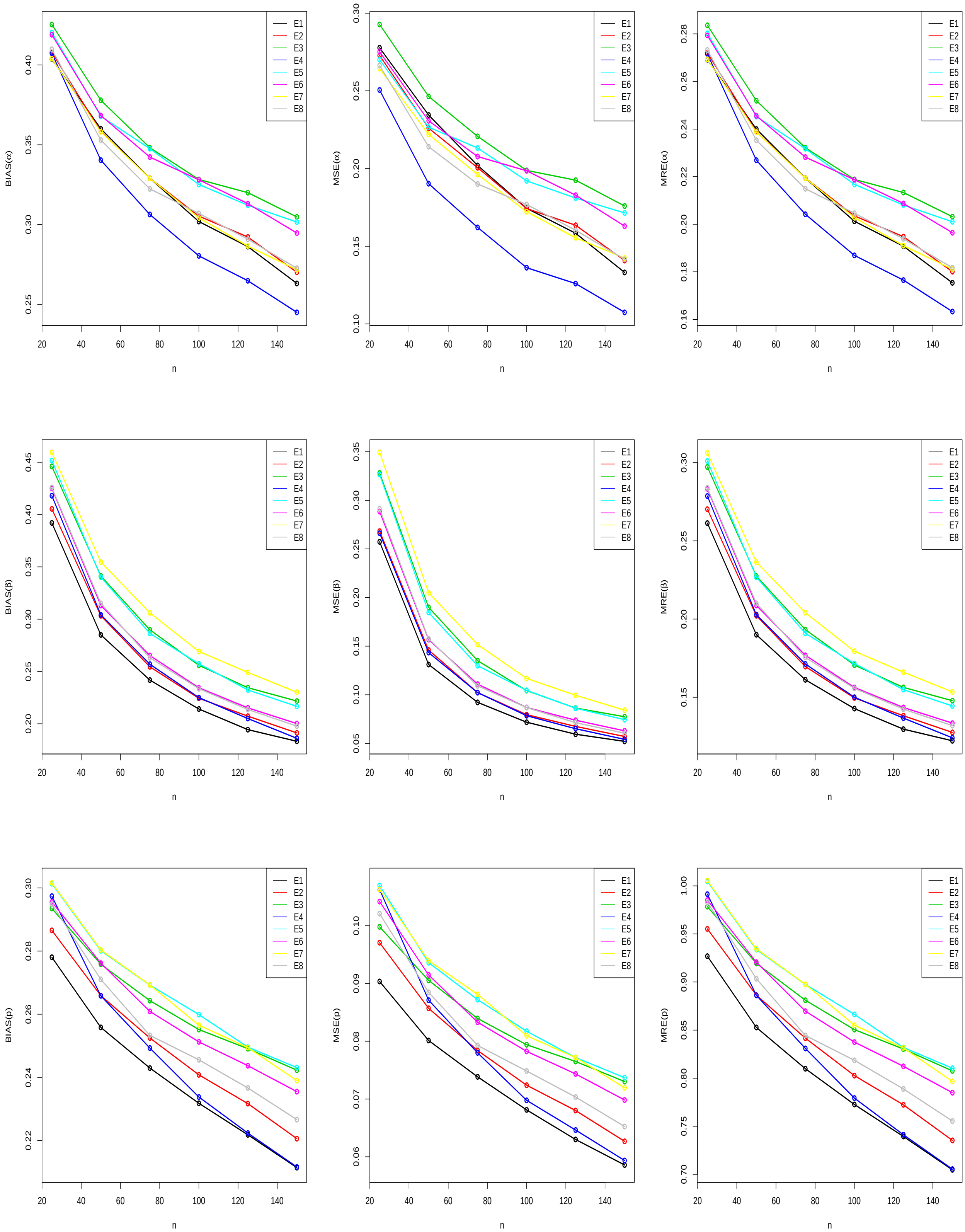

Figure 6. Graphical representation of BIAS, MSE, and MRE values in Table 4.

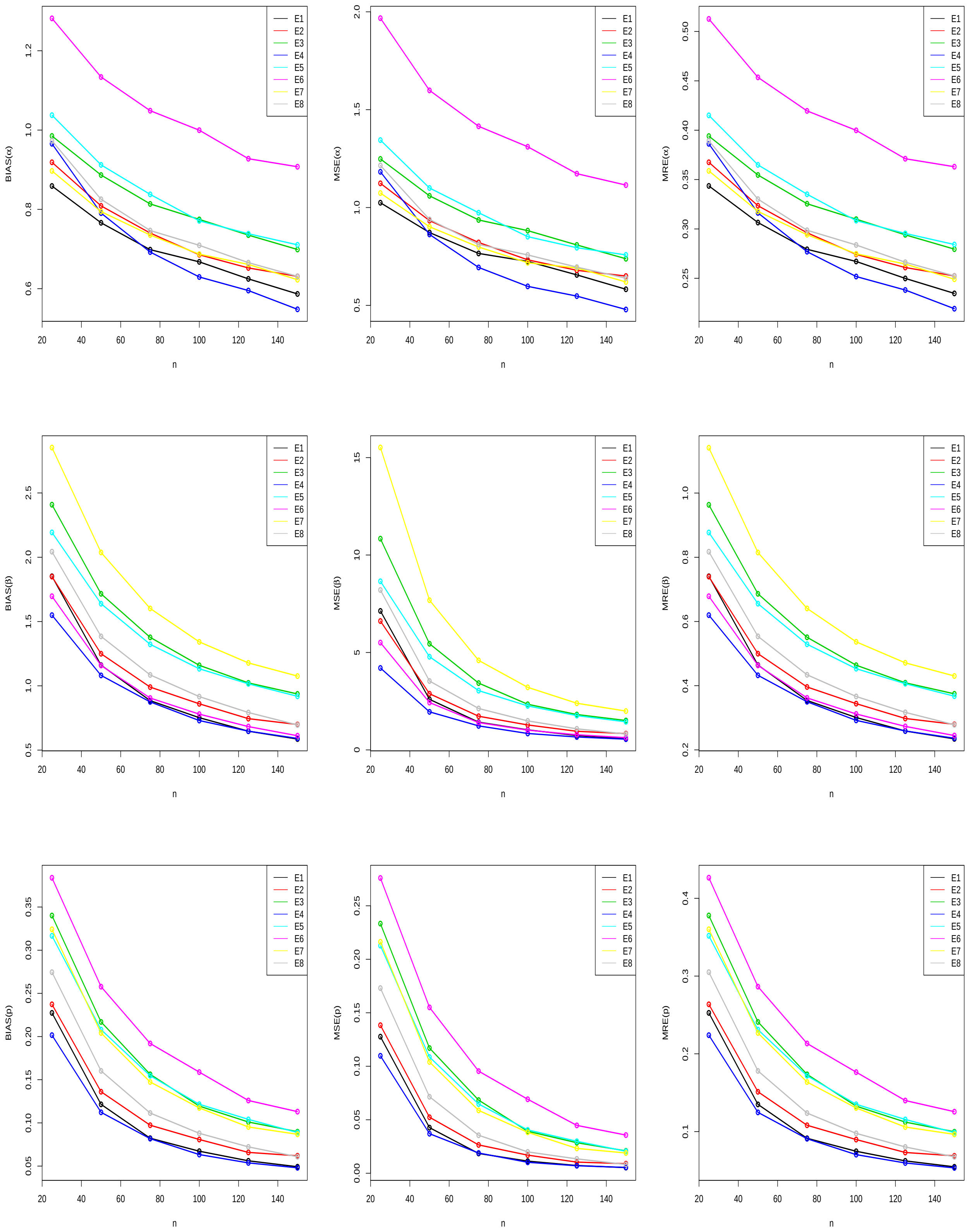

Figure 7. Graphical representation of BIAS, MSE, and MRE values in Table 5.

Table 6. Partial and overall ranks of all the methods of estimation of the proposed distribution by various values of model parameters.

Parameter	n	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8
$\alpha = 0.25,\ \beta = 0.5,\ p = 0.75$	25	4.0	2.0	8.0	1.0	5.0	7.0	6.0	3.0
	50	2.0	3.0	8.0	1.0	6.0	5.0	7.0	4.0
	75	2.0	3.0	8.0	1.0	6.0	5.0	7.0	4.0
	100	2.5	2.5	8.0	1.0	7.0	5.0	6.0	4.0
	125	2.0	3.0	8.0	1.0	6.5	5.0	6.5	4.0
	150	2.0	3.0	8.0	1.0	6.0	5.0	7.0	4.0
$\alpha = 0.75,\ \beta = 1.5,\ p = 0.25$	25	1.0	3.0	5.0	2.0	7.0	6.0	8.0	4.0
	50	1.0	2.0	5.0	3.0	7.0	6.0	8.0	4.0
	75	1.0	3.0	7.5	2.0	5.0	6.0	7.5	4.0
	100	1.0	3.0	6.0	2.0	7.0	5.0	8.0	4.0
	125	1.0	2.5	6.0	2.5	7.5	5.0	7.5	4.0
	150	1.0	3.0	6.0	2.0	5.0	7.0	8.0	4.0
$\alpha = 1.5,\ \beta = 0.75,\ p = 0.5$	25	1.0	2.0	8.0	3.0	7.0	5.5	5.5	4.0
	50	2.0	3.0	8.0	1.0	6.0	7.0	5.0	4.0
	75	2.0	3.5	8.0	1.0	6.0	7.0	5.0	3.5
	100	1.5	3.0	7.0	1.5	6.0	8.0	5.0	4.0
	125	1.0	3.0	6.0	2.0	8.0	7.0	5.0	4.0
	150	1.0	3.0	8.0	2.0	6.0	7.0	5.0	4.0
$\alpha = 1.5,\ \beta = 1.5,\ p = 0.3$	25	1.0	2.0	6.0	3.0	8.0	5.0	7.0	4.0
	50	2.0	3.0	8.0	1.0	6.0	5.0	7.0	4.0
	75	2.0	3.5	8.0	1.0	7.0	5.0	6.0	3.5
	100	1.0	3.0	7.0	2.0	8.0	5.0	6.0	4.0
	125	1.0	3.0	8.0	2.0	7.0	5.0	6.0	4.0
	150	1.0	3.0	8.0	2.0	7.0	5.0	6.0	4.0
$\alpha = 2.5,\ \beta = 2.5,\ p = 0.9$	25	2.0	3.0	6.0	1.0	7.5	7.5	4.0	5.0
	50	2.0	3.0	8.0	1.0	7.0	6.0	5.0	4.0
	75	2.0	3.0	8.0	1.0	7.0	6.0	5.0	4.0
	100	2.0	3.0	8.0	1.0	7.0	6.0	5.0	4.0
	125	2.0	3.0	7.0	1.0	8.0	6.0	5.0	4.0
	150	2.0	4.0	8.0	1.0	6.5	6.5	5.0	3.0
$\sum$ Ranks		49.0	87.0	218.5	47.0	200.0	176.5	184.0	118.0
Overall Rank		2	3	8	1	7	5	6	4

7. Real data analysis

The real-world data set examined in this section is intended to illustrate the flexibility of the suggested distribution. The used real data set consists of 50 observations on Burr (measured in millimeters), which Dasgupta introduced [23]. The dataset used in this study consists of observations recorded in the unit interval (0,1), representing proportional data. The nature of the data makes it particularly suitable for modeling with bounded distributions such as the UGEG, which is specifically introduced to handle unit interval data. Modeling such data using traditional unbounded distributions often results in poor fit or boundary violations; hence, the need for appropriate unit-based models.

While the UGEG distribution introduces additional parameters to enhance modeling flexibility, we acknowledge that this may increase model complexity. However, this trade-off is justified in applications where standard bounded models (e.g., beta, Kumaraswamy) fail to capture skewness or variable hazard structures. Practical scenarios where the UGEG model proves advantageous include:

- **Reliability data** exhibiting early-life or wear-out failures (thanks to flexible hazard rates),
- **Normalized performance scores** in quality control or education data bounded between 0 and 1,
- **Proportional biomedical measures**, such as body fat percentage or infection rates,
- **Bounded financial ratios**, such as debt-to-equity or profit margin proportions.

In such settings, the interpretability of the parameters can be handled via sensitivity plots, parameter profiling, or marginal effects in regression settings.

We use well-known models to compare with our proposed model. All compared models are defined as follows: $(0 < x < 1)$:

- UGEG distribution; its CDF is defined as follows:

$$F_{UGEG}(p_1, p_2, p_3) = 1 - \left(\frac{x^{p_2} - 1}{p_3 x^{p_2} - 1}\right)^{p_1}.$$

- Beta (B) distribution; its CDF is defined as follows:

$$F_B(p_1, p_2) = I_x(p_1, p_2).$$

- Log-Lindley (LL) [11] distribution; its CDF is defined as follows:

$$F_{LL}(p_1, p_2) = \frac{x^{p_1}(p_1(p_2 - \log(x)) + 1)}{p_1 p_2 + 1}.$$

- Power log-Lindley (PLL) [24] distribution; its CDF is defined as follows:

$$F_{PLL}(p_1, p_2, p_3) = \frac{x^{p_1 p_2}(p_3 p_2 - p_1 p_2 \log(x) + 1)}{p_3 p_2 + 1}.$$

- Cosine-sine (CS) [3] distribution; its CDF is defined as follows:

$$F_{CS}(p_1) = \frac{e^{p_1} - e^{p_1 - p_1 \sin\left(\frac{\pi x}{2}\right)}}{e^{p_1} - 1}.$$

- Power logarithmic (PL) [4] distribution; its CDF is defined as follows:

$$F_{PL}(p_1, p_2, p_3) = 1 - \frac{x^{p_1+1}\left((p_1 + 1)p_3 \log(x) - \left(1 - x^{-p_1-1}\right)((p_1 + 1)p_2 + p_3)\right)}{p_1 p_2 + p_2 + p_3}.$$

- Reduced Kies (RK) [25] distribution; its CDF is defined as follows:

$$F_{RK}(p_1) = 1 - e^{-\left(\frac{x}{1-x}\right)^{p_1}}.$$

- Transformed gamma (TG) [26] distribution; its CDF is defined as follows:

$$F_{TG}(p_1) = (1-x)^{p_1}(p_1 \log(1-x) - 1) + 1.$$

- Log-gamma (LG) [27] distribution; its CDF is defined as follows:

$$F_{LG}(p_1) = x^{p_1}(1 - p_1 \log(x)).$$

- Log-weighted power (LWP) [28] distribution; its CDF is defined as follows:

$$F_{LWP}(p_1, p_2) = x^{p_1}(1 - p_1 p_2 \log(x)).$$

- Transmuted power (TP) [28] distribution; its CDF is defined as follows:

$$F_{TP}(p_1, p_2) = x^{p_1}\left(p_2\left(1 - x^{p_1}\right) + 1\right).$$

To choose the optimal model for the real-world data set, we employ a variety of analytical criteria, such as Akaike information criteria (AIC), corrected Akaike information criteria (CAIC), and Hannan-Quinn information criteria (HQIC). Also, we use numerous goodness-of-fit statistics in this choice, such as Cramér-von Mises (CvM), Anderson Darling (AD), and Kolmogorov-Smirnov (KS) with its p-value (KS p-value).

As shown in Table 7, the analytical measures, as well as the MLE and related standard errors (SE), are supplied for the evaluated real data set. From this, we can infer that the UGEG model fits the data better than the other models. It is shown in Figure 8 that the suggested UGEG model is fitted to the Burr data set using estimated PDF, CDF, and SF plots as well as P-P plot. This demonstrated that the proposed model is suited to the Burr data set. The TTT plot and plot of the estimated HRF of the UGEG model are shown in Figure 9, which demonstrates that the HRF of the UGEG model is increasing. The goodness-of-fit of the proposed UGEG distribution is compared with other established models using several model selection and fit criteria. As shown in Table 7, the UGEG distribution achieves the lowest values for AIC, BIC, CAIC, and HQIC, indicating its superior balance between model fit and complexity. Furthermore, the KS, AD, and CvM statistics for the UGEG model are the smallest among all competitors, with the highest p-value, suggesting a better alignment with the empirical distribution. These results confirm that the UGEG distribution provides a more flexible and accurate fit to the data without overfitting. Therefore, the statistical evidence supports the claimed flexibility and practical relevance of the proposed model.

Table 7. Data analysis results for Burr real data set.

	UGEG	B	LL	PLL	CS	PL	RK	TG	LG	LWP	TP
$-l$	-56.25	-54.6067	-31.8265	-31.8265	-40.649	-31.8265	-11.6763	-53.5138	-31.8265	-31.8265	-30.3176
AIC	-106.5	-105.213	-59.653	-57.653	-79.298	-57.653	-21.3526	-105.028	-61.653	-59.653	-56.6353
CAIC	-105.978	-104.958	-59.3977	-57.1313	-79.2146	-57.1313	-21.2692	-104.944	-61.5697	-59.3977	-56.3799
HQIC	-104.316	-103.757	-58.1968	-55.4687	-78.5699	-55.4687	-20.6244	-104.3	-60.9249	-58.1968	-55.179
AD	0.646201	0.912344	6.56759	6.56759	4.72157	6.5676	21.7194	1.23799	6.56759	6.56759	6.79842
CvM	0.101619	0.153904	1.22677	1.22677	0.869979	1.22677	4.86516	0.236507	1.22677	1.22677	1.26438
KS	0.11041	0.141461	0.318138	0.318138	0.261222	0.318138	0.563342	0.171007	0.318138	0.318138	0.320945
KS p-value	0.575809	0.269698	0.0000804	0.0000804	0.00217545	0.00008	< 0.000001	0.107384	0.00008043	0.0000804	0.000067
$\widehat{p_1}$	11425.6	2.68257	1.00664	1.89694	3.56212	0.00664	0.736768	10.9387	1.00664	1.00664	0.733619
SE($\widehat{p_1}$)	125382	0.507179	0.0762406	0.645369	0.640514	0.100664	0.087676	1.09387	0.100664	0.173704	0.11218
$\widehat{p_2}$	2.03144	13.8658	1×10^{-9}	0.530664	–	0.003714	–	–	–	1.000	1.000
SE($\widehat{p_2}$)	0.265213	2.82802	0.129734	0.151324	–	27.6384	–	–	–	0.511507	0.631411
$\widehat{p_3}$	0.997376	–	–	1.03311×10^{-13}	–	2.32994×10^{6}	–	–	–	–	–
SE($\widehat{p_3}$)	0.028808	–	–	1.23461×10^{-7}	–	792681	–	–	–	–	–

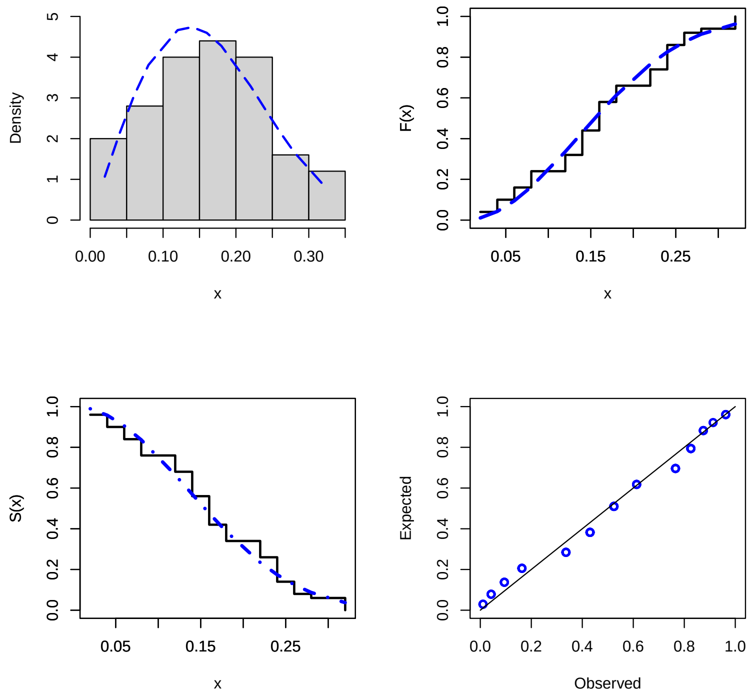

Figure 8. Histogram of the Burr data with the fitted PDF, CDF, SF, and P-P plots.

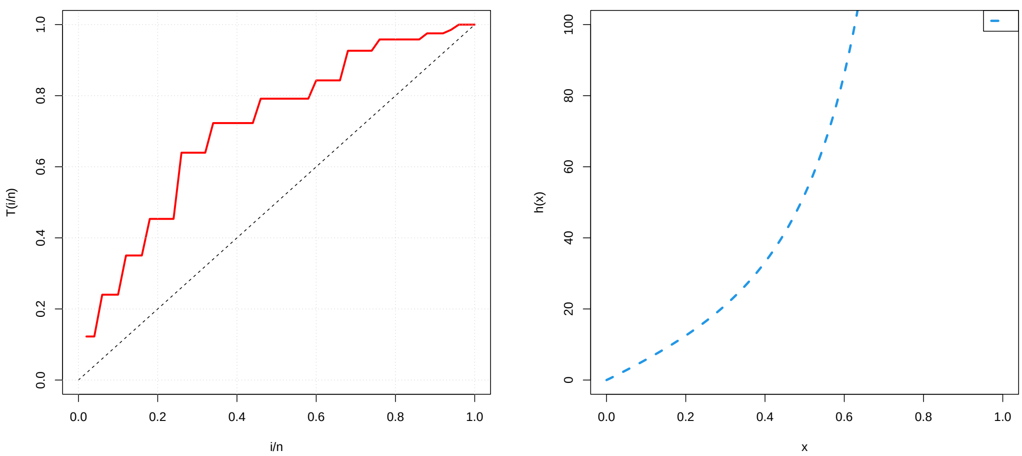

Figure 9. TTT plot and fitted HRF of UGEG model for the Burr data.

8. A novel quantile regression based on UGEG distribution

In this section, a new quantile regression as an alternative to beta and Kumaraswamy regression is introduced based on UGEG distribution. This new quantile regression is built using the quantile function given in Eq (2.5). The PDF and CDF of the UGEG distribution can be obtained with parameterization using the quantile function. Let $Q(u; \alpha, \beta, p) = M$ be the quantile function given in Eq (2.5), where $u \in (0, 1)$ denotes the desired quantile level (e.g., 0.25, 0.50, 0.75), and $M \in (0, 1)$ is the corresponding quantile value of the response variable. The quantile function of the UGEG distribution is given by:

$$M = \left(\frac{(1-u)^{1/\alpha}-1}{p(1-u)^{1/\alpha}-1}\right)^{1/\beta}.$$

Solving this equation for β, we obtain a re-parameterization in terms of M, u, α, and p

$$\beta = \frac{\log\left((1-u)^{1/\alpha}-1\right)-\log\left(p\left(1-u\right)^{1/\alpha}-1\right)}{\log\left(M\right)}. \tag{8.1}$$

Then the CDF and PDF of the re-parametrized distribution are presented, respectively, as

$$F\left(y, a, p, M\right) = 1-\left(\frac{1-y^{\left\{\frac{\log\left((1-u)^{1/\alpha}-1\right)-\log\left(p(1-u)^{1/\alpha}-1\right)}{\log(M)}\right\}}}{1-py^{\left\{\frac{\log\left((1-u)^{1/\alpha}-1\right)-\log\left(p(1-u)^{1/\alpha}-1\right)}{\log(M)}\right\}}}\right)^{\alpha}, \tag{8.2}$$

and

$$\begin{aligned} f\left(y, a, p, M\right) \;&=\; \alpha\left\{\frac{\log\left((1-u)^{1/\alpha}-1\right)-\log\left(p\left(1-u\right)^{1/\alpha}-1\right)}{\log\left(M\right)}\right\}\left(1-p\right)y^{\left\{\frac{\log\left((1-u)^{1/\alpha}-1\right)-\log\left(p(1-u)^{1/\alpha}-1\right)}{\log(M)}\right\}-1} \\ &\times\left(1-y^{\left\{\frac{\log\left((1-u)^{1/\alpha}-1\right)-\log\left(p(1-u)^{1/\alpha}-1\right)}{\log(M)}\right\}}\right)^{\alpha-1}\left(1-py^{\left\{\frac{\log\left((1-u)^{1/\alpha}-1\right)-\log\left(p(1-u)^{1/\alpha}-1\right)}{\log(M)}\right\}}\right)^{-(\alpha+1)}, \end{aligned} \tag{8.3}$$

where $\alpha > 0$ and $p \in (0, 1)$ are the UGEG model parameters. It is noted that the random variable Y is denoted by $Y\tilde{}QUGEG\left(a, p, M, u\right)$.

After the definition of the QUGEG distribution, the step of creating a new regression model based on the QUGEG with the PDF given in Eq (8.3) can be started. Let $y_1, y_2, \ldots, y_n$ such that y_i is an realization of $Y\tilde{}QUGEG\left(a, p, M_i, u\right)$ for $i = 1, 2, \ldots, n$, where a, p, and M_i are unknown parameters, and u is known. The following is the proposed quantile regression model:

$$g\left(M_i\right) = \mathbf{x}_i\gamma^{\mathrm{T}}, \tag{8.4}$$

where $\gamma = \left(\gamma_0, \gamma_1, \ldots, \gamma_p\right)$ are the unknown regression parameter vector, $\mathbf{x}_i = \left(\mathbf{1}, \mathbf{x}_{i1}, \mathbf{x}_{i2}, \ldots, \mathbf{x}_{ip}\right)$ known ith vector of the covariates, and g is a link function. In the simulation study and real data analysis, u=0.5 was taken; that is, covariates are linked to the conditional median of the response variable. Because the QUGEG distribution has a domain (0,1) interval, we apply the following logit-link function:

$$g\left(M_i\right) = \log\left(\frac{M_i}{1-M_i}\right), i = 1, 2, \ldots, n. \tag{8.5}$$

8.1. Estimation of regression parameters using the maximum likelihood approach

In this section, the maximum likelihood estimation method is introduced for the estimation of the unknown regression parameters and model parameters. It is obtained via Eq (8.5)

$$M_i = \frac{\exp(\mathbf{x}_i \gamma^{\mathrm{T}})}{1 + \exp(\mathbf{x}_i \gamma^{\mathrm{T}})}. \tag{8.6}$$

Let $Y_1, Y_2, \ldots, Y_n$ be a random sample of size n from the $QUGEG(a, p, M_i, u)$ distribution with realizations $y_1, y_2, \ldots, y_n$, where the M_i is given in (8.6) for $i = 1, 2, \ldots, n$.Then the log-likelihood function is given by

$$\begin{aligned}
\ell(\Psi) &= n\log(\alpha) + n\log(1-p) + \sum_{i=1}^{n} \log\left\{\frac{\log\left((1-u)^{1/\alpha}-1\right) - \log\left(p(1-u)^{1/\alpha}-1\right)}{\log(M_i)}\right\} \\
&\quad + \sum_{i=1}^{n} \log(y_i)\left\{\frac{\log\left((1-u)^{1/\alpha}-1\right) - \log\left(p(1-u)^{1/\alpha}-1\right)}{\log(M_i)}\right\} - \sum_{i=1}^{n}\log(y_i) \\
&\quad + (\alpha-1)\sum_{i=1}^{n} \log\left(1 - y_i^{\left\{\frac{\log\left((1-u)^{1/\alpha}-1\right) - \log\left(p(1-u)^{1/\alpha}-1\right)}{\log(M_i)}\right\}}\right) \\
&\quad - (\alpha+1)\sum_{i=1}^{n} \log\left(1 - p y_i^{\left\{\frac{\log\left((1-u)^{1/\alpha}-1\right) - \log\left(p(1-u)^{1/\alpha}-1\right)}{\log(M_i)}\right\}}\right),
\end{aligned} \tag{8.7}$$

where $\Psi = (a, p, \gamma)$ is the unknown parameter vector. It is important to note that for $u = 0.5$, it is the same as modeling the conditional median. The maximum likelihood estimators (MLEs) of the Ψ, say $\widehat{\Psi} = \left(\widehat{a}, \widehat{p}, \gamma_1, \gamma_2, \ldots, \gamma_p\right)$ is achieved by maximizing the $\ell(\Psi)$ given in Eq (8.7) with respect to a, p and γ. Because (8.7) contains a nonlinear function based on parameters, the log-likelihood function can be maximized using the **optim** function in R software. Based on some regularity conditions, asymptotic distribution of $\left(\widehat{\Psi} - \Psi\right)$ is multivariate normal $N_{p+3}\left(0, J^{-1}\right)$ where J is the expected information matrix. In practice, it is generally used instead of J with the observed information matrix. The observed information matrix may be calculated using any program. The **optim** function in R may be used to determine the asymptotic standard errors of estimates based on the observed Fisher Information matrix.

8.2. Simulation studies for the new quantile regression model

In this subsection, a simulation study is carried out to see the behavior of MLEs of the QUGEG regression model parameters based on 5000 trials. In the simulation, the number of covariates is considered as two and generated from a multivariate normal distribution. Three correlation matrices for covariates $(\mathbf{x}_{i1}, \mathbf{x}_{i2})$ are chosen as

$$\boldsymbol{\rho}_1 = \begin{pmatrix} 1 & 0.1 \\ 0.1 & 1 \end{pmatrix}, \ \boldsymbol{\rho}_2 = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}, \text{ and } \boldsymbol{\rho}_3 = \begin{pmatrix} 1 & 0.9 \\ 0.9 & 1 \end{pmatrix}.$$

The multicollinearity effect on the QUGEG regression analysis can also be shown. In addition, six different scenarios are considered in Table 8.

Table 8. Different scenarios for multicollinearity effect on the QUGEG regression analysis.

Scenario	Correlation matrix	γ_0	γ_1	γ_2	α	p
1	$\boldsymbol{\rho}_1$	1	1	1	0.7	0.9
2	$\boldsymbol{\rho}_2$	1	1	1	0.7	0.9
3	$\boldsymbol{\rho}_3$	1	1	1	0.7	0.9
4	$\boldsymbol{\rho}_1$	0.5	2	1	1.5	0.7
5	$\boldsymbol{\rho}_2$	0.5	2	1	1.5	0.7
6	$\boldsymbol{\rho}_3$	0.5	2	1	1.5	0.7

The response variable(y_i) can be obtained as

$$y_i = \left(\frac{(1-\tau_i)^{\frac{1}{\alpha}} - 1}{p\,(1-\tau_i)^{\frac{1}{\alpha}} - 1} \right)^{\frac{1}{\beta_i}},$$

where $\tau_i \sim U\,(0,1)$ and $\beta_i = \frac{\log\left((1-u)^{1/\alpha}-1\right)-\log\left(p(1-u)^{1/\alpha}-1\right)}{\log(M_i)}$ for $i = 1, 2, \ldots, n$. The simulation is conducted with $u = 0.5$, and the sample size is selected as $50, 75, 100, 200, 250, 500$ and 1000. For six simulation scenarios, it has been tackled with the true parameter values of $\gamma_0 = 1, \gamma_1 = 1, \gamma_2 = 1, \alpha = 0.7, p = 0.9$ and $\gamma_0 = 0.5, \gamma_1 = 2, \gamma_2 = 1, \alpha = 1.5, p = 0.7$ with the following regression structure:

$$\operatorname{logit}(M_i) = \gamma_0 + \gamma_1 \mathbf{x}_{i1} + \gamma_2 \mathbf{x}_{i2}, i = 1, 2, \ldots, n.$$

The bias and mean square errors (MSEs) of the MLEs of the QUGEG regression model and the coverage probabilities (CPs) and mean lengths (MLs) of the MLEs based confidence intervals are reported in Table 9. From Table 9, the bias and MSEs decrease when the sample size increases and approach zero. It can also be determined that as the sample size increased, the CPs values approached the nominal level of 0.95, and the MLs values decreased.

Table 9. Average bias and MSEs for MLEs, CPs, and MLs of the MLEs based confidence intervals for different scenarios.

		Bias					MSEs					CPs					MLs				
Scenario	n	γ_0	γ_1	γ_2	α	p	γ_0	γ_1	γ_2	α	p	γ_0	γ_1	γ_2	α	p	γ_0	γ_1	γ_2	α	p
1	50	-0.0405	0.0100	0.0103	0.0907	-0.1194	0.2847	0.0893	0.0912	0.1065	0.0583	0.9376	0.9252	0.9274	0.9684	0.9312	1.9966	1.0831	1.0831	0.9142	0.5264
	75	-0.0311	0.0122	0.0015	0.0414	-0.0934	0.1926	0.0578	0.0583	0.0351	0.0394	0.9318	0.9280	0.9328	0.9630	0.9390	1.6205	0.8770	0.8759	0.6328	0.4536
	100	-0.0204	0.0086	0.0009	0.0319	-0.0670	0.1371	0.0410	0.0432	0.0219	0.0240	0.9408	0.9342	0.9246	0.9604	0.9454	1.3919	0.7555	0.7554	0.5267	0.3839
	200	-0.0091	0.0031	0.0008	0.0119	-0.0307	0.0634	0.0189	0.0196	0.0089	0.0066	0.9462	0.9450	0.9436	0.9456	0.9544	0.9821	0.5330	0.5323	0.3498	0.2596
	250	-0.0113	0.0036	0.0015	0.0110	-0.0234	0.0525	0.0153	0.0152	0.0068	0.0049	0.9472	0.9450	0.9470	0.9520	0.9542	0.8776	0.4751	0.4754	0.3109	0.2256
	500	-0.0031	-0.0014	0.0008	0.0062	-0.0110	0.0247	0.0072	0.0074	0.0032	0.0017	0.9512	0.9496	0.9448	0.9502	0.9572	0.6177	0.3348	0.3353	0.2161	0.1479
	1000	-0.0038	0.0004	0.0007	0.0032	-0.0050	0.0123	0.0036	0.0038	0.0015	0.0007	0.9508	0.9512	0.9460	0.9514	0.9574	0.4359	0.2363	0.2364	0.1512	0.1000
2	50	-0.0461	0.0132	0.0093	0.0832	-0.1265	0.2697	0.1193	0.1253	0.0832	0.0626	0.9288	0.9312	0.9180	0.9594	0.9372	1.8938	1.2473	1.2488	0.8853	0.5328
	75	-0.0439	0.0205	0.0052	0.0434	-0.0914	0.1619	0.0779	0.0780	0.0361	0.0372	0.9472	0.9284	0.9224	0.9604	0.9470	1.5323	1.0083	1.0091	0.6361	0.4533
	100	-0.0287	0.0101	0.0037	0.0273	-0.0705	0.1207	0.0542	0.0522	0.0216	0.0249	0.9456	0.9360	0.9432	0.9562	0.9534	1.3222	0.8719	0.8701	0.5229	0.3906
	200	-0.0118	0.0036	-0.0013	0.0165	-0.0300	0.0559	0.0259	0.0250	0.0087	0.0068	0.9500	0.9416	0.9476	0.9542	0.9526	0.9271	0.6116	0.6121	0.3527	0.2576
	250	-0.0086	0.0030	0.0020	0.0116	-0.0241	0.0448	0.0208	0.0213	0.0069	0.0049	0.9486	0.9436	0.9362	0.9484	0.9574	0.8271	0.5462	0.5465	0.3112	0.2263
	500	-0.0017	0.0007	0.0006	0.0048	-0.0112	0.0228	0.0100	0.0099	0.0031	0.0017	0.9466	0.9422	0.9488	0.9518	0.9542	0.5838	0.3858	0.3858	0.2154	0.1482
	1000	-0.0048	0.0005	0.0019	0.0020	-0.0058	0.0110	0.0048	0.0048	0.0015	0.0007	0.9508	0.9514	0.9488	0.9470	0.9530	0.4125	0.2721	0.2720	0.1508	0.1006
3	50	-0.0503	0.0058	0.0195	0.0979	-0.1182	0.2488	0.4692	0.4635	0.1326	0.0581	0.9354	0.9264	0.9248	0.9696	0.9330	1.8305	2.4629	2.4665	0.9294	0.5224
	75	-0.0217	-0.0045	0.0091	0.0405	-0.0914	0.1530	0.2965	0.2969	0.0319	0.0375	0.9414	0.9292	0.9248	0.9634	0.9486	1.4812	1.9925	1.9936	0.6310	0.4528
	100	-0.0302	0.0150	0.0001	0.0316	-0.0684	0.1123	0.2080	0.2085	0.0230	0.0243	0.9470	0.9356	0.9420	0.9542	0.9480	1.2795	1.7146	1.7152	0.5275	0.3867
	200	-0.0101	0.0027	0.0007	0.0146	-0.0300	0.0533	0.1012	0.1004	0.0088	0.0065	0.9492	0.9418	0.9434	0.9554	0.9568	0.8977	1.2088	1.2081	0.3518	0.2581
	250	-0.0148	0.0030	0.0011	0.0095	-0.0252	0.0439	0.0797	0.0808	0.0069	0.0050	0.9492	0.9458	0.9434	0.9508	0.9580	0.8032	1.0801	1.0796	0.3103	0.2283
	500	-0.0070	0.0013	0.0022	0.0050	-0.0114	0.0210	0.0384	0.0375	0.0031	0.0017	0.9522	0.9494	0.9488	0.9500	0.9518	0.5661	0.7606	0.7603	0.2155	0.1482
	1000	0.0001	0.0019	-0.0023	0.0026	-0.0056	0.0104	0.0183	0.0187	0.0015	0.0007	0.9480	0.9512	0.9526	0.9502	0.9516	0.3993	0.5365	0.5367	0.1510	0.1004
4	50	0.0084	0.0113	0.0050	1.0109	-0.0405	0.1300	0.0388	0.0337	13.0538	0.0637	0.9406	0.9336	0.9366	0.9786	0.8534	1.3657	0.7339	0.6957	8.4299	0.7106
	75	0.0033	0.0066	0.0027	0.4036	-0.0505	0.0826	0.0241	0.0215	2.3821	0.0528	0.9412	0.9392	0.9424	0.9700	0.8906	1.1073	0.5912	0.5592	3.6062	0.7031
	100	0.0006	0.0012	0.0020	0.2419	-0.0519	0.0605	0.0176	0.0150	0.8549	0.0432	0.9462	0.9444	0.9468	0.9640	0.9090	0.9510	0.5065	0.4778	2.4274	0.6904
	200	-0.0008	0.0016	0.0001	0.0949	-0.0315	0.0311	0.0087	0.0075	0.1450	0.0239	0.9416	0.9452	0.9462	0.9606	0.9262	0.6696	0.3551	0.3359	1.2910	0.5523
	250	0.0010	0.0017	0.0006	0.0609	-0.0283	0.0242	0.0069	0.0061	0.0926	0.0188	0.9484	0.9452	0.9420	0.9544	0.9324	0.6000	0.3174	0.3011	1.0969	0.4958
	500	-0.0031	0.0008	0.0006	0.0281	-0.0165	0.0119	0.0034	0.0030	0.0390	0.0085	0.9490	0.9462	0.9488	0.9502	0.9450	0.4232	0.2241	0.2119	0.7345	0.3412
	1000	-0.0012	0.0009	0.0006	0.0142	-0.0070	0.0059	0.0017	0.0014	0.0173	0.0037	0.9500	0.9438	0.9512	0.9506	0.9442	0.2993	0.1584	0.1496	0.5066	0.2333
5	50	0.0165	0.0076	0.0018	1.1089	-0.0323	0.1189	0.0502	0.0476	14.5847	0.0609	0.9370	0.9412	0.9332	0.9838	0.8380	1.3065	0.8352	0.8053	9.2338	0.7007
	75	0.0086	0.0020	0.0023	0.4431	-0.0532	0.0770	0.0321	0.0290	2.9322	0.0552	0.9412	0.9358	0.9360	0.9746	0.8852	1.0597	0.6740	0.6463	3.8985	0.7017
	100	-0.0035	0.0047	-0.0002	0.2349	-0.0522	0.0565	0.0235	0.0214	0.7543	0.0449	0.9448	0.9366	0.9392	0.9646	0.9052	0.9145	0.5788	0.5551	2.3501	0.6864
	200	-0.0035	0.0017	0.0017	0.0749	-0.0384	0.0279	0.0111	0.0102	0.1287	0.0246	0.9476	0.9452	0.9460	0.9618	0.9328	0.6435	0.4053	0.3885	1.2610	0.5621
	250	-0.0052	0.0043	0.0004	0.0673	-0.0266	0.0220	0.0089	0.0080	0.0964	0.0186	0.9490	0.9464	0.9482	0.9562	0.9306	0.5750	0.3622	0.3476	1.1054	0.4929
	500	-0.0016	0.0009	-0.0005	0.0246	-0.0154	0.0109	0.0042	0.0040	0.0374	0.0083	0.9508	0.9488	0.9464	0.9522	0.9428	0.4060	0.2557	0.2449	0.7306	0.3394
	1000	-0.0019	0.0007	0.0001	0.0129	-0.0077	0.0054	0.0022	0.0020	0.0174	0.0038	0.9456	0.9466	0.9462	0.9520	0.9462	0.2869	0.1804	0.1728	0.5059	0.2336
6	50	0.0121	0.0021	0.0067	1.0267	-0.0437	0.1090	0.1932	0.1911	12.2395	0.0631	0.9382	0.9308	0.9298	0.9786	0.8524	1.2725	1.6085	1.5948	8.8073	0.7165
	75	0.0053	-0.0008	0.0044	0.4528	-0.0526	0.0717	0.1163	0.1125	4.1559	0.0527	0.9422	0.9400	0.9402	0.9732	0.8802	1.0322	1.2929	1.2777	3.9037	0.7070
	100	-0.0036	-0.0012	0.0050	0.2521	-0.0515	0.0542	0.0829	0.0831	1.1222	0.0450	0.9420	0.9466	0.9438	0.9660	0.9078	0.8934	1.1118	1.1021	2.4514	0.6840
	200	-0.0069	0.0001	0.0047	0.0704	-0.0391	0.0267	0.0408	0.0409	0.1253	0.0240	0.9450	0.9526	0.9440	0.9562	0.9370	0.6287	0.7802	0.7716	1.2562	0.5636
	250	-0.0025	-0.0001	0.0010	0.0567	-0.0302	0.0219	0.0323	0.0316	0.0901	0.0189	0.9434	0.9488	0.9462	0.9550	0.9414	0.5628	0.6951	0.6882	1.0920	0.4972
	500	-0.0011	0.0023	-0.0012	0.0245	-0.0141	0.0102	0.0159	0.0158	0.0375	0.0080	0.9522	0.9494	0.9430	0.9530	0.9482	0.3974	0.4921	0.4869	0.7308	0.3382
	1000	0.0002	0.0013	-0.0014	0.0120	-0.0075	0.0051	0.0077	0.0076	0.0173	0.0037	0.9534	0.9482	0.9530	0.9560	0.9470	0.2807	0.3469	0.3432	0.5054	0.2333

8.3. *Data application for the new quantile regression model*

In this subsection, a practical data application is carried out to observe the usability of the new regression model. The data set used in this study is originally obtained from the Organisation for Economic Co-operation and Development (OECD) database, which can be accessed at https://stats.oecd.org/. Specifically, the data are related to the Better Life Index under the theme of Social Protection and Well-being, with the reference year starting from 2017. Detailed information regarding this data set is also available in Korkmaz et al. [29] and has additionally been utilized in Korkmaz [13]. The beta (B) regression model by [30], Kumaraswamy (Kw) regression model by [31], and log-extended exponential geometric (LEEG) regression model by [32] are considered for comparing the new model. In all models u is taken as 0.5, and so the median is modelled. This application aims to relate the percentage of the educational attainment values of the OECD countries (variable y) with the percentage of the voter turnout (variable x_1), homicide rate (variable x_2), and life satisfaction (variable x_3). The regression model is presented as based on M_i

$$\log\mathtt{it}\,(M_i) = \gamma_0 + \gamma_1 x_{i1} + \gamma_2 x_{i2} + \gamma_3 x_{i3},$$

where $i = 1, 2, \ldots, 38$ and M_i represents the median for the QUGEG, Kw, and LEEG models, whereas M_i denotes the mean for the B model. The MLEs, standard error(SE), log-likelihood values $(\widehat{\ell})$, and Akaike information criterion (AIC) are calculated for all models. The results are given in Table 10. From Table 10, it is evident that the proposed quantile regression model based on the QUGEG distribution achieves the highest log-likelihood value $(\widehat{\ell})$ and the lowest AIC value among all the considered models, including the classical beta regression. These results clearly indicate the superior goodness-of-fit of the QUGEG model to the bounded data under study. Furthermore, the QUGEG model provides a higher level of flexibility in capturing various shapes of conditional distributions, such as skewness and heavy tails, which are often observed in real-world bounded data but cannot be fully addressed by classical beta regression. In addition, the QUGEG model offers a more accurate estimation of extreme quantiles, which is particularly important when the focus is on modeling the lower or upper tails of the response variable. Therefore, it can be concluded that the QUGEG regression model not only outperforms the classical beta regression in terms of overall fit statistics but also provides a more comprehensive and robust modeling framework for bounded data. This makes it a valuable alternative in applications where classical models may fail to adequately describe the underlying data characteristics. In addition, all covariate parameters in the QUGEG regression model are shown to be statistically significant at the 5% level. The parameter γ_3 has a positive effect on the median response, but parameters γ_1 and γ_2 have a negative effect on the median response. It can be concluded that an increase in voter turnout and homicide rate decreases the percentage of educational attainment. However, it is seen that the increase in life satisfaction also increases the percentage of educational attainment.

Table 10. The results of fitted regression models.

		QUGEG			**B**			**Kw**			**LEEG**	
	Estimate	SE	p-value	Estimate	SE	p-value	Estimate	SE	p-value	Estimate	SE	p-value
γ_0	0.7749	1.2809	0.5452	0.9615	0.9685	0.3208	1.6247	1.1740	0.1664	0.3275	1.0754	0.7607
γ_1	-2.4195	1.0696	0.0237	-2.9211	1.0176	0.0041	-4.1197	1.3892	0.0030	-4.0917	1.4520	0.0048
γ_2	-0.0604	0.0273	0.0269	-0.0470	0.0178	0.0084	-0.0404	0.0168	0.0159	-0.0477	0.0145	0.0010
γ_3	0.3814	0.1637	0.0198	0.3794	0.1492	0.0110	0.4237	0.2546	0.0960	0.6214	0.1745	<0.0001
α	3.9370	4.5543	0.3873	11.5900	2.6100	<0.0001	6.2167	1.0787	<0.0001	7.8378	1.7365	<0.0001
p	0.2722	1.3073	0.8351									
$\widehat{\ell}$		33.2720			30.9024			29.4339			28.6480	
AIC		-54.5441			-51.8048			-48.8677			-47.2961	

9. Conclusions

A new flexible distribution is proposed in the literature as an alternative to beta, Kumaraswamy, and other unit distributions. The hazard rate function of the new distribution has a bathtub shape, U-shape, J-shape, or increasing shape. Many statistical features of the UGEG distribution are explored, including moments, Bonferroni and Lorenz curves, entropies, etc. Several estimation methods are considered to obtain point estimates for the unknown parameters of the UGEG distribution. A new regression model is constructed by the quantile function of the UGEG distribution. Extensive Monte Carlo simulations are conducted to observe the maximum likelihood technique's behavior in distribution and regression parameter cases. Real data applications show that the UGEG distribution and the QUGEG regression model are good alternatives to the models in the literature. While the current study does not include a formal robustness analysis, the structure of the UGEG distribution suggests that it can accommodate skewness, heavy tails, and variability in hazard behavior. These features implicitly contribute to model robustness. In future work, we intend to investigate the performance of the model under outlier contamination and local distributional shifts, using robust estimation techniques or influence function analysis.

Author contributions

Ahmed M. T. Abd El-Bar, Ahmed R. El-Saeed, Kadir Karakaya, and Ahmed M. Gemeay: Conceptualization, Methodology, Validation, Writing-original draft, Writing-review & editing. All authors of this article have contributed equally. All authors have worked equally to write and review the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare no conflict of interest.

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