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Research article

Dynamical analysis of a fractional-order Cournot–Bertrand duopoly model with time delays

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Abstract: This paper investigates a fractional-order Cournot–Bertrand duopoly model with time delays. Employing stability theory for fractional-order delayed dynamical systems and Hopf bifurcation (HB) analysis, we rigorously derive stability criteria for equilibrium points and HB thresholds across six distinct scenarios. Theoretical and numerical results demonstrate that both fractional order and delay length significantly influence the model's dynamical properties. Enterprises should account for memory effects and decision delays in market information to construct monitoring mechanisms, while regulators must track corporate decision-making strategies and market dynamics to establish early-warning systems. Such systems can prevent market imbalance risks through real-time monitoring of key parameters.

Keywords: fractional-order evolutionary; Cournot–Bertrand duopoly model; asymptotic stability; Hopf bifurcation; time delay
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1. Introduction

In economic issues, output and price are eternal themes. In 1975, Bylka and Komar [1] first put forward the concept of mixed competition between output and price. In 1984, Singh and Vives [2] presented a duopoly monopoly model of mixed competition between output and price, namely the Cournot–Bertrand mixed model. In 2000, Häckner [3] constructed a Cournot–Bertrand mixed model for different products. In 2013, Ma and Pu [4] considered a Cournot–Bertrand duopoly model with heterogeneous goods and used nonlinear theories and methods to conduct an in-depth analysis of the evolutionary behavior of the model. In 2014, Wang and Ma [5] gave a Cournot–Bertrand mixed model with different expectations and found that under the same conditions, the stability region under the Cournot–Bertrand system is larger than that under the Cournot or Bertrand system. Up to now, scholars have achieved numerous results in the research on the Cournot–Bertrand game model [6, 7].

However, these traditional models are often based on idealized assumptions and ignore many key factors in reality. With the increasing complexity and diversification of economic activities, the time delay phenomenon has become an important factor that cannot be ignored. The time delay phenomenon is prevalent in nature and various systems. In biological systems, time delay is manifested in various stages of species growth and their interactions, such as physiological time delay, predation time delay, signal time delay, etc., and it is one of the main factors that cause system degradation and present nonlinear characteristics that inhibit or promote system behaviors [8, 9]. In the actual business environment, enterprises cannot complete the adjustment of output or price instantaneously. The preparation cycle of the production process, the delay in information transmission, and the time delay in decision implementation all make the current decisions of enterprises closely related to the past market status. This time delay effect may lead to increased volatility in market dynamics. The equilibrium state predicted by traditional models may be difficult to achieve or maintain in reality, which greatly limits their ability to explain and predict the real market. In recent years, researchers have conducted in-depth discussions on the existence, uniqueness, positivity, and boundedness of solutions to time delay systems [10]; focused on studying the system stability problems, bifurcation behaviors, and control strategy problems caused by time delay; and have given corresponding system analyses and interpretations [11, 12]. The above research results are all obtained based on the relevant conclusions of the integer-order differential equation theory.

As is well known, integer-order calculus shows certain limitations when describing specific practical problems, while fractional-order calculus breaks through these limitations and can delicately depict the memory and hereditary characteristics of the system. With the continuous improvement of its basic theories in aspects such as the existence, uniqueness, stability, and HB of solutions [13, 14], as well as its unique advantages in the stable region [15, 16], it has been widely applied in various scientific research fields, such as electricity [17], medicine [18], finance [19], and neural networks [20]. Furthermore, there have been numerous achievements in the research of numerical solution methods for fractional differential equations [21, 22].

Based on the discrete form of fractional-order differential equations, Qian et al. [23], in view of the long-term memory characteristic of price fluctuations, proposed a new fractional-order discrete dynamic system for price games. Xin et al. [24] constructed a discrete fractional-order Cournot game. In this game, participants can make full use of their historical information to make decisions. Considering the problem of the long memory effect, Xin et al. [25] extended the Bertrand duopoly monopoly game model to a fractional-order form. They studied the local stability of the Nash equilibrium and, through numerical simulation experiments, revealed complex dynamic phenomena such as bifurcation and chaos in game behaviors. In the research on the Cournot–Bertrand competition with uncertain demand, Quadir [26] proved that information sharing is the dominant strategy for enterprises making output decisions, while not sharing information is the dominant strategy for enterprises making price decisions. Zhu et al. [27] fully considered a mixed competition model that is more in line with the complex real economic market and analyzed the general stability conditions of four equilibrium points by using eigenvalues and the Jury criterion. Gurcan et al. [28] deeply explored a valuable Cournot duopoly model, which is described by the Caputo fractional-order differential equation with piecewise constant variables. In this model, through detailed analysis of the piecewise constant variables with specific properties, new perspectives for observing the competitive behavior of the Cournot duopoly and market dynamics are provided. Al-khedhairi et al. [29] explored a fractional-order difference Cournot–Bertrand competitive duopoly game based on the Caputo operator. Al-khedhairi [30] and Culda et al. [31] used fractional-order differential equations to deal with the memory effect problems in continuous-time and discrete-time game scenarios. In these games, the memory effect is of great significance for game results and dynamics.

Combining the time delay and fractional-order theories can more accurately describe real-world phenomena in different disciplines, such as population reproduction cycles [32], virus incubation periods [33], and information transmission time delays [34], etc., and can further deepen people's understanding of the inherent complexity of systems [35, 36]. In the field of ecological dynamics, researchers have deeply explored the stability and HB phenomena of the fractional-order predator-prey system with two different time delays [37,38], and found that the stability will decrease as the fractional order and the other time delay increase, that is, the HB will occur earlier [39]. Xu et al. [40] conducted an in-depth analysis on the fractional-order neural network model with multiple time delays. The research results show that time delay will not only lead to the loss of system stability but also trigger complex dynamic phenomena such as periodic solutions, bifurcations, and chaos [41,42]. Furthermore, the application of fractional-order delay calculus in economics and finance has attracted increasing attention from scholars; see [43–45].

Reference [5] considered a mixed Cournot–Bertrand duopoly game model where two enterprises take output and price as decision variables, respectively, and studied the complexity of the Cournot–Bertrand duopoly game model with different expectations. However, since enterprises consider not only the current influence of each other's strategies but also the memory of past market history and the limitations of their own development trajectories when making decisions. For example, when formulating output or price strategies, information such as the long-term changes in market share accumulated in the past, the trend of cost changes, and the historical strategies of competitors will be incorporated into the current decision-making process according to certain weights, and fractional calculus can precisely describe this complex memory-dependent relationship. Moreover, when making decisions, enterprises often exhibit a certain lag. Therefore, integrating fractional calculus and time delay into the Cournot–Bertrand duopoly game model is of great practical significance.

Based on this idea, this paper will conduct a fractional-order generalized Cournot–Bertrand model with two unequal time delays. Unlike existing studies that mostly rely on integer-order derivatives and equal time delay settings [5, 24–27, 30], this model incorporates fractional derivatives and unequal time delays into the Cournot–Bertrand mixed game framework, thereby constructing a dynamically coupled mechanism for two-dimensional competition in output and price. It not only reveals dynamic behaviors and stability conditions not covered by traditional models but also provides a more general theoretical paradigm for analyzing the complex mechanisms of real-world market competition.

The structure of this paper is as follows: In Section 2, the fractional-order Cournot–Bertrand duopoly model with time delays is introduced. The basic concepts and related lemmas of the Caputo fractional-order differential equation are introduced in Section 3. The existence, non-negativity, and boundedness of solutions are studied in Section 4. In Section 5, we studied the evolution stability and global asymptotic stability of the equilibrium points. In Section 6, numerical simulations are used to verify the impact of parameters on stability strategies. Moreover, we conduct a sensitivity analysis of the key parameters on the model's behavior based on Theorem 5.5. Conclusions and policy suggestions are given in Section 7.

2. Fractional-order Cournot–Bertrand duopoly model with time delays

Suppose enterprise 1 chooses output as the decision variable and influences market supply by controlling output. Enterprise 2 chooses price as the decision variable and influences market demand by controlling price. Then the following model can be obtained from [5]:

$$\begin{cases} p_1(t) = 1 - d - q_1(t) + dp_2(t) + d^2 q_1(t), \\ q_2(t) = 1 - dq_1(t) - p_2(t), \end{cases} \tag{2.1}$$

where, q_i represents the output of products, p_i represents the price, and d indicates the level of difference between the two products. When $d \in (0, 1)$, it represents that two products are substitutes, and when the value of d is greater, the degree of substitution is higher. When $d \in (-1, 0)$, it means that products are complementary to each other; the smaller the value of d, the stronger their complementarity. When $d = 0$, it means that the two are independent of each other. Therefore, the profit functions of enterprises are as follows:

$$\begin{cases} \Pi_1(t) = q_1(t)[1 - c - d - q_1(t) + dp_2(t) + d^2 q_1(t)], \\ \Pi_2(t) = [p_2(t) - c][1 - dq_1(t) - p_2(t)]. \end{cases} \tag{2.2}$$

The first enterprise achieves maximization by adjusting output. The second enterprise realizes its maximum profit by adjusting price. In this way, we established a duopoly game. In the actual market, we suppose that two participants do not have complete knowledge of the market. They follow a limited correction process based on the estimation of marginal profit and continuously adjust their output proportionally according to their profit changes. This is called gradient dynamics and can be described as follows:

$$\begin{cases} \dfrac{dq_1}{dt} = v_1 q_1 \dfrac{\partial \Pi_1}{\partial q_1}, \\ \dfrac{dp_2}{dt} = v_2 p_2 \dfrac{\partial \Pi_2}{\partial p_2}, \end{cases} \tag{2.3}$$

where v_1 and v_2 are speed parameters that represent output adjustments. In a real market environment, the realization of profits between two enterprises is influenced by many factors. To describe the memory and genetic characteristics of the profit realization process, it is proposed to introduce a fractional-order system to depict their relationships. Combining Eqs (2.1)–(2.3), the fractional-order duopoly Cournot–Bertrand model for boundedly rational enterprises is as follows:

$$\begin{cases} \dfrac{d^\alpha q_1}{dt^\alpha} = v_1 q_1(t)[1 - c - d + dp_2(t) - 2q_1(t) + 2d^2 q_1(t)], \\ \dfrac{d^\alpha p_2}{dt^\alpha} = v_2 p_2(t)[1 + c - dq_1(t) - 2p_2(t)], \quad \alpha \in (0, 1). \end{cases} \tag{2.4}$$

Further, if both participants have diverse self-feedback time delays: specifically, enterprise 1, which achieves profits through constant output, is subject to the self-feedback time delay τ_1, while enterprise 2, which secures profits via constant pricing, is affected by the self-feedback time delay τ_2. The fractional-order delayed duopoly Cournot–Bertrand model for boundedly rational enterprises is as follows:

$$\begin{cases} \dfrac{d^\alpha q_1}{dt^\alpha} = v_1 q_1(t)[1 - c - d + dp_2(t) - 2q_1(t-\tau_1) + 2d^2 q_1(t-\tau_1)], \\ \dfrac{d^\alpha p_2}{dt^\alpha} = v_2 p_2(t)[1 + c - dq_1(t) - 2p_2(t-\tau_2)]. \end{cases} \tag{2.5}$$

3. Preliminary

In this section, some concepts and properties of fractional-order derivatives will be introduced.

Definition 3.1. [46] Let $\alpha > 0$, $n-1 < \alpha < n$, $n \in \mathbb{N}_+$. The Caputo's fractional derivative of order α for a function $f(t)$ with initial value $f(t_0)$ is given as follows:

$$ {}_{t_0}^{C}D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{1+\alpha-n}} d\tau,$$

where $\Gamma(\alpha) = \displaystyle\int_0^{+\infty} x^{\alpha-1} e^{-x} dx$.

Lemma 3.1. [47] Given the following fractional-order model: $D^\alpha g = Kg$, $g(0) = g_0$ where $\alpha \in (0,1)$, $g \in \mathbb{R}^n$, and $K \in \mathbb{R}^{n\times n}$. Let λ_i $(i = 1, 2, \cdots, n)$ be the root of the characteristic equation of $D^\alpha g = Kg$, the system is locally asymptotically stable on the condition that $|arg(\lambda_i)| > \frac{\alpha\pi}{2}$ $(i = 1, 2, ..., n)$ and all critical eigenvalues satisfying $|arg(\lambda_i)| = \frac{\alpha\pi}{2}$ $(i = 1, 2, ..., n)$ have geometric multiplicity one.

Lemma 3.2. [48] Let $g(t) \in C[t_0, \infty)$ satisfy

$$D^\alpha g(t) \le -\chi_1 g(t) + \chi_2, \quad g(t_0) = g_{t_0},$$

where $\alpha \in (0,1)$, $(\chi_1,\chi_2) \in \mathbb{R}^2$, $\chi_1 \ne 0$, and $t_0 \ge 0$ is the initial time, then

$$g(t) \le \Big(g(t_0) - \frac{\chi_2}{\chi_1}\Big) E_\alpha[-\chi_1 (t-t_0)^\alpha] + \frac{\chi_2}{\chi_1}.$$

4. Existence, non-negativity, and boundedness analysis

In this part, we analyze the existence, uniqueness, non-negativity, and boundedness of system (2.5).

Theorem 4.1. There exists a constant $t^* \in \mathbb{R}_+$, denoted Λ=$\{(q_1(t), p_2(t)) \in \mathbb{R}^2$: $\max\{|q_1(t)|, |p_2(t)|\} \le a$, $a \in \mathbb{R}_+$, $t \in (-\tau, t^*)$, $\tau = \max\ (\tau_1, \tau_2)\}$. System (2.5) with initial value (q_{10}, p_{20}) owns a unique solution $M = (q_1(t), p_2(t)) \in \Lambda$.

Proof. Construct the following mapping:

$$g(M) = (g_1(M), g_2(M)), \tag{4.1}$$

where

$$\begin{cases} g_1(M) = v_1 q_1(t)[1 - c - d + dp_2(t) - 2q_1(t-\tau_1) + 2d^2 q_1(t-\tau_1)], \\ g_2(M) = v_2 p_2(t)[1 + c - dq_1(t) - 2p_2(t-\tau_2)]. \end{cases} \tag{4.2}$$

$\forall M, \tilde{M} \in \Lambda$, one has

$$\begin{aligned}
\|g(M) - g(\tilde{M})\| = & \Big|v_1 q_1(t)[1 - c - d + dp_2(t) - 2q_1(t - \tau_1) + 2d^2 q_1(t - \tau_1)] \\
& - v_1 \tilde{q}_1(t)[1 - c - d + d\tilde{p}_2(t) - 2\tilde{q}_1(t - \tau_1) + 2d^2 \tilde{q}_1(t - \tau_1)]\Big| \\
& + \Big|v_2 p_2(t)[1 + c - dq_1(t) - 2p_2(t - \tau_2)] \\
& - v_2 \tilde{p}_2(t)[1 + c - d\tilde{q}_1(t) - 2\tilde{p}_2(t - \tau_2)]\Big| \\
= & \Big|v_1(1 - c - d)(q_1(t) - \tilde{q}_1(t)) + v_1 d[q_1(t)p_2(t) - \tilde{q}_1(t)\tilde{p}_2(t)] \\
& - 2v_1[q_1(t)q_1(t - \tau_1) - \tilde{q}_1(t)\tilde{q}_1(t - \tau_1)] \\
& + 2v_1 d^2[q_1(t)q_1(t - \tau_1) - \tilde{q}_1(t)\tilde{q}_1(t - \tau_1)]\Big| \\
& + \Big|v_2(1 + c)[p_2(t) - \tilde{p}_2(t)] - 2v_2[p_2(t)p_2(t - \tau_2) \\
& - \tilde{p}_2(t)\tilde{p}_2(t - \tau_2)] - v_2 d[p_2(t)q_1(t) - \tilde{p}_2(t)\tilde{q}_1(t)]\Big| \\
\leq & [v_1(1 - c - d) + 2v_1 a + 2av_1 d^2 + v_1 da]\Big|q_1(t) - \tilde{q}_1(t)\Big| \\
& + v_1 da\Big|p_2(t) - \tilde{p}_2(t)\Big| + v_2 da\Big|q_1(t) - \tilde{q}_1(t)\Big| \\
& + [v_2(1 - c) + 2v_2 a + v_2 da]\Big|p_2(t) - \tilde{p}_2(t)\Big| \\
= & [v_1(1 - c - d) + 2v_1(a + d^2) + da(v_1 + v_2)]\Big|q_1(t) - \tilde{q}_1(t)\Big| \\
& + [v_2(1 + c) + a(2v_2 + v_2 d + v_1 d)]\Big|p_2(t) - \tilde{p}_2(t)\Big| \\
= & B_1\Big|q_1(t) - \tilde{q}_1(t)\Big| + B_2\Big|p_2(t) - \tilde{p}_2(t)\Big| \\
\leq & B\|M - \tilde{M}\|,
\end{aligned}$$

where

$$\begin{cases} B_1 = v_1(1 - c - d) + 2v_1 a(1 + d^2) + da(v_1 + v_2), \\ B_2 = v_2(1 + c) + a(2v_2 + v_2 d + v_1 d), \\ B = \max\{B_1, B_2\}. \end{cases} \tag{4.3}$$

This implies that $g(M)$ satisfies the Lipschitz condition. Consequently, system (2.5) has a unique solution.

Theorem 4.2. ① The solution of model (2.5) with any given initial value is non-negative; ② If $\tau_1 = \tau_2 = 0$ and there exist c_1, c_2 ($c_i > 0, i = 1, 2$) such that $\min_{t\in\mathbb{R}}\{q_1(t)\} > c_1$ and $\min_{t\in\mathbb{R}}\{p_2(t)\} > c_2$, the solution of model (2.5) with any given initial value is uniformly bounded.

Proof. Let $M(t_0) = (q_1(t_0), p_2(t_0))$ be the initial value of model (2.5). Suppose that there exist *a* constant t_* satisfying $t_0 < t < t_*$ such that

$$\begin{cases} q_1(t) > 0, \quad t_0 < t < t_*, \\ q_1(t_*) = 0, \\ q_1(t_*^+) < 0. \end{cases} \tag{4.4}$$

From Eq (2.5),

$$D^\alpha q_1(t)|_{q_1(t)=0} = 0, \quad t > t_*, \tag{4.5}$$

which contradicts $q_1(t_*^+) < 0$.

Define the mapping as follows:

$$\theta(t) = q_1(t) + p_2(t). \tag{4.6}$$

Then

$$\begin{aligned}
D^{\alpha}\theta(t)+d\theta(t) &= D^{\alpha}q_1(t)+D^{\alpha}p_2(t)+d\theta(t)\\
&= v_1q_1(t)[1-c-d+dp_2(t)-2q_1(t)+2d^2q_1(t)]\\
&\quad +v_2p_2(t)[1+c-dq_1(t)-2p_2(t)]+dq_1(t)+dp_2(t)\\
&\le v_1q_1(t)[1-c-d+dp_2(t)-2q_1(t)+2d^2q_1(t)]\\
&\quad +v_2p_2(t)[1+c+dq_1(t)-2p_2(t)]+dq_1(t)+dp_2(t)\\
&= 2(d^2-1)v_1\Big[q_1(t)+\frac{(1-c-d)v_1+d}{4(d^2-1)v_1}\Big]^2\\
&\quad -2(d^2-1)v_1\Big[\frac{(1-c-d)v_1+d}{4(d^2-1)v_1}\Big]^2+2v_2\Big[\frac{(1+c)v_2+d}{4v_2}\Big]^2\\
&\quad -2v_2\Big[p_2(t)-\frac{(1+c)v_2+d}{4v_2}\Big]^2+dq_1(t)p_2(t)(v_1+v_2)\\
&\le 2v_2\Big[\frac{(1+c)v_2+d}{4v_2}\Big]^2-2(d^2-1)v_1\Big[\frac{(1-c-d)v_1+d}{4(d^2-1)v_1}\Big]^2\\
&\quad +dq_1(t)p_2(t)(v_1+v_2),
\end{aligned}$$

consequently,

$$D^{\alpha}\theta(t)\le -d\theta(t)+2v_2\Big[\frac{(1+c)v_2+d}{4v_2}\Big]^2-2(d^2-1)v_1\Big[\frac{(1-c-d)v_1+d}{4(d^2-1)v_1}\Big]^2+dq_1(t)p_2(t)(v_1+v_2). \tag{4.7}$$

According to Lemma 3.2, it follows that:

$$\theta(t)\le\Big[\theta(t_0)-\frac{\beta_2}{\beta_1}\Big]E_{\alpha}[-\beta_1(t-t_0)^{\alpha}]+\frac{\beta_2}{\beta_1}, \tag{4.8}$$

namely

$$\theta(t)\longrightarrow\frac{\beta_2}{\beta_1},\quad \text{as}\quad t\longrightarrow\infty, \tag{4.9}$$

where $\beta_1=d,\beta_2=2v_2\Big[\frac{(1+c)v_2+d}{4v_2}\Big]^2-2(d^2-1)v_1\Big[\frac{(1-c-d)v_1+d}{4(d^2-1)v_1}\Big]^2+dq_1(t)p_2(t)(v_1+v_2)$. Therefore, the solution of system (2.5) is bounded.

5. Stability analysis

System (2.5) has four non-negative equilibrium points: $E_1=(0,0)$, $E_2=\Big(0,\frac{1+c}{2}\Big)$, $E_3=\Big(\frac{1-c-d}{2(1-d^2)},0\Big)$, $E_4=(q_1^*,p_2^*)$, where

$$\begin{cases} q_1^*=\dfrac{2+cd-2c-d}{4-3d^2},\\ p_2^*=\dfrac{2+2c+cd-d-d^2-2cd^2}{4-3d^2}. \end{cases} \tag{5.1}$$

5.1. Evolutionary stability analysis

Let (κ_1) $sgn\{2+cd-d-2c\}$=$sgn\{4-3d^2\}$ $=$ $sgn\{2+2c+cd-d-d^2-2cd^2\}$. The linearization system of (2.5) is as follows:

$$D^{\alpha}m(t)=N_1m(t)+N_2m(t-\tau_1)+N_3m(t-\tau_2), \tag{5.2}$$

where

$$\begin{cases} m(t)=\begin{bmatrix} q_1(t) \\ p_2(t) \end{bmatrix}, \\ N_1=\begin{bmatrix} 0 & e_1 \\ e_2 & 0 \end{bmatrix}, \\ N_2=\begin{bmatrix} e_3 & 0 \\ 0 & 0 \end{bmatrix}, \\ N_3=\begin{bmatrix} 0 & 0 \\ 0 & e_4 \end{bmatrix}, \end{cases} \tag{5.3}$$

$$\begin{cases} e_1=dv_1q_1^*, \\ e_2=-dv_2p_2^*, \\ e_3=(2d^2-2)v_1q_1^*, \\ e_4=-2v_2p_2^*. \end{cases} \tag{5.4}$$

The characteristic equation of Eq (5.2) is

$$det\begin{bmatrix} k^{\alpha}-e_3e^{-k\tau_1} & -e_1 \\ -e_2 & k^{\alpha}-e_4e^{-k\tau_2} \end{bmatrix}=0. \tag{5.5}$$

Namely

$$\Pi_1(k)+\Pi_2(k)e^{-k\tau_1}+\Pi_3(k)e^{-k\tau_2}+\Pi_4(k)e^{-k(\tau_1+\tau_2)}=0, \tag{5.6}$$

where

$$\begin{cases} \Pi_1(k)=k^{2\alpha}+a_1, \\ \Pi_2(k)=a_2k^{\alpha}, \\ \Pi_3(k)=a_3k^{\alpha}, \\ \Pi_4(k)=a_4, \end{cases} \tag{5.7}$$

$$\begin{cases} a_1=-e_1e_2, \\ a_2=-e_3, \\ a_3=-e_4, \\ a_4=e_3e_4. \end{cases} \tag{5.8}$$

Next, we will consider six cases of Eq (5.6).

① $\tau_1=\tau_2=0$, set $k^{\alpha}=b$, Eq (5.6) takes

$$\lambda^2+b_1\lambda+b_2=0. \tag{5.9}$$

Obviously if (κ_2) $b_1 > 0, b_2 > 0$ ($b_1 = a_2 + a_3, b_2 = a_1 + a_4$) holds, then both roots λ_1, λ_2 of Eq (5.9) satisfy $|arg(\lambda_1)| > \dfrac{\alpha\pi}{2}$ and $|arg(\lambda_2)| > \dfrac{\alpha\pi}{2}$. According to Lemma 3.1, $E_4(q_1^*, p_2^*)$ is an evolutionarily stable strategy (ESS).

② $\tau_1 > 0, \tau_2 = 0$, Eq (5.6) takes

$$\Pi_1(k) + \Pi_3(k) + (\Pi_2(k) + \Pi_4(k))e^{-k\tau_1} = 0, \tag{5.10}$$

that is

$$k^{2\alpha} + a_3 k^\alpha + a_1 + (a_2 k^\alpha + a_4)e^{-k\tau_1} = 0. \tag{5.11}$$

Assume that $k = i\psi = \psi(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$ $(\psi > 0)$ is the root of Eq (5.11). Then

$$\begin{cases} Z_1\cos\psi\tau_1 + Z_2\sin\psi\tau_1 = Z_3, \\ Z_2\cos\psi\tau_1 - Z_1\sin\psi\tau_1 = Z_4, \end{cases} \tag{5.12}$$

where

$$\begin{cases} Z_1 = a_2\psi^\alpha\cos\frac{\alpha\pi}{2} + a_4, \\ Z_2 = a_2\psi^\alpha\sin\frac{\alpha\pi}{2}, \\ Z_3 = -(\psi^{2\alpha}\cos\alpha\pi + a_3\psi^\alpha\cos\frac{\alpha\pi}{2} + a_1), \\ Z_4 = -(\psi^{2\alpha}\sin\alpha\pi + a_3\psi^\alpha\sin\frac{\alpha\pi}{2}). \end{cases} \tag{5.13}$$

From Eq (5.12), one obtains

$$Z_1^2 + Z_2^2 = Z_3^2 + Z_4^2, \tag{5.14}$$

which leads to

$$\psi^{4\alpha} + \gamma_1\psi^{3\alpha} + \gamma_2\psi^{2\alpha} + \gamma_3\psi^\alpha + \gamma_4 = 0, \tag{5.15}$$

where

$$\begin{cases} \gamma_1 = 2a_3\cos\frac{\alpha\pi}{2}, \\ \gamma_2 = a_3^2 - a_2^2 + 2a_1\cos\alpha\pi, \\ \gamma_3 = 2(a_1a_3 - a_2a_4)\cos\frac{\alpha\pi}{2}, \\ \gamma_4 = a_1^2 - a_4^2. \end{cases} \tag{5.16}$$

Let $\psi^{4\alpha} + \gamma_1\psi^{3\alpha} + \gamma_2\psi^{2\alpha} + \gamma_3\psi^\alpha + \gamma_4 = \theta_1(\psi)$, assume that ($\kappa_3$) $|a_1| < |a_4|$, then Eq (5.15) has at least one root $\psi > 0$. It can be concluded as follows.

Lemma 5.1. *(i) For $\tau_2 = 0$, if $\gamma_n > 0$ $(n = 1, 2, 3, 4)$, then Eq (5.15) has no positive real roots; this means that $E_4(q_1^*, p_2^*)$ is an ESS for arbitrary $\tau_1 > 0$. (ii) When $\tau_1 = \tau_{1m}$ $(m = 0, 1, 2, ...)$, if (κ_3) holds, then Eq (5.15) has a pair of pure imaginary roots $\pm i\psi_0$, where*

$$\tau_{1m} = \frac{1}{\psi_0}\Big[\arccos\Big(\frac{Z_1Z_3 + Z_2Z_4}{Z_1^2 + Z_2^2}\Big) + 2m\pi\Big], \quad \psi_0 > 0, \tag{5.17}$$

where ψ_0 represents the zero point of $\theta_1(\psi)$=0.

Assume (κ_4) $L_{1R}L_{2R} + L_{1I}L_{2I} > 0$,

where

$$\begin{cases} L_{1R} = \alpha\psi_0^{\alpha-1}[2\psi_0^{\alpha}\cos\frac{(2\alpha-1)\pi}{2} + a_3\cos\frac{(\alpha-1)\pi}{2} + a_2\cos(\frac{(\alpha-1)\pi}{2} - \psi_0\tau_1)], \\ L_{1I} = \alpha\psi_0^{\alpha-1}[2\psi_0^{\alpha}\sin\frac{(2\alpha-1)\pi}{2} + a_3\sin\frac{(\alpha-1)\pi}{2} + a_2\sin(\frac{(\alpha-1)\pi}{2} - \psi_0\tau_1)], \\ L_{2R} = \psi_0[\sin\psi_0\tau_1(a_2\psi_0^{\alpha}\cos\frac{\alpha\pi}{2} + a_4) - a_2\psi_0^{\alpha}\cos\psi_0\tau_1\sin\frac{\alpha\pi}{2}], \\ L_{2I} = \psi_0[\cos\psi_0\tau_1(a_2\psi_0^{\alpha}\cos\frac{\alpha\pi}{2} + a_4) + a_2\psi_0^{\alpha}\sin\psi_0\tau_1\sin\frac{\alpha\pi}{2}]. \end{cases} \tag{5.18}$$

Lemma 5.2. *Let $k(\tau_1) = \rho_1(\tau_1) + i\rho_2(\tau_1)$ be the root of Eq (5.11) and $\tau_1 = \tau_{10}$ such that $\rho_1(\tau_{10}) = 0$, $\rho_2(\tau_{10}) = \xi_0$, then $Re(\frac{dk}{d\tau_1})|_{\tau_1=\tau_{10},\psi=\psi_0} > 0$.*

Proof. According to Eq (5.11),

$$\left(\frac{dk}{d\tau_1}\right)^{-1} = \frac{L_1(k)}{L_2(k)} - \frac{\tau_1}{k}, \tag{5.19}$$

where

$$\begin{cases} L_1(k) = 2\alpha k^{2\alpha-1} + a_3\alpha k^{\alpha-1} + a_2\alpha k^{\alpha-1}e^{-k\tau_1}, \\ L_2(k) = k(a_2k^{\alpha} + a_4)e^{-k\tau_1}. \end{cases} \tag{5.20}$$

Then

$$Re\Big[(\frac{dk}{d\tau_1})^{-1}\Big]_{\tau_1=\tau_{10},\psi=\psi_0} = Re\Big[\frac{L_1(k)}{L_2(k)}\Big]_{\tau_1=\tau_{10},\psi=\psi_0} = \frac{L_{1R}L_{2R} + L_{1I}L_{2I}}{L_{2R}^2 + L_{2I}^2}. \tag{5.21}$$

Since $L_{1R}L_{2R} + L_{1I}L_{2I} > 0$, one gets $Re\Big[(\frac{dk}{d\tau_1})^{-1}\Big]_{\tau_1=\tau_{10},\psi=\psi_0} > 0$.

From Lemmas 5.1 and 5.2, it yields the following result.

Theorem 5.1. When $\tau_1 > 0,\ \tau_2 = 0$, if (κ_1)–(κ_4) are simultaneously satisfied and $\tau_1 \in (0, \tau_{10})$, then $E_4(q_1^*, p_2^*)$ is an ESS. When $\tau_1 = \tau_{10}$, system (2.5) generates the HB phenomenon at $E_4(q_1^*, p_2^*)$.

③ When $\tau_1 = 0,\ \tau_2 > 0$, Eq (5.6) takes:

$$\Pi_1(k) + \Pi_2(k) + (\Pi_3(k) + \Pi_4(k))e^{-k\tau_2} = 0, \tag{5.22}$$

that is,

$$k^{2\alpha} + a_2k^{\alpha} + a_1 + (a_3k^{\alpha} + a_4)e^{-k\tau_2} = 0. \tag{5.23}$$

Let $k = i\omega = \omega(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$ $(\omega > 0)$ be the root of Eq (5.23). Then

$$\begin{cases} B_1\cos\omega\tau_2 + B_2\sin\omega\tau_2 = B_3, \\ B_2\cos\omega\tau_2 - B_1\sin\omega\tau_2 = B_4, \end{cases} \tag{5.24}$$

$$\begin{cases} B_1 = a_3\omega^{\alpha}\cos\frac{\alpha\pi}{2} + a_4, \\ B_2 = a_3\omega^{\alpha}\sin\frac{\alpha\pi}{2}, \\ B_3 = -(\omega^{2\alpha}\cos\alpha\pi + a_2\omega^{\alpha}\cos\frac{\alpha\pi}{2} + a_1), \\ B_4 = -(\omega^{2\alpha}\sin\alpha\pi + a_2\omega^{\alpha}\sin\frac{\alpha\pi}{2}). \end{cases} \tag{5.25}$$

From Eq (5.24), one obtains

$$B_1^2 + B_2^2 = B_3^2 + B_4^2, \tag{5.26}$$

which leads to

$$\theta_2(\omega) = \omega^{4\alpha} + \delta_1\omega^{3\alpha} + \delta_2\omega^{2\alpha} + \delta_3\omega^{\alpha} + \delta_4, \tag{5.27}$$

where

$$\begin{cases} \delta_1 = 2a_2 \cos\frac{\alpha\pi}{2}, \\ \delta_2 = a_2^2 + 2a_1\cos\alpha\pi - a_3^2, \\ \delta_3 = 2(a_1a_2 - a_3a_4)\cos\frac{\alpha\pi}{2}, \\ \delta_4 = a_1^2 - a_4^2. \end{cases} \tag{5.28}$$

Let

$$\theta_2(\omega) = \omega^{4\alpha} + \delta_1\omega^{3\alpha} + \delta_2\omega^{2\alpha} + \delta_3\omega^{\alpha} + \delta_4 = 0. \tag{5.29}$$

Assume that (κ_5) $|a_1| < |a_4|$, then Eq (5.29) has at least one root $\omega > 0$. Thus, the following conclusion can be obtained.

Lemma 5.3. *(i) For $\tau_1 = 0$, if $\delta_n > 0$ $(n = 1, 2, 3, 4)$, then Eq (5.29) has no positive real roots; this means that $E_4(q_1^*, p_2^*)$ is an ESS for arbitrary $\tau_2 > 0$. (ii) When $\tau_2 = \tau_{2m}$ $(m = 0, 1, 2, ...)$, if (κ_5) holds, then Eq (5.29) has a pair of pure imaginary roots $\pm\, i\omega_0$ where*

$$\tau_{2m} = \frac{1}{\omega_0}\Big[\arccos\big(\frac{B_1B_3 + B_2B_4}{B_1^2 + B_2^2}\big) + 2m\pi\Big], \quad \omega_0 > 0, \tag{5.30}$$

where ω_0 is the zero point of $\theta_2(\omega) = 0$.

Assume that (κ_6) $J_{1R}J_{2R} + J_{1I}J_{2I} > 0$,
where

$$\begin{cases} J_{1R} = 2\alpha\omega_0^{2\alpha-1}\cos\frac{(2\alpha-1)\pi}{2} + a_2\alpha\omega_0^{\alpha-1}\cos\frac{(\alpha-1)\pi}{2} + a_3\alpha\omega_0^{\alpha-1}\cos\Big[\frac{(\alpha-1)\pi}{2} - \tau_{20}\Big], \\ J_{1I} = 2\alpha\omega_0^{2\alpha-1}\sin\frac{(2\alpha-1)\pi}{2} + a_2\alpha\omega_0^{\alpha-1}\sin\frac{(\alpha-1)\pi}{2} + a_3\alpha\omega_0^{\alpha-1}\sin\Big[\frac{(\alpha-1)\pi}{2} - \tau_{20}\Big], \\ J_{2R} = \omega_0\big[(a_3\omega_0^{\alpha}\cos\frac{\alpha\pi}{2} + a_4)\sin\omega_0\tau_{20} - a_3\omega_0^{\alpha}\sin\frac{\alpha\pi}{2}\cos\omega_0\tau_{20}\big], \\ J_{2I} = \omega_0\big[(a_3\omega_0^{\alpha}\cos\frac{\alpha\pi}{2} + a_4)\cos\omega_0\tau_{20} + a_3\omega_0^{\alpha}\sin\frac{\alpha\pi}{2}\sin\omega_0\tau_{20}\big]. \end{cases} \tag{5.31}$$

Lemma 5.4. *Let $k(\tau_2) = \iota_1(\tau_2) + i\iota_2(\tau_2)$ be the root of Eq (5.23) and $\tau_2 = \tau_{20}$ such that $\iota_1(\tau_{20}) = 0$, $\iota_2(\iota_{20}) = \xi_0$, then $Re(\frac{dk}{d\tau_2})|_{\tau_2=\tau_{20},\omega=\omega_0} > 0$.*

Proof. According to Eq (5.23), we have

$$\left(\frac{dk}{d\tau_2}\right)^{-1} = \frac{J_1(k)}{J_2(k)} - \frac{\tau_2}{k}, \tag{5.32}$$

where

$$\begin{cases} J_1(k) = 2\alpha k^{2\alpha-1} + a_2\alpha k^{\alpha-1} + a_3\alpha k^{\alpha-1}e^{-k\tau_2}, \\ J_2(k) = k(a_3k^{\alpha} + a_4)e^{-k\tau_2}. \end{cases} \tag{5.33}$$

Then

$$Re\left[\left(\frac{dk}{d\tau_2}\right)^{-1}\right]_{\tau_2=\tau_{20},\omega=\omega_0} = Re\left[\frac{J_1(k)}{J_2(k)}\right]_{\tau_2=\tau_{20},\omega=\omega_0} = \frac{J_{1R}J_{2R}+J_{1I}J_{2I}}{J_{2R}^2+J_{2R}^2}. \tag{5.34}$$

Since $J_{1R}J_{2R} + J_{1I}J_{2I} > 0$, one gets $Re\Big[(\frac{dk}{d\tau_2})^{-1}\Big]_{\tau_2=\tau_{20},\omega=\omega_0} > 0$.

From Lemmas 5.3 and 5.4, it yields the following result.

Theorem 5.2. When $\tau_1 = 0,\ \tau_2 > 0$, if (κ_1), (κ_2), (κ_5), and (κ_6) are simultaneously satisfied and $\tau_2 \in (0, \tau_{20})$, then $E_4(q_1^*, p_2^*)$ is an ESS. When $\tau_2 = \tau_{20}$, then system (2.5) generates the HB phenomenon at $E_4(q_1^*, p_2^*)$.

④ When $\tau_1 > 0,\ \tau_2 \in (0, \tau_{20})$, regard τ_1 as a variable, then Eq (5.6) takes

$$\Pi_1(k) + \Pi_2(k)e^{-k\tau_1} + \Pi_3(k)e^{-k\tau_2} + \Pi_4(k)e^{-k(\tau_1+\tau_2)} = 0. \tag{5.35}$$

Assume that $k = i\varphi = \varphi(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$ $(\varphi > 0)$ is the root of Eq (5.35). Then

$$\begin{cases} D_1\cos\varphi\tau_1 + D_2\sin\varphi\tau_1 = D_3, \\ D_2\cos\varphi\tau_1 - D_1\sin\varphi\tau_1 = D_4, \end{cases} \tag{5.36}$$

where

$$\begin{cases} D_1 = a_2\varphi^\alpha\cos\frac{\alpha\pi}{2} + a_4\cos\varphi\tau_2, \\ D_2 = a_2\varphi^\alpha\sin\frac{\alpha\pi}{2} - a_4\sin\varphi\tau_2, \\ D_3 = -(\varphi^{2\alpha}\cos\alpha\pi + a_1 + a_3\varphi^\alpha\cos(\frac{\alpha\pi}{2} - \varphi\tau_2)), \\ D_4 = -(\varphi^{2\alpha}\sin\alpha\pi + a_3\varphi^\alpha\sin(\frac{\alpha\pi}{2} - \varphi\tau_2)). \end{cases} \tag{5.37}$$

One obtains

$$D_1^2 + D_2^2 = D_3^2 + D_4^2, \tag{5.38}$$

which leads to

$$\varphi^{4\alpha} + \epsilon_1\varphi^{3\alpha} + \epsilon_2\varphi^{2\alpha} + \epsilon_3\varphi^\alpha + \epsilon_4 = 0, \tag{5.39}$$

where

$$\begin{cases} \epsilon_1 = 2a_3\cos(\frac{\alpha\pi}{2} + \varphi\tau_2), \\ \epsilon_2 = a_3^2 - a_2^2 + 2a_1\cos\alpha\pi, \\ \epsilon_3 = 2(a_1a_3\cos(\frac{\alpha\pi}{2} - \varphi\tau_2) - a_2a_4\cos(\frac{\alpha\pi}{2} + \varphi\tau_2)), \\ \epsilon_4 = a_1^2 - a_4^2. \end{cases} \tag{5.40}$$

Define

$$\theta_3(\varphi) = \varphi^{4\alpha} + \epsilon_1\varphi^{3\alpha} + \epsilon_2\varphi^{2\alpha} + \epsilon_3\varphi^\alpha + \epsilon_4. \tag{5.41}$$

Assume that (κ_7) $\epsilon_4 < 0$; then Eq (5.39) has at least one positive $\varphi > 0$, and we can get the following conclusions.

Lemma 5.5. *(i) For $\tau_1 > 0$ and $\tau_2 \in (0, \tau_{20})$, if $\epsilon_n > 0$ $(n = 1, 2, 3, 4)$, then Eq (5.39) has no positive real roots; this means that $E_4(q_1^*, p_2^*)$ is an ESS for arbitrary $\tau_1 > 0$, $\tau_2 \in (0, \tau_{20})$. (ii) If $\epsilon_4 < 0$, $\tau_1 = \tau_{10*}^k$ $(k = 0, 1, 2, ...)$ holds, then Eq (5.39) has a pair of pure imaginary roots $\pm i\varphi_0$ where*

$$\tau_{10*}^k = \frac{1}{\varphi_0}\Big[\arccos\Big(\frac{D_1D_3 + D_2D_4}{D_1^2 + D_2^2}\Big) + 2k\pi\Big], \quad \varphi_0 > 0, \tag{5.42}$$

where φ_0 is the zero point of $\theta_3(\varphi) = 0$.

Denote

$$\tau_{10*} = \tau_{10*}^0,\ \varphi_0 = \varphi|_{\tau_1=\tau_{10*}}. \tag{5.43}$$

Next, we make the following assumptions:

$$(\kappa_8) \quad T_{1R}T_{2R} + T_{1I}T_{2I} > 0, \tag{5.44}$$

where

$$\begin{cases} T_{1R} = 2\alpha\varphi_0^{2\alpha-1}\cos\frac{2\alpha-1}{2}\pi + \alpha\varphi_0^{\alpha-1}[(a_2\cos\varphi_0\tau_{10*} + a_3\cos\varphi_0\tau_2)\cos\frac{(\alpha-1)\pi}{2} \\ \qquad +(a_2\sin\varphi_0\tau_{10*} + a_3\sin\varphi_0\tau_2)\sin\frac{(\alpha-1)\pi}{2}] - \tau_2[a_3\varphi_0^\alpha\cos(\frac{\alpha\pi}{2} - \varphi_0\tau_2) \\ \qquad +a_4\cos\varphi_0(\tau_{10*} + \tau_2)], \\ T_{1I} = 2\alpha\varphi_0^{2\alpha-1}\sin\frac{2\alpha-1}{2}\pi + \alpha\varphi_0^{\alpha-1}[(a_2\cos\varphi_0\tau_{10*} + a_3\cos\varphi_0\tau_2)\sin\frac{(\alpha-1)\pi}{2} \\ \qquad -(a_2\sin\varphi_0\tau_{10*} + a_3\sin\varphi_0\tau_2)\cos\frac{(\alpha-1)\pi}{2}] - \tau_2[a_3\varphi_0^\alpha\cos(\frac{\alpha\pi}{2} + \varphi_0\tau_2) \\ \qquad -a_4\sin\varphi_0(\tau_{10*} + \tau_2)], \\ T_{2R} = \varphi_0[a_2\varphi_0^\alpha\sin(\varphi_0\tau_{10*} - \frac{\alpha\pi}{2}) + a_4\sin\varphi_0(\tau_{10*} + \tau_2)], \\ T_{2I} = \varphi_0[a_2\varphi_0^\alpha\cos(\frac{\alpha\pi}{2} - \varphi_0\tau_{10*}) + a_4\cos\varphi_0(\tau_{10*} + \tau_2)]. \end{cases} \tag{5.45}$$

Lemma 5.6. *Let $k(\tau_1) = \sigma_1(\tau_1) + i\sigma_1(\tau_1)$ be the root of Eq (5.35), and for $\tau_1 = \tau_{10*}$, $\tau_2 \in (0, \tau_{20})$, $\sigma_1(\tau_{10*}) = 0$ and $\sigma_2(\sigma_{10*}) = \varphi_0$, then $Re(\frac{dk}{d\tau_1})|_{\tau_1=\tau_{10*},\varphi=\varphi_0} > 0$.*
Proof. From Eq (5.35),

$$\Big(\frac{dk}{d\tau_1}\Big)^{-1} = \frac{T_1(k)}{T_2(k)} - \frac{\tau_1}{k}, \tag{5.46}$$

where

$$\begin{cases} T_1(k) = 2\alpha k^{2\alpha-1} + (a_2e^{-k\tau_1} + a_3e^{-k\tau_2})\alpha k^{\alpha-1} - \tau_2\Big[a_3k^\alpha e^{-k\tau_2} + a_4e^{-k(\tau_1+\tau_2)}\Big], \\ T_2(k) = k\Big[a_2k^\alpha e^{-k\tau_1} + a_4e^{-k(\tau_1+\tau_2)}\Big]. \end{cases} \tag{5.47}$$

Then

$$Re\Big[\Big(\frac{dk}{d\tau_1}\Big)^{-1}\Big]_{\tau_1=\tau_{10*},\varphi=\varphi_0} = Re\Big[\frac{T_1(k)}{T_2(k)}\Big]_{\tau_1=\tau_{10*},\varphi=\varphi_0} = \frac{T_{1R}T_{2R} + T_{1I}T_{2I}}{T_{2R}^2 + T_{2I}^2}. \tag{5.48}$$

Since $T_{1R}T_{2R} + T_{1I}T_{2I} > 0$, it follows that $Re[(\frac{dk}{d\tau_1})^{-1}]_{\tau_1=\tau_{10*},\varphi=\varphi_0} > 0$.

From Lemmas 5.5 and 5.6, the following result holds.

Theorem 5.3. When $\tau_1 > 0,\ \tau_2 \in (0, \tau_{20})$, if (κ_1), (κ_2), (κ_7), and (κ_8) are simultaneously satisfied and $\tau_1 \in (0, \tau_{10*})$, then $E_4(q_1^*, p_2^*)$ is an ESS. When $\tau_1 = \tau_{10*}^k$, the system (2.5) generates the HB phenomenon at the $E_4(q_1^*, p_2^*)$.

⑤ $\tau_2 > 0, \tau_1 \in (0, \tau_{10})$, regard τ_2 as a variable. Then Eq (5.6) takes

$$\Pi_1(k) + \Pi_2(k)e^{-k\tau_1} + \Pi_3(k)e^{-k\tau_2} + \Pi_4(k)e^{-k(\tau_1+\tau_2)} = 0. \tag{5.49}$$

Assume that $k = i\phi = \phi(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$ $(\phi > 0)$ is the root of Eq (5.49), then

$$\begin{cases} P_1\cos\phi\tau_2 + P_2\sin\phi\tau_2 = P_3, \\ P_2\cos\phi\tau_2 - P_1\sin\phi\tau_2 = P_4, \end{cases} \tag{5.50}$$

$$\begin{cases} P_1 = a_3\phi^\alpha\cos\frac{\alpha\pi}{2} + a_4\cos\phi\tau_1, \\ P_2 = a_3\phi^\alpha\sin\frac{\alpha\pi}{2} - a_4\sin\phi\tau_1, \\ P_3 = -[\phi^{2\alpha}\cos\alpha\pi + a_2\phi^\alpha\cos(\frac{\alpha\pi}{2} - \phi\tau_1) + a_1], \\ P_4 = -[\phi^{2\alpha}\sin\alpha\pi + a_2\phi^\alpha\sin(\frac{\alpha\pi}{2} - \phi\tau_1)]. \end{cases} \tag{5.51}$$

One obtains

$$P_1^2 + P_2^2 = P_3^2 + P_4^2, \tag{5.52}$$

which leads to

$$\phi^{4\alpha} + \nu_1\phi^{3\alpha} + \nu_2\phi^{2\alpha} + \nu_3\phi^\alpha + \nu_4 = 0, \tag{5.53}$$

where

$$\begin{cases} \nu_1 = 2a_2\cos(\frac{\alpha\pi}{2} + \phi\tau_1), \\ \nu_2 = a_2^2 - a_3^2 + 2a_1\cos\alpha\pi, \\ \nu_3 = 2[a_1a_2\cos(\frac{\alpha\pi}{2} - \phi\tau_1) - a_3a_4\cos(\frac{\alpha\pi}{2} + \phi\tau_1)], \\ \nu_4 = a_1^2 - a_4^2. \end{cases} \tag{5.54}$$

Define

$$\theta_4(\phi) = \phi^{4\alpha} + \nu_1\phi^{3\alpha} + \nu_2\phi^{2\alpha} + \nu_3\phi^\alpha + \nu_4. \tag{5.55}$$

Assume that (κ_9) $\nu_4 < 0$; then Eq (5.53) has at least one root $\phi > 0$. From the above analysis, the following conclusions can be obtained.

Lemma 5.7. *(i) For $\tau_2 > 0$ and $\tau_1 \in (0, \tau_{10})$, if $\nu_n > 0$ $(n = 1, 2, 3, 4)$, then Eq (5.53) has no positive real roots; this means that $E_4(q_1^*, p_2^*)$ is an ESS for arbitrary $\tau_2 > 0$, $\tau_1 \in (0, \tau_{10})$. (ii) If $\nu_4 < 0$, $\tau_2 = \tau_{20*}^i$ $(i = 0, 1, 2...)$ holds, then Eq (5.53) has a pair of pure imaginary roots $\pm\, i\phi_0$ where*

$$\tau_{20*}^i = \frac{1}{\phi_0}\Big[\arccos\Big(\frac{P_2P_3 + P_1P_4}{P_1^2 + P_2^2}\Big) + 2i\pi\Big], \quad \phi_0 > 0, \tag{5.56}$$

where ϕ_0 represents the zero point of $\theta_4(\phi) = 0$.

Denote

$$\tau_{20*} = \tau_{20*}^{0},\ \phi_0 = \phi|_{\tau_2=\tau_{20*}}. \tag{5.57}$$

Next, we make the following assumptions:

$$\kappa(10) \quad X_{1R}X_{2R} + X_{1I}X_{2I} > 0, \tag{5.58}$$

where

$$\begin{cases} X_{1R} = 2\alpha\phi_0^{2\alpha-1}\cos\frac{2\alpha-1}{2}\pi + \big[(a_2\cos\phi_0\tau_1 + a_3\cos\phi_0\tau_{20*})\cos\frac{\alpha-1}{2}\pi \\ \qquad +(a_2\sin\phi_0\tau_1 + a_3\sin\phi_0\tau_{20*})\sin\frac{\alpha-1}{2}\pi\big]\alpha\phi^{\alpha-1} - \tau_1\big[(a_2\phi_0^{\alpha}\cos\frac{\alpha\pi}{2} \\ \qquad +a_4\cos\phi_0\tau_{20*})\cos\phi_0\tau_1 + (a_2\phi_0^{\alpha}\sin\frac{\alpha\pi}{2} - a_4\sin\phi_0\tau_{20*})\sin\phi_0\tau_1\big], \\ X_{1I} = 2\alpha\phi_0^{2\alpha-1}\sin\frac{2\alpha-1}{2}\pi + \big[(a_2\cos\phi_0\tau_1 + a_3\cos\phi_0\tau_1)\sin\frac{\alpha-1}{2}\pi \\ \qquad -(a_2\sin\phi_0\tau_1 + a_3\sin\phi_0\tau_{20*})\cos\frac{\alpha-1}{2}\pi\big]\alpha\phi^{\alpha-1} - \tau_1\big[(a_2\phi_0^{\alpha}\sin\frac{\alpha\pi}{2} \\ \qquad -a_4\sin\phi_0\tau_{20*})\cos\phi_0\tau_1 - (a_2\phi_0^{\alpha}\cos\frac{\alpha\pi}{2} + a_4\cos\phi_0\tau_{20*})\sin\phi_0\tau_1\big], \\ X_{2R} = \phi_0\big[a_3\phi_0^{\alpha}\sin(\phi_0\tau_{20*} - \frac{\alpha\pi}{2}) + a_4\sin\phi_0(\tau_1+\tau_{20*})\big], \\ X_{2I} = \phi_0\big[a_3\phi_0^{\alpha}\cos(\frac{\alpha\pi}{2} - \phi_0\tau_{20*}) + a_4\cos\phi_0(\tau_1+\tau_{20*})\big]. \end{cases} \tag{5.59}$$

Lemma 5.8. *Let $k(\tau_2) = \xi_1(\tau_2) + i\xi_2(\tau_2)$ be the root of Eq (5.49) and $\tau_2 = \tau_{20*}$, $\tau_1 \in (0, \tau_{10})$ such that $\xi_1(\tau_{20*}) = 0$, $\xi_2(\xi_{20*}) = \phi_0$, then $Re(\frac{dk}{d\tau_2})|_{\tau_2=\tau_{20*},\phi=\phi_0} > 0$.*

Proof. According to Eq (5.49),

$$\left(\frac{dk}{d\tau_2}\right)^{-1} = \frac{X_1(k)}{X_2(k)} - \frac{\tau_2}{k}, \tag{5.60}$$

where

$$\begin{cases} X_1(k) = 2\alpha k^{2\alpha-1} + (a_2e^{-k\tau_1} + a_3e^{-k\tau_2})\alpha k^{\alpha-1} - \tau_1e^{-k\tau_1}(a_2k^{\alpha} + a_4e^{-k\tau_2}), \\ X_2(k) = k\big[a_3k^{\alpha}e^{-k\tau_2} + a_4e^{-k(\tau_1+\tau_2)}\big]. \end{cases} \tag{5.61}$$

Then

$$Re\left[\left(\frac{dk}{d\tau_2}\right)^{-1}\right]_{\tau_2=\tau_{20*},\phi=\phi_0} = Re\left[\frac{X_1(k)}{X_2(k)}\right]_{\tau_2=\tau_{20*},\phi=\phi_0} = \frac{X_{1R}X_{2R} + X_{1I}X_{2I}}{X_{2R}^2 + X_{2I}^2}. \tag{5.62}$$

Since $X_{1R}X_{2R} + X_{1I}X_{2I} > 0$, one gets $Re[(\frac{dk}{d\tau_2})^{-1}]_{\tau_2=\tau_{20*},\phi=\phi_0} > 0$.

From Lemmas 5.7 and 5.8, the following result holds.

Theorem 5.4. When $\tau_2 > 0$, $\tau_1 \in (0, \tau_{10})$, if (κ_1), (κ_2), (κ_9), and (κ_{10}) are simultaneously satisfied and $\tau_2 \in (0, \tau_{20*})$, then $E_4(q_1^*, p_2^*)$ is an ESS. When $\tau_2 = \tau_{20*}^{i}$, system (2.5) generates the HB phenomenon at $E_4(q_1^*, p_2^*)$.

⑥ When $\tau_1 = \tau_2 = \tau$, then Eq (5.6) takes

$$\Pi_1(k) + \Pi_2(k)e^{-k\tau} + \Pi_3(k)e^{-k\tau} + \Pi_4(k)e^{-2k\tau} = 0, \tag{5.63}$$

which leads to

$$\Pi_1(k)e^{k\tau} + \Pi_2(k) + \Pi_3(k) + \Pi_4(k)e^{-k\tau} = 0, \tag{5.64}$$

that is

$$(k^{2\alpha} + a_1)e^{k\tau} + (a_2 + a_3)k^{\alpha} + a_4e^{-k\tau} = 0. \tag{5.65}$$

Assume that $k = i\upsilon = \upsilon(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$ $(\upsilon > 0)$ is the root of Eq (5.64). Then

$$\begin{cases} G_1\cos\upsilon\tau - G_2\sin\upsilon\tau = K_1, \\ G_2\cos\upsilon\tau + G_3\sin\upsilon\tau = K_2, \end{cases} \tag{5.66}$$

$$\begin{cases} G_1 = \upsilon^{2\alpha}\cos\alpha\pi + a_1 + a_4, \\ G_2 = \upsilon^{2\alpha}\sin\alpha\pi, \\ G_3 = \upsilon^{2\alpha}\cos\alpha\pi + a_1 - a_4, \\ K_1 = -\upsilon^{\alpha}(a_2 + a_3)\cos\frac{\alpha\pi}{2}, \\ K_2 = -\upsilon^{\alpha}(a_2 + a_3)\sin\frac{\alpha\pi}{2}. \end{cases} \tag{5.67}$$

From Eq (5.66),

$$\begin{cases} \cos\upsilon\tau = \dfrac{G_3K_1 + G_2K_2}{G_2^2 + G_1G_3}, \\ \sin\upsilon\tau = \dfrac{G_1K_2 - G_2K_1}{G_2^2 + G_1G_3}. \end{cases} \tag{5.68}$$

One obtains

$$(G_3K_1 + G_2K_2)^2 + (G_1K_2 - G_2K_1)^2 = (G_2^2 + G_1G_3)^2. \tag{5.69}$$

In Eq (5.67), set

$$\begin{cases} \mu_1 = \cos\alpha\pi, \\ \mu_2 = a_1 + a_4, \\ \mu_3 = \sin\alpha\pi, \\ \mu_4 = a_1 - a_4, \\ \mu_5 = -(a_2 + a_3)\cos\frac{\alpha\pi}{2}, \\ \mu_6 = -(a_2 + a_3)\sin\frac{\alpha\pi}{2}, \end{cases} \tag{5.70}$$

then

$$\begin{cases} G_1 = \mu_1\upsilon^{2\alpha} + \mu_2, \\ G_2 = \mu_3\upsilon^{2\alpha}, \\ G_3 = \mu_1\upsilon^{2\alpha} + \mu_4, \\ K_1 = \mu_5\upsilon^{\alpha}, \\ K_2 = \mu_6\upsilon^{\alpha}. \end{cases} \tag{5.71}$$

According to Eqs (5.69) and (5.71), we obtain

$$\eta_1 \upsilon^{8\alpha} + \eta_2 \upsilon^{6\alpha} + \eta_3 \upsilon^{4\alpha} + \eta_4 \upsilon^{2\alpha} + \eta_5 = 0, \tag{5.72}$$

where

$$\begin{cases} \eta_1 = (\mu_1^2 + \mu_3^2)^2, \\ \eta_2 = 2(\mu_1^2 + \mu_3^2)(\mu_1\mu_4 + \mu_1\mu_2) - (\mu_1\mu_5 + \mu_3\mu_6)^2 - (\mu_1\mu_6 - \mu_3\mu_5)^2, \\ \eta_3 = (\mu_1\mu_4 + \mu_1\mu_2)^2 + 2[\mu_2\mu_4(\mu_1^2 + \mu_3^2) - \mu_4\mu_5(\mu_1\mu_5 + \mu_3\mu_6) \\ \qquad -\mu_2\mu_6(\mu_1\mu_6 - \mu_3\mu_5)], \\ \eta_4 = 2\mu_2\mu_4(\mu_1\mu_4 + \mu_1\mu_2) - (\mu_4\mu_5)^2 - (\mu_2\mu_6)^2, \\ \eta_5 = (\mu_2\mu_4)^2. \end{cases} \tag{5.73}$$

Define

$$\theta_5(\upsilon) = \eta_1 \upsilon^{8\alpha} + \eta_2 \upsilon^{6\alpha} + \eta_3 \upsilon^{4\alpha} + \eta_4 \upsilon^{2\alpha} + \eta_5, \tag{5.74}$$

since $\eta_1, \eta_5 > 0$, it is possible that $\theta_5(\upsilon) = 0$ may or may not have a positive root. When $\theta_5(\upsilon) = 0$ does not have a positive root, it is evident that $E_4(q_1^*, p_2^*)$ is an ESS for arbitrary $\tau > 0$. If $\theta_5(\upsilon) = 0$ has a positive root, we can obtain the following result.

Lemma 5.9. *(i) For $\tau > 0$, if $\eta_n > 0$ $(n = 2, 3, 4)$, then Eq (5.65) has no positive real roots; this means that $E_4(q_1^*, p_2^*)$ is an ESS for arbitrary $\tau > 0$. (ii) When $\tau = \tau_n$ $(n = 0, 1, 2, ...)$, if Eq (5.72) has positive real roots, then Eq (5.65) has a pair of pure imaginary roots $\pm i\upsilon_0$ where*

$$\tau_n = \frac{1}{\upsilon_0}\Big[\arccos\Big(\frac{G_3K_1 + G_2K_2}{G_2^2 + G_1K_2}\Big) + 2n\pi\Big], \quad \upsilon_0 > 0, \tag{5.75}$$

where $n = 0, 1, 2...$, and υ_0 represents the zero point of $\theta_5(\upsilon) = 0$.

Next, we make the following assumption:

$$\kappa(11) \quad V_{1R}V_{2R} + V_{1I}V_{2I} > 0, \tag{5.76}$$

where

$$\begin{cases} V_{1R} = 2\alpha\upsilon_0^{2\alpha-1}\cos(\upsilon_0\tau_0 + \frac{2\alpha-1}{2}\pi) + (a_2 + a_3)\alpha\upsilon_0^{\alpha-1}\cos\frac{\alpha-1}{2}\pi, \\ V_{1I} = 2\alpha\upsilon_0^{2\alpha-1}\sin(\upsilon_0\tau_0 + \frac{2\alpha-1}{2}\pi) + (a_2 + a_3)\alpha\upsilon_0^{\alpha-1}\sin\frac{\alpha-1}{2}\pi, \\ V_{2R} = \upsilon_0[\sin\upsilon_0\tau_0(\upsilon_0^{2\alpha}\cos\alpha\pi + a_1 + a_4) + \upsilon_0^{2\alpha}\cos\upsilon_0\tau_0\sin\alpha\pi], \\ V_{2I} = \upsilon_0[\cos\upsilon_0\tau_0(a_4 - a_1 - \upsilon_0^{2\alpha}\cos\alpha\pi) + \upsilon_0^{2\alpha}\sin\upsilon_0\tau_0\sin\alpha\pi]. \end{cases} \tag{5.77}$$

Lemma 5.10. *Let $k(\tau) = o_1(\tau_2) + io_2(\tau)$ be the root of Eq (5.65) at $\tau = \tau_0$ such that $o_1(\tau_0) = 0$, $o_2(\tau_0) = \delta_0$, then $Re(\frac{dk}{d\tau})|_{\tau=\tau_0,\upsilon=\upsilon_0} > 0$.*

Proof. According to Eq (5.65),

$$\Big(\frac{dk}{d\tau}\Big)^{-1} = \frac{V_1(k)}{V_2(k)} - \frac{\tau}{k}, \tag{5.78}$$

where

$$\begin{cases} V_1(k) = 2\alpha k^{2\alpha-1}e^{k\tau} + (a_2 + a_3)\alpha k^{\alpha-1}, \\ V_2(k) = ka_4e^{-k\tau} - k(k^{2\alpha} + a_1)e^{k\tau}. \end{cases} \tag{5.79}$$

Then

$$Re\left[\left(\frac{dk}{d\tau}\right)^{-1}\right]_{\tau=\tau_0,\upsilon=\upsilon_0} = Re\left[\frac{V_1(k)}{V_2(k)}\right]_{\tau=\tau_0,\upsilon=\upsilon_0} = \frac{V_{1R}V_{2R} + V_{1I}V_{2I}}{V_{2R}^2 + V_{2I}^2}. \tag{5.80}$$

Since $V_{1R}V_{2R} + V_{1I}V_{2I} > 0$, one gets $Re[(\frac{dk}{d\tau})^{-1}]_{\tau=\tau_0,\upsilon=\upsilon_0} > 0$.

From Lemmas 5.9 and 5.10, the following result holds.

Theorem 5.5. When $\tau_1 = \tau_2 = \tau > 0$, if (κ_1), (κ_2), and (κ_{11}) are simultaneously satisfied and $\tau \in (0, \tau_0)$, $E_4(q_1^*, p_2^*)$ is an ESS. When $\tau = \tau_0$, the system (2.5) generates the HB phenomenon at $E_4(q_1^*, p_2^*)$.

5.2. Global asymptotic stability analysis

In this section, the global asymptotic stability of $E_4(q_1^*, p_2^*)$ will be researched.

Theorem 5.6. *$E_4(q_1^*, p_2^*)$ of system (2.5) is globally asymptotically stable if $\kappa(12)$ holds, where*

$$\begin{aligned} \kappa(12) \quad & \frac{(1+c)^2 + (1-c+d)^2}{8} + 2d^2q_1^2 - q_1^*[1 - c - d + dp_2(t) \\ & -2q_1(1) + 2d^2q_1(t)] - p_2^*[1 + c - dq_1(t) - 2p_2(t)] < 0. \end{aligned} \tag{5.81}$$

Proof. Set:

$$\Theta(q_1, p_2) = \frac{1}{v_1}\Big[q_1(t) - q_1^* - q_1^* \ln\frac{q_1(t)}{q_1^*}\Big] + \frac{1}{v_2}\Big[p_2(t) - p_2^* - p_2^* \ln\frac{p_2(t)}{p_2^*}\Big], \tag{5.82}$$

which satisfies $\Theta(q_1, p_2) > 0$ for $q_1 \neq q_1^*$, $p_2 \neq p_2^*$ and $\Theta(q_1^*, p_2^*) = 0$.
Then

$$\begin{aligned} \frac{d^\alpha\Theta(q_1, p_2)}{dt^\alpha} &\leq \frac{1}{v_1}\frac{q_1(t) - q_1^*}{q_1(t)}\frac{d^\alpha q_1}{dt^\alpha} + \frac{1}{v_2}\frac{p_2(t) - p_2^*}{p_2(t)}\frac{d^\alpha p_2}{dt^\alpha} \\ &= (q_1 - q_1^*)[1 - c - d + dp_2(t) - 2q_1(t) + 2d^2q_1(t)] \\ &\quad + (p_2 - p_2^*)[1 + c - dq_1(t) - 2p_2(t)] \\ &= -[\sqrt{2}p_2 - \frac{\sqrt{2}}{4}(1+c)]^2 - [\sqrt{2}q_1 - \frac{\sqrt{2}}{4}(1-c-d)]^2 \\ &\quad + \frac{(1+c)^2 + (1-c+d)^2}{8} + 2d^2q_1^2 - q_1^*[1 - c - d + dp_2(t) \\ &\quad - 2q_1(1) + 2d^2q_1(t)] - p_2^*[1 + c - dq_1(t) - 2p_2(t)]. \end{aligned}$$

From Lyapunov's stable theory, if $\kappa(12)$ holds, then $E_4(q_1^*, p_2^*)$ is globally asymptotically stable and also an ESS.

6. Numerical simulation and sensitivity analysis

6.1. Numerical simulation

Utilizing Matlab R2022a, this section will conduct numerical simulation experiments on the dynamic characteristics of system (2.5) to verify the correctness of the conclusions established in this paper.

Example 6.1. Set $c = 0.35$, $d = 0.75$, $\alpha = 0.82$, $v_1 = 2.35$, $v_2 = 4.5$, and system (2.5) becomes the following form:

$$\begin{cases} D^{0.82}q_1(t) = 2.35q_1(t)[-0.1 + 0.75p_2(t) - 0.875q_1(t-\tau_1)], \\ D^{0.82}p_2(t) = 4.5p_2(t)[1.35 - 0.75q_1(t) - 2p_2(t-\tau_2)]. \end{cases} \tag{6.1}$$

Obviously, $E_4(q_1^*, p_2^*) = E_4(0.3514,0.5432)$.

① $\tau_1 > 0, \tau_2 = 0$.
By simple calculation, it can be obtained:

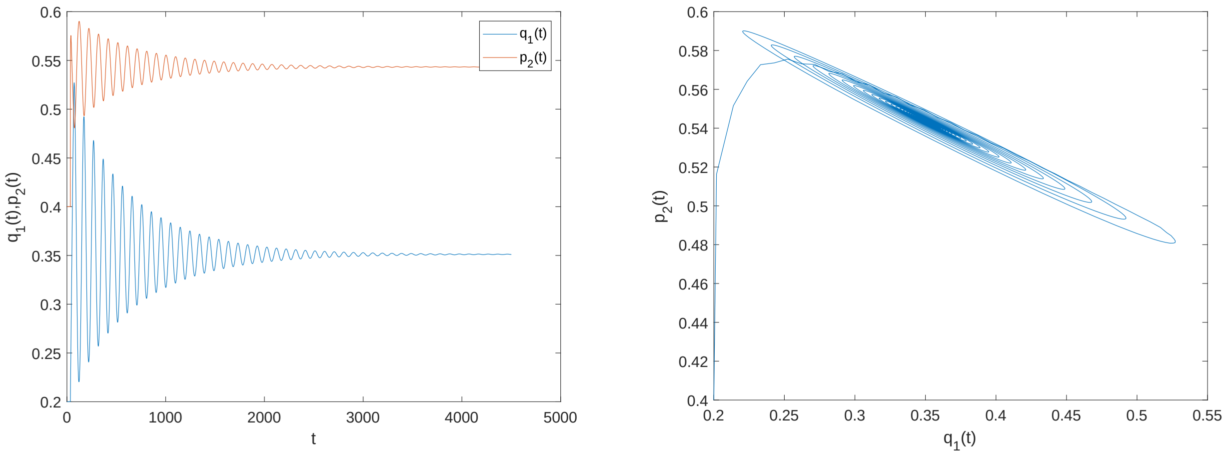
$$\begin{aligned} &(\kappa_1) \quad 2 - 2c - d + cd > 0,\ 4 - 3d^2 > 0,\ 2 + 2c - d - cd - d^2 - 2cd^2 > 0; \\ &(\kappa_2) \quad b_1 = 5.6117 > 0,\ b_2 = 4.6675 > 0; \\ &(\kappa_3) \quad \gamma_4 = (a_2 + a_6)^2 - (a_4 + a_7)^2 = -11.1879 < 0; \\ &(\kappa_4) \quad L_{1R}L_{2R} + L_{1I}L_{2I} = 9.1866 > 0,\ \frac{Z_1Z_3 + Z_2Z_4}{Z_1^2 + Z_2^2} = -0.5574 \in (-1, 1); \\ &\psi_0 = 0.5947 \text{ and } \tau_{10} = 3.6355. \end{aligned}$$

From Theorem 5.1, if $\tau_1 > \tau_{10}$, the state trajectory of system (2.5) is asymptotically stable; otherwise, it is unstable.

Case 1: Set $\tau_1 = 3.3355 < \tau_{10}$, $\tau_2 = 0$. The corresponding simulation diagram can be referred to in Figure 1. It can be clearly seen from Figure 1 that $E_4(0.3514,0.5432)$ is asymptotically stable.

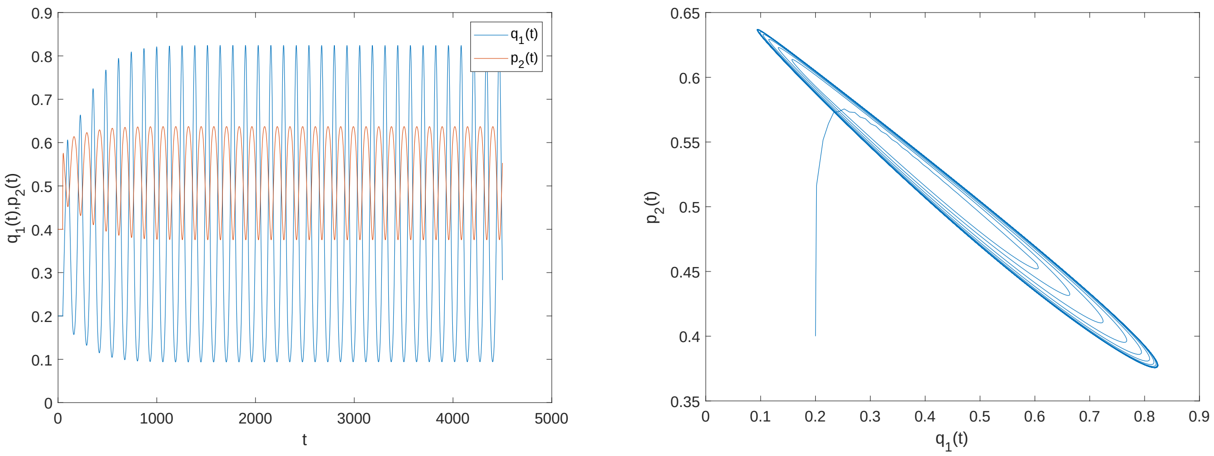
Case 2: Set $\tau_1 = 4.6355 > \tau_{10}$, $\tau_2 = 0$. The corresponding simulation diagram can be referred to in Figure 2. It can be clearly seen from Figure 2 that the state trajectory exhibits periodic oscillation.

The simulation results of cases 1 and 2 indicate that the conclusion of Theorem 5.1 is correct.



(a) (b)

Figure 1. Simulation diagrams of system (6.1) when $c = 0.35$, $d = 0.75$, $v_1 = 2.35$, $v_2 = 4.5$, $\alpha = 0.82$, $\tau_1 = 3.3355 < \tau_{10} = 3.6355$ and (a) curves of $q_1(t)$ and $p_2(t)$ as functions of time t; (b) evolution curves for $q_1(t)$ and $p_2(t)$.



(a) (b)

Figure 2. Simulation diagrams of system (6.1) when $c = 0.35$, $d = 0.75$, $v_1 = 2.35$, $v_2 = 4.5$, $\alpha = 0.82$, $\tau_1 = 4.6355 > \tau_{10} = 3.6355$ and (a) curves of $q_1(t)$ and $p_2(t)$ as functions of time t; (b) evolution curves for $q_1(t)$ and $p_2(t)$.

Example 6.2. Set $c = 0.85$, $d = 0.5$, $\alpha = 0.82$, $v_1 = 0.58$, $v_2 = 1.2$, and system (2.5) becomes the following form:

$$\begin{cases} D^{0.82}q_1(t) = 0.58q_1(t)[-0.35 + 0.75p_2(t) - 1.5q_1(t-\tau_1)], \\ D^{0.82}p_2(t) = 1.2p_2(t)[1.85 - 0.75q_1(t) - 2p_2(t-\tau_2)]. \end{cases} \quad (6.2)$$

Obviously, $E_4(q_1^*, p_2^*) = E_4(0.0692, 0.9077)$.

② $\tau_2 > 0, \tau_1 = 0.$

By simple calculation, it can be obtained:

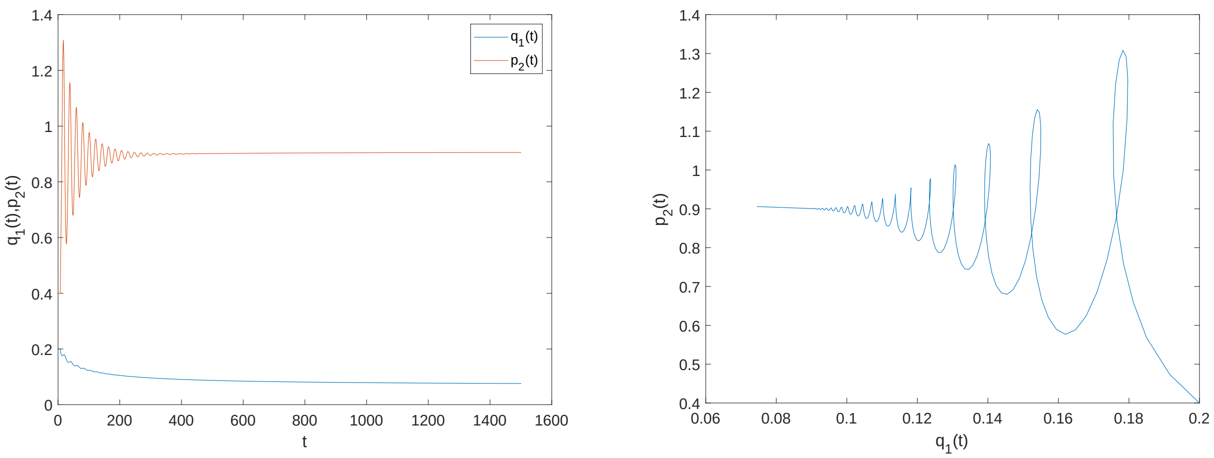
$$
\begin{aligned}
&(\kappa_1) \quad 2-2c-d+cd>0,\ 4-3d^2>0,\ 2+2c-d-cd-d^2-2cd^2>0;\\
&(\kappa_2) \quad b_1=2.2387>0,\ b_2=0.1421>0;\\
&(\kappa_5) \quad \delta_4=(a_2+a_4)^2-(a_6+a_7)^2=-0.0171<0;\\
&(\kappa_6) \quad J_{1R}J_{2R}+J_{1I}J_{2I}=30.2129>0,\ \frac{B_1B_3+B_2B_4}{B_1^2+B_2^2}=-0.2802\in(-1,1).\\
&\omega_0=2.5905 \text{ and } \tau_{20}=0.7160.
\end{aligned}
$$

From Theorem 5.2, if $\tau_2 > \tau_{20}$, the state trajectory of system (2.5) is asymptotically stable; otherwise, it is unstable.

Case 1: Set $\tau_2 = 0.63 < \tau_{20}$, $\tau_1 = 0$. The corresponding simulation diagram can be referred to in Figure 3. It can be clearly seen from Figure 3 that E_4(0.0692,0.9077) is asymptotically stable.

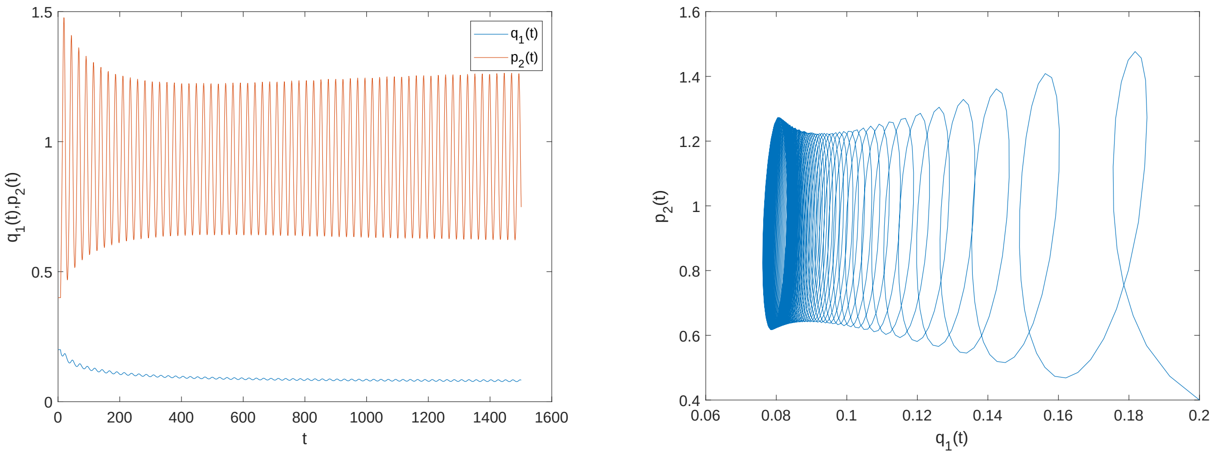
Case 2: Set $\tau_2 = 0.74 > \tau_{20}$, $\tau_1 = 0$. The corresponding simulation diagram can be referred to in Figure 4. It can be clearly seen from Figure 4 that the state trajectory exhibits periodic oscillation.

The simulation results of cases 1 and 2 indicate that the conclusion of Theorem 5.2 is correct.



(a) (b)

Figure 3. Simulation diagrams of system (6.2) when $c = 0.85$, $d = 0.5$, $v_1 = 0.58$, $v_2 = 1.2$, $\alpha = 0.82$, $\tau_2 = 0.63 < \tau_{20} = 0.7160$ and (a) curves of $q_1(t)$ and $p_2(t)$ as functions of time t; (b) evolution curves for $q_1(t)$ and $p_2(t)$.



(a) (b)

Figure 4. Simulation diagrams of system (6.2) when $c = 0.85$, $d = 0.5$, $v_1 = 0.58$, $v_2 = 1.2$, $\alpha = 0.82$, $\tau_2 = 0.74 > \tau_{20} = 0.7160$ and (a) curves of $q_1(t)$ and $p_2(t)$ as functions of time t; (b) evolution curves for $q_1(t)$ and $p_2(t)$.

Example 6.3. Set $c = 0.35$, $d = 0.75$, $\alpha = 0.92$, $v_1 = 3.35$, $v_2 = 4.5$, and system (2.5) becomes the following form:

$$\begin{cases} D^{0.92}q_1(t) = 3.35q_1(t)[-0.1 + 0.75p_2(t) - 0.875q_1(t-\tau_1)], \\ D^{0.92}p_2(t) = 4.5p_2(t)[1.35 - 0.75q_1(t) - 2p_2(t-\tau_2)]. \end{cases} \tag{6.3}$$

Obviously, $E_4(q_1^*, p_2^*) = E_4(0.3514{,}0.5432)$.

③ $\tau_1 > 0$, $\tau_2 \in (0, \tau_{20})$.

By simple calculation, it can be obtained:

$$(\kappa_1) \quad 2 - 2c - d + cd > 0,\ 4 - 3d^2 > 0,\ 2 + 2c - d - cd - d^2 - 2cd^2 > 0;$$

$$(\kappa_2) \quad b_1 = 5.9191 > 0,\ b_2 = 6.6539 > 0;$$

$$(\kappa_7) \quad \epsilon_4 = -22.7354 < 0;$$

$$(\kappa_8) \quad T_{1R}T_{2R} + T_{1I}T_{2I} = 19.5616 > 0,\ \frac{D_1D_3 + D_2D_4}{D_1^2 + D_2^2} = -0.4320 \in (-1, 1).$$

$$\varphi_0 = 0.9489 \text{ and } \tau_{10*} = 2.1263.$$

From Theorem 5.3, if $\tau_1 > \tau_{10*}$, the state trajectory of system (2.5) is asymptotically stable; otherwise, it is unstable.

Case 1: Set $\tau_1 = 1.85 < \tau_{10*}$, $\tau_2 = 0.15$. The corresponding simulation diagram can be referred to in Figure 5. It can be clearly seen from Figure 5 that $E_4(0.3514{,}0.5432)$ is asymptotically stable.

Case 2: Set $\tau_1 = 3.15 > \tau_{10*}$, $\tau_2 = 0.15$. The corresponding simulation diagram can be referred to in Figure 6. It can be clearly seen from Figure 6 that the state trajectory exhibits periodic oscillation.

The simulation results of cases 1 and 2 indicate that the conclusion of Theorem 5.3 is correct.

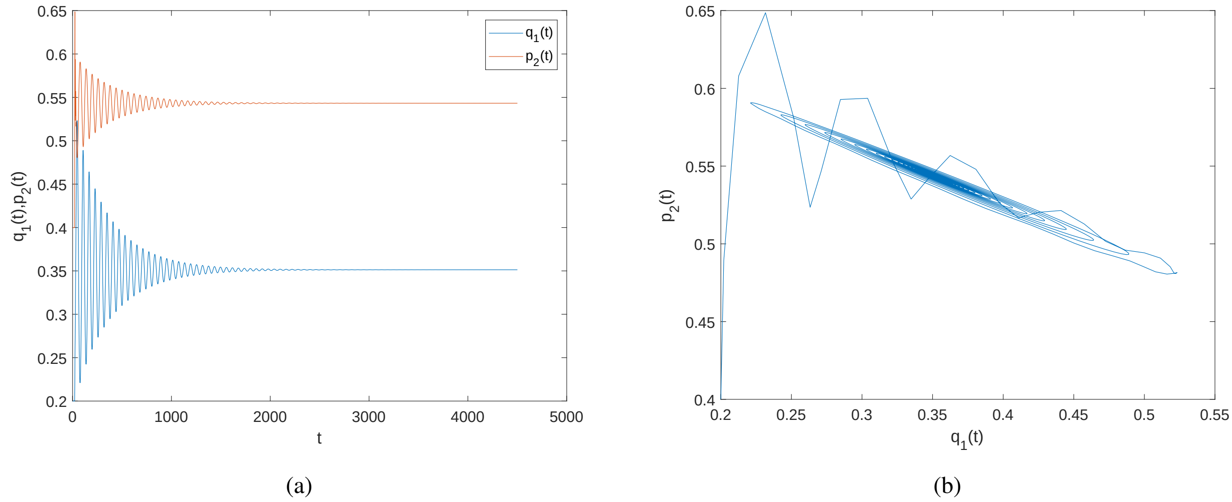


Figure 5. Simulation diagrams of system (6.3) when $c = 0.35$, $d = 0.75$, $v_1 = 3.35$, $v_2 = 4.5$, $\alpha = 0.92$, $\tau_2 = 0.15$, $\tau_1 = 1.85 < \tau_{10*} = 2.1263$ and (a) curves of $q_1(t)$ and $p_2(t)$ as functions of time t; (b) evolution curves for $q_1(t)$ and $p_2(t)$.

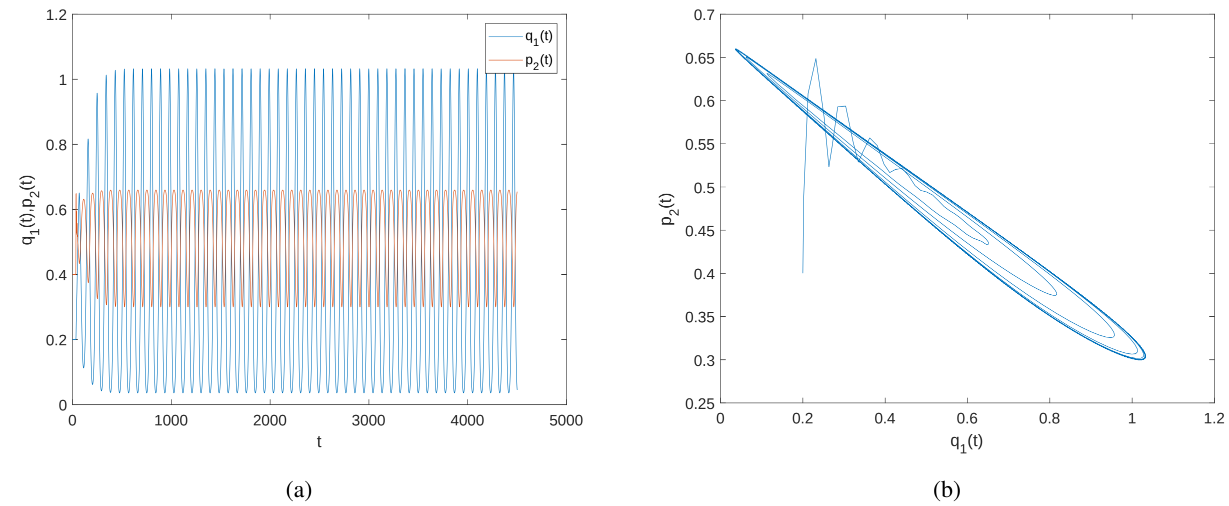


Figure 6. Simulation diagrams of system (6.3) when $c = 0.35$, $d = 0.75$, $v_1 = 3.35$, $v_2 = 4.5$, $\alpha = 0.92$, $\tau_2 = 0.15$, $\tau_1 = 3.15 > \tau_{10*} = 2.1263$ and (a) curves of $q_1(t)$ and $p_2(t)$ as functions of time t; (b) evolution curves for $q_1(t)$ and $p_2(t)$.

Example 6.4. Set $c = 0.35$, $d = 0.75$, $\alpha = 0.85$, $v_1 = 2.35$, $v_2 = 2.5$, and system (2.5) becomes the following form:

$$\begin{cases} D^{0.85}q_1(t) = 2.35q_1(t)[-0.1 + 0.75p_2(t) - 0.875q_1(t-\tau_1)], \\ D^{0.85}p_2(t) = 2.5p_2(t)[1.35 - 0.75q_1(t) - 2p_2(t-\tau_2)]. \end{cases} \tag{6.4}$$

Obviously, $E_4(q_1^*, p_2^*) = E_4(0.3514,0.5432)$.

④ $\tau_2 > 0, \tau_1 \in (0, \tau_{10})$.
By simple calculation, it can be obtained:

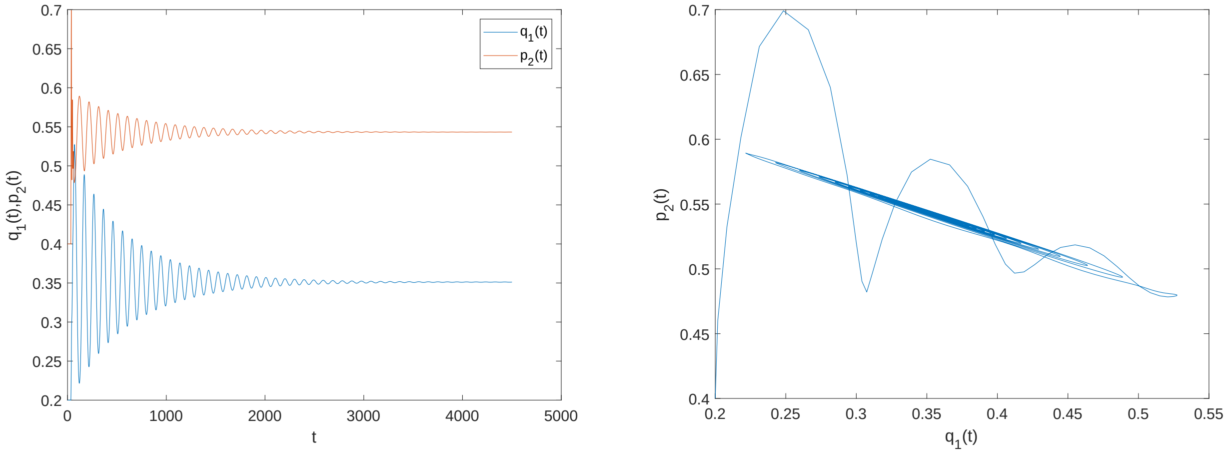
$$(\kappa_1) \quad 2 - 2c - d + cd > 0,\ 4 - 3d^2 > 0,\ 2 + 2c - d - cd - d^2 - 2cd^2 > 0;$$
$$(\kappa_2) \quad b_1 = 3.4387 > 0,\ b_2 = 2.5931 > 0;$$
$$(\kappa_9) \quad v_4 = -3.4531 < 0;$$
$$(\kappa_{10}) \quad X_{1R}X_{2R} + X_{1I}X_{2I} = 96.8974 > 0,\ \frac{P_2P_3 + P_1P_4}{P_1^2 + P_2^2} = 0.6241 \in (-1, 1).$$
$$\phi_0 = 3.4627 \text{ and } \tau_{20*} = 0.5297.$$

From Theorem 5.4, if $\tau_2 > \tau_{20*}$, the state trajectory of system (2.5) is asymptotically stable; otherwise, it is unstable.

Case 1: Set $\tau_2 = 0.25 < \tau_{20*}$, $\tau_1 = 3.2$. The corresponding simulation diagram can be referred to in Figure 7. It can be clearly seen from Figure 7 that $E_4(0.3514,0.5432)$ is asymptotically stable.

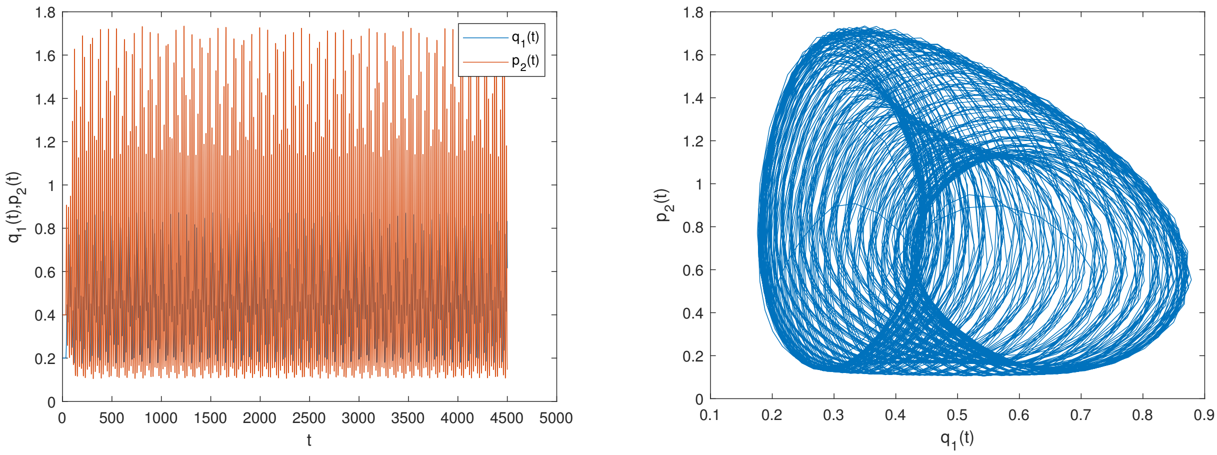
Case 2: Set $\tau_2 = 0.56 > \tau_{20*}$, $\tau_1 = 3.2$. The corresponding simulation diagram can be referred to in Figure 8. It can be clearly seen from Figure 8 that the state trajectory exhibits periodic oscillation.

The simulation results of cases 1 and 2 indicate that the conclusion of Theorem 5.4 is correct.



(a) (b)

Figure 7. Simulation diagrams of system (6.4) when $c = 0.35$, $d = 0.75$, $v_1 = 2.35$, $v_2 = 2.5$, $\alpha = 0.85$, $\tau_1 = 3.2$, $\tau_2 = 0.25 < \tau_{20*} = 0.5297$ and (a) curves of $q_1(t)$ and $p_2(t)$ as functions of time t; (b) evolution curves for $q_1(t)$ and $p_2(t)$.



(a) (b)

Figure 8. Simulation diagrams of system (6.4) when $c = 0.35$, $d = 0.75$, $v_1 = 2.35$, $v_2 = 2.5$, $\alpha = 0.85$, $\tau_1 = 3.2$, $\tau_2 = 0.56 > \tau_{20*} = 0.5297$ and (a) curves of $q_1(t)$ and $p_2(t)$ as functions of time t; (b) evolution curves for $q_1(t)$ and $p_2(t)$.

Example 6.5. Set $c = 0.35$, $d = 0.25$, $\alpha = 0.9$, $v_1 = 0.75$, $v_2 = 0.69$, and system (2.5) becomes the following form:

$$\begin{cases} D^{0.9}q_1(t) = 0.75q_1(t)[0.4 + 0.25p_2(t) - 1.875q_1(t-\tau_1)], \\ D^{0.9}p_2(t) = 0.69p_2(t)[1.35 - 0.25q_1(t) - 2p_2(t-\tau_2)]. \end{cases} \tag{6.5}$$

Obviously, $E_4(q_1^*, p_2^*) = E_4(0.2984{,}0.6377)$.

⑤ $\tau_1 = \tau_2 = \tau$.

By simple calculation, it can be obtained:

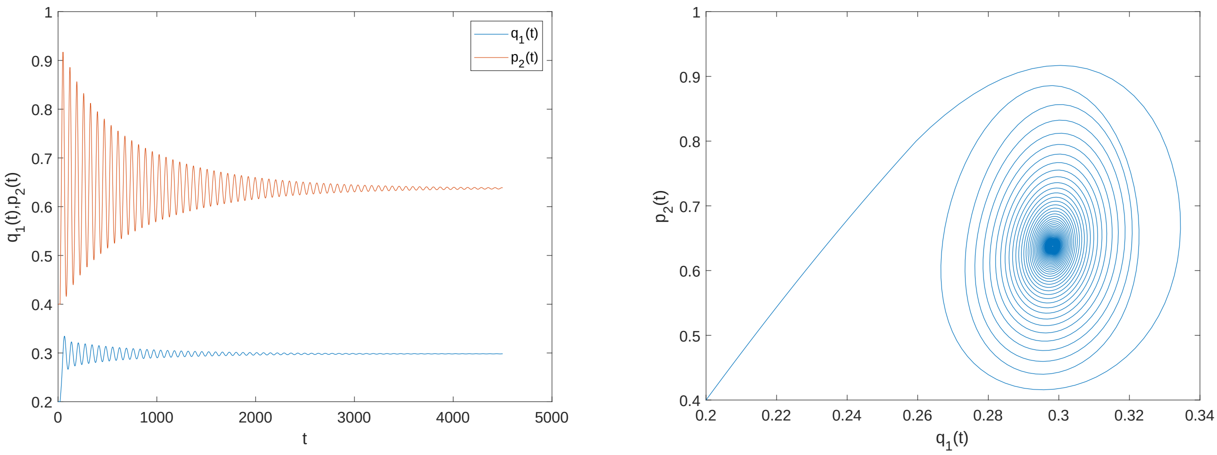
$$\begin{aligned} &(\kappa_1) \quad 2 - 2c - d + cd > 0,\ 4 - 3d^2 > 0,\ 2 + 2c - d - cd - d^2 - 2cd^2 > 0; \\ &(\kappa_2) \quad b_1 = 1.2996 > 0,\ b_2 = 0.3754 > 0; \\ &(\kappa_{11}) \quad V_{1R}V_{2R} + V_{1I}V_{2I} = 0.1642 > 0,\ \frac{G_3K_1 + G_2K_2}{G_2^2 + G_1K_2} = -0.1578 \in (-1, 1). \\ &\upsilon_0 = 0.8812 \text{ and } \tau_0 = 1.9624. \end{aligned}$$

From Theorem 5.5, if $\tau > \tau_0$, the state trajectory of system (2.5) is asymptotically stable; otherwise, it is unstable.

Case 1: Set $\tau = 1.85 < \tau_0$. The corresponding simulation diagram can be referred to in Figure 9. It can be clearly seen from Figure 9 that $E_4(0.2984, 0.6377)$ is asymptotically stable.

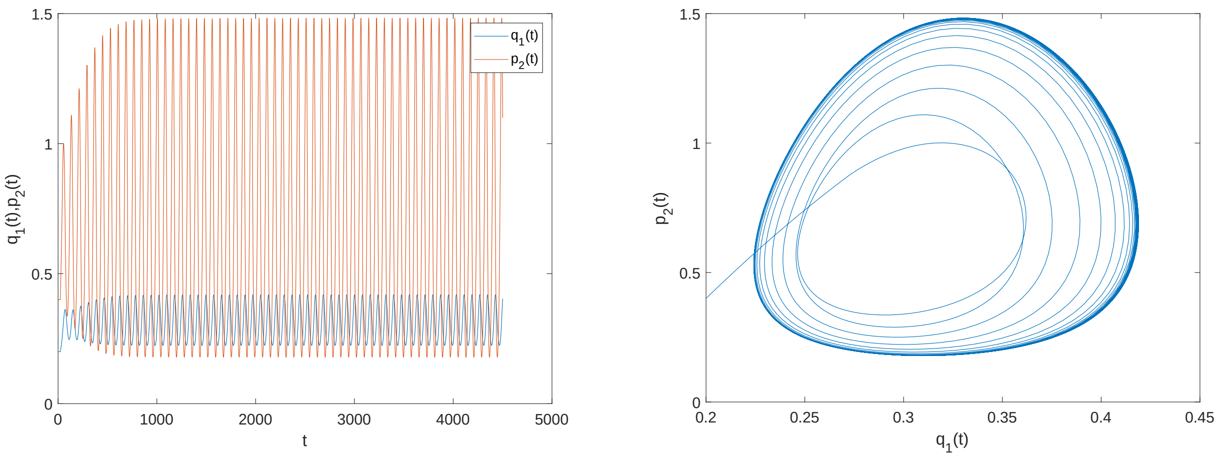
Case 2: Set $\tau = 2.1661 > \tau_0$. The corresponding simulation diagram can be referred to in Figure 10. It can be clearly seen from Figure 10 that the state trajectory exhibits periodic oscillation.

The simulation results of cases 1 and 2 indicate that the conclusion of Theorem 5.5 is correct.



(a) (b)

Figure 9. Simulation diagrams of system (6.5) when $c = 0.35$, $d = 0.25$, $v_1 = 0.75$, $v_2 = 0.69$, $\alpha = 0.9$, $\tau = 1.85 < \tau_0 = 1.9624$ and (a) curves of $q_1(t)$ and $p_2(t)$ as functions of time t; (b) evolution curves for $q_1(t)$ and $p_2(t)$.



(a) (b)

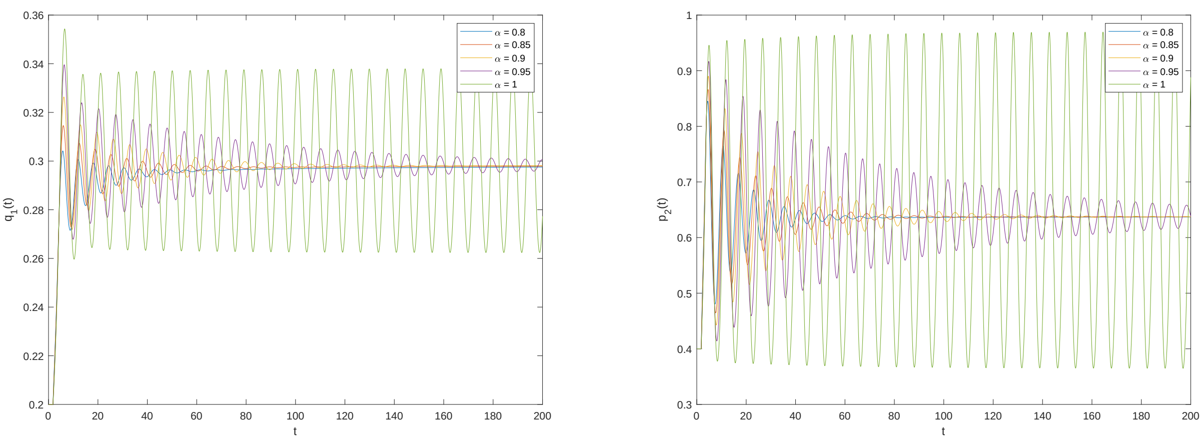
Figure 10. Simulation diagrams of system (6.5) when $c = 0.35$, $d = 0.25$, $v_1 = 0.75$, $v_2 = 0.69$, $\alpha = 0.9$, $\tau = 2.1661 > \tau_0 = 1.9624$ and (a) curves of $q_1(t)$ and $p_2(t)$ as functions of time t; (b) evolution curves for $q_1(t)$ and $p_2(t)$.

Remark 6.1. For Theorem 5.1, the parameters are set as $c = 0.35$, $d = 0.75$, $v_1 = 2.35$, $v_2 = 4.5$, and $\alpha = 0.82$. Through calculation, we obtain $\psi_0 = 0.5947$ and $\tau_{10} = 3.6355$. Numerical simulations are performed with $\tau = 3.3355$ and 4.6355, and the results are shown in Figures 1 and 2. Figure 1 presents the time series and phase diagram of system (6.1) when $\tau_2 = 0$ and $\tau_1 = 3.3355 < \tau_{10}$. It can be seen that the positive equilibrium point $E_4(0.3514, 0.5432)$ of system (6.1) is locally asymptotically stable, meaning that the output of enterprise 1 converges to 0.3514 and the price of enterprise 2 converges to 0.5432. Figure 2 shows the time series of system (6.1) when $\tau_2 = 0$ and $\tau_1 = 4.6355 > \tau_{10}$, as well as the phase diagram under the corresponding parameters. It indicates that an HB occurs near the positive equilibrium point $E_4(0.3514, 0.5432)$ of system (6.1), where the output of enterprise 1 oscillates periodically around 0.3514 and the price of enterprise 2 oscillates periodically around 0.5432. Similarly, for Theorems 5.2–5.5, under the set parameters, when the system time delay is less than the HB threshold, the positive equilibrium points are all in a locally asymptotically stable state, and the output of enterprise 1 and the price of enterprise 2 will converge to q_1^* and p_2^* respectively. When the time delay exceeds this threshold, a HB will occur near the positive equilibrium point, and the output of enterprise 1 and the price of enterprise 2 will exhibit periodic oscillations around their respective equilibrium values.

6.2. Sensitivity analysis

In this section, we conduct a sensitivity analysis of the key parameters on the model's behavior based on Theorem 5.5. Theorems 5.1–5.4 can be discussed in a similar manner, so we will not elaborate on them here.

Remark 6.2. It can be seen from Figure 11 that when the fractional order approaches an integer order, in terms of inventory management, enterprises can arrange inventory replenishment and consumption based on more stable laws, making output fluctuations show periodicity. In terms of price fluctuations, market information transmission becomes smoother, and prices fluctuate around the law of value, reflecting the stability and maturity of the market. That is, output and prices shift from complex and irregular fluctuations to periodic fluctuations, which indicates that the operation mechanism of the economic system tends to be orderly.



(a) (b)

Figure 11. Set $c = 0.35$, $d = 0.25$, $v_1 = 0.75$, $v_2 = 0.69$, $\tau_1 = \tau_2 = 1.75$. The influence of different α values on the system: (a) shows the influence of different α values on q_1, and (b) shows the influence of different α values on p_2.

Remark 6.3. It can be seen from Figure 12(a) and (b) that when the cost decreases, the output of enterprise 1 will increase, and the product price of enterprise 2 will decrease. Similarly, when the cost increases, the output of enterprise 1 will decrease, and the price of enterprise 2 will increase. It can be seen from Figure 12(c) and (d) that the correlation of products is negatively correlated with the equilibrium point of the system; that is, the larger the correlation coefficient d, the lower the output of enterprise 1 and the price of enterprise 2.

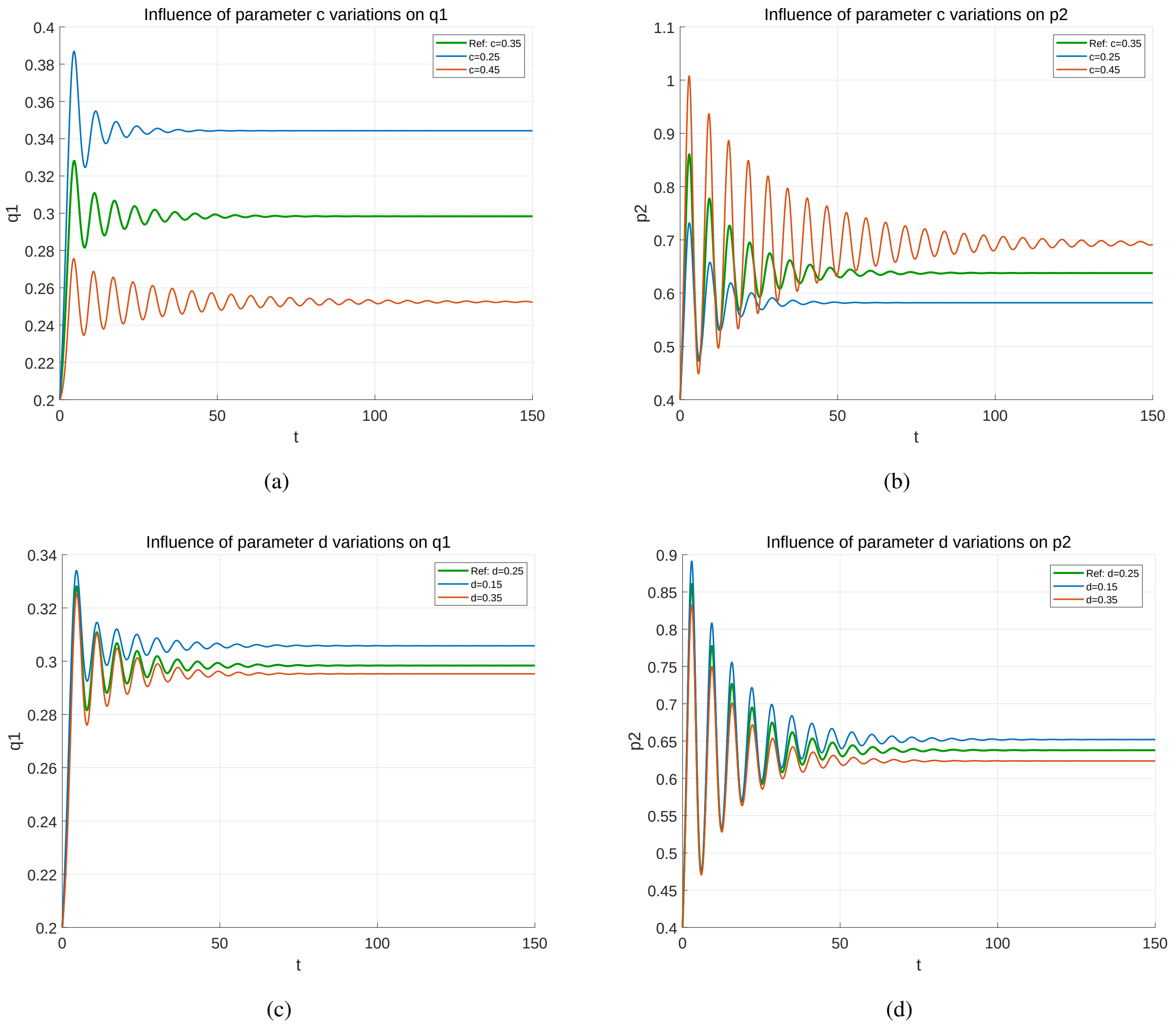


Figure 12. Set $c = 0.35$, $d = 0.25$, $v_1 = 0.75$, $v_2 = 0.69$, $\tau_1 = \tau_2 = 1.55$. Explore the influence of changes in parameters c and d on the system, where (a) and (b) illustrate the impact of parameter c on q_1 and p_2 of the system, and (c) and (d) show the influence of parameter d on q_1 and p_2 of the system.

Remark 6.4. Based on Example 6.5, when $\tau_1 = \tau_2 = 1.45 < \tau_0$, a 5% perturbation is applied to the initial values to assess the system's stability and anti-interference ability, thereby testing the robustness of the economic system. As can be seen from Figure 13, in a stable system, when the initial values of q_1 and p_2 are perturbed by ±5%, although the system variables fluctuate and deviate in the initial and intermediate stages, over time, the system, through its internal adjustment mechanism, makes q_1 and p_2 converge to their original stable trajectories.

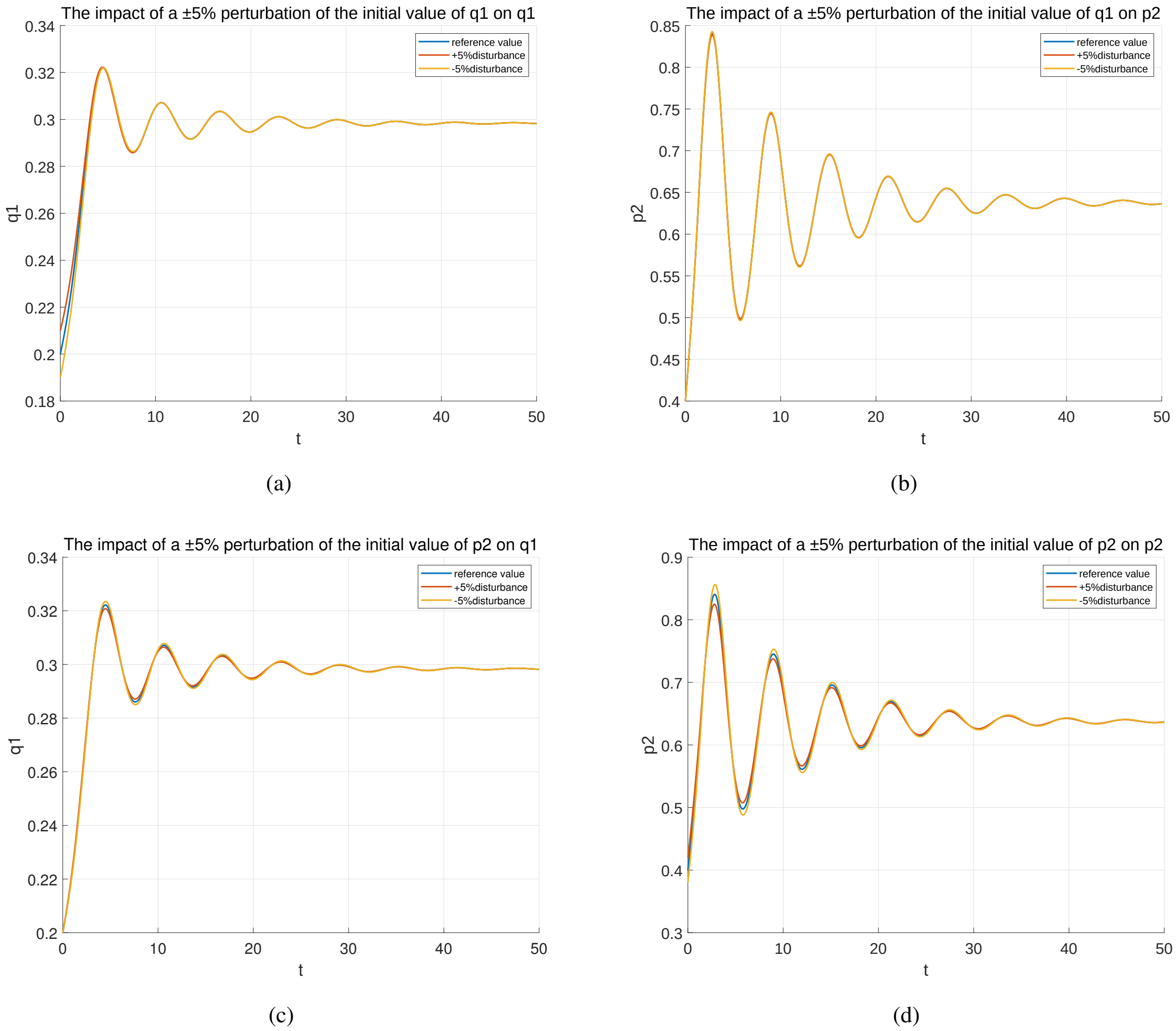
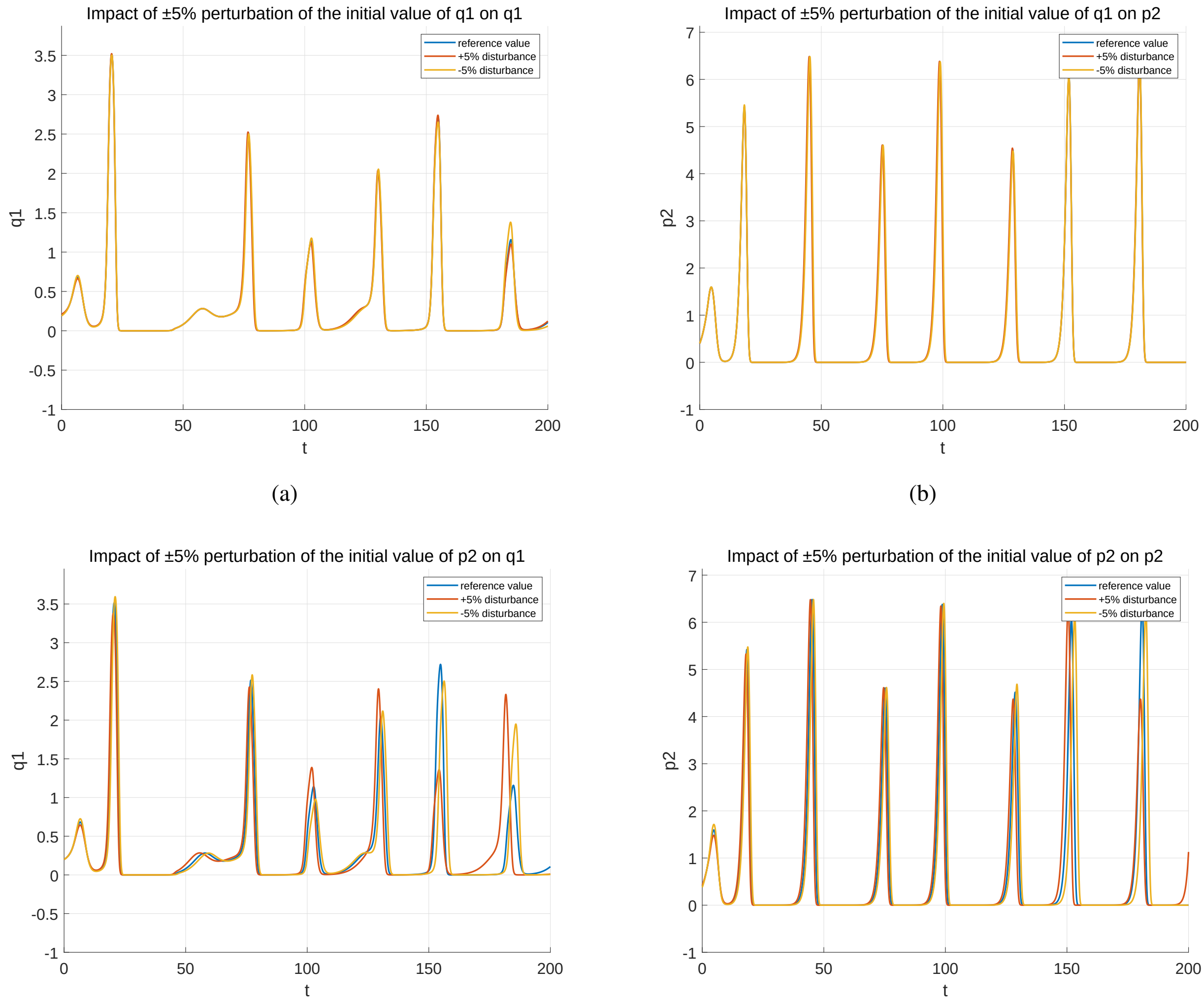


Figure 13. Set $c = 0.35$, $d = 0.25$, $v_1 = 0.75$, $v_2 = 0.69$, $\tau_1 = \tau_2 = 1.45$. Apply positive and negative 5% perturbations to the initial values [0.2, 0.4], respectively, to explore the influence of initial-value changes on the behavior of the model.

Remark 6.5. Based on Example 6.5, when $\tau_1 = \tau_2 = 3.5 > \tau_0$, from Figure 14, it can be seen that after applying a ±5% perturbation to the initial values in the unstable system, their trajectories deviate from the stable path from the initial stage. In the medium term, the amplitude of fluctuations continues to expand without a convergence trend and finally completely breaks away from the original stable trajectory.



(a) (b) (c) (d)

Figure 14. Set $c = 0.35$, $d = 0.25$, $v_1 = 0.75$, $v_2 = 0.69$, $\tau_1 = \tau_2 = 3.5$. Apply positive and negative 5% perturbations to the initial values [0.2, 0.4], respectively, to explore the influence of initial-value changes on the behavior of the model.

7. Conclusions

In this paper, a delayed fractional-order Cournot–Bertrand game model is established, the existence, uniqueness, non-negativity, boundedness, and dynamic properties are deeply researched. By using fractional-order stability and HB theory, the stability and HB criteria are derived in the following six cases: (i) $\tau_1 = \tau_2 = 0$; (ii) $\tau_1 > 0, \tau_2 = 0$; (iii) $\tau_1 = 0, \tau_2 > 0$; (iv) $\tau_1 > 0$, τ_2 is a positive constant; (v) $\tau_2 > 0$, τ_1 is a positive constant; (vi) $\tau_1 = \tau_2 = \tau$.

Based on the theoretical analysis and numerical simulation results, the following conclusions can be obtained: The fractional order and time delay significantly affect the dynamic properties of the fractional-order differential system. Compared with the traditional integer-order delayed model, it exhibits more complex dynamic behaviors. In terms of stability, time delay and fractional order jointly determine the stability of the equilibrium point. There are specific combinations of time delay and fractional order that cause the system to oscillate or even become unstable. From an economic perspective, this means that in the decision-making process, enterprises should not only consider the current market information, but also pay attention to the memory of past information (fractional-order characteristics) and delayed decision feedback, because these factors will greatly change the evolutionary stability and dynamic evolution path of market competition equilibrium. In addition, through the bifurcation analysis system (2.5), the process of the system changing from stable to unstable with the change of parameters is revealed, providing a theoretical basis for understanding the sudden change phenomenon in market competition.

Furthermore, in light of the aforesaid analysis, this paper offers the following suggestions to enterprise decision-makers and market regulators. For enterprises, when formulating production or pricing strategies, they should fully recognize the impact of market information memory and decision-making delay. Build a more perfect market dynamic monitoring mechanism. They should not only pay attention to the current behavior of competitors and market demand but also mine the potential value of market information in the past period through data analysis so as to predict market trends more accurately. For market regulators, since both the fractional order and time delay may cause unstable fluctuations in the market, regulatory authorities need to closely monitor the decision-making models of enterprises in the industry and market change dynamics. Establish a market risk early warning system based on current and historical data, and timely detect potential market imbalance risks by monitoring the changes of key parameters (such as enterprise decision-making delay, market memory intensity, etc.).

Author contributions

Nengfa Wang: Conceptualization, Formal analysis, Methodology, Writing–original draft, Writing–review & editing, Supervision; Kai Gu: Formal analysis, Writing–original draft, Writing–review & editing, Software, Validation; Zixin Liu: Formal analysis, Writing–review & editing, Software, Validation, Supervision; Changjin Xu: Software, Validation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest in this paper.

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