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Research article

Certain novel local fractional half-discrete Hilbert-type inequalities with nonhomogeneous kernels

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Abstract: By employing weight coefficient methods and parameterization techniques, this paper investigates certain novel local fractional half-discrete Hilbert-type inequalities with nonhomogeneous kernels on fractal sets. Furthermore, both the canonical equivalences and its degenerate forms are presented as applications. The main results of this paper can be seen as generalizations and extensions of classical half-discrete Hilbert-type inequalities to the realm of local fractional calculus.

Keywords: Hilbert's inequality; half-discrete Hilbert-type inequalities; local fractional calculus; nonhomogeneous kernels; conjugate parameters
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1. Introduction

Let p and q be a pair of positive conjugate parameters, i.e., $1/p + 1/q = 1$ with $p > 1$. The discrete and integral forms of well-known Hardy-Hilbert inequalities (see [1]) are stated as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mathfrak{a}_m\mathfrak{b}_n}{m+n} \le \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\|\mathfrak{a}\|_{l^p}\|\mathfrak{b}\|_{l^q} \text{ and } \int_{\mathbb{R}_+^2}\frac{\mathfrak{f}(x)\mathfrak{g}(y)}{x+y}dxdy \le \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\|\mathfrak{f}\|_{L^p}\|\mathfrak{g}\|_{L^q}, \tag{1.1}$$

where nonnegative sequences $\mathfrak{a} = (\mathfrak{a}_m)_{m=1}^{\infty} \in l^p := \{\mathfrak{a}|\|\mathfrak{a}\|_{l^p} = (\sum_{m=1}^{\infty}\mathfrak{a}_m^p)^{1/p} < \infty\}$, $\mathfrak{b} = (\mathfrak{b}_n)_{n=1}^{\infty} \in l^q$ with $\|\mathfrak{a}\|_{l^p}, \|\mathfrak{b}\|_{l^q} > 0$, and nonnegative functions $\mathfrak{f} \in L^p(\mathbb{R}_+) = \{\mathfrak{f}|\|\mathfrak{f}\|_{L^p} = (\int_0^{\infty}\mathfrak{f}^p(x)dx)^{1/p} < \infty\}$, $\mathfrak{g} \in L^q(\mathbb{R}_+)$ with $\|\mathfrak{f}\|_{L^p}, \|\mathfrak{g}\|_{L^q} > 0$. The constant $\pi/\sin(\pi/p)$, appearing on the right-hand side of (1.1), is the best possible. When $p = q = 2$, the previous inequalities (1.1) are also commonly referred to as the famous Hilbert's inequalities (HIs) with the best constant π.

Because HIs have a wide range of applications in the various fields of mathematics, they have attracted widespread attention from scholars around the world. A large number of generalizations and extensions of HIs have been established, covering various aspects such as different weighted functions, integration domains, types and dimensionality of integrals, etc. The reader can refer to [2–4] and the references cited therein. For example, some local fractional discrete HIs and Hilbert-Pachpatte-type integral inequalities on the fractal set have been studied in the references [5, 6], respectively. By introducing some independent parameters, Yang [7, 8] investigated some extensions of integral and discrete Hardy-Hilbert-type inequalities with nonnegative homogeneous functions, respectively. By using the differential weighted functions and multiple parameters, Krnić and Pečarić [9,10] established some extensions of the celebrated HI and their corresponding equivalent formulations in both integral and discrete frameworks, respectively. With the construction of different nonhomogeneous kernels and parameters, Rassias and Yang [11–13] proposed several novel HIs featuring optimal constant factors associated with the hypergeometric function, the extended Riemann zeta function, and the extended Hurwitz-zeta function, respectively. Using weight kernel functions and techniques from real analysis, You et al. [14, 15] explored some new HIs involving the best factors associated with special constants (Euler, Bernoulli, and Catalan numbers) and the higher-order derivatives of the cotangent and cosecant functions across the entire plane, respectively. Using the symmetry principle and the Euler-Maclaurin summation formula, Chen et al. [16] developed appropriate weight coefficients to create a more precise extended Hardy-Hilbert's inequality with the optimal constant factor involving multiple parameters. In the works [17, 18], Adiyasuren utilized the homogeneous functions to develop HIs that incorporate both a Hardy operator and geometric and harmonic operators, respectively.

On the other hand, Hardy, Littlewood, and Pólya claimed several results concerning half-discrete Hilbert-type inequalities (HDHIs) with nonhomogeneous kernels [1, Theorem 351]. From that time onward, numerous academics have widely acknowledged the importance of HDHIs. Additional results concerning HDHIs can be found in references [19–21] and the sources cited therein. For example, using weight function methods and real analysis techniques, Rassias and Yang [22, 23] provided some HDHIs featuring the best possible constant factors related to the Euler–Mascheroni constant and the Riemann zeta function, respectively. By employing the method of weight coefficients, You derived several HDHIs involving hyperbolic functions such as tangent, cotangent, secant, and cosecant [24, 25], and encompassing both homogeneous and nonhomogeneous cases across the entire plane [26, 27]. Through the application of Hermite-Hadamard inequalities, Hong et al. [28] developed some higher-accuracy multidimensional HDHIs with differential-based homogeneous kernel functions. By employing the sophisticated weight functions and real analysis techniques, Wang et al. [29], Liao and Yang [30], and Peng et al. [31] investigated various reverse forms of HDHIs with both homogeneous and nonhomogeneous kernels, respectively. Adiyasuren et al. [32] and Krnić et al. [33] developed various new HDHIs with a general homogeneous kernel for non-conjugate exponents, respectively.

Following Yang's introduction of properties and theorems related to local fractional derivatives and integrals [34, 35], numerous findings in the field of local fractional calculus (LFC) have emerged, see [36–38]. For instance, Baleanu et al. [39] considered some HIs via Cantor-type higher dimensional spherical coordinates on a fractal set. In [40], Batbold et al. obtained a unified treatment of fractal HIs with a general kernel and weight functions. A single-parameter fractal Bullen-type inequality and a series of parameterized inequalities for locally fractional differentiable generalized (s, P)-convex and concave functions were derived in [41]. By integrating generalized convexity properties of differentiable mappings with some elementary inequalities, Butt and Khan [42] established a series of novel parameterized inequalities within fractal-fractional frameworks. Within the framework of

LFC, novel Hermite-Hadamard-type inequalities were formulated for generalized harmonic convex functions, extending classical results to non-differentiable spaces in [43]. In [44], Ge-JiLe et al. utilized fractal set techniques to develop Hermite-Hadamard-type inequalities and related variants involving Raina's function. Exploiting the LFC and weighted function method, a fractal HI with the optimal constant and its equivalent version are proposed in [45]. In [46], Krnić and Vuković achieved a comprehensive approach to multidimensional local fractional HIs with the best constants. Using the real-analysis techniques on the fractal set, Liu and Liu [47] developed a general local fractional HI involving a hyperbolic cosecant kernel.

To the best of the authors' knowledge, there has been no research on local fractional HDHIs so far. To bridge this theoretical gap, this study will employ weight function methodologies combined with LFC theory to construct some innovative local fractional HDHIs featuring a nonhomogeneous kernel. As principal research outcomes, both the canonical equivalence and its degenerate forms will be also systematically derived. The primary results of this paper can be viewed as generalizations and extensions of classical HDHIs to the realm of LFC.

2. Preliminaries

For the convenience of readers, this section concisely cites the foundational theoretical concepts of LFC. For the details of LFC, the readers can refer to the literatures [34, 35].

For $0 < \alpha \leq 1$, let $\mathbb{R}^\alpha$ be an α-type fractal set of real line numbers. We endow the fractal real number set $\mathbb{R}^\alpha$ with binary operations satisfying closure under: $a^\alpha + b^\alpha := (a+b)^\alpha$ (fractal addition) and $a^\alpha \cdot b^\alpha = a^\alpha b^\alpha := (ab)^\alpha$ (fractal multiplication), preserving the topological group structure of $\mathbb{R}$ under α-scaling transformations. It is natural to see that under the application of binary operations, $\mathbb{R}^\alpha$ is a field where 0^α and 1^α denote the additive and multiplicative identities, respectively.

Local fractional continuity emerges as an indispensable theoretical prerequisite for constructing consistent local fractional derivatives on $\mathbb{R}^\alpha$, particularly in addressing non-differentiable functions inherent to fractal analysis. The definition of local fractional continuity can be formally established through the non-differentiable measure criterion in fractal space $\mathbb{R}^\alpha$. A non-differentiable function $F : \mathbb{R} \to \mathbb{R}^\alpha$ is termed locally fractional continuous at x_0, if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - x_0| < \delta$ satisfies the condition $|F(x) - F(x_0)| < \varepsilon^\alpha$. Throughout this work, the symbol $C_\alpha(I)$ denotes the collection of all local fractional continuous functions defined on the interval I.

The local fractional derivative of F of order α at the point $x = x_0$ is constructed via the fractal limit process:

$$F^{(\alpha)}(x_0) = \left. \frac{d^\alpha F(x)}{dx^\alpha} \right|_{x=x_0} = \lim_{x \to x_0} \frac{\Gamma(1+\alpha)(F(x) - F(x_0))}{(x - x_0)^\alpha},$$

where Γ denotes the classical Gamma function. Equivalently, we can reexpress $F^{(\alpha)}(x) = D_x^\alpha F(x)$. Furthermore, for every $x \in I$, if $F^{((k+1)\alpha)}(x) = \underbrace{D_x^\alpha \ldots D_x^\alpha}_{k+1} F(x)$ is well-defined, then F is contained within $D_{(k+1)\alpha}(I)$, i.e., $F \in D_{(k+1)\alpha}(I)$, $k = 0, 1, 2, \ldots$. Let $F \in D_\alpha(I)$. Then, F is an increasing function (or a decreasing function) if and only if $F^{(\alpha)}(x) \geq 0$ (or $F^{(\alpha)}(x) \leq 0$) for $x \in I$.

For a class of locally fractional continuous functions, the local fractional integral can be properly defined. For any $F \in C_\alpha[a, b]$, let $P = \{x_0, x_1, \ldots, x_N\}$, $N \in \mathbb{N}$, be a partition of interval $[a, b]$ satisfying $a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b$. Moreover, for this partition P, let $\Delta x_j = x_{j+1} - x_j$, $j = 0, \ldots, N-1$, and $\Delta x = \max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_{N-1}\}$. Under these conditions, the local fractional integral of F on the interval $[a, b]$ of order α (denoted by ${}_aI_b^{(\alpha)}F(x)$) is introduced by

$$ {}_aI_b^{(\alpha)}F(x) = \frac{1}{\Gamma(1+\alpha)}\int_a^b F(x)(dx)^\alpha = \frac{1}{\Gamma(1+\alpha)}\lim_{\Delta x\to 0}\sum_{j=0}^{N-1} F(x_j)(\Delta x_j)^\alpha. $$

If for any $x \in [a, b]$, the integral ${}_aI_x^{(\alpha)}F(x)$ exists, then $F(x) \in I_x^{(\alpha)}[a, b]$.

Adopting an analogous strategy to the Riemann integral, a local fractional analogue of the Newton-Leibniz formula can be established within the framework of LFC. In other words, if $f = F^{(\alpha)} \in C_\alpha[a, b]$, then ${}_aI_b^{(\alpha)}f(x) = F(b) - F(a)$. For instance, if $f(x) = x^\gamma$, $\gamma > 0$, then

$$ {}_aI_b^{(\alpha)}x^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\alpha)}(b^{\gamma+\alpha} - a^{\gamma+\alpha}). $$

Additionally, the following formal definition is hereby formulated (see [48]).

Definition 2.1. *For $F : I \subseteq \mathbb{R} \to \mathbb{R}^\alpha$, the function F is said to be a generalized convex function on I, if for any $x_1, x_2 \in I$ and $\lambda \in [0, 1]$, the following inequality holds:*

$$ F(\lambda x_1 + (1-\lambda)x_2) \le \lambda^\alpha F(x_1) + (1-\lambda)^\alpha F(x_2). \tag{2.1} $$

Let $F \in D_{2\alpha}(I)$. Then, F is a generalized convex function (or a generalized concave function) if and only if $F^{(2\alpha)}(x) \ge 0$ $(or\ F^{(2\alpha)}(x) \le 0)$ for $x \in I$.

Finally, utilizing the LFC framework, a fractal Hermite-Hadamard-type inequality was rigorously derived by Mo et al. [48] for a generalized convex function in fractal spaces. Namely, let $F \in I_x^{(\alpha)}[a, b]$ be a generalized convex function on $[a, b]$ with $a < b$. Then,

$$ F\left(\frac{a+b}{2}\right) \le \frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\ {}_aI_b^{(\alpha)}F(x) \le \frac{F(a)+F(b)}{2^\alpha}. \tag{2.2} $$

3. Main results

This section starts by recalling the classical Hölder's inequality [49]: If $\theta_j \ge 0, \Theta_{ij} \ge 0$ $(i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m)$ and $\sum_{j=1}^m \theta_j = 1$, then $\sum_{i=1}^n \prod_{j=1}^m \Theta_{ij}^{\theta_j} \le \prod_{j=1}^m (\sum_{i=1}^n \Theta_{ij})^{\theta_j}$. Also, recall the two-variable local fractal Hölder's inequality [50]: Let $1/p + 1/q = 1$ with $p > 1$ and $h, \Phi, \Psi \in C_\alpha(\mathbb{R}_+^2)$ be nonnegative functions. If $0 < \sum_{m,n=1}^\infty h(m,n)\Phi^p(m,n) < +\infty$ and $0 < \sum_{m,n=1}^\infty h(m,n)\ \Psi^q(m,n) < +\infty$, then the following inequality holds $\sum_{m,n=1}^\infty h(m,n)\Phi(m,n)\Psi(m,n) \le (\sum_{m,n=1}^\infty h(m,n)\ \Phi^p(m,n))^{1/p}(\sum_{m,n=1}^\infty h(m,n)\Psi^q(m,n))^{1/q}$. Along the previous two inequalities, we can obtain the following lemma about the half-discrete fractal Hölder's inequality without proofs. This lemma plays a crucial role in the proof of the main results.

Lemma 3.1. *Let $n_0 \in \mathbb{N}$, $\sum_{i=1}^3 p_i = 1$ with $p_i > 1$, $i = 1, 2, 3$, and let h, $F_i, \in C_\alpha(\mathbb{R}_+^2)$, $i = 1, 2, 3$, be nonnegative functions. If*

$$ 0 < \frac{1}{\Gamma(1+\alpha)}\int_0^\infty \sum_{n=n_0}^\infty h(x,n)F_i^{p_i}(x,n)(dx)^\alpha < \infty, \quad i = 1, 2, 3, $$

then the following inequality holds:

$$\frac{1}{\Gamma(1+\alpha)}\int_0^\infty \sum_{n=n_0}^{\infty} h(x,n)\prod_{i=1}^{3} F_i(x,n)(dx)^\alpha \le \prod_{i=1}^{3}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty \sum_{n=n_0}^{\infty} h(x,n)F_i^{p_i}(x,n)(dx)^\alpha\right)^{\frac{1}{p_i}}. \tag{3.1}$$

Assume that p and q are real numbers so that

$$p>1,\ q>1,\ \frac{1}{p}+\frac{1}{q}\ge 1, \tag{3.2}$$

and let $p' = p/(p-1)$ and $q' = q/(q-1)$ respectively be their conjugate exponents. Then, $1/p+1/p' = 1$ and $1/q + 1/q' = 1$. Additionally, define

$$\lambda = \frac{1}{p'}+\frac{1}{q'}, \tag{3.3}$$

and observe that $0 < \lambda \le 1$ holds for all p and q as in (3.2). Specifically, the equality $\lambda = 1$ holds in (3.3) if and only if $q = p'$, that is, only if p and q are mutually conjugate. Alternatively, we have $0 < \lambda < 1$, and such parameters p and q will be referred to as non-conjugate exponents.

We shall use the earlier lemma to support the main result.

Theorem 3.1. *Let p, q, and λ be real parameters satisfying* (3.2) *and* (3.3), $n_0 \in \mathbb{N}$, *and let $(a_n)_{n\ge n_0}$ be a nonnegative real sequence. If φ, ψ, $f \in C_\alpha(\mathbb{R}_+)$, and $K \in C_\alpha(\mathbb{R}_+^2)$ is a nonnegative decreasing function in both variables on $\mathbb{R}_+^2$, then the following inequalities hold and are equivalent:*

$$\frac{1}{\Gamma(1+\alpha)}\int_0^\infty \sum_{n=n_0}^{\infty} K^\lambda(x,n)f(x)a_n^\alpha(dx)^\alpha \le \left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty (\varphi\omega_1 f)^p(x)(dx)^\alpha\right)^{\frac{1}{p}}\left(\sum_{n=n_0}^{\infty}(\psi\omega_2)^q(n)a_n^{\alpha q}\right)^{\frac{1}{q}}, \tag{3.4}$$

and

$$\left(\sum_{n=n_0}^{\infty}(\psi\omega_2)^{-q'}(n)\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty K^\lambda(x,n)f(x)(dx)^\alpha\right)^{q'}\right)^{\frac{1}{q'}} \le \left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty (\varphi\omega_1 f)^p(x)(dx)^\alpha\right)^{\frac{1}{p}}, \tag{3.5}$$

where

$$\omega_1^{q'}(x) := \sum_{n=n_0}^{\infty} K(x,n)\psi^{-q'}(n) \quad \textit{and} \quad \omega_2^{p'}(n) := \frac{1}{\Gamma(1+\alpha)}\int_0^\infty K(x,n)\varphi^{-p'}(x)(dx)^\alpha. \tag{3.6}$$

Proof. The left-hand side of inequality (3.4) can be rewritten as follows:

$$L := \frac{1}{\Gamma(1+\alpha)}\int_0^\infty \sum_{n=n_0}^{\infty} K^\lambda(x,n)f(x)a_n^\alpha(dx)^\alpha = \frac{1}{\Gamma(1+\alpha)}\int_0^\infty \sum_{n=n_0}^{\infty}\left(K(x,n)\psi^{-q'}(n)(\varphi^p\omega_1^{p-q'}f^p)(x)\right)^{\frac{1}{q'}}$$
$$\times\left(K(x,n)\varphi^{-p'}(x)(\psi^q\omega_2^{q-p'})(n)a_n^{\alpha q}\right)^{\frac{1}{p'}}\left((\varphi\omega_1 f)^p(x)(\psi\omega_2)^q(n)a_n^{\alpha q}\right)^{1-\lambda}(dx)^\alpha.$$

Applying the half-discrete Hölder's inequality (3.1) to the previous relation with the conjugate parameters $q', p', 1/(1-\lambda) > 1$ leads to the following result:

$$L \le \left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty \left(\sum_{n=n_0}^{\infty} K(x,n)\psi^{-q'}(n)\right)(\varphi^p\omega_1^{p-q'}f^p)(x)(dx)^\alpha\right)^{\frac{1}{q'}}$$

$$\times\left(\sum_{n=n_0}^{\infty}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}K(x,n)\varphi^{-p'}(x)(dx)^{\alpha}\right)(\psi^q\omega_2^{q-p'})(n)a_n^{\alpha q}\right)^{\frac{1}{p'}}$$
$$\times\left(\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}(\varphi\omega_1 f)^p(x)(dx)^{\alpha}\right)^{1-\lambda}\left(\sum_{n=n_0}^{\infty}(\psi\omega_2)^q(n)a_n^{\alpha q}\right)^{1-\lambda}.$$

Lastly, we obtain (3.4) using the definitions of functions ω_1, ω_2 and the Fubini theorem.

The equivalence of inequalities (3.4) and (3.5) will now be demonstrated. Assume for the purposes of this discussion that inequality (3.4) is true. Specifying the sequence $(a_n)_{n\in\mathbb{N}}$ by

$$a_n^{\alpha}=(\psi\omega_2)^{-q'}(n)\left(\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}K^{\lambda}(x,n)f(x)(dx)^{\alpha}\right)^{q'-1},$$

and using (3.4), we have

$$\begin{aligned}
&\sum_{n=n_0}^{\infty}(\psi\omega_2)^{-q'}(n)\left(\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}K^{\lambda}(x,n)f(x)(dx)^{\alpha}\right)^{q'}\\
=&\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}\sum_{n=n_0}^{\infty}K^{\lambda}(x,n)f(x)a_n^{\alpha}(dx)^{\alpha}\le\left(\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}(\varphi\omega_1 f)^p(x)(dx)^{\alpha}\right)^{\frac{1}{p}}\left(\sum_{n=n_0}^{\infty}(\psi\omega_2)^q(n)a_n^{\alpha q}\right)^{\frac{1}{q}}\\
=&\left(\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}(\varphi\omega_1 f)^p(x)(dx)^{\alpha}\right)^{\frac{1}{p}}\left(\sum_{n=n_0}^{\infty}(\psi\omega_2)^{-q'}(n)\left(\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}K^{\lambda}(x,n)f(x)(dx)^{\alpha}\right)^{q'}\right)^{\frac{1}{q}},
\end{aligned}$$

that is, we get (3.5).

On the other hand, assume that inequality (3.5) is true. The discrete local fractional Hölder's inequality (see also [50]) is then used to derive

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}\sum_{n=n_0}^{\infty}K^{\lambda}(x,n)f(x)a_n^{\alpha}(dx)^{\alpha}\\
&=\sum_{n=n_0}^{\infty}(\psi\omega_2)^{-1}(n)\left(\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}K^{\lambda}(x,n)f(x)(dx)^{\alpha}\right)(\psi\omega_2)(n)a_n^{\alpha}\\
&\le\left(\sum_{n=n_0}^{\infty}(\psi\omega_2)^{-q'}(n)\left(\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}K^{\lambda}(x,n)f(x)(dx)^{\alpha}\right)^{q'}\right)^{\frac{1}{q'}}\left(\sum_{n=n_0}^{\infty}(\psi\omega_2)^q(n)a_n^{\alpha q}\right)^{\frac{1}{q}}\\
&\le\left(\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}(\varphi\omega_1 f)^p(x)(dx)^{\alpha}\right)^{\frac{1}{p}}\left(\sum_{n=n_0}^{\infty}(\psi\omega_2)^q(n)a_n^{\alpha q}\right)^{\frac{1}{q}},
\end{aligned}$$

which implies (3.4). Hence, inequalities (3.4) and (3.5) are equivalent. □

We assume that $h\in C_{\alpha}(\mathbb{R})$ is a nonnegative function in the following. Additionally, we define

$$k(\eta)=\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}h(t)t^{-\alpha\eta}(dt)^{\alpha},\tag{3.7}$$

under the assumption $k(\eta)<\infty$.

Furthermore, we take into account some weight functions that encompass real variable differentiable functions. More specifically, we present the definition and notation as follows.

Definition 3.1. *Let $r > 0$. We denote by $\mathcal{H}(r)$ the set of all nonnegative differentiable functions $u : \mathbb{R}_+ \to \mathbb{R}$ satisfying the following conditions:*

(i) u is an increasing function on $\mathbb{R}_+$ and $\lim\limits_{x\to+\infty} u(x) = +\infty$.

(ii) $\frac{[u'(x)]^\alpha}{[u(x)]^{\alpha r}}$ is a decreasing and generalized convex function on $\mathbb{R}_+$

The following lemma is easily obtained by using the LFC.

Lemma 3.2. *Let $r > 0$, and let $u, v \in \mathcal{H}(r)$. If $h : \mathbb{R}_+ \longrightarrow \mathbb{R}^\alpha$ is a nonnegative function such that $h(u(x)v(y))$ is a decreasing and generalized convex function in both variables on $\mathbb{R}_+^2$, then,*

$$h(u(x)v(y))\frac{[v'(y)]^\alpha}{[v(y)]^{\alpha r}} \quad \text{and} \quad h(u(x)v(y))\frac{[u'(x)]^\alpha}{[u(x)]^{\alpha r}}$$

are decreasing and generalized convex functions on $\mathbb{R}_+$ for any fixed x or y, respectively.

Proof. For the sake of proof, we set $H(y) := h(u(x)v(y))$ and $V(y) := \frac{[v'(y)]^\alpha}{[v(y)]^{\alpha r}}$. It follows from Definition 3.1 and the assumptions that $H(y)$ and $V(y)$ are nonnegative decreasing and generalized convex functions. Then, $H^{(\alpha)}(y) \le 0$, $H^{(2\alpha)}(y) \ge 0$, $V^{(\alpha)}(y) \le 0$, and $V^{(2\alpha)}(y) \ge 0$. Furthermore, by using the product rule for the local fractional derivative, we can observe

$$\frac{\partial^\alpha}{\partial y^\alpha}\left[h(u(x)v(y))\frac{[v'(y)]^\alpha}{[v(y)]^{\alpha r}}\right] = \frac{d^\alpha}{dy^\alpha}[H(y)V(y)] = H^{(\alpha)}(y)V(y) + H(y)V^{(\alpha)}(y) \le 0,$$
$$\frac{\partial^{2\alpha}}{\partial y^{2\alpha}}\left[h(u(x)v(y))\frac{[v'(y)]^\alpha}{[v(y)]^{\alpha r}}\right] = \frac{d^{2\alpha}}{dy^{2\alpha}}[H(y)V(y)] = H^{(2\alpha)}(y)V(y) + 2H^{(\alpha)}(y)V^{(\alpha)}(y) + H(y)V^{(2\alpha)}(y) \ge 0,$$

which imply that $h(u(x)v(y))\frac{[v'(y)]^\alpha}{[v(y)]^{\alpha r}}$ is a decreasing and generalized convex function with respect to y for any fixed x. Following the same way, we can easily obtain the proof for the function $h(u(x)v(y))\frac{[u'(x)]^\alpha}{[u(x)]^{\alpha r}}$ with respect to x for any fixed y. □

The following lemma is employed for proving our main result (see [50]).

Lemma 3.3. *If $f \in I_x^{(\alpha)}(\mathbb{R}_+)$, $f^{(\alpha)}(t) \le 0$, $f^{(2\alpha)}(t) \ge 0$ $(t \in (1/2, \infty))$, then we have*

$$\frac{1}{\Gamma(1+\alpha)}\int_1^\infty f(t)(dt)^\alpha \le \frac{1}{\Gamma(1+\alpha)}\sum_{n=1}^\infty f(n) \le \frac{1}{\Gamma(1+\alpha)}\int_{\frac{1}{2}}^\infty f(t)(dt)^\alpha. \tag{3.8}$$

We are now prepared to present and prove our general result.

Theorem 3.2. *Let $A_1, A_2 \in \mathbb{R}_+$, and p, q, λ be real parameters satisfying* (3.2) *and* (3.3). *Suppose that u is a nonnegative increasing differentiable function such that $u(\infty) = \infty$, $v \in \mathcal{H}(qA_2)$, and $h : \mathbb{R}_+ \mapsto \mathbb{R}^\alpha$ is defined as in Lemma 3.2. Then, the following inequalities hold and are equivalent:*

$$\frac{1}{\Gamma(1+\alpha)}\int_0^\infty\sum_{n=1}^\infty h^\lambda(u(x)v(n))f(x)a_n^\alpha(dx)^\alpha \le C_1\left(\frac{1}{\Gamma(1+\alpha)}\int_0^\infty[u(x)]^{\alpha p(A_1+A_2)-\frac{\alpha p}{q'}}[u'(x)]^{\alpha(1-p)}f^p(x)(dx)^\alpha\right)^{\frac{1}{p}}$$
$$\times\left(\sum_{n=1}^\infty[v(n)]^{\alpha q(A_1+A_2)-\frac{\alpha q}{p'}}[v'(n)]^{\alpha(1-q)}a_n^{\alpha q}\right)^{\frac{1}{q}}, \tag{3.9}$$

and

$$\left(\sum_{n=1}^{\infty}[v(n)]^{-\alpha q'(A_1+A_2)-\frac{\alpha q'}{p'}}[v'(n)]^{-\alpha}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}h^{\lambda}(u(x)v(n))f(x)\right)^{q'}(dx)^{\alpha}\right)^{\frac{1}{q'}}$$
$$\leq C_1\left(\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}[u(x)]^{\alpha p(A_1+A_2)-\alpha\frac{\alpha p}{q'}}[u'(x)]^{\alpha(1-p)}f^p(x)(dx)^{\alpha}\right)^{\frac{1}{p}}, \quad (3.10)$$

where $C_1=\Gamma^{\frac{1}{q'}}(1+\alpha)k^{\frac{1}{p'}}(p'A_1)k^{\frac{1}{q'}}(q'A_2)$.

Proof. Let the functions $K(x,y)=h(u(x)v(y))$, $(\varphi\circ u)(x)=[u(x)]^{\alpha A_1}[u'(m)]^{-\frac{\alpha}{p'}}$, $(\psi\circ v)(n)=[v(n)]^{\alpha A_2}\cdot[v'(n)]^{-\frac{\alpha}{q'}}$, and $n_0=1$ in inequality (3.4). Clearly, these substitutions are well defined, since u and v are injective functions. Thus, in this setting we have

$$\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}\sum_{n=1}^{\infty}h(u(x)v(n))f(x)a_n^{\alpha}(dx)^{\alpha}\leq\left(\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}[u(x)]^{\alpha pA_1}[u'(x)]^{\alpha(1-p)}(\omega_1\circ u)^p(x)f(x)(dx)^{\alpha}\right)^{\frac{1}{p}}$$
$$\times\left(\sum_{n=1}^{\infty}[v(n)]^{\alpha qA_2}[v'(n)]^{\alpha(1-q)}(\omega_2\circ v)(n)a_n^{\alpha q}\right)^{\frac{1}{q}}, \quad (3.11)$$

where

$$(\omega_1\circ u)(x)=\left(\sum_{n=1}^{\infty}\frac{h(u(x)v(n))[v'(n)]^{\alpha}}{[v(n)]^{\alpha q'A_2}}\right)^{\frac{1}{q'}}\text{ and }(\omega_2\circ v)(n)=\left(\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}\frac{h(u(x)v(n))[u'(x)]^{\alpha}}{[u(x)]^{\alpha p'A_1}}(dx)^{\alpha}\right)^{\frac{1}{p'}}.$$

Applying Lemmas 3.2 and 3.3, we get

$$(\omega_1\circ u)^{q'}(x)\leq\Gamma(1+\alpha)\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}\frac{h(u(x)v(y))}{[v(y)]^{\alpha q'A_2}}[v'(y)]^{\alpha}(dy)^{\alpha}.$$

Furthermore, by using the substitution $t=u(x)v(y)$, we obtain

$$\begin{aligned}(\omega_1\circ u)^{q'}(x)&\leq\Gamma(1+\alpha)[u(x)]^{\alpha q'A_2-\alpha}\frac{1}{\Gamma(1+\alpha)}\int_{u(x)v(0)}^{\infty}h(t)t^{-\alpha q'A_2}(dt)^{\alpha}\\&\leq\Gamma(1+\alpha)[u(x)]^{\alpha q'A_2-\alpha}\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}h(t)t^{-\alpha q'A_2}(dt)^{\alpha}\\&=\Gamma(1+\alpha)k(q'A_2)[u(x)]^{\alpha q'A_2-\alpha},\end{aligned}\quad (3.12)$$

where we used the definition of the function $k(\cdot)$.

By the similar arguments as for the function $\omega_2\circ u$, we get

$$(\omega_2\circ v)^{p'}(n)\leq[v(n)]^{\alpha p'A_1-\alpha}\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}h(t)t^{-\alpha p'A_1}(dt)^{\alpha}[v(n)]^{\alpha p'A_1-\alpha}k(p'A_1). \quad (3.13)$$

Finally, relations (3.12) and (3.13) yield the inequality (3.9).

On the other hand, if we rewrite inequality (3.5) with the same functions as in the proof of inequality (3.9), after using estimates (3.12) and (3.13), we easily get (3.10). This completes the proof. □

Now, in order to present our main result, we define the integral

$$k(\beta; r_1, r_2) = \frac{1}{\Gamma(1+\alpha)} \int_{r_1}^{r_2} h(t) t^{-\alpha\beta} (dt)^\alpha, \quad 0 \le r_1 < r_2 \le \infty, \tag{3.14}$$

where the arguments β, r_1, and r_2 are selected appropriately such that (3.14) converges. In addition, if $r_1 = 0$ and $r_2 = \infty$, then the integral $k(\beta; 0, \infty)$ will be denoted by $k(\beta)$, as in (3.7).

Theorem 3.3. *Let $A_1, A_2 \in \mathbb{R}_+$, and p, q, λ be real parameters satisfying* (3.2) *and* (3.3). *Let the functions h, v be defined as in Theorem 3.2. Suppose that $u : [n_0 - 1, \infty) \longrightarrow \mathbb{R}$, $n_0 \in \mathbb{N}$, is a nonnegative increasing differentiable function such that $u(\infty) = \infty$. Then, the following inequality holds:*

$$\frac{1}{\Gamma(1+\alpha)} \int_0^\infty \sum_{n=n_0}^{\infty} h^\lambda(u(x)v(n)) f(x) a_n^\alpha (dx)^\alpha \le C_2 \Big(\frac{1}{\Gamma(1+\alpha)} \int_0^\infty [u(x)]^{\alpha p(A_1+A_2) - \frac{\alpha p}{q'}} [u'(x)]^{\alpha(1-p)}$$
$$\times k^{\frac{p}{q'}} \Big(q'A_2; u(x) v\Big(n_0 - \frac{1}{2}\Big), \infty \Big) f^p(x) (dx)^\alpha \Big)^{\frac{1}{p}} \Big(\sum_{n=n_0}^{\infty} [v(n)]^{\alpha q(A_1+A_2) - \frac{\alpha q}{p'}} [v'(n)]^{\alpha(1-q)} a_n^{\alpha q} \Big)^{\frac{1}{q}}, \tag{3.15}$$

where $C_2 = \Gamma^{\frac{1}{p'}}(1+\alpha) k^{\frac{1}{p'}}(p'A_1)$.

Proof. Since the function $h(u(x)v(y))[v(y)]^{-\alpha q' A_2}[v'(y)]^\alpha$ is convex on $[n_0 - 1/2, \infty)$ for any fixed $x \in \mathbb{R}_+$, applying the generalized Hermite-Hadamard inequality, i.e., the left inequality in (2.2), to unit intervals $[n - 1/2, n + 1/2]$, yields the following inequalities:

$$\frac{h(u(x)v(n))[v'(n)]^\alpha}{[v(n)]^{\alpha q' A_2}} \le \Gamma(1+\alpha) \frac{1}{\Gamma(1+\alpha)} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{h(u(x)v(y))[v'(y)]^\alpha}{[v(y)]^{\alpha q' A_2}} (dy)^\alpha,$$

where $n = n_0, n_0 + 1, \ldots$.

Now, summing these inequalities, we have

$$(\omega_1 \circ u)^{q'}(x) = \sum_{n=n_0}^{\infty} \frac{h(u(x)v(n))[v'(n)]^\alpha}{[v(n)]^{\alpha q' A_2}} \le \Gamma(1+\alpha) \frac{1}{\Gamma(1+\alpha)} \int_{n_0-\frac{1}{2}}^{\infty} \frac{h(u(x)v(y))[v'(y)]^\alpha}{[v(y)]^{\alpha q' A_2}} (dy)^\alpha,$$

and the change of variable $t = u(x)v(y)$, and the definition (3.14) yield

$$(\omega_1 \circ u)^{q'}(x) \le \Gamma(1+\alpha)[u(x)]^{\alpha q' A_2 - \alpha} k \left(q'A_2; u(x) v \left(n_0 - \frac{1}{2} \right), \infty \right). \tag{3.16}$$

By using the substitution $t = u(x)v(y)$ (see also the proof of Theorem 3.2), we obtain

$$(\omega_2 \circ v)^{p'}(n) = \frac{1}{\Gamma(1+\alpha)} \int_0^\infty \frac{h(u(x)v(n))[u'(x)]^\alpha}{[u(x)]^{\alpha p' A_1}} (dx)^\alpha \le [v(n)]^{\alpha p' A_1 - \alpha} k(p'A_1). \tag{3.17}$$

It follows from (3.11), (3.16), and (3.17) that we get (3.15). □

Now, we suppose that $h : \mathbb{R}_+ \mapsto \mathbb{R}^\alpha$ is a nonnegative decreasing and generalized convex function on $\mathbb{R}_+$. Let $u, v : \mathbb{R}_+ \mapsto \mathbb{R}$ be nonnegative functions such that v is an increasing function on $\mathbb{R}_+$. By using the chain rule for local fractional derivative, then, we have the following results:

$$\frac{\partial^\alpha}{\partial y^\alpha} [h(u(x)v(y))] = h^{(\alpha)}(u(x)v(y))(u(x))^\alpha (v'(y))^\alpha \le 0, \tag{3.18}$$

and, similarly,

$$\frac{\partial^{2\alpha}}{\partial y^{2\alpha}}\left[h(u(x)v(y))\right] = h^{(2\alpha)}(u(x)v(y))(u(x))^{2\alpha}(v'(y))^{2\alpha} + h^{(\alpha)}(u(x)v(y))(u(x))^{\alpha}\frac{d^{\alpha}}{dy^{\alpha}}[(v'(y))^{\alpha}]. \tag{3.19}$$

By putting $\overline{h}(y) = (1+y)^{-\alpha s}$, $s > 0$, and $v(y) = y^b$, $0 < b < 1$, we easily obtain

$$\overline{h}^{(\alpha)}(y) = -\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s-1)\alpha)}\frac{1}{(1+y)^{\alpha(s+1)}}, \tag{3.20}$$

$$\overline{h}^{(2\alpha)}(y) = \frac{\Gamma(1+(s+1)\alpha)}{\Gamma(1+(s-1)\alpha)}\frac{1}{(1+y)^{\alpha(s+2)}}, \tag{3.21}$$

and

$$\frac{d^{\alpha}}{dy^{\alpha}}\left[(v'(y))^{\alpha}\right] = \frac{d^{\alpha}}{dy^{\alpha}}\left[b^{\alpha}y^{\alpha(b-1)}\right] = -\frac{\Gamma(1+(1-b)\alpha)}{\Gamma(1-b\alpha)}\frac{b^{\alpha}}{y^{\alpha(2-b)}}. \tag{3.22}$$

Applying (3.20)–(3.22) to (3.19), we obtain

$$\frac{\partial^{\alpha}}{\partial y^{\alpha}}[\overline{h}(u(x)v(y))] \leq 0 \ \text{ and } \ \frac{\partial^{2\alpha}}{\partial y^{2\alpha}}[\overline{h}(u(x)v(y))] \geq 0.$$

Furthermore, we will check that for the function $v(y) = y^b$, $0 < b < 1$, then $v \in \mathcal{H}(r)$, $r > 0$. That is, we can count

$$\frac{[v'(y)]^{\alpha}}{[v(y)]^{\alpha r}} = \frac{(by^{b-1})^{\alpha}}{y^{\alpha br}} = b^{\alpha}y^{\alpha(b-1-br)},$$

and, consequently,

$$\frac{d^{\alpha}}{dy^{\alpha}}\left[\left(\frac{1}{y}\right)^{\alpha(br+1-b)}\right] = -\frac{\Gamma(1+(br+1-b)\alpha)}{\Gamma(1+(br-b)\alpha)}\frac{1}{y^{\alpha(br+2-b)}},$$

$$\frac{d^{2\alpha}}{dy^{2\alpha}}\left[\left(\frac{1}{y}\right)^{\alpha(br+1-b)}\right] = \frac{\Gamma(1+(br+2-b)\alpha)}{\Gamma(1+(br-b)\alpha)}\frac{1}{y^{\alpha(br+3-b)}}.$$

Finally, by employing Lemma 3.2, we obtain that the function $\overline{h}(u(x)v(y))\frac{[v'(y)]^{\alpha}}{[v(y)]^{\alpha r}}$ is a nonnegative decreasing and generalized convex function on $\mathbb{R}_+$ for any $x > 0$, if $r > 0$ and $v(y) = y^b$, $0 < b < 1$.

The following application of Theorem 3.3 refers to the functions $\overline{h}(t) = (1+t)^{-\alpha s}$, $s > 0$, $u(x) = x^a$, $a > 0$, and $v(y) = y^b$, $0 < b < 1$. In this case, the weight function appearing in (3.15) may be expressed in terms of the incomplete fractal Beta function defined by

$$B_{\alpha,r}(c,d) = \frac{1}{\Gamma(1+\alpha)}\int_r^{\infty}\frac{u^{\alpha(c-1)}}{(1+u)^{\alpha(c+d)}}(du)^{\alpha}, \quad r > 0.$$

Corollary 3.1. *Let $A_1, A_2 \in \mathbb{R}_+$, and p, q, λ be real parameters satisfying* (3.2) *and* (3.3). *If* $0 < b < 1$ *and* $a > 0$, *then the following inequality holds:*

$$\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}\sum_{n=1}^{\infty}\frac{f(x)a_n^{\alpha}}{(1^{\alpha}+x^{\alpha a}n^{\alpha b})^{\lambda s}}(dx)^{\alpha} \leq C_3\left(\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}x^{\alpha ap(A_1+A_2-\lambda)+\alpha(p-1)}f^p(x)(dx)^{\alpha}\right)^{\frac{1}{p}}$$
$$\times\left(\sum_{n=1}^{\infty}n^{\alpha bq(A_1+A_2-\lambda)+\alpha(q-1)}a_n^{\alpha q}\right)^{\frac{1}{q}}, \tag{3.23}$$

where $C_3 = \Gamma^{\frac{1}{q'}}(1+\alpha)a^{-\frac{\alpha}{p'}}b^{-\frac{\alpha}{q'}}B^{\frac{1}{p'}}_{\alpha,\frac{n^b}{2^a}}(1-p'A_1, s-1+p'A_1)$.

Proof. Before, we have proved that $\overline{h}(u(x)v(y))\frac{[v'(y)]^{\alpha}}{[v(y)]^{\alpha r}}$ is a nonnegative decreasing and generalized convex function on $\mathbb{R}_+$ for any $x > 0$, if $r > 0$ and $v(y) = y^b$, $0 < b < 1$. From the definition of the incomplete fractal Beta function, by setting $n_0 = 1$, we have

$$k\Big(p'A_2; \frac{n^b}{2^a}, \infty\Big) = \frac{1}{\Gamma(1+\alpha)}\int_{\frac{n^b}{2^a}}^{\infty} \frac{u^{-\alpha p' A_2}}{(1+u)^{\alpha s}}(du)^{\alpha} = B_{\alpha,\frac{n^b}{2^a}}(1-p'A_2, s-1+p'A_2).$$

Finally, from (3.15), we get (3.23). □

Remark 3.1. By choosing the appropriate functions in Theorems 3.2 and 3.3, the main results degenerate into the local fractional form of HDHI obtained by the authors [1, Theorem 351], Peng et al. [31], and Krnić et al. [33]. By setting the suitable parameters in Corollary 3.1, the inequality (3.23) reduces to the local fractional form of HDHI given by Yang [51].

4. Conclusions

In this paper, we investigated some novel local fractional HDHIs with nonhomogeneous kernels using weight coefficient methods and parameterization techniques. As applications, both the canonical equivalences and their degenerate forms have also been presented. The primary results of this paper have generalized and extended classical HDHIs to LFC. Subsequently, we have not addressed the problem of determining the optimal constants in this paper. Consequently, as part of our future research objectives, we intend to explore various other local fractional HDHIs and investigate the optimality of their associated constants.

Author contributions

Xiaohong Zuo: conceptualization, writing-original draft, funding acquisition; Predrag Vuković: writing-review and editing, investigation, funding acquisition; Wengui Yang: formal analysis, writing-review and editing, supervision. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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