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Research article

Analysis of a stochastic food chain model with nonlinear prey refuge and Allee effect driven by Ornstein-Uhlenbeck process

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Abstract: In this paper, we investigated a food chain model driven by the Ornstein-Uhlenbeck process, incorporating the Holling type II functional response, nonlinear prey refuge, and the Allee effect in the top predator. First, the biological significance of the Ornstein-Uhlenbeck process was illustrated, and its rationality was explained. Subsequently, the existence and uniqueness of the global solution of the model were established, and its ultimate boundedness was analyzed. Then, by constructing a Lyapunov function and applying Itô's formula, the existence of the stationary distribution of the model was demonstrated. Furthermore, the conditions for the system extinction were provided. Finally, numerical simulations were conducted to verify the theoretical results and confirm the validity of the conclusions.

Keywords: Ornstein-Uhlenbeck process; nonlinear prey refuge; Allee effect; stationary distribution; extinction
Mathematics Subject Classification: 60H10, 60H30, 92D25, 92D40

1. Introduction

The predator-prey model is an important mathematical tool for studying species interactions in ecosystems. By analyzing its dynamic behavior, researchers can uncover long-term evolutionary trends of species. The model proposed by Lotka and Volterra [1, 2] laid the foundation for the study of predator-prey dynamics, and subsequent models, such as the Leslie-Gower [3] model, have enhanced the realism of ecological modeling by better capturing the complexities of natural systems. However, with the advancement of ecology, the original linear models have proven insufficient in explaining newly discovered biological phenomena. To more accurately describe biological interactions, the functional response have been introduced. In 1965, based on experimental observations, biomathematician Holling proposed the Holling type II functional response, which describes the average feeding rate of a predator when it spends time searching for prey and additional time, exclusive of searching, processing each captured prey [4–6].

The saturation effect of predation rate is captured by the Holling type II functional response, but the influence of predator population density or the active avoidance behavior of prey is not considered. In natural ecosystems, population density is crucial for species survival. When predator numbers decline below a critical threshold, further population decline can be caused by reduced cooperation among individuals and lower reproductive success, a phenomenon known as the Allee effect, which was first proposed by Allee in 1931 [7,8]. In fact, the Allee effect is widespread in natural populations, with numerous causes. For example, at low population densities, challenges such as difficult mate-finding, habitat alteration, foraging difficulty, anti-predator defense, and biological invasion can induce Allee effects [9–11]. In ecological research, understanding the influence of the Allee effect is of great importance, as this phenomenon can significantly increase the probability of both local and global species extinctions. Consequently, the main significance of the Allee effect lies in its provision of valuable insights for protecting endangered species and illuminating the evolutionary processes that have facilitated the development of current biodiversity [10]. Many researchers explored the impact of the Allee effect on prey-predator systems, with the effect considered in prey populations [12–14], predator populations [15], or both [16]. Notably, most researchers focus on Allee effects in prey populations. However, predator populations typically have lower densities than prey, making them more susceptible to Allee effects. Despite this, research on Allee effects in predators remains limited. In summary, investigating the Allee effect in top predators within food chain models is reasonable and significant.

In addition, studying the evasion strategies of prey and the predation strategies of predators is a key concern in ecology. A growing body of research has shown that prey can assess the risk of predation and adopt corresponding strategies to alter their behaviors and survival mechanisms, thereby increasing their chances of survival [17]. Among these, prey refuges represent a simple yet effective defensive mechanism. Moreover, in reality, prey will balance the cost of defense and the risk of predation by adjusting the intensity of refuge use. They adopt induced defenses to cope with fluctuating predation risks, which helps save costs [18]. Therefore, it is reasonable to consider this process from a nonlinear perspective.

Studies have shown that prey refuges not only affect the foraging efficiency of predators but can also alter predator population dynamics, interacting with the Allee effect. Molla et al. [19] studied a predator-prey model incorporating the Allee effect and nonlinear prey refuge, revealing richer bifurcation behaviors in the system. Rahman et al. [20] further combined these two factors and found that the Allee effect and prey refuge play a stabilizing role in system dynamics behaviors. Nonlinear prey refuges have been extensively studied and have significant effects on species coexistence [19–22]. Building on these ideas, in 2023, Wu [23] proposed and analyzed a food chain model incorporating the Holling II functional response, the Allee effect, and nonlinear prey refuge, formulated as follows:

$$\begin{cases} dx(t) = x(t)\left[r(t)\left(1 - \frac{x(t)}{K}\right) - \frac{m_1 y(t)(1-\delta y(t))}{1+a_1 x(t)(1-\delta y(t))}\right]dt, \\ dy(t) = y(t)\left[-d_1(t) + \frac{c_1 m_1 x(t)(1-\delta y(t))}{1+a_1 x(t)(1-\delta y(t))} - \frac{m_2 z(t)}{1+a_2 y(t)}\right]dt, \\ dz(t) = z(t)\left[-d_2(t) + \frac{c_2 m_2 y(t) z(t)}{(1+a_2 y(t))(z(t)+h)}\right]dt. \end{cases} \tag{1.1}$$

In model (1.1), $x(t)$, $y(t)$, and $z(t)$ represent the population densities of the prey, the intermediate predator, and the top predator at time t, respectively. The parameter K denotes the environmental carrying capacity of the prey, while r represents its intrinsic growth rate. The natural mortality rates of the intermediate predator and the top predator are given by d_1 and d_2, respectively. Additionally, c_1 and c_2 denote the conversion efficiencies of the intermediate predator and the top predator, respectively. The functional responses of the intermediate predator and the top predator follow the Holling type II functional response forms: $G_1(x) = \frac{m_1 x(1-\delta y)}{1+a_1 x(1-\delta y)}$ and $G_2(y) = \frac{m_2 y}{1+a_2 y}$, where the intensity of the nonlinear prey refuge is assumed to depend on the number of intermediate predators. Here, a_1 and a_2 are the half-saturation constants for the prey and the intermediate predator, respectively, while m_1 and m_2 represent the search rates of the intermediate predator and the top predator, respectively. Parameter δ is the prey refuge coefficient, where δy indicates that the refuge intensity is proportional to the number of predators. As a result, δxy represents the number of prey escaping predation, and the available prey for the intermediate predator is given by $x - \delta xy$ [20, 21]. Moreover, the model incorporates a weak Allee effect in the growth function of the top predator, expressed as $A(z) = \frac{z}{z+h}$. Parameter h is used to quantify the strength of the Allee effect, with $h > 0$ defined as the Allee effect constant. All parameters in the model are positive, except for r, which can be either positive or negative.

In the real ecological systems, the growth of the species is inevitably influenced by environmental noise, and ecological populations are subject to disturbances caused by stochastic environmental fluctuations. As May [24] pointed out, environmental noise can affect various parameters of a population system, including the intrinsic growth rate, environmental carrying capacity, competition coefficients, and other system parameters. Given that population parameters are highly susceptible to environmental noise, we assume that parameters r, d_1, and d_2 in model (1.1) are stochastic variables, denoted as $r(t)$, $d_1(t)$, and $d_2(t)$, respectively. According to the literature, there are two commonly used methods to model parameter variations when environmental conditions are changed. The first method assumes that parameters are perturbed by a linear function of Gaussian white noise [25]. However, in a stochastically varying environment, modeling parameter perturbations by a linear function of Gaussian white noise is unreasonable. In the following discussion, mathematical methods are used to demonstrate this inadequacy. Thus, assume that the intrinsic growth rate and mortality rates are governed by the following equations:

$$r(t) = \bar{r} + \frac{\eta_1\, dB_1(t)}{dt}, d_1(t) = \bar{d}_1 + \frac{\eta_2\, dB_2(t)}{dt}, \text{ and } d_2(t) = \bar{d}_2 + \frac{\eta_3\, dB_3(t)}{dt},$$

where $\bar{r}$, $\bar{d}_1$, and $\bar{d}_2$ represent the long-term average levels of $r(t)$, $d_1(t)$, and $d_2(t)$, respectively. The terms $B_i(t)$, where $i = 1, 2, 3$, denote three independent standard Brownian motions defined on a complete probability space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\}$, where filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies the usual conditions [26]. Furthermore, $\eta_i > 0$ for $i = 1, 2, 3$ represents the noise intensity of $B_i(t)$. For any time interval $[0, t]$, let $\langle r(t)\rangle$, $\langle d_1(t)\rangle$, and $\langle d_2(t)\rangle$ denote the time-averaged values of the aforementioned parameters. There exists

$$\begin{aligned}
\langle r(t)\rangle &:= \frac{1}{t}\int_0^t r(s)\,ds = \bar{r} + \frac{\eta_1 B_1(t)}{t} \sim \mathbb{N}\left(\bar{r}, \frac{\eta_1^2}{t}\right),\\
\langle d_1(t)\rangle &:= \frac{1}{t}\int_0^t d_1(s)\,ds = \bar{d}_1 + \frac{\eta_2 B_2(t)}{t} \sim \mathbb{N}\left(\bar{d}_1, \frac{\eta_2^2}{t}\right), \qquad (1.2)\\
\langle d_2(t)\rangle &:= \frac{1}{t}\int_0^t d_2(s)\,ds = \bar{d}_2 + \frac{\eta_3 B_3(t)}{t} \sim \mathbb{N}\left(\bar{d}_2, \frac{\eta_3^2}{t}\right),
\end{aligned}$$

where $\mathbb{N}(\cdot, \cdot)$ represents a one-dimensional normal distribution. Taking the intrinsic growth rate as an example, it is evident that the variance of the time-averaged growth rate $\langle r(t)\rangle$ over the interval

$[0, t]$ is given by $\frac{\eta_1^2}{t}$. As $t \to 0^+$, the variance approaches infinity. This implies that the random fluctuations in the growth rate $r(t)$ become extremely large over very short time intervals. Therefore, we consider an alternative approach to modeling stochastic perturbations by employing the mean-reverting Ornstein-Uhlenbeck process to simulate environmental fluctuations. However, the direct application of the Ornstein-Uhlenbeck process to mortality rates may result in values less than zero, which contradicts the fundamental definition of mortality. Thus, we assume that $\ln d_1(t)$ and $\ln d_2(t)$ are perturbed by the Ornstein-Uhlenbeck process [27–29]. For convenience, we define $g_1(t) = r(t)$, $g_2(t) = \ln d_1(t)$, and $g_3(t) = \ln d_2(t)$. Based on the literature, the specific formulation is given as follows [30–32]:

$$dg_i(t) = \beta_i \left[\bar{g}_i - g_i(t)\right] dt + \sigma_i dB_i(t), i = 1, 2, 3, \tag{1.3}$$

where, in the equations, $\beta_i > 0$ and $\sigma_i > 0$ $(i = 1, 2, 3)$ represent the regression rate and the fluctuation intensity, respectively. As stated by Mao [32], by performing stochastic integral operations, we can obtain the unique solution in the following form:

$$g_i(t) = \bar{g}_i + \left[g_i(0) - \bar{g}_i\right] e^{-\beta_i t} + \sigma_i \int_0^t e^{-\beta_i (t-s)} dB_i(s), i = 1, 2, 3, \tag{1.4}$$

where $g_i(0)$ is the initial value of the Ornstein-Uhlenbeck process $g_i(t)$. The above equation indicates that the variable

$$g_i(t) \sim \mathbb{N}\left(\bar{g}_i + \left[g_i(0) - \bar{g}_i\right] e^{-\beta_i t}, \frac{\sigma_i^2}{2\beta_i}\left(1 - e^{-2\beta_i t}\right)\right).$$

Furthermore, it is evident that the expectation and variance of $g_i(t)$ satisfy

$$\lim_{t\to 0^+} \mathbb{E}[g_i(t)] = g_i(0), \lim_{t\to 0^+} \mathbb{VAR}[g_i(t)] = 0, \lim_{t\to\infty} \mathbb{E}[g_i(t)] = \bar{g}_i, \lim_{t\to\infty} \mathbb{VAR}[g_i(t)] = \frac{\sigma_i^2}{2\beta_i}.$$

Therefore, within certain time intervals, the fluctuation of $r(t)$ remains within a relatively small range, which aligns with the continuous perturbation characteristics of environmental noise. This demonstrates that employing the Ornstein-Uhlenbeck process to model random disturbances is reasonable [28].

Based on the above analysis, we accordingly modify model (1.1) to obtain the following stochastic model:

$$\begin{cases} dx(t) = x(t)\left[g_1(t)\left(1 - \frac{x(t)}{K}\right) - \frac{m_1 y(t)(1-\delta y(t))}{1+a_1 x(t)(1-\delta y(t))}\right] dt, \\ dy(t) = y(t)\left[-e^{g_2(t)} + \frac{c_1 m_1 x(t)(1-\delta y(t))}{1+a_1 x(t)(1-\delta y(t))} - \frac{m_2 z(t)}{1+a_2 y(t)}\right] dt, \\ dz(t) = z(t)\left[-e^{g_3(t)} + \frac{c_2 m_2 y(t) z(t)}{(1+a_2 y(t))(z(t)+h)}\right] dt, \\ dg_1(t) = \beta_1 \left[\bar{g}_1 - g_1(t)\right] dt + \sigma_1 dB_1(t), \\ dg_2(t) = \beta_2 \left[\bar{g}_2 - g_2(t)\right] dt + \sigma_2 dB_2(t), \\ dg_3(t) = \beta_3 \left[\bar{g}_3 - g_3(t)\right] dt + \sigma_3 dB_3(t). \end{cases} \tag{1.5}$$

Investigating the dynamic properties of system (1.5) is necessary. Researchers have either focused on the impacts of partial properties in the models mentioned above [33, 34] or considered only models of two species [35], with limited research on systems similar to system (1.5). Here, we construct a coupled model featuring the "Ornstein-Uhlenbeck process driving, Holling type II functional response, nonlinear refuges, and Allee effect in top predators", systematically studying and revealing the synergistic effects of these four factors on the food chain model. This fills the gap in theories regarding stochastic food chain models with multiple complex effect couplings. Furthermore, the dynamic properties studied here belong to classic research directions [36–39], and the findings provide critical theoretical references for ecology and conservation biology to validate interspecific interaction mechanisms [40].

Next, in Section 3, the existence of a unique global solution for system (1.5) is proved, along with its ultimate boundedness, the existence of the stationary distribution, and extinction. Finally, in Section 4, numerical simulations are performed to verify the theoretical results and demonstrate the validity of these findings.

2. Preliminaries

To simplify the proof, we define two necessary sets: $\mathbb{G}_n = (-n,n) \times (-n,n) \times (-n,n)$ and $\mathbb{R}^n_+ = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_k > 0, 0 \leq k \leq n\}$, where $\|\cdot\|$ represents the Euclidean norm. Consider the following stochastic differential equation:

$$dX(t) = \xi(t, X(t))dt + \sum_{j=1}^{n} \varsigma_j(t, X(t))dB_j(t). \tag{2.1}$$

According to Khasminskii [41], the existence of the stationary solution to the system (1.5) for arbitrary initial conditions can be established by the following lemma.

Lemma 2.1. (Khasminskii) *Let the vectors* $\xi(s,x), \varsigma_1(s,x), \ldots, \varsigma_l(s,x)$ *(with* $s \in [t_0, T]$, $x \in \mathbb{R}^m$*) be continuous functions of* (s,x)*, and suppose that there exists some constants* M *such that the following conditions hold in the entire domain:*

$$\begin{aligned} |\xi(s,x) - \xi(s,y)| + \sum_{j=1}^{m} \left|\varsigma_j(s,x) - \varsigma_j(s,y)\right| &\leq M|x-y|, \\ |\xi(s,x)| + \sum_{j=1}^{m} \left|\varsigma_j(s,x)\right| &\leq M(1+|x|). \end{aligned} \tag{2.2}$$

Additionally, there exists a non-negative function $V_*(x)$*, such that*

$$\mathcal{L}V_*(x) \leq -1, \quad \forall x \in \mathbb{R}^m \backslash \mathbb{H}, \tag{2.3}$$

where $\mathbb{H}$ *is a compact subset of* $\mathbb{R}^m$*. Then, the Markov process* (2.1) *has at least one stationary solution* $X(t)$*, which has a stationary distribution* $\omega(\cdot)$ *on* $\mathbb{R}^m$*.*

Lemma 2.2. (Strong Law of Large Numbers [42]) *Let* $M = \{M_t\}_{t\geq 0}$ *be a real-valued, continuous local martingale that starts from zero at* $t = 0$*. Then, almost surely, the following hold:* $\lim_{t\to\infty}\langle M, M\rangle_t = \infty$, $\lim_{t\to\infty} \frac{M_t}{\langle M,M\rangle_t} = 0$, $\limsup_{t\to\infty} \frac{\langle M,M\rangle_t}{M_t} < \infty$, $\lim_{t\to\infty} \frac{M_t}{t} = 0$*. More generally, if* $S = \{S_t\}_{t\geq 0}$ *is a continuous, adapted, and increasing process satisfying* $\lim_{t\to\infty} S_t = \infty$ *and* $\int_0^\infty \frac{d\langle M,M\rangle_t}{(1+S_t)^2} < \infty$*, then it follows that* $\lim_{t\to\infty} \frac{M_t}{S_t} = 0$ *a.s.*

Lemma 2.3. (Itô's formula [32]) *Let $x(t)$ be an Itô process on $t \geq 0$ with the stochastic differential*

$$dx(t) = f(t)\,dt + g(t)\,dB_t,$$

where $f \in L^1(\mathbb{R}_+;\mathbb{R})$ and $g \in L^2(\mathbb{R}_+;\mathbb{R})$. Let $V \in C^{2,1}(\mathbb{R}\times\mathbb{R}_+;\mathbb{R})$. Then, $V(x(t),t)$ is again an Itô process with the stochastic differential given by

$$dV(x(t),t) = \left[V_t(x(t),t) + V_x(x(t),t)f(t) + \frac{1}{2}V_{xx}(x(t),t)g^2(t)\right]dt + V_x(x(t),t)g(t)\,dB_t \quad a.s.$$

3. Results

3.1. Existence and uniqueness of the global solution

Theorem 3.1. *For any initial value $(x(0),y(0),z(0),g_i(0)) \in \mathbb{R}^3_+ \times \mathbb{R}^3$, system (1.5) admits a unique global solution $(x(t),y(t),z(t),g_i(t)) \in \mathbb{R}^3_+ \times \mathbb{R}^3$ for $t > 0$, and will remain in $\mathbb{R}^3_+ \times \mathbb{R}^3$ with probability one, where $i = 1,2,3$.*

Proof. It is evident that the coefficients of model (1.5) satisfy the local Lipschitz condition. Therefore, for any given initial values $(x(0),y(0),z(0),g_i(0))$, system (1.5) has a unique local solution $(x(t),y(t),z(t),g_i(t))$ on the interval $[0,\tau_e)$, where $i = 1,2,3$ and τ_e represents the explosion time [32]. To demonstrate that the solution is global, it is sufficient to show that $\tau_e = \infty$.

Let n be large enough such that each component of $\ln x(0)$, $\ln y(0)$, $\ln z(0)$ and $g_i(0)$ ($i = 1,2,3$) lies in the interval $[-n,n]$, and denote this n by n_0. For $n \in \mathbb{Z}$ and $n \geq n_0$, define the stopping time τ_n by

$$\tau_n = \inf\left\{t \in [0,\tau_e] : \ln x(t) \notin (-n,n)\ \text{ or} \ln y\,(t) \notin (-n,n)\ \text{ or } \ln z\,(t) \notin (-n,n)\ \text{ or} \, g_i\,(t) \notin (-n,n)\right\},$$

where i=1,2,3. Here, as n increases, τ_n increases. Let $\tau_\infty = \lim\limits_{n\to\infty} \tau_n$, then $\tau_\infty \leq \tau_e$. Therefore, it suffices to prove almost surely that $\tau_\infty = \infty$. Suppose this assertion is false, then there exist constants $T > 0$ and $\varepsilon \in (0,1)$ such that there exists an integer $n_1 \geq n_0$ for which

$$P\{\tau_n \leq T\} \geq \varepsilon, \forall n \geq n_1. \tag{3.1}$$

For any $t \leq \tau_n$, define a non-negative Lyapunov function $V(x(t),y(t),z(t),g_i(t))$ on $\mathbb{R}^3_+ \times \mathbb{R}^3 \to \mathbb{R}$

$$V(x,y,z,g_i) = x - 1 - \ln x + y - 1 - \ln y + z - 1 - \ln z + \frac{{g_1}^4}{4} + \frac{{g_2}^4}{4} + \frac{{g_3}^4}{4}. \tag{3.2}$$

According to the Itô's formula,

$$dV = \mathcal{L}Vdt + \sigma_1 {g_1}^3 dB_1(t) + \sigma_2 {g_2}^3 dB_2(t) + \sigma_3 {g_3}^3 dB_3(t), \tag{3.3}$$

where

$$\begin{aligned}\mathcal{L}V = {} & (x-1)\left[g_1\left(1 - \frac{x}{K}\right) - \frac{m_1 y(1-\delta y)}{1 + a_1 x(1-\delta y)}\right] + (y-1)\left[-e^{g_2} + \frac{c_1 m_1 x(1-\delta y)}{1 + a_1 x(1-\delta y)} - \frac{m_2 z}{1 + a_2 y}\right] \\ & + (z-1)\left[-e^{g_3} + \frac{c_2 m_2 yz}{(1 + a_2 y)(z+h)}\right] + \sum_{i=1}^{3}\left(\frac{3}{2}{g_i}^2{\sigma_i}^2 + {g_i}^3\beta_i(\bar{g}_i - g_i)\right).\end{aligned}$$

By the calculation, we can obtain

$$\begin{aligned}\mathcal{L}V \leqslant &\left(|g_1| + \frac{g_1}{K}\right)x - \frac{|g_1|}{K}x^2 - \left(e^{g_2} - m_1 - \frac{c_1 m_1}{a_1}\right)y - m_1\delta y^2 - \left(e^{g_3} - m_2 - \frac{c_2 m_2}{a_2}\right)z\\ &+ |g_1| + e^{g_2} + e^{g_3} + \sum_{i=1}^{3}\left(\frac{3}{2}{g_i}^2{\sigma_i}^2 + \beta_i\bar{g}_i{g_i}^3 - \beta_i{g_i}^4\right)\\ \leqslant &\Pi_0 < \infty,\end{aligned}$$

where

$$\begin{aligned}\Pi_0 = \sup_{(x,y,z,g_i)\in\mathbb{R}_+^3\times\mathbb{R}^3} &\left\{\left(|g_1| + \frac{g_1}{K}\right)x - \frac{|g_1|}{K}x^2 - \left(e^{g_2} - m_1 - \frac{c_1 m_1}{a_1}\right)y - m_1\delta y^2 - \left(e^{g_3} - m_2 - \frac{c_2 m_2}{a_2}\right)z\right.\\ &\left.+ |g_1| + e^{g_2} + e^{g_3} + \sum_{i=1}^{3}\left(\frac{3}{2}{g_i}^2{\sigma_i}^2 + \beta_i\bar{g}_i{g_i}^3 - \beta_i{g_i}^4\right)\right\}.\end{aligned}$$

Furthermore,

$$dV \leqslant \Pi_0 dt + \sigma_1{g_1}^3 dB_1(t) + \sigma_2{g_2}^3 dB_2(t) + \sigma_3{g_3}^3 dB_3(t). \tag{3.4}$$

Integrating inequality (3.4) from 0 to $\tau_n \wedge T$, we get

$$\int_0^{\tau_n\wedge T} dV \leqslant \int_0^{\tau_n\wedge T} \Pi_0 dt + \int_0^{\tau_n\wedge T} [\sigma_1{g_1}^3 dB_1(t) + \sigma_2{g_2}^3 dB_2(t) + \sigma_3{g_3}^3 dB_3(t)]. \tag{3.5}$$

Taking the expectation of both sides, we obtain

$$\mathbb{E}\left[V(x(\tau_n \wedge T), y(\tau_n \wedge T), z(\tau_n \wedge T), g_i(\tau_n \wedge T))\right] \leq V(x(0), y(0), z(0), g_i(0)) + \Pi_0 T. \tag{3.6}$$

For $n \geqslant n_1$, let $\Omega_n = \{\tau_n \leq T\}$. From Eq (3.1), we have $P(\Omega_n) \geq \varepsilon$. Note that for each $\omega \in \Omega_n$, there exists an n such that $\ln x(\tau_n, \omega)$, $\ln y(\tau_n, \omega)$, $\ln z(\tau_n, \omega)$, and $g_i(\tau_n, \omega)$ equal $-n$ or n.

$$\begin{aligned}V(x(0), y(0), z(0), g_i(0)) + \Pi_0 T &\geq \mathbb{E}[(I_{\Omega_n(\omega)}V(x(\tau_n,\omega), y(\tau_n,\omega), z(\tau_n,\omega), g_i(\tau_n,\omega))]\\ &\geq \varepsilon\min\left\{e^{-n} - 1 + n, e^{n} - 1 - n, \frac{n^4}{4}\right\}, i = 1,2,3,\end{aligned}$$

where $I_{\Omega_{\rm n}(\omega)}$ represents the indicator function of Ω_n. As $n \to \infty$, we have

$$\infty > V(x(0), y(0), z(0), g_i(0)) + \Pi_0 T = \infty, i = 1,2,3, \tag{3.7}$$

which implies that $\tau_\infty = \infty$ almost surely, and the proof is complete.

3.2. Ultimate boundedness

In natural ecosystems, resource availability is limited, preventing populations from growing indefinitely. As time progresses, population density will reach a stable level than increasing without bounds. Therefore, it is crucial to establish the theoretical ultimate boundedness of system (1.5). To achieve this, we first introduce the concept of stochastic ultimate boundedness [43].

Definition 3.1. *[43] System* (1.5) *is said to be stochastically ultimately bounded if for any $\varepsilon \in (0,1)$, there is a positive constant $\varpi = \varpi(\omega)$, such that for any initial value (x_0, y_0, z_0, g_{i0}), where $i = 1,2,3$, and the solution of system* (1.5) *has the property that*

$$\limsup_{t\to\infty} P\left(\sqrt{x^2+y^2+z^2} > \varpi\right) < \varepsilon. \tag{3.8}$$

Lemma 3.1. *For any initial value $(x(0), y(0), z(0), g_i(0)) \in \mathbb{R}^3_+ \times \mathbb{R}^3$, where $i = 1,2,3$, the solution of the system* (1.5) *has the property*

$$\limsup_{t\to\infty} \mathbb{E}\left[|(x,y,z)|^q\right] \le Q(q), \tag{3.9}$$

where $q \in (0,1)$, and $Q(q)$ is a positive constant independent of the initial value $(x(0), y(0), z(0), g_i(0))$.

Proof. Define a Lyapunov function V_1 on $\mathbb{R}^3_+ \times \mathbb{R}^3 \to \mathbb{R}$,

$$V_1 = \frac{x^q(t)}{q} + \frac{y^q(t)}{q} + \frac{z^q(t)}{q} + \sum_{i=1}^{3} \frac{{g_i}^{2q+2}(t)}{2q+2}.$$

By applying Itô's formula to function V_1, it can be derived that

$$dV_1 = \mathcal{L}V_1 dt + \sigma_1 {g_1}^{2q+1} dB_1(t) + \sigma_2 {g_2}^{2q+1} dB_2(t) + \sigma_3 {g_3}^{2q+1} dB_3(t),$$

where

$$\begin{aligned}\mathcal{L}V_1 =& x^q\left[g_1\left(1-\frac{x}{K}\right) - \frac{m_1 y(1-\delta y)}{1+a_1 x(1-\delta y)}\right] + y^q\left[-e^{g_2} + \frac{c_1 m_1 x(1-\delta y)}{1+a_1 x(1-\delta y)} - \frac{m_2 z}{1+a_2 y}\right]\\ &+ z^q\left[-e^{g_3} + \frac{c_2 m_2 y z}{(1+a_2 y)(z+h)}\right] + \sum_{i=1}^{3}\left(\frac{2q+1}{2}{g_i}^{2q}{\sigma_i}^2 + \beta_i {g_i}^{2q+1}(\bar{g}_i - g_i)\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{L}V_1 \leqslant& |g_1|\, x^q - \frac{|g_1|}{K}x^{q+1} - \left(e^{g_2} - \frac{c_1 m_1}{a_1}\right)y^q - \left(e^{g_3} - \frac{c_2 m_2}{a_2}\right)z^q\\ &+ \sum_{i=1}^{3}\left(\frac{2q+1}{2}{g_i}^{2q}{\sigma_i}^2 + \beta_i \bar{g}_i {g_i}^{2q+1} - \beta_i {g_i}^{2q+2}\right).\end{aligned} \tag{3.10}$$

Let $\eta = q\min\{\beta_1, \beta_2, \beta_3\}$. With the help of Itô formula again, we have

$$\begin{aligned}d(e^{\eta t}V_1) =& \eta e^{\eta t} V_1 dt + e^{\eta t} dV_1\\ =& e^{\eta t}(\eta V_1 + \mathcal{L}V_1)dt + e^{\eta t}\sum_{i=1}^{3}\sigma_i {g_i}^{2q+1} dB_i(t).\end{aligned} \tag{3.11}$$

Integrating both sides of Eq (3.11) from 0 to t and taking the expected value, we find

$$\mathbb{E}(e^{\eta t}V_1) = \mathbb{E}\left(V_1\left(x(0), y(0), z(0), g_i(0)\right)\right) + \int_0^t \mathbb{E}\left(e^{\eta s}(\eta V_1 + \mathcal{L}V_1)\right) ds, i = 1,2,3. \tag{3.12}$$

Combining with Eq (3.10), we have

$$\begin{aligned}\eta V_1 + \mathcal{L}V_1 \leqslant & \frac{\eta x^q}{q} + \frac{\eta y^q}{q} + \frac{\eta z^q}{q} + \sum_{i=1}^{3} \frac{\eta {g_i}^{2q+2}}{2q+2} + |g_1|\, x^q - \frac{|g_1|}{K} x^{q+1} - \left(e^{g_2} - \frac{c_1 m_1}{a_1}\right) y^q \\ & - \left(e^{g_3} - \frac{c_2 m_2}{a_2}\right) z^q + \sum_{i=1}^{3} \left(\frac{2q+1}{2} {g_i}^{2q} {\sigma_i}^2 + \beta_i \bar{g}_i {g_i}^{2q+1} - \beta_i {g_i}^{2q+2}\right) \\ \leqslant & \sup_{(x,y,z,g_1,g_2)\in\mathbb{R}_+^3\times\mathbb{R}^3} \left\{ (|g_1| + \beta_1)\, x^q - \frac{|g_1|}{K} x^{q+1} - \left(e^{g_2} - \beta_2 - \frac{c_1 m_1}{a_1}\right) y^q - \left(e^{g_3} - \beta_3 - \frac{c_2 m_2}{a_2}\right) z^q \right. \\ & \left. + \sum_{i=1}^{3} \left((q+1)\, {g_i}^{2q} {\sigma_i}^2 + \beta_i \bar{g}_i {g_i}^{2q+1} - \frac{\beta_i}{2} {g_i}^{2q+2} \right) \right\} := \kappa_1(q).\end{aligned} \tag{3.13}$$

Substituting Eq (3.13) into Eq (3.12), we get

$$\mathbb{E}(e^{\eta t} V_1) \leqslant \mathbb{E}\left(V_1\left(x(0), y(0), z(0), g_i(0)\right)\right) + \mathbb{E}\int_0^t e^{\eta s} \kappa_1(q)\, ds.$$

Then,

$$e^{\eta t}\mathbb{E}V_1 \leqslant \mathbb{E}\left(V_1\left(x(0), y(0), z(0), g_i(0)\right)\right) + \frac{e^{\eta t} - 1}{\eta} \kappa_1(q).$$

Further,

$$\begin{aligned}\limsup_{t\to\infty} \mathbb{E}\left[|(x,y,z)|^q\right] \leqslant & 3^{\frac{q}{2}} q \limsup_{t\to\infty} \mathbb{E}\left(V_1(x,y,z,\mathrm{g}_i)\right) \\ \leqslant & 3^{\frac{q}{2}} q \lim_{t\to\infty} \mathbb{E}\left[\frac{V_1(x(0), y(0), z(0), g_i(0))}{e^{\eta t}} + \frac{e^{\eta t} - 1}{\eta e^{\eta t}} \kappa_1(q)\right] \\ = & 3^{\frac{q}{2}} \frac{q \kappa_1(q)}{\eta} := \kappa_2(q).\end{aligned} \tag{3.14}$$

By setting $Q(q) = \kappa_2(q)$, Lemma 3.1 is proved.

Theorem 3.2. *The solution of system* (1.5) *is stochastic and ultimately bounded.*

Proof. Based on the above, we can conclude that when $q = \frac{1}{2}$, $Q(q)$ satisfies $\limsup\limits_{t\to\infty} \mathbb{E}\sqrt{|(x,y,z)|} \leqslant Q(q)$. Applying Chebyshev's inequality, for any $\varepsilon > 0$, let $\varpi = \frac{\sqrt{3}\kappa_1\left(\frac{1}{2}\right)^2}{4\varepsilon^2\eta^2}$, we can obtain

$$P\left(|(x,y,z)| > \varpi\right) \leqslant \frac{\mathbb{E}\left[\sqrt{|(x,y,z)|}\right]}{\sqrt{\varpi}}.$$

From the above equation, we can derive that $\limsup\limits_{t\to\infty} P\left(|(x,y,z)| > \varpi\right) \leqslant \frac{Q}{\frac{Q}{\varepsilon}} = \varepsilon$. According to Definition 3.1, Theorem 3.2 is proved.

3.3. *Existence of a stationary distribution*

In biological systems, understanding their long-term dynamics under stochastic influences is crucial. A key aspect of this analysis is to determine whether the system admits a stationary distribution, which characterizes its long-term probabilistic behavior. Establishing sufficient conditions for the existence of a stationary distribution helps in predicting the persistence and stability of populations over time. By Theorem 3.1, it is easy to know that there is a globally unique solution to the system (1.5), so the description of $\mathbb{R}^m$ in Lemma 2.1 should be changed to $\mathbb{R}_+^3 \times \mathbb{R}^3$.

Theorem 3.3. *Let N be a positive number satisfying $N \in \left(\max\left\{0, \frac{2+\Pi_1}{\sum_{i=1}^3 \bar{g}_i}\right\}, \min\left\{\frac{1}{4m_1}, \frac{1}{4m_2}\right\}\right)$, where*

$$\begin{aligned}\Pi_1 = \sup_{(x,y,z,g_i)\in\mathbb{R}_+^3\times\mathbb{R}^3} &\left\{\left(|g_1| + \frac{g_1}{K}\right)x - e^{g_2}y - e^{g_3}z - \frac{|g_1|}{K}x^2 + \frac{1}{2}x^{\frac{4}{3}} + \frac{1}{2}y + \frac{1}{2}z + N\left(e^{g_2} + e^{g_3} + g_2 + g_3\right)\right.\\ &\left. - \frac{(1-c_1)\,m_1xy\,(1-\delta y)}{1+a_1x\,(1-\delta y)} - \frac{m_2yz}{1+a_2y} + \frac{c_2m_2yz^2}{(1+a_2y)\,(z+h)} + \sum_{i=1}^{3}\left(\frac{3}{2}{g_i}^2{\sigma_i}^2 + \beta_i\bar{g}_i{g_i}^3 - \frac{1}{2}\beta_i{g_i}^4\right)\right\},\end{aligned}$$

with $i = 1,2,3$. Then, for any initial value $(x(0), y(0), z(0), g_i(0)) \in \mathbb{R}_+^3 \times \mathbb{R}^3$, system (1.5) *admits a stationary distribution on $\mathbb{R}_+^3 \times \mathbb{R}^3$.*

Proof. Define the Lyapunov function V_2 on $\mathbb{R}_+^3 \times \mathbb{R}^3 \to \mathbb{R}$

$$V_2 = N\left[-\ln x - \ln y - \ln z - \sum_{i=1}^{3}\frac{g_i}{\beta_i}\right] + x + y + z + \sum_{i=1}^{3}\frac{g_i^4}{4}.$$

Applying Itô formula to function V_2, and using the definition of N yields

$$\begin{aligned}\mathcal{L}V_2 =& (x-N)\left[g_1\left(1-\frac{x}{K}\right) - \frac{m_1y\,(1-\delta y)}{1+a_1x\,(1-\delta y)}\right] + (y-N)\left[-e^{g_2} + \frac{c_1m_1x\,(1-\delta y)}{1+a_1x\,(1-\delta y)} - \frac{m_2z}{1+a_2y}\right]\\ &+ (z-N)\left[-e^{g_3} + \frac{c_2m_2yz}{(1+a_2y)\,(z+h)}\right] - N\sum_{i=1}^{3}(\bar{g}_i - g_i) + \sum_{i=1}^{3}\left(\frac{3}{2}{g_i}^2{\sigma_i}^2 + {g_i}^3\beta_i\,(\bar{g}_i - g_i)\right)\\ \leqslant& -N\sum_{i=1}^{3}\bar{g}_i + \left(|g_1| + \frac{g_1}{K}\right)x - e^{g_2}y - e^{g_3}z - \frac{|g_1|}{K}x^2 + \frac{1}{2}x^{\frac{4}{3}} + \frac{1}{2}y + \frac{1}{2}z\\ &+ N\left(e^{g_2} + e^{g_3} + g_2 + g_3\right) - \frac{(1-c_1)\,m_1xy\,(1-\delta y)}{1+a_1x\,(1-\delta y)} - \frac{m_2yz}{1+a_2y}\\ &+ \frac{c_2m_2yz^2}{(1+a_2y)\,(z+h)} + \sum_{i=1}^{3}\left(\frac{3}{2}{g_i}^2{\sigma_i}^2 + \beta_i\bar{g}_i{g_i}^3 - \frac{1}{2}\beta_i{g_i}^4\right)\\ &+ N\left(\frac{m_1y\,(1-\delta y)}{1+a_1x\,(1-\delta y)} + \frac{m_2z}{1+a_2y} - \frac{c_1m_1x\,(1-\delta y)}{1+a_1x\,(1-\delta y)} - \frac{c_2m_2yz}{(1+a_2y)\,(z+h)}\right)\\ &- \frac{1}{2}x^{\frac{4}{3}} - \frac{1}{2}y - \frac{1}{2}z - \frac{1}{2}\sum_{i=1}^{3}\beta_i{g_i}^4.\end{aligned}$$

Then,

$$\begin{aligned}\mathcal{L}V_2 \leqslant & -2+N\left(\frac{m_1y(1-\delta y)}{1+a_1x(1-\delta y)}+\frac{m_2z}{1+a_2y}-\frac{c_1m_1x(1-\delta y)}{1+a_1x(1-\delta y)}-\frac{c_2m_2yz}{(1+a_2y)(z+h)}\right)\\ & -\frac{1}{2}x^{\frac{4}{3}}-\frac{1}{2}y-\frac{1}{2}z-\frac{1}{2}\sum_{i=1}^{3}\beta_i {g_i}^4.\end{aligned} \tag{3.15}$$

From the given formulation of $V_2(x, y, z, g_1, g_2, g_3)$, it is evident that as x and y approach infinity, the function $V_2(x, y, z, g_1, g_2, g_3)$ will also diverge to infinity. Therefore, there exists a point $(x^0, y^0, z^0, g_1^0, g_2^0, g_3^0)$ within $\mathbb{R}_+^3 \times \mathbb{R}^3$ where $V_2(x, y, z, g_1, g_2, g_3)$ reaches its minimum value. Taking into account the previous analysis and the functional conditions outlined in Lemma 2.1, we can define a non-negative C^2 – function $V_3(x, y, z, g_1, g_2, g_3)$, given explicitly by

$$V_3(x,y,z,g_1,g_2,g_3)=V_2(x,y,z,g_1,g_2,g_3)-V_2\left(x^0,y^0,z^0,g_1^0,g_2^0,g_3^0\right).$$

By applying the Itô formula, it is shown that for the function $V_2(x, y, z, g_1, g_2, g_3)$ under consideration, the addition of a constant term $V_2(x^0, y^0, z^0, g_1^0, g_2^0, g_3^0)$ does not affect the final expression. Consequently, the functions $V_2(x, y, z, g_1, g_2, g_3)$ and $V_3(x, y, z, g_1, g_2, g_3)$ are governed by the same operator. Therefore, the following inequality holds

$$\begin{aligned}\mathcal{L}V_3 \leqslant & -2+N\left(\frac{m_1y(1-\delta y)}{1+a_1x(1-\delta y)}+\frac{m_2z}{1+a_2y}-\frac{c_1m_1x(1-\delta y)}{1+a_1x(1-\delta y)}-\frac{c_2m_2yz}{(1+a_2y)(z+h)}\right)\\ & -\frac{1}{2}x^{\frac{4}{3}}-\frac{1}{2}y-\frac{1}{2}z-\frac{1}{2}\sum_{i=1}^{3}\beta_i {g_i}^4.\end{aligned} \tag{3.16}$$

We consider the closed set $\mathbb{H}_\varepsilon$ defined as

$$\mathbb{H}_\varepsilon=\left\{(x,y,z,g_i)\in\mathbb{R}_+^3\times\mathbb{R}^3\,|x\in\left[\varepsilon^3,\frac{1}{\varepsilon^3}\right],y\in\left[\varepsilon^4,\frac{1}{\varepsilon^4}\right],z\in\left[\varepsilon^4,\frac{1}{\varepsilon^4}\right],g_i\in\left[-\frac{1}{\varepsilon},\frac{1}{\varepsilon}\right]\right\},$$

and denote Π_2 as

$$\begin{aligned}\Pi_2=\sup_{(x,y,z,g_i)\in\mathbb{R}_+^3\times\mathbb{R}^3}\Bigg\{ & N\left(\frac{m_1y(1-\delta y)}{1+a_1x(1-\delta y)}+\frac{m_2z}{1+a_2y}-\frac{c_1m_1x(1-\delta y)}{1+a_1x(1-\delta y)}-\frac{c_2m_2yz}{(1+a_2y)(z+h)}\right)\\ & -\frac{1}{4}x^{\frac{4}{3}}-\frac{1}{4}y-\frac{1}{4}z-\frac{1}{4}\sum_{i=1}^{3}\beta_i {g_i}^4\Bigg\}.\end{aligned}$$

Let $\varepsilon \in (0, 1)$ be a sufficiently small number such that the following inequalities hold:

$$-2+\Pi_2-\frac{\min\{1,\beta_i\}}{4}\left(\frac{1}{\varepsilon}\right)^4\leqslant -1, i=1,2,3. \tag{3.17}$$

$$-2+Nm_j\varepsilon^4\leqslant -1, j=1,2. \tag{3.18}$$

Simply, we divide the complement of the closed set into nine distinct regions, specifically given by $\left(\mathbb{R}_+^3 \times \mathbb{R}^3\right) \backslash \mathbb{H}_\varepsilon = \bigcup_{k=1}^{9} \mathbb{H}_{k,\varepsilon}^c$, where

$$\begin{aligned}
\mathbb{H}_{1,\varepsilon}^c &= \left\{(x,y,z,g_i) \in \mathbb{R}_+^3 \times \mathbb{R}^3 \,\middle|\, x \in \left(\frac{1}{\varepsilon^3}, \infty\right)\right\},\\
\mathbb{H}_{2,\varepsilon}^c &= \left\{(x,y,z,g_i) \in \mathbb{R}_+^3 \times \mathbb{R}^3 \,\middle|\, y \in \left(\frac{1}{\varepsilon^4}, \infty\right)\right\},\\
\mathbb{H}_{3,\varepsilon}^c &= \left\{(x,y,z,g_i) \in \mathbb{R}_+^3 \times \mathbb{R}^3 \,\middle|\, z \in \left(\frac{1}{\varepsilon^4}, \infty\right)\right\},\\
\mathbb{H}_{j,\varepsilon}^c &= \left\{(x,y,z,g_i) \in \mathbb{R}_+^3 \times \mathbb{R}^3 \,\middle|\, |g_i| \in \left(\frac{1}{\varepsilon}, \infty\right)\right\}, \ j = 4,5,6;\ i = 1,2,3,\\
\mathbb{H}_{7,\varepsilon}^c &= \left\{(x,y,z,g_i) \in \mathbb{R}_+^3 \times \mathbb{R}^3 \,\middle|\, x \in \left(0, \varepsilon^3\right)\right\},\\
\mathbb{H}_{8,\varepsilon}^c &= \left\{(x,y,z,g_i) \in \mathbb{R}_+^3 \times \mathbb{R}^3 \,\middle|\, y \in \left(0, \varepsilon^4\right)\right\},\\
\mathbb{H}_{9,\varepsilon}^c &= \left\{(x,y,z,g_i) \in \mathbb{R}_+^3 \times \mathbb{R}^3 \,\middle|\, z \in \left(0, \varepsilon^4\right)\right\}.
\end{aligned}$$

Next, we prove that for $(x,y,z,d_i) \in \left(\mathbb{R}_+^3 \times \mathbb{R}^3\right) \setminus \mathbb{H}_\varepsilon$, the inequality $\mathcal{L}V_3 \leqslant -1$ holds. Based on the partition of the complement set outlined above, we establish this by considering six distinct cases.

Case 1. If $(x,y,z,g_i) \in \mathbb{H}_{1,\varepsilon}^c$, the corresponding results can be derived by combining Eqs (3.16) and (3.17), which lead to the following inequality:

$$\mathcal{L}V_3 \leqslant -2 + \Pi_2 - \frac{1}{4}x^{\frac{4}{3}} \leqslant -2 + \Pi_2 - \frac{1}{4}\left(\frac{1}{\varepsilon}\right)^4 \leqslant -1.$$

Case 2. If $(x,y,z,g_i) \in \mathbb{H}_{2,\varepsilon}^c$ and $(x,y,z,d_i) \in \mathbb{H}_{3,\varepsilon}^c$, consequently, from Eqs (3.16) and (3.17), we can obtain

$$\mathcal{L}V_3 \leqslant -2 + \Pi_2 - \frac{1}{4}\min\{y,z\} \leqslant -2 + \Pi_2 - \frac{1}{4}\left(\frac{1}{\varepsilon}\right)^4 \leqslant -1.$$

Case 3. If $(x,y,z,g_i) \in \mathbb{H}_{j,\varepsilon}^c$, according to Eqs (3.16) and (3.17), we can obtain

$$\mathcal{L}V_3 \leqslant -2 + \Pi_2 - \frac{\beta_i}{4}{g_i}^4 \leqslant -2 + \Pi_2 - \frac{\beta_i}{4}\left(\frac{1}{\varepsilon}\right)^4 \leqslant -1, j = 4,5,6; i = 1,2,3.$$

Case 4. If $(x,y,z,g_i) \in \mathbb{H}_{7,\varepsilon}^c$, from Eq (3.16), it follows that

$$\mathcal{L}V_3 \leqslant -2 - \left(\frac{1}{4} - Nm_1\right)y - \left(\frac{1}{4} - Nm_2\right)z \leqslant -2 \leqslant -1.$$

Case 5. If $(x,y,z,g_i) \in \mathbb{H}_{8,\varepsilon}^c$, from Eqs (3.16) and (3.18), we deduce

$$\mathcal{L}V_3 \leqslant -2 + Nm_1 y - \left(\frac{1}{4} - Nm_2\right)z \leqslant -2 + Nm_1\varepsilon^4 \leqslant -1.$$

Case 6. If $(x, y, z, g_i) \in \mathbb{H}^c_{9,\varepsilon}$, from Eqs (3.16) and (3.18), we derive

$$\mathcal{L}V_3 \leqslant -2 - \left(\frac{1}{4} - Nm_1\right) y + Nm_2 z \leqslant -2 + Nm_2\varepsilon^4 \leqslant -1.$$

Hence, we can prove the existence of a sufficiently small constant ε, such that for all $(x, y, z, g_i) \in \left(\mathbb{R}^3_+ \times \mathbb{R}^3\right) \setminus \mathbb{H}_\varepsilon$, the inequality $\mathcal{L}V_3(x, y, z, g_i) \leq -1$ holds, where ε satisfies the following condition:

$$\varepsilon \leqslant \min\left\{1, \sqrt[4]{\frac{1}{Nm_1}}, \sqrt[4]{\frac{1}{Nm_2}}\right\},$$

for any $\Pi_2 \leqslant 1$. Additionally, for any $\Pi_2 > 1$, ε satisfies that

$$\varepsilon \leqslant \min\left\{1, \sqrt[4]{\frac{\min\{1, \beta_1, \beta_2, \beta_3\}}{4(\Pi_2 - 1)}}\right\}.$$

3.4. Extinction

Theorem 3.4. *We define*

$$\begin{aligned}
\varphi_1(t) &= \bar{r} - \varepsilon\frac{\phi_1^2}{4\eta_1}, \bar{\varphi}_1 = \lim_{t\to\infty}\frac{1}{t}\int_0^t \varphi_1(s)\,ds = \bar{r} - \varepsilon\frac{\phi_1^2}{4\eta_1},\\
\varphi_2(t) &= -\bar{d}_1 - \frac{\phi_2^2}{4\eta_2} + \frac{\phi_2^2}{4\eta_2}e^{-2\eta_2 t}, \bar{\varphi}_2 = \lim_{t\to\infty}\frac{1}{t}\int_0^t \varphi_2(s)\,ds = -\bar{d}_1 - \frac{\phi_2^2}{4\eta_2},\\
\varphi_3(t) &= -\bar{d}_2 - \frac{\phi_3^2}{4\eta_3} + \frac{\phi_3^2}{4\eta_3}e^{-2\eta_3 t}, \bar{\varphi}_3 = \lim_{t\to\infty}\frac{1}{t}\int_0^t \varphi_3(s)\,ds = -\bar{d}_2 - \frac{\phi_3^2}{4\eta_3}.
\end{aligned}$$

When $\bar{\varphi}_1 < 0, \bar{\varphi}_2 + \frac{c_1 m_1}{a_1} < 0, \bar{\varphi}_3 + \frac{c_2 m_2}{a_2} < 0$, *then x(t), y(t),z(t) are extinct.*

Proof. From Eq (1.2) and the definition of the Ornstein-Uhlenbeck process, we can obtain

$$\begin{aligned}
r &= \bar{r} + [r(0) - \bar{r}]\, e^{-\eta_1 t} + \phi_1\int_0^t e^{-\eta_1(t-s)}dB_1(s),\\
d_1 &= \bar{d}_1 + \left[d_1(0) - \bar{d}_1\right] e^{-\eta_2 t} + \phi_2\int_0^t e^{-\eta_2(t-s)}dB_2(s),\\
d_2 &= \bar{d}_2 + \left[d_2(0) - \bar{d}_2\right] e^{-\eta_3 t} + \phi_3\int_0^t e^{-\eta_3(t-s)}dB_3(s),
\end{aligned} \tag{3.19}$$

where η_i and ϕ_i are positive constants, $i = 1, 2, 3$; η_i denotes the speed of reversion; and ϕ_i represents the intensity of volatility. Equation (3.19) shows that $r(t)$, $d_1(t)$, and $d_2(t)$ are normally distributed as $\mathbb{N}(\mathbb{E}[r(t)], \mathbb{VAR}[r(t)])$, $\mathbb{N}(\mathbb{E}[d_1(t)], \mathbb{VAR}[d_1(t)])$ and $\mathbb{N}(\mathbb{E}[d_2(t)], \mathbb{VAR}[d_2(t)])$ over the interval $[0, t]$. We can deduce that

$$\begin{aligned}
\mathbb{E}[r(t)] &= \bar{r} + [r(0) - \bar{r}]\, e^{-\eta_1 t}, \mathbb{VAR}[r(t)] = \frac{\phi_1^2}{2\eta_1}\left(1 - e^{-2\eta_1 t}\right),\\
\mathbb{E}[d_1(t)] &= \bar{d}_1 + \left[d_1(0) - \bar{d}_1\right] e^{-\eta_2 t}, \mathbb{VAR}[d_1(t)] = \frac{\phi_2^2}{2\eta_2}\left(1 - e^{-2\eta_2 t}\right),\\
\mathbb{E}[d_2(t)] &= \bar{d}_2 + \left[d_2(0) - \bar{d}_2\right] e^{-\eta_3 t}, \mathbb{VA}\,R[d_2(t)] = \frac{\phi_3^2}{2\eta_3}\left(1 - e^{-2\eta_3 t}\right).
\end{aligned}$$

Consequently, the term $\phi_i \int_0^t e^{-\eta_i(t-s)} dB_i(s)$ follows a normal distribution $\mathbb{N}\left(0, \frac{\phi_i^2}{2\eta_i}\left(1 - e^{-2\eta_i t}\right)\right)$ for each $i = 1, 2, 3$. This can be equivalently expressed as $\frac{\phi_i}{\sqrt{2\eta_i}} \sqrt{1 - e^{-2\eta_i t}} \frac{dB_i(t)}{dt}$. Accordingly, we define $\gamma_i(t) = \frac{\phi_i}{\sqrt{2\eta_i}} \sqrt{1 - e^{-2\eta_i t}}$, where $B_i(t)$ represents a standard Brownian motion. Thus, Eq (3.19) can be expressed as follows:

$$
\begin{aligned}
r &= \bar{r} + \left[r(0) - \bar{r}\right] e^{-\eta_1 t} + \gamma_1 \frac{dB_1(t)}{dt}, \\
d_1 &= \bar{d}_1 + \left[d_1(0) - \bar{d}_1\right] e^{-\eta_2 t} + \gamma_2 \frac{dB_2(t)}{dt}, \\
d_2 &= \bar{d}_2 + \left[d_2(0) - \bar{d}_2\right] e^{-\eta_3 t} + \gamma_3 \frac{dB_3(t)}{dt}.
\end{aligned}
\tag{3.20}
$$

Subsequently, we refine system (1.1) accordingly.

$$
\begin{cases}
dx(t) = x(t)\left(\left(\bar{r} + \left[r(0) - \bar{r}\right] e^{-\eta_1 t}\right)\left(1 - \frac{x(t)}{K}\right) - \frac{m_1 y(t)(1-\delta y(t))}{1+a_1 x(t)(1-\delta y(t))}\right) dt + \left(1 - \frac{x(t)}{K}\right)\gamma_1 x(t) dB_1(t), \\
dy(t) = y(t)\left(-\left(\bar{d}_1 + \left[d_1(0) - \bar{d}_1\right] e^{-\eta_2 t}\right) + \frac{c_1 m_1 x(t)(1-\delta y(t))}{1+a_1 x(t)(1-\delta y(t))} - \frac{m_2 z(t)}{1+a_2 y(t)}\right) dt - \gamma_2 y(t) dB_2(t), \\
dz(t) = z(t)\left(-\left(\bar{d}_2 + \left[d_2(0) - \bar{d}_2\right] e^{-\eta_3 t}\right) + \frac{c_2 m_2 y(t) z(t)}{(1+a_2 y(t))(z(t)+h)}\right) dt - \gamma_3 z(t) dB_3(t).
\end{cases}
\tag{3.21}
$$

By applying Itô's formula to $\ln x(t)$, $\ln y(t)$, and $\ln z(t)$, and integrating from 0 to t, we obtain the following result.

$$
\begin{aligned}
\ln x(t) = &\ln x(0) + \int_0^t \left(\bar{r} + \left[r(0) - \bar{r}\right] e^{-\eta_1 s}\right)\left(1 - \frac{x(s)}{K}\right) ds - \int_0^t \frac{m_1 y(s)(1 - \delta y(s))}{1 + a_1 x(s)(1 - \delta y(s))} ds \\
&- \int_0^t \frac{\phi_1^2}{4\eta_1}\left(1 - e^{-2\eta_1 s}\right)\left(1 - \frac{x(s)}{K}\right)^2 ds + \int_0^t \left(1 - \frac{x(s)}{K}\right)\gamma_1(s) dB_1(s), \\
\ln y(t) = &\ln y(0) + \int_0^t \varphi_2(s) ds + \int_0^t \frac{c_1 m_1 x(s)(1 - \delta y(s))}{1 + a_1 x(s)(1 - \delta y(s))} ds \\
&- \frac{d_1(0) - \bar{d}_1}{\eta_2}\left(1 - e^{-\eta_2 t}\right) - \int_0^t \frac{m_2 z(s)}{1 + a_2 y(s)} ds - \int_0^t \gamma_2(s) dB_2(s), \\
\ln z(t) = &\ln z(0) + \int_0^t \varphi_3(s) ds + \int_0^t \frac{c_2 m_2 y(s) z(s)}{(1 + a_2 y(s))(z(s) + h)} ds \\
&- \frac{d_2(0) - \bar{d}_2}{\eta_3}\left(1 - e^{-\eta_3 t}\right) - \int_0^t \gamma_3(s) dB_3(s).
\end{aligned}
\tag{3.22}
$$

Considering the fact that $\bar{r}$ can be either positive or negative, we therefore discuss the cases $\bar{r} \leqslant 0$ and $\bar{r} > 0$ separately. When $\bar{r} > 0$, it follows that $0 \leqslant \lim\limits_{t\to\infty} \frac{1}{t} \bar{r} \int_0^t \left(1 - e^{-\eta_1 s}\right)\left(1 - \frac{x(s)}{K}\right) ds \leqslant \bar{r}$, and for any $\varepsilon > 0$, we have

$$
\frac{1}{t} \ln \frac{x(t)}{x(0)} \leqslant \frac{1}{t} \int_0^t \left(\bar{r} - \varepsilon \frac{\phi_1^2}{4\eta_1}\right) ds + \frac{r(0)}{t\eta_1}\left(1 - e^{-\eta_1 t}\right) + \frac{\varepsilon \phi_1^2}{4t\eta_1}\left(1 - \frac{1}{2\eta_1} e^{-2\eta_1 t}\right) + \frac{1}{t} \int_0^t \left(1 - \frac{x(s)}{K}\right)\gamma_1(s) dB_1(s).
$$

By Lemma 2.2, $\int_0^t \gamma_i(s) dB_i(s)$ satisfies the strong law of large numbers for martingales, which implies $\lim\limits_{t\to\infty} \frac{1}{t} \int_0^t \gamma_i(s) dB_i(s) = 0$. Hence, it follows that

$$
\limsup_{t\to\infty} \frac{\ln x(t)}{t} \leqslant \bar{\varphi}_1.
$$

When $\bar{r} \leqslant 0$, it is evident that

$$\limsup_{t\to\infty} \frac{\ln x(t)}{t} \leqslant 0$$

holds.

In the analysis of y and z, from Eq (3.22), we deduce

$$\begin{aligned} t^{-1} \ln \frac{y(t)}{y(0)} &\leqslant \bar{\varphi}_2 - \frac{d_1(0) - \bar{d}_1}{t\eta_2}\left(1 - e^{-\eta_2 t}\right) + \frac{c_1 m_1}{a_1} - \frac{1}{t}\int_0^t \gamma_2(s)\, dB_2(s), \\ t^{-1} \ln \frac{z(t)}{z(0)} &\leqslant \bar{\varphi}_3 - \frac{d_2(0) - \bar{d}_2}{t\eta_3}\left(1 - e^{-\eta_3 t}\right) + \frac{c_2 m_2}{a_2} - \frac{1}{t}\int_0^t \gamma_3(s)\, dB_3(s). \end{aligned} \tag{3.23}$$

By taking the upper limit for both sides of Eq (3.23), we obtain

$$\begin{aligned} \limsup_{t\to\infty} \frac{\ln y(t)}{t} &\leqslant \bar{\varphi}_2 + \frac{c_1 m_1}{a_1}, \\ \limsup_{t\to\infty} \frac{\ln z(t)}{t} &\leqslant \bar{\varphi}_3 + \frac{c_2 m_2}{a_2}. \end{aligned} \tag{3.24}$$

According to the definition of the Ornstein-Uhlenbeck process, if $\bar{\varphi}_1 < 0$, then $\lim\limits_{t\to\infty} x(t) = 0$. Similarly, when $\bar{\varphi}_2 + \frac{c_1 m_1}{a_1} < 0$ and $\bar{\varphi}_3 + \frac{c_2 m_2}{a_2} < 0$, it follows that $\lim\limits_{t\to\infty} y(t) = 0$ and $\lim\limits_{t\to\infty} z(t) = 0$. Theorem 3.4 is proved.

4. Numerical simulations

In this section, we perform numerical simulations to confirm our results. By applying the Euler-Maruyama method [44], we derive the discretized form of system (1.5) as follows:

$$\begin{cases} x^{j+1} = x^j + x^j\left(g_1^j\left(1 - \frac{x^j}{K}\right) - \frac{m_1 y^j\left(1-\delta y^j\right)}{1+a_1 x^j\left(1-\delta y^j\right)}\right)\Delta t, \\ y^{j+1} = y^j + y^j\left(-e^{g_2^j} + \frac{c_1 m_1 x^j\left(1-\delta y^j\right)}{1+a_1 x^j\left(1-\delta y^j\right)} - \frac{m_2 z^j}{1+a_2 y^j}\right)\Delta t, \\ z^{j+1} = z^j + z^j\left(-e^{g_3^j} + \frac{c_2 m_2 y^j z^j}{\left(1+a_2 y^j\right)\left(z^j+h\right)}\right)\Delta t, \\ g_1^{j+1} = g_1^j + \beta_1\left(\bar{g}_1 - g_1^j\right)\Delta t + \sigma_1 \sqrt{\Delta t}\,\omega_j, \\ g_2^{j+1} = g_2^j + \beta_2\left(\bar{g}_2 - g_2^j\right)\Delta t + \sigma_2 \sqrt{\Delta t}\,\xi_j, \\ g_3^{j+1} = g_3^j + \beta_3\left(\bar{g}_3 - g_3^j\right)\Delta t + \sigma_3 \sqrt{\Delta t}\,\zeta_j. \end{cases} \tag{4.1}$$

When $\Delta t > 0$ represents the time increment, ω_j, ξ_j, ζ_j are three independent stochastic variables following the standard Gaussian distribution $\mathbb{N}(0,1)$. Additionally, (x^j, y^j, z^j, g_i^j) corresponds to the values obtained at the jth iteration of the discretized Eq (4.1), where $i = 1,2,3$ and $j = 1,2,\ldots$. We use some combinations of biological parameters in Tables 1 and 2.

Table 1. List of biological parameters in the system (1.5).

Parameter	Description
$\bar{g}_1$	Average growth rate of Prey
$\bar{g}_2$	Average log-transformed natural mortality rate of the intermediate predator
$\bar{g}_3$	Average log-transformed natural mortality rate of the top predator
K	Environmental carrying capacity of the prey
m_1	Search rate of the intermediate predator
m_2	Search rate of the top predator
c_1	Conversion efficiency of the intermediate predator
c_2	Conversion efficiency of the top predator
a_1	Half-saturation constant for the prey
a_2	Half-saturation constant for the intermediate predator
h	Allee effect constant
δ	Prey refuge coefficient
β_1	The reversion speed of g_1
β_2	The reversion speed of g_2
β_3	The reversion speed of g_3
σ_1	The intensity of volatility of g_1
σ_2	The intensity of volatility of g_2
σ_3	The intensity of volatility of g_3

Based on the biological significance of the parameters in Table 1 and Jørgensen's real datasets [45], data on phytoplankton, zooplankton, and fish are selected, respectively, to form Table 2. Their clear predator-prey relationships reflect the energy transfer and interactions in the food chain model, making them suitable for verifying the properties proven earlier.

Table 2. Several combinations of biological parameters of system (1.5) in Table 1.

Combinations	Value
$(\mathcal{A}_1)$	$\bar{g}_1 = 0.32, \bar{g}_2 = -2.4, \bar{g}_3 = -4.4, K = 5, m_1 = 0.6, m_2 = 0.2, c_1 = 0.3, c_2 = 0.3, a_1 = 0.1,$ $a_2 = 0.12, h = 0.46, \delta = 0.2, \beta_1 = 0.5, \beta_2 = 0.5, \beta_3 = 0.5, \sigma_1 = 0.04, \sigma_2 = 0.04, \sigma_3 = 0.04$
$(\mathcal{A}_2)$	$\bar{g}_1 = 0.63, \bar{g}_2 = -3.5, \bar{g}_3 = -3.9, K = 4, m_1 = 1, m_2 = 0.2, c_1 = 0.7, c_2 = 0.2, a_1 = 0.02,$ $a_2 = 0.16, h = 0.1, \delta = 0.08, \beta_1 = 0.5, \beta_2 = 0.5, \beta_3 = 0.5, \sigma_1 = 0.04, \sigma_2 = 0.04, \sigma_3 = 0.04$
$(\mathcal{A}_3)$	$\bar{g}_1 = 0.58, \bar{g}_2 = -3.6, \bar{g}_3 - 3.2, K = 4, m_1 = 0.6, m_2 = 0.3, c_1 = 0.47, c_2 = 0.3, a_1 = 0.016,$ $a_2 = 0.1, h = 0.4, \delta = 0.16, \beta_1 = 0.5, \beta_2 = 0.5, \beta_3 = 0.5, \sigma_1 = 0.04, \sigma_2 = 0.04, \sigma_3 = 0.04$
$(\mathcal{A}_4)$	$\bar{g}_1 = -0.02,\ \bar{g}_2 = -4.6,\ \bar{g}_3 = -5.6,\ K = 4,\ m_1 = 0.6,\ m_2 = 0.2,\ c_1 = 0.4,\ c_2 = 0.2,\ a_1 = 0.1,$ $a_2 = 0.12,\ h = 0.06,\ \delta = 0.2,\ \beta_1 = 0.5,\ \beta_2 = 0.5,\ \beta_3 = 0.5,\ \sigma_1 = 0.04,\ \sigma_2 = 0.04,\ \sigma_3 = 0.04$
$(\mathcal{A}_5)$	$\bar{g}_1 = 0.12,\ \bar{g}_2 = -0.3,\ \bar{g}_3 = -7.6,\ K = 3,\ m_1 = 0.4,\ m_2 = 0.2,\ c_1 = 0.9,\ c_2 = 0.6,\ a_1 = 0.04,$ $a_2 = 0.4,\ h = 1.2,\ \delta = 0.9,\ \beta_1 = 0.5,\ \beta_2 = 0.5,\ \beta_3 = 0.5,\ \sigma_1 = 0.04,\ \sigma_2 = 0.04,\ \sigma_3 = 0.04$
$(\mathcal{A}_6)$	$\bar{g}_1 = 0.51,\ \bar{g}_2 = -1.2,\ \bar{g}_3 = -1.9,\ K = 4,\ m_1 = 0.6,\ m_2 = 0.1,\ c_1 = 0.6,\ c_2 = 0.1,\ a_1 = 0.02,$ $a_2 = 0.4,\ h = 0.5,\ \delta = 0.3,\ \beta_1 = 0.5,\ \beta_2 = 0.5,\ \beta_3 = 0.5,\ \sigma_1 = 0.04,\ \sigma_2 = 0.04,\ \sigma_3 = 0.04$

Example 4.1. *In Table 2, the parameter set* $(\mathscr{A}_1) - (\mathscr{A}_3)$ *is selected to represent the biological parameters of the system* (1.5)*. According to Theorem 3.1, the system* (1.5) *admits a unique global solution. By setting the maximum number of iterations to* $T_{\max} = 2000$*, the results shown in Figure 1 are obtained.*

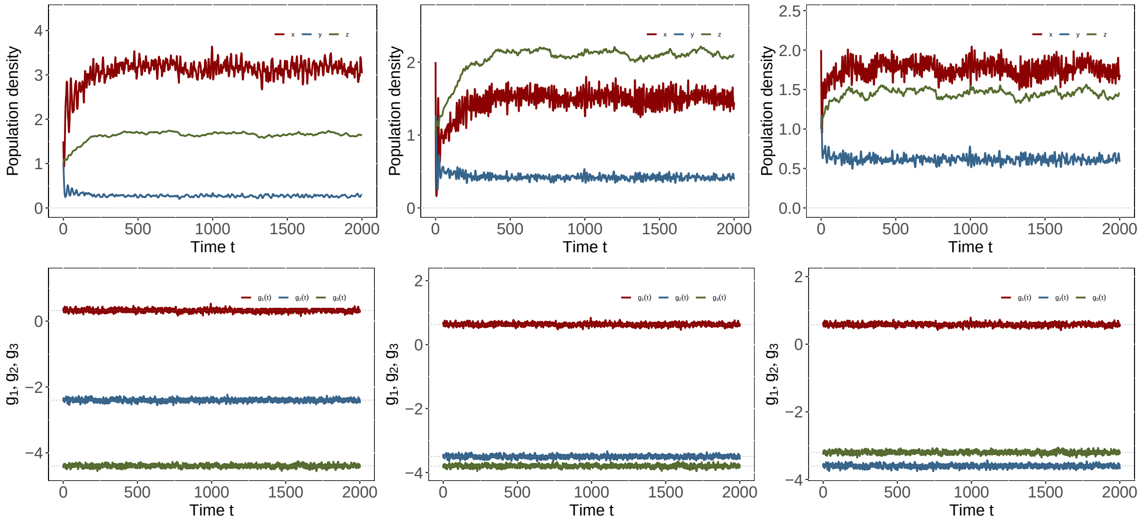

Figure 1. Numerical simulations are conducted to analyze the dynamics of prey, predator, and top predator in system (1.5). The graphical results support Theorem 3.1, indicating that system (1.5) admits a unique global solution. The parameters are determined by the combination $(\mathscr{A}_1) - (\mathscr{A}_3)$.

Remark 4.1. *Figure 1 shows that the growth rate of* x *and the mortality rates of* y *and* z *fluctuate around their mean values, reflecting the mean-reverting nature of the Ornstein–Uhlenbeck process. Moreover, different coefficient combinations yield distinct solutions, all of which are existing and unique. These results support the conclusion of Theorem 3.1.*

Example 4.2. *To more clearly verify the existence and uniqueness of the solution to system* (1.5)*, we conduct 100 simulations based on Theorem 3.1. As illustrated in Figure 2, all 100 simulation paths are generated under the same initial conditions and parameter settings, with slight stochastic perturbations. The simulation time is set to* $T_{\max} = 2000$.

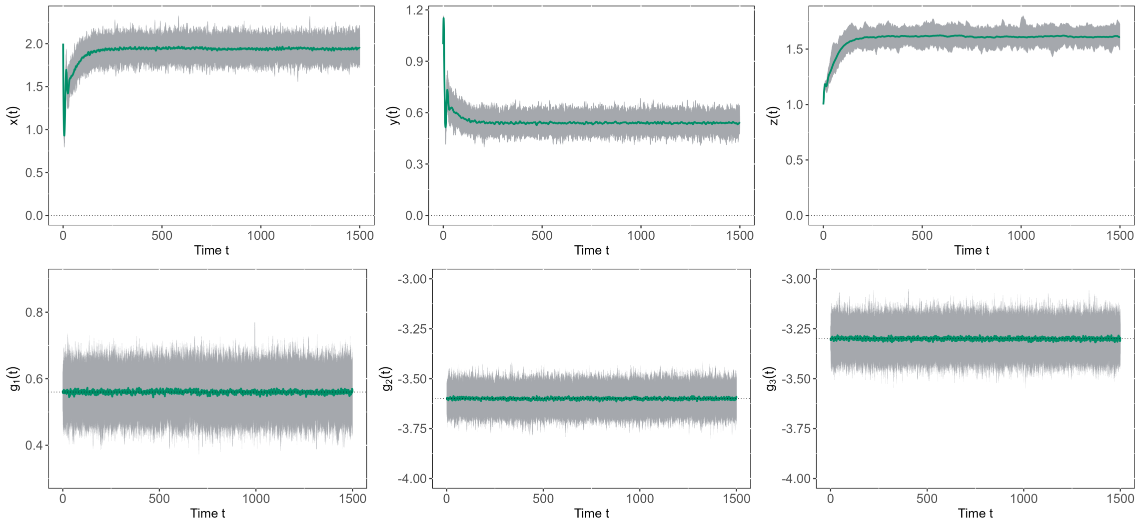

Figure 2. 100 path simulation figures.

Remark 4.2. *In Figure 2, the 100 simulation paths are displayed as gray lines, and the green solid line represents their average trajectory. It is observable that varying coefficient combinations result in distinct solutions, and each solution exists uniquely. This confirms the conclusion of Theorem 3.1.*

Example 4.3. *In Table 2, the parameter set* $(\mathcal{A}_1) - (\mathcal{A}_3)$ *is employed to specify the biological parameters of system* (1.5)*. According to Lemma 3.1, the qth order of the solutions to the system* (1.5) *are bounded, and the solutions themselves are ultimately bounded as well. By setting the total number of iterations to* $T_{\max} = 2000$*, the results presented in Figure 3 are obtained.*

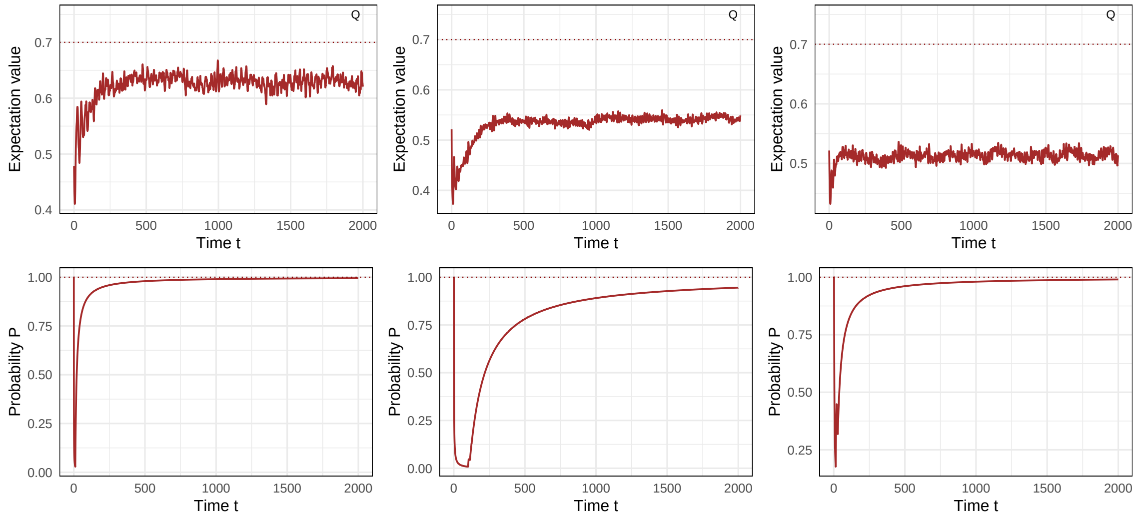

Figure 3. Numerical simulations are conducted to investigate the *qth* order moment dynamics of system (1.5). The results indicate that the solution remains bounded, confirming the ultimate boundedness of the system. The relevant parameters are selected based on the combination $(\mathcal{A}_1) - (\mathcal{A}_3)$.

Remark 4.3. *Figure 3 illustrates that the expected values associated with the three coefficient combinations* $(\mathcal{A}_1) - (\mathcal{A}_3)$ *lie below the finite upper bound* $Q(q)$*. As time* t *increases, the probability* P *tends to stabilize, eventually exceeding a fixed threshold. In particular, we have* $\limsup_{t\to\infty} P\left(\sqrt{x^2+y^2+z^2} \le \varpi\right) \ge 1-\varepsilon$*. This observation confirms the validity of Theorem 3.2. From a biological perspective, this is consistent with the fact that environmental resources are limited, preventing any population from growing indefinitely.*

Example 4.4. *In Table 2, parameter combination* $(\mathcal{A}_3)$ *is used to define the biological parameters of system* (1.5)*. Theorem 3.3 guarantees the existence of a stationary distribution for its solutions. Setting the iteration number to* $T = 2000$*, the simulation results shown in Figure 4 are obtained.*

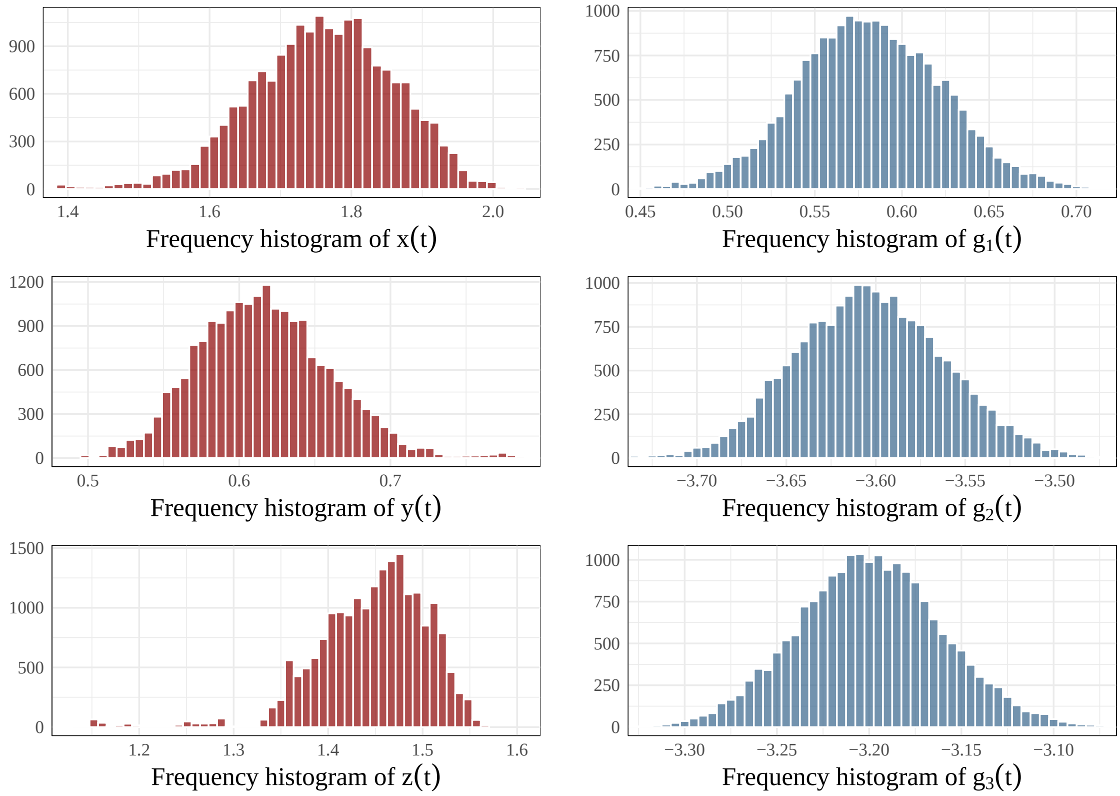

Figure 4. Numerical simulations are performed to examine the stationary distribution of system (1.5). As shown in Figure 4, the population $x(t)$ primarily ranges from 1.7 to 1.9, $y(t)$ from 0.55 to 0.65, and $z(t)$ from 1.4 to 1.5, with all populations concentrated in their respective middle ranges. The frequency histograms of these populations also exhibit a pattern of being high in the middle and low at both ends, approximating a normal distribution. This suggests that despite random environmental disturbances, the populations' growth states will stabilize. These results are based on parameters determined by combination $(\mathcal{A}_3)$.

Example 4.5. *To further verify Theorems 3.2 and 3.3, we conduct 100 path simulations using the same method as in the verification of Theorem 3.1, as shown in Figure 5. The left panel shows the simulation results for Theorem 3.2, and the right panel for Theorem 3.3. It is evident that both theorems hold true.*

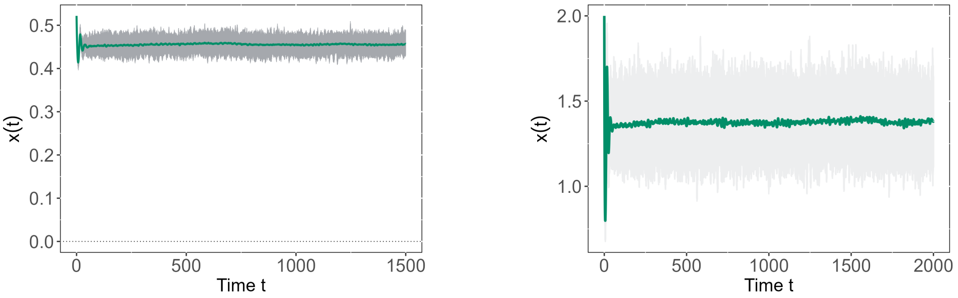

Figure 5. 100 path simulation figures.

Example 4.6. *In Table 2, the parameter combination* $(\mathcal{A}_4) - (\mathcal{A}_6)$ *is used to define the biological parameters of system* (1.5)*. As shown in Theorem 3.4, system* (1.5) *exhibits extinction. By setting the maximum number of iterations to* $T_{\max} = 2000$*, we obtain Figure 6.*

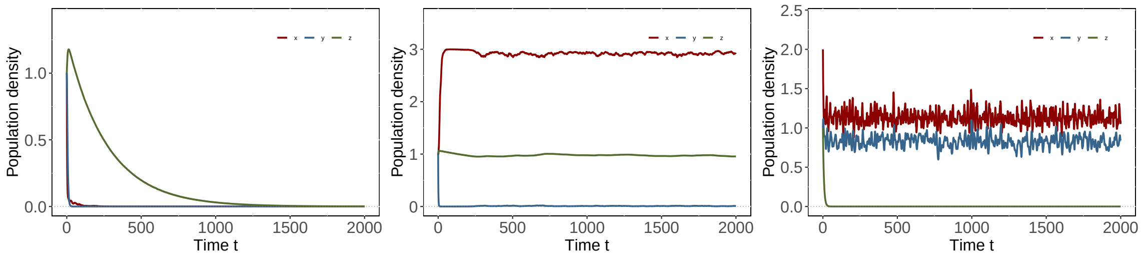

Figure 6. Numerical simulations are conducted to investigate the extinction of system (1.5). When $\bar{\varphi}_1 < 0$, the population of x becomes extinct, and when $\bar{\varphi}_2 + \frac{c_1 m_1}{a_1} < 0$, the population of y becomes extinct. Additionally, when $\bar{\varphi}_3 + \frac{c_2 m_2}{a_2} < 0$, the population of z becomes extinct. The relevant parameters are determined by the combination $(\mathcal{A}_4) - (\mathcal{A}_6)$.

5. Conclusions

In this paper, we primarily investigate the mathematical properties of a food chain model driven by the Ornstein-Uhlenbeck process, which incorporates the Holling-II functional response, nonlinear prey refuge, and the Allee effect in the top predator. First, the biological significance of the Ornstein-Uhlenbeck process is clarified, and its rationality is explained. Next, the existence and uniqueness of the global solution of the model are established, and its ultimate boundedness is studied. Then, by constructing a Lyapunov function and applying Itô's formula, the existence of the model's stationary distribution is proven. Furthermore, we conclude that the population will go extinct when the conditions $\bar{\varphi}_1 < 0$, $\bar{\varphi}_2 + \frac{c_1 m_1}{a_1} < 0$, and $\bar{\varphi}_3 + \frac{c_2 m_2}{a_2} < 0$ are satisfied. Finally, numerical simulations are conducted to verify the theoretical results and confirm the validity of the conclusions.

In conclusion, we provide an in-depth theoretical analysis of a food chain model based on the Ornstein-Uhlenbeck process, revealing the diversity and complexity of population dynamics when environmental randomness and ecological complexities are considered. By incorporating nonlinear effects and stochastic disturbances, the model more realistically reflects the random fluctuations in the ecosystem and the interactions between species, helping to identify potential extinction thresholds for populations. These findings can inform the design of effective intervention measures, such as habitat management or species conservation strategies, to support the long-term survival of vulnerable species. Overall, this research offers a new theoretical framework for ecological conservation and species management. However, the model needs improvement. Moreover, we employ the Ornstein-Uhlenbeck process to model environmental stochasticity, as its mean-reverting property and short-term dependence align with empirical observations of ecological fluctuations. In future work, researchers may incorporate semi-martingale, time delays, or lévy jumps to model abrupt changes or ecological lags, enhancing realism for extreme events or interaction delays.

Author contributions

Shujie Yang and Xiaohui Ai: Conceptualization, writing-original draft; Binghui Zhao: Software, visualization. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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