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Research article

Stably continuous semilattices in closure spaces

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Abstract: In this paper, we introduce the concept of S-closure spaces and demonstrate that they precisely generate stably continuous semilattices. Additionally, we define the notion of S-morphisms between S-closure spaces to represent Scott continuous functions between stably continuous semilattices. These developments establish an equivalence between the category of stably continuous semilattices and the category of S-closure spaces with S-morphisms as the morphisms. This result provides a method for representing stably continuous semilattices through the framework of closure spaces.

Keywords: categorical equivalence; stably continuous semilattices; generalized directed sets; S-closure spaces
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1. Introduction

Closure spaces are mathematical structures that formalize the concept of closure within a set. A closure space (Y, τ) consists of a set Y and an associated closure operator τ, where a closure operator is an expansive, monotonic, idempotent map on the powerset of Y. Closure spaces play a crucial role in characterizing various types of lattices [1–3]. Early foundational results can be traced back to Birkhoff's [4] representation theorem for finite distributive lattices and Stone's [5] duality for Boolean algebras. These seminal works have inspired extensive research into the connections between lattices and closure spaces, as seen in [6–9]. For instance, Winskel [10] demonstrated that completely distributive algebraic lattices are isomorphic to complete rings of sets. Edelman [11] introduced the notion of anti-exchange closures and showed that the closed sets of an anti-exchange closure generate a meet-distributive lattice under the inclusion order. Conversely, every meet-distributive lattice can be derived in this manner. Erné [12] conducted a systematic categorical study on representing various complete lattices using closure spaces. Guo and Li [13] introduced the concept of F-augmented closure spaces by enriching closure spaces with additional structures, thereby enabling the representation of algebraic domains. Recently, Li et al. [14] proposed the concept of continuous closure spaces and established a categorical equivalence with that of continuous domains. Wang et al. [15] further developed interpolative generalized closure spaces, which provide a framework for capturing continuous domains.

In this paper, we introduce the notion of S-closure spaces, which offers a novel method for representing stably continuous semilattices. The paper is organized as follows. Section 2 provides the necessary definitions and foundational results from domain theory. In Section 3, we propose the concept of S-closure spaces by augmenting interpolative generalized closure spaces with an additional condition. We then prove that S-closure spaces precisely characterize stably continuous semilattices and vice versa. In Section 4, we define S-morphisms between S-closure spaces, which leads to an equivalent category to that of stably continuous semilattices with Scott-continuous functions.

2. Preliminaries

Given a set A, we write $F \sqsubseteq A$ to indicate that F is a finite subset of A. Suppose $(L, \leq)$ is a poset. A subset E of L is directed, if it is non-empty and every pair of elements of E has an upper bound within E. We denote the least upper bound of E by $\bigvee E$ and the greatest lower bound of E by $\bigwedge E$. L is called a semilattice if it has a meet $x \wedge y$ for any $x, y \in L$. A poset L is a dcpo if every directed subset $E \subseteq L$ has a least upper bound $\bigvee E$ in L. Furthermore, L is a complete lattice if every subset of L has a sup in L.

Let L be a dcpo. We say x is way below y, denoted $x \ll y$, if for every directed subset E of L, then $y \leq \bigvee E$ implies $x \leq e$ for some $e \in E$. For any subset $X \subseteq L$, we write $\Downarrow X = \{b \in L \mid (\exists a \in X) b \ll a\}$. And $\Downarrow a$ for $\Downarrow\{a\}$. A dcpo L is called a continuous domain, if for every element $x \in L$, there exists a directed subset $E \subseteq \Downarrow x$ with $x = \bigvee E$. A continuous domain L that is also a complete lattice is called a continuous lattice. A dcpo is called a continuous semilattice if it is both a continuous domain and a semilattice. The way below relation $\ll$ in L is called multiplicative if $x \ll y, z$ implies $x \ll y \wedge z$. In this case, L is called a stably continuous semilattice.

Definition 2.1. [16] A function $h : L \to L'$ between dcpos L and L' is called Scott continuous if for all directed subsets E of L, we have $h(\bigvee E) = \bigvee_{e \in E} h(e)$.

Definition 2.2. [12] A closure space is a pair (Y, τ) consisting of a set Y and a closure operator τ on Y such that for any $B, B' \subseteq Y$,

(1) expansive: $B \subseteq \tau(B)$,
(2) monotonic: $B \subseteq B' \Rightarrow \tau(B) \subseteq \tau(B')$,
(3) idempotent: $\tau(B) = \tau(\tau(B))$.

Definition 2.3. [15] A generalized closure space is a pair $(Y, \langle\cdot\rangle)$ consisting of a set Y equipped with an operation $\langle\cdot\rangle : \mathcal{P}(Y) \to \mathcal{P}(Y)$ which is

(1) monotonic: $\langle B \rangle \subseteq \langle B' \rangle$ whenever $B \subseteq B' \subseteq Y$;
(2) sub-idempotent: $\langle\langle B \rangle\rangle \subseteq \langle B \rangle$ for all $B \subseteq Y$.

Definition 2.4. [15] Let $(Y, \langle\cdot\rangle)$ be a generalized closure space. Then $(Y, \langle\cdot\rangle)$ is said to be an interpolative generalized closure space (for short, an IG-closure space) provided that:

(In) $(\forall y \in Y, M \sqsubseteq \langle y\rangle) \Rightarrow (\exists z \in \langle y\rangle)(M \sqsubseteq \langle z\rangle)$.

Definition 2.5. [15] Suppose $(Y, \langle\cdot\rangle)$ is an IG-closure space. A subset U of Y is called a generalized directed set of $(Y, \langle\cdot\rangle)$ if, for any $M \sqsubseteq U$, there exists $y \in U$ such that $M \sqsubseteq \langle y\rangle \subseteq U$.

In the sequels, we use $\mathcal{G}(Y)$ to denote the family of all generalized directed sets of $(Y, \langle\cdot\rangle)$.

Theorem 2.6. *[15] Suppose $(Y, \langle\cdot\rangle)$ is an IG-closure space. Then $(\mathcal{G}(Y), \subseteq)$ is a continuous domain.*

By the proof of Wang and Li, ([15], Proposition 3.14.). Let $(L, \leqslant)$ be a continuous domain. Define

$$\langle A\rangle = \mathop{\downarrow\!\!\!\downarrow} A,$$

for all $A \subseteq L$. The following conclusion holds.

Theorem 2.7. *[15] Let $(L, \leqslant)$ be a continuous domain. Then $(L, \langle\cdot\rangle)$ is an IG-closure space, and $(L, \leqslant)$ is order-isomorphic to $(\mathcal{G}(L), \subseteq)$.*

3. S-closure space

In this section, we introduce the notion of an S-closure space in order to represent stably continuous semilattices.

Definition 3.1. An IG-closure space $(Y, \langle\cdot\rangle)$ is called an S-closure space, if, for any $y, z, y', z' \in Y$, the following hold:

(SC) $(\forall M \subseteq Y)(M \subseteq \langle y'\rangle, y' \in \langle y\rangle)(M \subseteq \langle z'\rangle, z' \in \langle z\rangle) \Rightarrow (\exists s \in Y)(M \subseteq \langle s\rangle, s \in \langle y\rangle, s \in \langle z\rangle)$.

Example 3.2. Consider a poset $(L, \leq)$ in Figure 1.

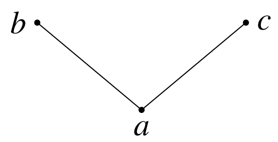

Figure 1. Hasse diagram of the poset L.

For all $A \subseteq L$, define

$$\langle A\rangle = \{x \in L \mid \exists a \in A, x \ll a\}.$$

Then it is easy to see that $(L, \langle\cdot\rangle)$ is an S-closure space.

Proposition 3.3. *Consider an S-closure space $(Y, \langle\cdot\rangle)$. Then $\langle A\rangle \subseteq \langle A'\rangle$ for any $A \subseteq \langle A'\rangle$.*

Proof. If $A \subseteq \langle A'\rangle$, then $\langle A\rangle \subseteq \langle\langle A'\rangle\rangle$ by part (1) of Definition 2.3. Again using part (2) of Definition 2.3, we have $\langle A\rangle \subseteq \langle A'\rangle$. □

Proposition 3.4. *Given an S-closure space $(Y, \langle\cdot\rangle)$.*

(1) For any $E \in \mathcal{G}(Y)$, it holds that $\{\langle y\rangle \mid y \in E\}$ is directed and

$$E = \bigcup\{\langle y\rangle \mid y \in E\}.$$

(2) Suppose $\{E_l \mid l \in L\}$ is a directed subset of $\mathcal{G}(Y)$, then

$$\bigvee_{l\in L} E_l = \bigcup_{l\in L} E_l.$$

(3) For any $E_1, E_2 \in \mathcal{G}(Y)$,

$$E_1 \ll E_2 \Leftrightarrow (\exists y \in E_2)E_1 \subseteq \langle y\rangle .$$

(4) For any $y \in Y$, $\langle y\rangle$ is an generalized directed set of $(Y, \langle\cdot\rangle)$,

$$\langle y\rangle = \bigcup\{\langle z\rangle \mid z \in \langle y\rangle\}.$$

Proof. (1) Given $y_1, y_2 \in E$, then $\{y_1, y_2\} \sqsubseteq E$. By Definition 2.5, there is $y \in E$ with $\{y_1, y_2\} \sqsubseteq \langle y\rangle \subseteq E$; from Proposition 3.3, it follows that $\langle y_1\rangle , \langle y_2\rangle \subseteq \langle y\rangle$. Hence the family $\{\langle y\rangle \mid y \in E\}$ is directed. Suppose $e \in E$; then $e \in \langle y\rangle \subseteq E$ for some $y \in E$ by Definition 2.5, which implies that $E \subseteq \bigcup\{\langle y\rangle \mid y \in E\}$. On the contrary, for any $e \in E$, $\langle e\rangle \subseteq E$. Therefore, $\bigcup\{\langle e\rangle \mid e \in E\} \subseteq E$.

(2) We only need to verify that $\bigcup_{l\in L} E_l \in \mathcal{G}(Y)$. For any $M \sqsubseteq \bigcup_{l\in L} E_l$, as $\{E_l \mid l \in L\}$ is directed and M is finite, we obtain that $M \sqsubseteq E_{l_0}$ for some $l_0 \in L$. For $M \sqsubseteq E_{l_0}$, by Definition 2.5, there is $y \in E_{l_0}$ such that $M \sqsubseteq \langle y\rangle \subseteq E_{l_0}$. Therefore, $\bigcup_{l\in L} E_l \in \mathcal{G}(Y)$. As a consequence, $\bigvee_{l\in L} E_l = \bigcup_{l\in L} E_l$.

(3) Suppose $E_2 \in \mathcal{G}(Y)$, then the family $\{\langle y\rangle \mid y \in E_2\}$ is directed, and $E_2 = \bigcup\{\langle y\rangle \mid y \in E_2\}$ by part (1). If $E_1 \ll E_2$, then there is $y \in E_2$ with $E_1 \subseteq \langle y\rangle$. Conversely, suppose $E_1 \subseteq \langle y\rangle$ for some $y \in E_2$. Let $\{S_l \mid l \in L\}$ be a directed subset of $\mathcal{G}(Y)$ and $E_2 \subseteq \bigvee_{l\in L} S_l$. Since $\bigvee_{l\in L} S_l = \bigcup_{l\in L} S_l$, there exists S_{l_0} such that $y \in S_{l_0}$ for some $l_0 \in L$, which implies that $E_1 \subseteq \langle y\rangle \subseteq S_{l_0}$. Thus $E_1 \ll E_2$.

(4) By Definition 2.4 and Proposition 3.3, we obtain that $\langle y\rangle$ is a generalized directed set of $(Y, \langle\cdot\rangle)$. From part (1), $\langle y\rangle = \bigcup\{\langle z\rangle \mid z \in \langle y\rangle\}$ follows. □

Proposition 3.5. *Let $(Y, \langle\cdot\rangle)$ be an S-closure space. For any $y, z \in Y$, the following holds:*

(SC1) $(\forall F \sqsubseteq Y)(F \sqsubseteq \langle y\rangle , F \sqsubseteq \langle z\rangle) \Rightarrow (\exists s \in Y)(F \sqsubseteq \langle s\rangle , s \in \langle y\rangle , s \in \langle z\rangle)$.

Proof. Suppose $F \sqsubseteq Y, F \sqsubseteq \langle y\rangle$, and $F \sqsubseteq \langle z\rangle$. By part (4) of Proposition 3.4, there are $u \in \langle y\rangle$ and $v \in \langle z\rangle$ with $F \sqsubseteq \langle u\rangle$ and $F \sqsubseteq \langle v\rangle$. According to Definition 3.1, then there is $s \in Y$ satisfying $F \sqsubseteq \langle s\rangle , s \in \langle y\rangle$, and $s \in \langle z\rangle$. Thus condition (SC1) holds. □

Proposition 3.6. *Consider an S-closure space $(Y, \langle\cdot\rangle)$. Then $E_1 \cap E_2 \in \mathcal{G}(Y)$ for any $E_1, E_2 \in \mathcal{G}(Y)$.*

Proof. Since $\emptyset \sqsubseteq E_1$ and $\emptyset \sqsubseteq E_2$, by part (1) of Proposition 3.4, there are $y \in E_1$ and $z \in E_2$ with $\emptyset \sqsubseteq \langle y\rangle$ and $\emptyset \sqsubseteq \langle z\rangle$. From Proposition 3.5, we have $\emptyset \sqsubseteq \langle s\rangle , s \in \langle y\rangle$, and $s \in \langle z\rangle$ for some $s \in Y$. Thus $s \in \langle y\rangle \cap \langle z\rangle \subseteq E_1 \cap E_2 \neq \emptyset$. Now, we prove that $E_1 \cap E_2$ satisfies condition (SG). For any $M \sqsubseteq E_1 \cap E_2$, as E_1 and E_2 are generalized directed sets, there exist $z \in E_1, t \in E_2$ with $M \sqsubseteq \langle z\rangle \subseteq E_1$ and $M \sqsubseteq \langle t\rangle \subseteq E_2$. Again using Proposition 3.5, we obtain that $M \sqsubseteq \langle e\rangle , e \in \langle z\rangle$, and $e \in \langle t\rangle$ for some $e \in Y$. It follows that $M \sqsubseteq \langle e\rangle \subseteq E_1 \cap E_2$ and $e \in E_1 \cap E_2$. Therefore, $E_1 \cap E_2 \in \mathcal{G}(Y)$. □

Theorem 3.7. *Given an S-closure space $(Y, \langle\cdot\rangle)$. Then $(\mathcal{G}(Y), \subseteq)$ is an stably continuous semilattice.*

Proof. First, we claim that $(\mathcal{G}(Y), \subseteq)$ is a continuous domain. Assume $E \in \mathcal{G}(Y)$. By Proposition 3.4, then $E = \bigcup\{\langle y \rangle \mid y \in E\}$, and the family $\{\langle y \rangle \mid y \in E\}$ is directed. For any $y \in E$, it is obvious that $\langle y \rangle \ll E$ by part (3) of Proposition 3.4. Thus $E = \bigvee\{\langle y \rangle \mid y \in E\}$. As a result, $(\mathcal{G}(Y), \subseteq)$ is a continuous domain.

Next, suppose $E_1, E_2 \in \mathcal{G}(Y)$. From Proposition 3.6, we have $E_1 \wedge E_2 = E_1 \cap E_2 \in \mathcal{G}(Y)$. Thus, $(\mathcal{G}(Y), \subseteq)$ is a semilattice.

Finally, we claim that the way-below relation $\ll$ is multiplicative. Suppose $E, E_1, E_2 \in \mathcal{G}(Y)$ with $E \ll E_1, E_2$, by part (3) of Proposition 3.4, we have $E \subseteq \langle t \rangle$ and $E \subseteq \langle s \rangle$ for some $t \in E_1, s \in E_2$. For $t \in E_1$ and $s \in E_2$, from Definition 2.5, there are $t' \in E_1$ and $s' \in E_2$ such that $t \in \langle t' \rangle \subseteq E_1$ and $s \in \langle s' \rangle \subseteq E_2$. Then, we obtain that $E \subseteq \langle e \rangle, e \in \langle t' \rangle$ and $e \in \langle s' \rangle$ for some $e \in Y$ by Definition 3.1. It implies that $E \subseteq \langle e \rangle$ and $e \in \langle t' \rangle \cap \langle s' \rangle \subseteq E_1 \cap E_2$. Thus $E \ll E_1 \wedge E_2$. □

Theorem 3.8. *Let $(L, \leqslant)$ be a stably continuous semilattice. Then there exists an S-closure space $(L, \langle \cdot \rangle)$ such that $L \cong \mathcal{G}(L)$.*

Proof. Suppose $(L, \leqslant)$ is a stably continuous semilattice. For any $B \subseteq L$, we define

$$\langle B \rangle = \mathop{\downarrow\!\!\!\downarrow} B.$$

It is routine to show that $(L, \langle \cdot \rangle)$ is a generalized closure space. We now prove that $(L, \langle \cdot \rangle)$ satisfies condition (In). For any $e \in L$ and $N \sqsubseteq \langle e \rangle$, we have $n \ll e$ for any $n \in N$. As $\mathop{\downarrow\!\!\!\downarrow} e$ is directed and N is finite, there is $z \in \mathop{\downarrow\!\!\!\downarrow} e$ with $N \sqsubseteq \mathop{\downarrow\!\!\!\downarrow} z$. Therefore, $N \sqsubseteq \langle z \rangle$ for some $z \in \langle e \rangle$. Thus, condition (In) follows.

Next, we check that $(L, \langle \cdot \rangle)$ is an S-closure space. From Definition 3.1, for any $u, v, u', v' \in L$ with $M \subseteq L$; suppose $M \subseteq \langle u' \rangle, M \subseteq \langle v' \rangle, u' \in \langle u \rangle$, and $v' \in \langle v \rangle$. Then $u' \ll u$ and $v' \ll v$ by the definition of $\langle \cdot \rangle$. Since the way-below relation $\ll$ is multiplicative, then $u' \wedge v' \ll u \wedge v$, it follows that $u' \wedge v' \ll s \ll u \wedge v$ for some $s \in L$. As a result, then $M \subseteq \langle s \rangle, s \in \langle u \rangle$, and $s \in \langle v \rangle$. Therefore, $(L, \langle \cdot \rangle)$ is an S-closure space.

Define

$$\phi : L \to \mathcal{G}(L), y \to \mathop{\downarrow\!\!\!\downarrow} y,$$

$$\psi : \mathcal{G}(L) \to L, E \to \bigvee E.$$

It is obvious that ϕ and ψ are well-defined, order-preserving and mutually inverse. Hence, $(L, \leqslant)$ is order isomorphic to $(\mathcal{G}(L), \subseteq)$. □

4. Categorical equivalence

In this section, we study the categorical equivalence between stably continuous semilattices and S-closure spaces.

Definition 4.1. Let $(Y, \langle \cdot \rangle)$ and $(Y', \langle \cdot \rangle')$ be two S-closure spaces. A relation $\Phi \subseteq Y \times Y'$ is an S-morphism from $(Y, \langle \cdot \rangle)$ to $(Y', \langle \cdot \rangle')$ if the following conditions hold:

(1) $y \in Y \Rightarrow (\exists y' \in Y') y \Phi y'$;
(2) $y \Phi y' \Rightarrow (\forall z' \in \langle y' \rangle') y \Phi z'$;
(3) $y \Phi y', y \in \langle z \rangle \Rightarrow z \Phi y'$;

(4) $y\Phi y', y' \in G' \sqsubseteq Y' \Rightarrow (\exists z \in \langle y \rangle, z' \in Y')(z\Phi z', G' \sqsubseteq \langle z' \rangle')$.

Proposition 4.2. *Let $\Phi : (Y, \langle\cdot\rangle) \to (Y', \langle\cdot\rangle')$ be an S-morphism. For any $y \in Y$ and $F \sqsubseteq Y'$:*

(1) $y\Phi F \Leftrightarrow (\exists z \in \langle y \rangle) z\Phi F$,
(2) $y\Phi F \Rightarrow (\exists z' \in Y')(y\Phi z', F \subseteq \langle z' \rangle')$,

where $y\Phi F$ means that $y\Phi f$ for any $f \in F$.

Proof. (1) Suppose $y\Phi F$. Then by condition (4) of Definition 4.1, we have $z\Phi z'$ and $F \sqsubseteq \langle z' \rangle'$ for some $z \in \langle y \rangle$ and $z' \in Y'$. For $z\Phi z'$, from condition (2) of Definition 4.1, $z\Phi \langle z' \rangle'$ follows. As $F \sqsubseteq \langle z' \rangle'$, we have $z\Phi F$. Conversely, if $z\Phi F$ for some $z \in \langle y \rangle$, then $y\Phi F$ by condition (3) of Definition 4.1.

(2) Suppose $y\Phi F$. Then there are $z \in \langle y \rangle$ and $z' \in Y'$ satisfying $z\Phi z'$ and $F \sqsubseteq \langle z' \rangle'$ by condition (4) of Definition 4.1, which implies that $y\Phi z'$ by condition (3) of Definition 4.1. □

Theorem 4.3. *Consider two S-closure spaces $(Y, \langle\cdot\rangle)$ and $(Y', \langle\cdot\rangle')$.*

(1) Suppose $\Phi : (Y, \langle\cdot\rangle) \to (Y', \langle\cdot\rangle')$ is an S-morphism. Define a map $\phi_\Phi : \mathcal{G}(Y) \to \mathcal{G}(Y')$ by

$$\phi_\Phi(E) = \{y' \in Y' \mid (\exists y \in E) y\Phi y'\}.$$

Then ϕ_Φ is Scott continuous.

(2) Suppose $\phi : \mathcal{G}(Y) \to \mathcal{G}(Y')$ is Scott continuous. Define $\Phi_\phi \subseteq Y \times Y'$ by

$$y\Phi_\phi y' \Leftrightarrow y' \in \phi(\langle y \rangle).$$

Then Φ_ϕ is an S-morphism.

(3) Moreover, $\phi = \phi_{\Phi_\phi}, \Phi = \Phi_{\phi_\Phi}$.

Proof. (1) First, we claim that ϕ_Φ is well-defined. Assume that $G \sqsubseteq \phi_\Phi(E)$. For any $g \in G$, there is $y_g \in E$ such that $y_g \Phi g$. As $\bigcup_{g\in G}\{y_g\} \sqsubseteq E$ and $E \in \mathcal{G}(Y)$, it follows that $\bigcup_{g\in G}\{y_g\} \sqsubseteq \langle z \rangle \subseteq E$ for some $z \in E$, which implies that $z\Phi G$ by condition (3) of Definition 4.1. For $z\Phi G$, from condition (2) of Proposition 4.2, there exists $z' \in Y'$ with $z\Phi z'$ and $G \sqsubseteq \langle z' \rangle'$. Therefore, $G \sqsubseteq \langle z' \rangle' \subseteq \phi_\Phi(E)$, that is, $\phi_\Phi(E) \in \mathcal{G}(Y)$.

It is obvious that ϕ_Φ is monotone. Suppose $\{E_l \mid l \in L\}$ is a directed subset of $\mathcal{G}(Y)$; then $\{\phi_\Phi(E_l) \mid l \in L\}$ is a directed set of $\mathcal{G}(Y')$. From part (2) of Proposition 3.4, we have $\bigvee_{l\in L} E_l = \bigcup_{l\in L} E_l$ and $\bigvee_{l\in L} \phi_\Phi(E_l) = \bigcup_{l\in L} \phi_\Phi(E_l)$. Now, we show that $\bigcup_{l\in K} \phi_\Phi(E_l) = \phi_\Phi(\bigcup_{l\in L} E_l)$. Suppose $y' \in \phi_\Phi(\bigcup_{l\in L} E_l)$; then there exists $y \in \bigcup_{l\in L} E_l$ such that $y\Phi y'$, which implies that $y \in E_{l_0}$ for some $l_0 \in L$. Hence $y' \in \phi_\Phi(E_{l_0}) \subseteq \bigcup_{l\in L} \phi_\Phi(E_l)$. As a result, we have $\phi_\Phi(\bigcup_{l\in L} E_l) \subseteq \bigcup_{l\in L} \phi_\Phi(E_l)$. Conversely, $\bigcup_{l\in L} \phi_\Phi(E_l) \subseteq \phi_\Phi(\bigcup_{l\in L} E_l)$ is obvious. Therefore, ϕ_Φ is Scott continuous.

(2) We show that Φ_ϕ satisfies Definition 4.1.

For Condition (1): If $y \in Y$, then $\phi(\langle y \rangle) \in \mathcal{G}(Y')$. Since $\phi(\langle y \rangle) \neq \emptyset$, there is $y' \in \phi(\langle y \rangle)$, $y\Phi_\phi y'$ follows.

For Condition (2): If $y\Phi_\phi y'$, then $y' \in \phi(\langle y \rangle)$. As $\phi(\langle y \rangle) \in \mathcal{G}(Y')$, we have $\langle y' \rangle' \subseteq \phi(\langle y \rangle)$. Hence $y\Phi_\phi \langle y' \rangle'$.

For Condition (3): If $y\Phi_\phi y'$ and $y \in \langle z \rangle$, then $y' \in \phi(\langle y \rangle)$ with $\langle y \rangle \subseteq \langle z \rangle$. As ϕ is monotone, we have $y' \in \phi(\langle y \rangle) \subseteq \phi(\langle z \rangle)$, which implies that $z\Phi_\phi y'$.

For Condition (4): If $y\Phi_\phi y'$ and $y' \in G' \sqsubseteq Y'$, then $G' \sqsubseteq \phi(\langle y\rangle)$. As $\langle y\rangle = \bigcup\{\langle z\rangle \mid z \in \langle y\rangle\}$ and $\{\langle z\rangle \mid z \in \langle y\rangle\}$ is directed, we have $\phi(\langle y\rangle) = \phi(\bigcup\{\langle z\rangle \mid z \in \langle y\rangle\}) = \bigcup\{\phi(\langle z\rangle) \mid z \in \langle y\rangle\}$, which implies that $G' \sqsubseteq \phi(\langle z\rangle)$ for some $z \in \langle y\rangle$. From Definition 2.5, we obtain that $G' \sqsubseteq \langle z'\rangle' \subseteq \phi(\langle z\rangle)$ for some $z' \in \phi(\langle z\rangle)$. Therefore, $z\Phi_\phi z'$ and $G' \subseteq \langle z'\rangle'$ for some $z \in \langle y\rangle$ and $z' \in Y'$.

(3) For any $E \in \mathcal{G}(Y)$. Then

$$\begin{aligned}\phi_{\Phi_\phi}(E) &= \{s' \in Y' \mid (\exists s \in E)s\Phi_\phi s'\}\\ &= \{s' \in Y' \mid (\exists s \in E)s' \in \phi(\langle s\rangle)\}\\ &= \bigcup\{\phi(\langle s\rangle) \mid s \in E\}\\ &= \phi(\bigcup\{\langle s\rangle \mid s \in E\})\\ &= \phi(E).\end{aligned}$$

This implies $\phi = \phi_{\Phi_\phi}$.

For any $s \in Y$ and $s' \in Y'$, it follows that

$$\begin{aligned}s\Phi_{\phi_\Phi} s' &\Leftrightarrow s' \in \phi_\Phi(\langle s\rangle)\\ &\Leftrightarrow (\exists t \in \langle s\rangle)t\Phi s'\\ &\Leftrightarrow s\Phi s'.\end{aligned}$$

This implies $\Phi = \Phi_{\phi_\Phi}$. □

Theorem 4.4. *Let $(L, \leq)$ and $(L', \leq')$ be two stably continuous semilattices.*

(1) Suppose $\psi : L \to L'$ is Scott continuous. Define a relation Ω_ψ by

$$y\Omega_\psi y' \Leftrightarrow y' \ll' \psi(y).$$

Then Ω_ψ is an S-morphism.

(2) Suppose $\Omega : (L, \langle\cdot\rangle) \to (L', \langle\cdot\rangle')$ is an S-morphism. Define a function $\psi_\Omega : L \to L'$ by

$$\psi_\Omega(y) = \bigvee\{y' \in L' \mid (\exists z \in L)(z \in \mathord{\downarrow\!\!\downarrow} y, z\Omega y')\}.$$

Then ψ_Ω is Scott continuous.

(3) Moreover, $\psi = \psi_{\Omega_\psi}, \Omega = \Omega_{\psi_\Omega}$.

Proof. (1) We show that Ω_ψ is an S-morphism as follows.

For Condition (1): Assume that $y \in L$, then $\psi(y) \in L'$. As L' is a continuous domain, we have $\mathord{\downarrow\!\!\downarrow}\psi(y) \neq \emptyset$, which implies that there exists $y' \in \mathord{\downarrow\!\!\downarrow}\psi(y)$. $y\Omega_\psi y'$ follows.

For Condition (2): Suppose $y\Omega_\psi y'$, then $y' \ll' \psi(y)$. Because $\langle y'\rangle' = \mathord{\downarrow\!\!\downarrow} y'$, we have $\mathord{\downarrow\!\!\downarrow} y' \subseteq \mathord{\downarrow\!\!\downarrow}\psi(y)$. Hence $y\Omega_\psi z'$ for any $z' \in \langle y'\rangle'$.

For Condition (3): If $y\Omega_\psi y'$ and $y \in \langle z\rangle$, then we have $y \ll z$ with $y' \ll' \psi(y)$. As ψ is Scott continuous, we have $y' \ll' \psi(y) \ll' \psi(z)$. Therefore, $z\Omega_\psi y'$.

For Condition (4): Suppose $y\Omega_\psi y'$ and $y' \in G' \sqsubseteq Y'$. Then $y' \ll' \psi(y)$ for any $y' \in G'$, which implies that $G \ll' z' \ll' \psi(y)$ for some $z' \in L'$ by the interpolation property of $\ll'$. Note that $\psi(y) = \psi(\bigvee \mathord{\downarrow\!\!\downarrow} y)$, which means that $z' \ll' \psi(z)$ for some $z \ll y$. As a result, we have $z \in \langle y \rangle$ and $z' \in L'$ with $z\Omega_\psi z'$ and $G' \sqsubseteq \langle z' \rangle'$.

(2) As L' is a stably continuous semilattice, the function ψ_Ω is well-defined. For any $s, t \in L$ with $s \leq t$, we have $\psi_\Omega(s) \leqslant \psi_\Omega(t)$, that is, ψ_Ω is order-preserving. Suppose $E \subseteq L$ is directed. Then we claim that $\psi_\Omega(\bigvee E) = \bigvee \psi_\Omega(E)$. If $s \in \{s \in L' \mid (\exists e \in L)(e \in \twoheaddownarrow(\bigvee E), e\Omega s)\}$, then $e \in \twoheaddownarrow(\bigvee E)$ and $e\Omega s$ for some $e \in L$, which means that $e \ll e'$ for some $e' \in E$. Thus $s \in \{s \in L' \mid (\exists e \in L)(e \in \twoheaddownarrow e', e\Omega s)\}$. Then we obtain that $\psi_\Omega(\bigvee E) \leq \bigvee \psi_\Omega(E)$. For the opposite, it is obvious that $\bigvee \psi_\Omega(E) \leq \psi_\Omega(\bigvee E)$. Therefore, ψ_Ω is Scott continuous.

(3) For any $s \in L$,

$$\begin{aligned}
\psi_{\Omega_\psi}(s) &= \bigvee \{s' \in L' \mid (\exists t \in L)(t \in \twoheaddownarrow s, t\Omega_\psi s')\} \\
&= \bigvee \{s' \in L' \mid (\exists t \in L)(t \in \twoheaddownarrow s, s' \ll' \psi(t))\} \\
&= \bigvee \{s' \in L' \mid s' \ll' \bigvee \psi(\twoheaddownarrow s)\} \\
&= \bigvee \{s' \in L' \mid s' \ll' \psi(s)\} \\
&= \psi(s).
\end{aligned}$$

This implies that $\psi = \psi_{\Omega_\psi}$.

For any $s \in L$ and $s' \in L'$,

$$\begin{aligned}
s\Omega_{\psi_\Omega} s' &\Leftrightarrow s' \ll' \psi_\Omega(s) \\
&\Leftrightarrow s' \ll' \bigvee \{t' \in L' \mid (\exists t \in L)(t \in \twoheaddownarrow s, t\Omega t')\} \\
&\Leftrightarrow (\exists t' \in L', \exists t \in L)(t \in \twoheaddownarrow s, t\Omega t', s' \ll t') \\
&\Leftrightarrow (\exists t \in L)(t \in \langle s \rangle, t\Omega s') \\
&\Leftrightarrow s\Omega s'.
\end{aligned}$$

This implies that $\Omega = \Omega_{\psi_\Omega}$. □

Let $(Y, \langle \cdot \rangle)$ be an S-closure space. Define a relation $\mathrm{id_Y} \subseteq \mathrm{Y} \times \mathrm{Y}$ by

$$s \; \mathrm{id_Y} \; t \Leftrightarrow t \in \langle s \rangle .$$

Suppose $(Y, \langle \cdot \rangle), (Y', \langle \cdot \rangle')$ and $(Y'', \langle \cdot \rangle'')$ are S-closure spaces. Let $\Phi : (Y, \langle \cdot \rangle) \to (Y', \langle \cdot \rangle')$ and $\Omega : (Y', \langle \cdot \rangle') \to (Y'', \langle \cdot \rangle'')$ be S-morphisms. Define a relation $\Omega \circ \Phi \subseteq Y \times Y''$ as follows:

$$s(\Omega \circ \Phi)s'' \Leftrightarrow (\exists s' \in Y')(s\Phi s', s'\Omega s'').$$

It is a routine check that $\mathrm{id_Y}$ is an S-morphism from $(Y, \langle \cdot \rangle)$ to $(Y, \langle \cdot \rangle)$ and $\Omega \circ \Phi$ is an S-morphism from $(Y, \langle \cdot \rangle)$ to $(Y'', \langle \cdot \rangle'')$.

For convenience, we use **SC** to denote the category of S-closure spaces with S-morphisms, and **SCS** to denote the category of stably continuous semilattices with Scott continuous functions.

From Theorem 3.7, for any $(Y, \langle \cdot \rangle)$, define a map $\xi_o : \mathbf{SC}_o \to \mathbf{SCS}_o$ by

$$\xi_o(Y) = \mathcal{G}(Y).$$

From Theorem 4.3, for any S-morphism $\Phi : (Y, \langle \cdot \rangle) \to (Y', \langle \cdot \rangle')$, define a map $\xi_a : \mathbf{SC}_a \to \mathbf{SCS}_a$ by

$$\xi_a(\Phi) = \phi_\Phi.$$

Proposition 4.5. $\xi : \mathbf{SC} \to \mathbf{SCS}$ *is a functor.*

Proof. Let $(Y, \langle\cdot\rangle)$ be an S-closure space. Suppose $E \in \mathcal{G}(Y)$. Then

$$\begin{aligned}
\xi_a(\mathrm{id}_Y)(E) &= \phi_{\mathrm{id}_Y}(E) \\
&= \{s' \in Y \mid (\exists s \in E) s \mathrm{id}_Y s'\} \\
&= \{s' \in Y \mid (\exists s \in E) s' \in \langle s\rangle\} \\
&= \bigcup\{\langle s\rangle \mid \exists s \in E\} \\
&= E.
\end{aligned}$$

It implies that $\xi_a(\mathrm{id}_Y) = \mathrm{id}_{\xi_o(Y)}$.

Suppose $(Y, \langle\cdot\rangle), (Y', \langle\cdot\rangle')$ and $(Y'', \langle\cdot\rangle'')$ are S-closure spaces. Let $\Phi : (Y, \langle\cdot\rangle) \to (Y', \langle\cdot\rangle')$ and $\Omega : (Y', \langle\cdot\rangle') \to (Y'', \langle\cdot\rangle'')$ be S-morphisms. If $E \in \mathcal{G}(Y)$ and $s'' \in Y''$, then

$$\begin{aligned}
s'' \in \xi_a(\Omega \circ \Phi)(E) &\Leftrightarrow s'' \in \phi_{\Omega\circ\Phi}(E) \\
&\Leftrightarrow (\exists s \in E) s(\Omega \circ \Phi)s'' \\
&\Leftrightarrow (\exists s \in E, \exists s' \in Y')(s\Phi s', s'\Omega s'') \\
&\Leftrightarrow (\exists s' \in Y')(s' \in \phi_\Phi(E), s'\Omega s'') \\
&\Leftrightarrow (\exists s' \in Y')(s' \in \xi_a(\Phi)(E), s'\Omega s'') \\
&\Leftrightarrow s'' \in \phi_\Omega(\xi_a(\Phi)(E)) \\
&\Leftrightarrow s'' \in \xi_a(\Omega)(\xi_a(\Phi)(E)).
\end{aligned}$$

It implies that $\xi_a(\Omega \circ \Phi) = \xi_a(\Omega) \circ \xi_a(\Phi)$. □

Theorem 4.6. *The cateory* **SC** *is equivalent to* **SCS***.*

Proof. According to Theorem 3.7, it suffices to prove that ξ is full and faithful.

Suppose $(Y, \langle\cdot\rangle)$ and $(Y', \langle\cdot\rangle')$ are S-closure spaces. For a Scott continuous function $\phi : \mathcal{G}(Y) \to \mathcal{G}(Y')$. From Theorem 4.3, we get an S-morphism $\Phi_\phi : Y \to Y'$ with $\phi_{\Phi_\phi} = \phi$, which implies that $\xi_a(\Phi_\phi) = \phi_{\Phi_\phi} = \phi$. Hence ξ is full.

Let $\Phi, \Omega : (Y, \langle\cdot\rangle) \to (Y', \langle\cdot\rangle')$ be two S-morphisms such that $\phi_\Phi = \phi_\Omega$. For any $s \in Y$ and $s' \in Y'$, we have that

$$\begin{aligned}
s\Phi s' &\Leftrightarrow (\exists t \in \langle s\rangle) t\Phi s' \\
&\Leftrightarrow s' \in \phi_\Phi(\langle s\rangle) \\
&\Leftrightarrow s' \in \phi_\Omega(\langle s\rangle) \\
&\Leftrightarrow (\exists t \in \langle s\rangle) t\Omega s' \\
&\Leftrightarrow s\Omega s'.
\end{aligned}$$

It implies that $\Phi = \Omega$. Therefore, ξ is faithful. □

5. Conclusions

This paper introduces the concept of S-closure spaces and establishes a direct correspondence between stably continuous semilattices and S-closure spaces. We demonstrate that every stably continuous semilattice induces a family of generalized directed sets within S-closure spaces. Furthermore, we define S-morphisms between S-closure spaces, thereby constructing a category that is equivalent to the category of stably continuous semilattices equipped with Scott-continuous functions.

Use of Generative-AI tools declaration

The author declares he have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no competing financial interest.

References

1. M. Erné, Algebraic ordered sets and their generalizations, In: *Algebras and orders*, Dordrecht: Springer, 1993, 113–192. http://dx.doi.org/10.1007/978-94-017-0697-1_3
2. M. Erné, General stone duality, *Topol. Appl.*, **137** (2004), 125–158. http://dx.doi.org/10.1016/S0166-8641(03)00204-9
3. J. B. Wright, E. G. Wagner, J. W. Thatcher, A uniform approach to inductive posets and inductive closure, *Theor. Comput. Sci.*, **7** (1978), 57–77. https://doi.org/10.1016/0304-3975(78)90040-3
4. G. Birkhoff, Rings of sets, *Duke Math. J.*, **3** (1937), 443–454. http://dx.doi.org/10.1215/S0012-7094-37-00334-X
5. M. H. Stone, The theory of representations for Boolean algebras, *Trans. Amer. Math. Soc.*, **40** (1936), 37–111. http://dx.doi.org/10.1090/s0002-9947-1936-1501865-8
6. H. A. Priestley, Ordered topological spaces and the representation of distributive lattices, *P. Lond. Math. Soc.*, **s3-24** (1972), 507–530. http://dx.doi.org/10.1112/plms/s3-24.3.507
7. F. Ranzato, Closures on CPOs form complete lattices, *Inf. Comput.*, **152** (1999), 236–249. http://dx.doi.org/10.1006/inco.1999.2801
8. M. Wu, L. Guo, Q. Li, New representations of algebraic domains and algebraic L-domains via closure systems, *Semigroup Forum*, **103** (2021), 700–712. http://dx.doi.org/10.1007/s00233-021-10209-7

9. D. Zhao, Closure spaces and completions of posets, *Semigroup Forum*, **90** (2015), 545–555. http://dx.doi.org/10.1007/s00233-015-9692-6
10. G. Winskel, *A representation of completely distributive algebraic lattices*, Carnegie-Mellon University, 1983.
11. P. H. Edelman, Meet-distributive lattices and the anti-exchange closure, *Algebr. Univ.*, **10** (1980), 290–299. http://dx.doi.org/10.1007/BF02482912
12. M. Erné, Closure, In: *Beyond topology: Contemporary mathematics*, Providence: American Methematical Society, **486** (2009), 163–238.
13. L. Guo, Q. Li, The categorical equivalence between algebraic domains and F-augmented closure spaces, *Order*, **32** (2015), 101–116. http://dx.doi.org/10.1007/s11083-014-9318-8
14. Q. Li, L. Wang, L. Yao, A representation of continuous lattices based on closure spaces, *Quaest. Math.*, **44** (2021), 1513–1528. http://dx.doi.org/10.2989/16073606.2020.1808864
15. L. Wang, Q. Li, The categorical equivalence between domains and interpolative generalized closure spaces, *Stud. Logica*, **111** (2023), 187–215. http://dx.doi.org/10.1007/s11225-022-10024-3
16. G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, D. S. Scott, *Continuous lattices and domains*, New York: Cambridge University Press, 2003. http://dx.doi.org/10.1017/CBO9780511542725