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Research article

Generalizations of Herz-Morrey spaces and boundedness of the Calderón-Zygmund operators

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Abstract: In this paper, we use the grand variable Herz-Morrey spaces, and our main objective is to prove the boundedness of multilinear Calderón-Zygmund operators on the product of grand variable Herz-Morrey spaces. We first define the Lebesgue spaces with variable exponents and some basic lemmas including Hölder's inequality for Lebesgue spaces. To prove the boundedness of multilinear Calderón-Zygmund operators on grand variable Herz-Morrey spaces, we use the well known Hölder's inequality and Minkonwski's inequality. We split the summation of grand variable Herz-Morrey spaces to find the $L^{p(\cdot)}(\mathbb{R}^n)$ estimate of the characteristic function under some conditions. After finding estimate of each term, we conclude that multilinear Calderón-Zygmund operators are bounded on grand variable Herz-Morrey spaces. Our results generalize some results on variable Herz spaces and grand variable Herz spaces.

Keywords: multilinear Calderón-Zygmund operators; grand Herz-Morrey spaces; grand Herz spaces
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1. Introduction

Variable exponent function spaces have been extensively studied due to their many important applications. One example of a work in this field is [1]. Understanding variable exponent function spaces is necessary for studying applied mathematics problems and partial differential equations. In particular, Izuki [2] points out that variable Herz spaces are a generalization of classical Herz spaces. The Herz-Morrey spaces $M\dot{K}^{\alpha,\lambda}_{q(\cdot),p}(\mathbb{R}^n)$ generalize the idea of variable Herz spaces $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$. The original definition of these function spaces was given in [3]. Lu and Zhu [4] investigated the Morrey-Herz spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ in more detail and demonstrated the boundedness of integral operators on these spaces. Most recently, Abdalmonem et al. [5] showed the boundedness of intrinsic square functions S_β on generalized variable Herz-Morrey spaces $MK^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)$. For variable exponent function spaces see [6–9].

Izuki and Noi [10, 11] introduced the concept of weighted variable Herz spaces $\dot{K}^{\alpha,r}_{s(\cdot)}(w)$. Subsequently the boundedness of homogeneous fractional operators in weighted variable Herz spaces was examined in [12].

The generalization of Herz spaces are grand variable Herz spaces; see [13, 14] for boundedness results in these spaces. Grand weighted Herz spaces $\dot{K}^{\alpha,p),\theta}_{q(\cdot)}(w)$ were defined and the boundedness of integral operators in these spaces was demonstrated in [15]. The proof of fractional integral operators in grand weighted variable Herz-Morrey spaces was demonstrated in [16]. see also [17–19]. In [20], Meskhi introduced the concept of grand Morrey spaces denoted by $L^{p,\theta,\lambda}$, which represent a generalization of Morrey spaces. The author demonstrated how to establish the boundedness of a certain class of integral operators within these grand Morrey spaces. This class of integral operators includes well-known operators such as the Hardy-Littlewood maximal function, Calderón-Zygmund singular integrals, and potentials. For more results on variable exponent function spaces see [21–25]. Motivated by above papers, we prove the boundedness of the multilinear Calderón-Zygmund operator (MCZ shortly) in grand variable Herz-Morrey spaces $M\dot{K}^{a(\cdot),\beta}_{u,v(\cdot)}(\mathbb{R}^n)$.

There are different sections of this paper. Apart from introduction, one section is for preliminaries. Another section deals with the boundedness results of MCZ operators on grand variable Herz-Morrey (GVHM shortly) spaces.

2. Basic definitions and lemmas

We are introducing some necessary definitions and lemmas as a foundation for our main results. For more details see [26–28].

Definition 2.1. Let $\mathcal{F}$ be a measurable subset in $\mathbb{R}^n$ and a measurable function $q(\cdot) : \mathcal{F} \to [1, \infty)$. Then

(a) Now Lebesgue space with variable exponent $L^{q(\cdot)}(\mathcal{F})$ is defined by

$$L^{q(\cdot)}(\mathcal{F}) = \left\{ \mathbf{g} \textit{ is measurable} : \int_{\mathcal{F}} \left(\frac{|\mathbf{g}(i)|}{\Gamma} \right)^{q(i)} di < \infty \textit{ where } \Gamma \textit{ is a constant} \right\}.$$

Norm in $L^{q(\cdot)}(\mathcal{F})$ can be defined as,

$$\|\mathbf{g}\|_{L^{q(\cdot)}(\mathcal{F})} = \inf \left\{ \Gamma > 0 : \int_{\mathcal{F}} \left(\frac{|\mathbf{f}(i)|}{\Gamma} \right)^{q(i)} di \le 1 \right\}.$$

(b) The space $L^{q(\cdot)}_{\mathrm{loc}}(\mathcal{F})$ can be defined as

$$L^{q(\cdot)}_{\mathrm{loc}}(\mathcal{F}) := \left\{ \mathbf{g} : \mathbf{g} \in L^{q(\cdot)}(K) \textit{ for all compact subsets } K \subset \mathcal{F} \right\}.$$

We use these notations below in this paper:

(i) If $\mathbf{g} \in L^1_{\text{loc}}(\mathcal{F})$, now the Hardy-Littlewood maximal operator $\mathfrak{M}$ can be given as

$$\mathfrak{M}\mathbf{g}(i) := \sup_{r>0} \frac{1}{r^n} \int_{B(i,r)} |\mathbf{g}(i)| di \quad (i \in \mathcal{F}),$$

where $B(i,r) := \{x \in \mathcal{F} : |i - x| < r\}$.

(ii) Assume that $q(\cdot)$ be a measurable function, then the set $\mathfrak{B}(\mathcal{F})$ is consists of those measurable functions satisfying the following properties:

$$\mathbf{p}_- := \operatorname{ess}\inf_{x\in\mathcal{F}} q(i) > 1, \quad \mathbf{p}_+ := \operatorname{ess}\sup_{i\in\mathcal{F}} q(i) < \infty. \tag{2.1}$$

(iii) Assume that $q \in \mathfrak{B}(\mathcal{F})$ then $\mathfrak{B}^{\log} = \mathfrak{B}^{\log}(\mathcal{F})$ is the class of functions satisfying (2.1) and log-condition given by,

$$|\pi(i_1) - \pi(i_2)| \leq \frac{C(\pi)}{-\ln|i_1 - i_2|}, \quad |i_1 - i_2| \leq \frac{1}{2}, \quad i_1, i_2 \in \mathcal{F}. \tag{2.2}$$

(iv) For an unbounded set $\mathcal{F}$ in $\mathbb{R}^n$, $\mathfrak{B}_\infty(\mathcal{F})$ is the subset of $\mathfrak{B}(\mathcal{F})$ and values are in $[1,\infty)$ satisfying the following condition:

$$|\pi(i) - \pi_\infty| \leq \frac{C}{\ln(e + |i|)}, \tag{2.3}$$

where $\pi_\infty \in (1,\infty)$. $\mathfrak{B}_{0,\infty}(\mathcal{F})$ is the subset of $\mathfrak{B}(\mathcal{F})$ satisfying the following condition

$$|\pi(i) - \pi_0| \leq \frac{C}{\ln|i|}, |i| \leq \frac{1}{2}. \tag{2.4}$$

(v) For an unbounded set $\mathcal{F}$ in $\mathbb{R}^n$, then $\mathfrak{B}_\infty(\mathcal{F})$ and $\mathfrak{B}_{0,\infty}(\mathcal{F})$ are the subsets of $\mathfrak{B}(\mathcal{F})$.

(vi) Consider an unbounded set $\mathcal{F}$ in $\mathbb{R}^n$ then $\mathfrak{B}^{\log}_\infty(\mathcal{F})$ is the set of exponent $q \in \mathfrak{B}_\infty(\mathcal{F})$ satisfying condition (2.1). $\mathfrak{B}_\infty(\mathcal{F})$ satisfy the conditions (2.2) and (2.3) with value in $[1,\infty]$.

(vii) $q(\cdot) \in \mathcal{F}$ then $\mathcal{B}(\mathcal{F})$ is the collection of $q(\cdot)$ where $\mathfrak{M}$ is bounded on $L^{q(\cdot)}(\mathcal{F})$.

(viii) $\mathcal{S}_t = \mathcal{S}(0, 2^t) = \{z_1 \in \mathbb{R}^n : |z_1| < 2^t\}$ for all $l \in \mathbb{Z}$. $F_t = \mathcal{S}_t \setminus \mathcal{S}_{t-1}$, $\mathbf{1}_{F_t} = \mathbf{1}_t$.

Lemma 2.2. *[29] Let $D > 1$ and $p \in \mathfrak{B}_{0,\infty}(\mathbb{R}^n)$. Then*

$$\frac{1}{t_0} s^{\frac{n}{p(0)}} \leq \|\mathbf{1}_{R_{s,D_s}}\|_{p(\cdot)} \leq t_0 s^{\frac{n}{p(0)}}, \text{ if } 0 < s \leq 1 \tag{2.5}$$

and

$$\frac{1}{t_\infty} s^{\frac{n}{p_\infty}} \leq \|\mathbf{1}_{R_{s,D_s}}\|_{p(\cdot)} \leq t_\infty s^{\frac{n}{p_\infty}}, \text{ if } s \geq 1, \tag{2.6}$$

respectively, where $t_0 \geq 1$ and $t_\infty \geq 1$ and depending on D but independent of s.

Lemma 2.3. *[1] Let $\mathcal{F} \subseteq \mathbb{R}^n$, and $1 \leq \boldsymbol{p}_-(\mathcal{F}) \leq \boldsymbol{p}_+(\mathcal{F}) \leq \infty$. Then*

$$\|\boldsymbol{g}\boldsymbol{f}\|_{L^{r(\cdot)}(\mathcal{F})} \leq \|\boldsymbol{g}\|_{L^{p(\cdot)}(\mathcal{F})}\|\boldsymbol{f}\|_{q(\cdot)(\mathcal{F})}$$

holds, where $\boldsymbol{g} \in L^{p(\cdot)}(\mathcal{F})$, $\boldsymbol{f} \in q(\cdot)(\mathcal{F})$ and $\frac{1}{r(i)} = \frac{1}{p(i)} + \frac{1}{q(i)}$ for every $i \in \mathcal{F}$.

Now we will define GVHM spaces.

Definition 2.4. Let $r \in [1, \infty)$, $\theta > 0$, $0 \leq \beta < \infty$, and $a(\cdot), s(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$. The norm of homogeneous GVHM is defined as:

$$M\dot{K}^{a(\cdot),r),\theta}_{\beta,s(\cdot)}(\mathbb{R}^n) = \left\{g \in L^{s(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|g\|_{M\dot{K}^{a(\cdot),r),\theta}_{\beta,s(\cdot)}(\mathbb{R}^n)} < \infty\right\},$$

where

$$\|g\|_{M\dot{K}^{a(\cdot),r),\theta}_{\beta,s(\cdot)}(\mathbb{R}^n)} = \sup_{\zeta>0}\sup_{l_o\in\mathbb{Z}} 2^{-l_0\beta}\left(\zeta^\theta \sum_{k=-\infty}^{l_0} \|2^{ka(\cdot)} g\mathbf{1}_k\|^{u(1+\zeta)}_{s(\cdot)}\right)^{\frac{1}{u(1+\zeta)}}.$$

For further details regarding the contents of this section we refer the reader to [1, 30, 31].

3. Boundedness of multilinear Calderón-Zygmund operators

3.1. Notations

(1) $\mathbb{N}$ is the set of natural numbers, and $\mathbb{N}_{k_0}$ is the set of positive integers from 0 to k_0.

(2) $\mathbb{Z}$ is the set of integers.

(3) $\mathbb{Z}_-$ is the set of negative integers.

(4) $n,^- m := \{n, n+1, n+2, ..., m-1, m\}$ for $n, m \in \mathbb{Z}$ and $n < m$.

(5) $f \leq g$ means that $f \leq C_g$ and $f \simeq g$ means that $f \leq g \leq f$.

Consider the multilinear operator T of the form

$$T(g_1, g_2, ..., g_m)(z) = \int_{(\mathbb{R}^n)^m} K(z, t_1, ..., t_m)(g_1(t_1)...g_m(t_m)dt_1...dt_m, \tag{3.1}$$

where $z \notin \bigcap_{j=1}^m spt(g_j)$ and $g_1, ..., g_m \in L^\infty_c(\mathbb{R}^n)$, the space of compactly supported functions. Let $K(z, t_1, t_2, ..., t_m)$ is a locally integrable functions defined away from the diagonal $z = t_1 = ... = t_m \in (\mathbb{R}^n)^{m+1}$, satisfying the following estimate:

$$\left|K\Big(z, t_1, ..., t_m\Big)\right| \leq \frac{c}{\Big(|z-t_1| + |z-t_2| + ... + |z-t_m|\Big)^{mn}}, \tag{3.2}$$

for some $c > 0$ and all $(z, t_1, ..., t_m) \in (\mathbb{R}^n)^{m+1}$ with $z \neq t_j$ for some j. For somoothness, suppose that for some $\epsilon > 0$,

$$\left|K\Big(z, t_1, ..., t_m\Big) - K\Big(z', t_1, ..., t_m\Big)\right| \leq \frac{c|z-z'|^\epsilon}{\Big(|z-t_1| + |z-t_2| + ... + |z-t_m|\Big)^{mn+\epsilon}}, \tag{3.3}$$

provided that $|z-z'| \le \frac{1}{2}\ \max\{|z-t_1|,...,|z-t_m|\}$ and

$$\left|K\Big(z,t_1,...,t_m\Big)-K\Big(z',t_1,...,t_m\Big)\right| \le \frac{c|t_j-t_j'|^{\epsilon}}{\Big(|z-t_1|+|z-t_2|+...+|z-t_m|\Big)^{mn+\epsilon}}, \tag{3.4}$$

whenever $|t_j-t_j'| \le 1/2\ \max\{|z-t_1|,...,|z-t_m|\}$ for all $j \in 1,^{-} m$.

These kernels are called Calderón-Zygmund kernels and those functions which satisfy (3.2)–(3.4) with parameters m, c and ϵ is denoted by $m-CZK(c,\epsilon)$.

Any operator T be as in (3.1) is m-linear Calderón-Zygmund operator, if

(a) The kernel belongs to $m-CZK(c,\epsilon)$ class.

(b) T is bounded from $L^{s_1}\times L^{s_2}\times...\times L^{s_m}$ to L^s for some $1<s_1,s_2,...,s_m<\infty$ and $1/s=1/s_1+...+1/s_m$.

The following lemma is given in [32].

Lemma 3.1. *Let $s_i \in \mathfrak{B}(\mathbb{R}^n)$, $i \in 1,^{-} m$, $s \in \mathfrak{B}_0^{\log}(\mathbb{R}^n)$ with $1/s(x) = 1/s_1(x)+...+1/s_m(x)$ and $(s/t)' \in \mathfrak{B}(\mathbb{R}^n)$ for some $0<t<s^-$, then the m-linear Calderón-Zygmund operator T is bounded on the product of variable exponents Lebesgue spaces.*

$$\|T(\overrightarrow{g})\|_{s(\cdot)} \le \prod_{i=1}^{m} \|g_i\|_{s_i(\cdot)}, \tag{3.5}$$

where C is the constant not depending on $\overrightarrow{g}=(g_1,...,g_m)$.

Now we will prove boundedness of MCZ operator on GVHM spaces.

Theorem 3.2. *Let $s, s_i \in \mathfrak{B}_{0,\infty}(\mathbb{R}^n)$ for $i \in 1,^{-} m$ such that $s_i(0) \le s_i(\infty)$, $1/s(x) = 1/s_1(x)+...+1/s_m(x)$, $1<s_i^- \le s_i^+ <\infty$, and $(s/t)' \in \mathfrak{B}(\mathbb{R}^n)$ for some $0<t<s^-$. Let $\theta>0$, $1<o_i<\infty$ and $\alpha_i \in \mathfrak{B}_{0,\infty}(\mathbb{R}^n)$ be a* log*-Hölder continuous both at the region and at infinity for $i \in 1,^{-} m$ with*

$$\frac{n}{s_i(\infty)} < \alpha_i(0) \le \alpha_i(\infty) < n\left(1-\frac{1}{s_i(0)}\right). \tag{3.6}$$

Suppose that $\alpha(z) = \sum\limits_{i=1}^{m}\alpha_i(z)$, $\lambda = \sum\limits_{i=1}^{m}\lambda_i$ for $0<\lambda_i<\infty$ and $1/r = \sum\limits_{i=1}^{m} 1/o_i$. Then, the m-linear Calderón-Zygmund operator T is bounded on the product of GVHM spaces. Moreover,

$$\|T(\overrightarrow{f})\|_{M\dot{K}^{\alpha(\cdot),r),\theta}_{\lambda,s(\cdot)}(\mathbb{R}^n)} \le \prod_{i=1}^{m} \|f_i\|_{M\dot{K}^{\alpha_i(\cdot),o_i),\theta}_{\lambda_i,s_i(\cdot)}(\mathbb{R}^n)}, \tag{3.7}$$

where C is the constant not depending on $\overrightarrow{f}=(f_1,...,f_m)$.

Proof. We restrict ourselves to $m=2$, the general case following in a similar manner. Defining $f_\varphi(z) := f_1(z)\mathbf{1}_{R_\varphi}(z)$ and $f_\sigma(z) := f_2(z)\mathbf{1}_{R_\sigma}(z)$. We decompose the component functions of $\overrightarrow{f}$ as

$$f_1(z) = \sum_{\varphi\in\mathbb{Z}} f_\varphi(z), f_2(z) = \sum_{\sigma\in\mathbb{Z}} f_\sigma(z). \tag{3.8}$$

We divide $\mathbb{Z}$ into the following sets

$$
\begin{aligned}
E_p &:= \{n \in \mathbb{Z} \mid n \le p-2\},\\
F_p &:= \{n \in \mathbb{Z} \mid p-1 \le n \le p+1\},\\
G_p &:= \{n \in \mathbb{Z} \mid n \ge p+2\},
\end{aligned}
$$

and for X and Y arbitrary subsets of $\mathbb{Z}$, we define

$$
v_p^{\alpha}(X,Y) := \left\| 2^{p\alpha(\cdot)} \sum_{\varphi\in X} \sum_{\sigma\in Y} T(f_\varphi, f_\sigma)\mathbf{1}_p \right\|_{s(\cdot)}, \ \textit{and}\ v_p(X,Y) := v_p^0(X,Y). \tag{3.9}
$$

From the decay condition at the origin and (3.8), we have

$$
\begin{aligned}
\|T(f_1,f_2)\|_{M\dot{K}_{\lambda,s(\cdot)}^{\alpha(\cdot),r),\theta}(\mathbb{R}^n)} &= \sup_{\zeta>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\zeta^\theta \sum_{p=-\infty}^{k_0} \left\| 2^{p\alpha(\cdot)} T(f_1,f_2)\mathbf{1}_p \right\|_{s(\cdot)}^{r(1+\zeta)} \right)^{\frac{1}{r(1+\zeta)}}\\
&\le \sup_{\zeta>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\zeta^\theta \sum_{p=-\infty}^{k_0} \left\| \sum_{\varphi\in\mathbb{Z}} \sum_{\sigma\in\mathbb{Z}} 2^{p\alpha(\cdot)} T(f_\varphi,f_\sigma)\mathbf{1}_p \right\|_{s(\cdot)}^{r(1+\zeta)} \right)^{\frac{1}{r(1+\zeta)}}\\
&\le \sum_{j=1}^{9} I_j,
\end{aligned}
$$

where

$$
I_j := \sup_{\zeta>0} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\zeta^\theta \sum_{p=-\infty}^{k_0} \Gamma_j^{r(1+\zeta)} \right)^{\frac{1}{r(1+\zeta)}}, \tag{3.10}
$$

$$
\begin{aligned}
\Gamma_1 &= v_p^\alpha(E_p,E_p), & \Gamma_2 &= v_p^\alpha(E_p,F_p), & \Gamma_3 &= v_p^\alpha(E_p,G_p),\\
\Gamma_4 &= v_p^\alpha(F_p,E_p), & \Gamma_5 &= v_p^\alpha(F_p,F_p), & \Gamma_6 &= v_p^\alpha(F_p,G_p),\\
\Gamma_7 &= v_p^\alpha(G_p,E_p), & \Gamma_8 &= v_p^\alpha(G_p,F_p), & \Gamma_9 &= v_p^\alpha(G_p,G_p).
\end{aligned}
$$

We will estimate $I_1, I_2, I_3, I_4, I_5, I_6$, and I_9, as I_4, I_7, and I_8 can be estimated similarly to I_2, I_3, and I_6, respectively. For estimation of I_1: splitting $\mathbb{Z} = \mathbb{Z}_- \cup \mathbb{N}_0$ and

(i) $2^{p\alpha(z)} \simeq 2^{p\alpha(0)}$ $\quad (z \in R_p$ and $p < 0)$,

(ii) $2^{p\alpha(z)} \simeq 2^{p\alpha(\infty)}$ $\quad (z \in R_p$ and $p \ge 0)$.

We get

$$
\begin{aligned}
I_1 \le &\sup_{\zeta>0} \sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda} \left(\zeta^\theta \sum_{p\in\mathbb{Z}_-} 2^{p\alpha(0)r(1+\zeta)} v_p(E_p,E_p)^{r(1+\zeta)} \right)^{\frac{1}{r(1+\zeta)}}\\
&+\sup_{\zeta>0} \sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda} \left(\zeta^\theta \sum_{p\in\mathbb{N}_{k_0}} 2^{p\alpha(\infty)r(1+\zeta)} v_p(E_p,E_p)^{r(1+\zeta)} \right)^{\frac{1}{r(1+\zeta)}}
\end{aligned}
$$

$$+\sup_{\zeta>0}\sup_{k_0\in\mathbb{Z}^-\cup\{0\}} 2^{-k_0\lambda}\left(\zeta^\theta\sum_{p=-\infty}^{k_0} 2^{p\alpha(0)r(1+\zeta)}v_p(E_p,E_p)^{r(1+\zeta)}\right)^{\frac{1}{r(1+\zeta)}}$$
$$=:I_{11}+I_{12}+I_{11}'.$$

For $\varphi,\sigma\in E_p$, $z\in R_p$, $t_1\in R_\varphi$ and $t_2\in R_\sigma$, we have

$$|z-t_1|\geq ||z|-|t_1||\geq 2^{p-1}-2^{\sigma}\geq 2^{p-1}-2^{p-2}\geq 2^p, \tag{3.11}$$

$$|z-t_2|\geq ||z|-|t_2||\geq 2^{p-1}-2^{\sigma}\geq 2^{p-1}-2^{p-2}\geq 2^p, \tag{3.12}$$

as a result

$$\begin{aligned}\left|T(f_\varphi,f_\sigma)(z)\right| &\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f_\varphi(t_1)||f_\sigma(t_2)|}{(|z-t_1|+||z-t_2|)^{2n}}dt_1dt_2\\ &\leq 2^{-2pn}\int_{\mathbb{R}^n}|f_\varphi(t_1)|dt_1\int_{\mathbb{R}^n}|f_\sigma(t_2)|dt_2.\end{aligned}$$

From Hölder's inequality, $1/s(z)=1/s_1(z)+1/s_2(z)$, and Lemma 2.2, we obtain

$$\begin{aligned}v_p(E_p,E_p) &\leq \sum_{\varphi\in E_p}\sum_{\sigma\in E_p}2^{-2pn}\|f_\varphi\|_{s_1(\cdot)}\|\mathbf{1}_\varphi\|_{s_1'(\cdot)}\|f_\sigma\|_{s_2(\cdot)}\|\mathbf{1}_\sigma\|_{s_2'(\cdot)}\|\mathbf{1}_p\|_{s(\cdot)}\\ &\leq \sum_{\varphi\in E_p}\sum_{\sigma\in E_p}2^{-2pn}\|f_\varphi\|_{s_1(\cdot)}2^{\varphi n/s_1'(0)}\|f_\sigma\|_{s_2(\cdot)}2^{\varphi n/s_2'(0)}2^{kn/s(0)}\\ &\leq \left(\sum_{\varphi\in E_p}2^{(p-\varphi)(n/s_1(0)-n)}\|f_\varphi\|_{s_1(\cdot)}\right)\left(\sum_{\sigma\in E_p}2^{(p-\sigma)(n/s_2(0)-n)}\|f_\sigma\|_{s_2(\cdot)}\right).\end{aligned}$$

Taking into account the estimate of $v_p(E_p,E_p)$, the inequality $1/r=1/o_1+1/o_2$, $\lambda=\lambda_1+\lambda_2$ and Hölder's inequality yields

$$\begin{aligned}I_{11} \leq& \sup_{\zeta>0}\sup_{k_0\in\mathbb{Z}^+}2^{-k_0\lambda_1}\left[\zeta^\theta\sum_{p\in\mathbb{Z}_-}\left(\sum_{\varphi\in E_p}2^{(p-\varphi)(n/s_1(0)-n)+p\alpha_1(0)}\|f_\varphi\|_{s_1(\cdot)}\right)^{(1+\zeta)o_1}\right]^{\frac{1}{(1+\zeta)o_1}}\\ &\times\sup_{\zeta>0}\sup_{k_0\in\mathbb{Z}^+}2^{-k_0\lambda_2}\left[\zeta^\theta\sum_{p\in\mathbb{Z}_-}\left(\sum_{\sigma\in E_p}2^{(p-\sigma)(n/s_2(0)-n)+p\alpha_2(0)}\|f_\sigma\|_{s_2(\cdot)}\right)^{o_2(1+\zeta)}\right]^{\frac{1}{o_2(1+\zeta)}}\\ =&:I_{111}I_{112}.\end{aligned}$$

If $2^{-(1+\zeta)o_1}<2^{-o_1}$, then Fubini's theorem and Hölder's inequality yield.
Let $b_1=n/s_1(0)-n+\alpha_1(0)<0$ we get,

$$I_{111}\leq \sup_{\zeta>0}\sup_{k_0\in\mathbb{Z}^+}2^{-k_0\lambda_1}\Bigg[\zeta^\theta\sum_{p\in\mathbb{Z}_-}\left(\sum_{\varphi\in E_p}2^{b_1(p-\varphi)\frac{(1+\zeta)o_1}{2}}2^{\varphi\alpha_1(0)(1+\zeta)o_1}\|f_\varphi\|_{s_1(\cdot)}^{(1+\zeta)o_1}\right)^{(1+\zeta)o_1}$$

$$
\begin{aligned}
&\times\left(\sum_{\varphi\in E_p} 2^{b_1(p-\varphi)\frac{((1+\zeta)o_1)'}{2}}\right)^{\frac{(1+\zeta)o_1}{((1+\zeta)o_1)'}}\Bigg]^{\frac{1}{(1+\zeta)o_1}}\\
\leq&\sup_{\zeta>0}\sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda_1}\left[\zeta^\theta\sum_{\varphi=-\infty}^{-1} 2^{\varphi\alpha_1(0)(1+\zeta)o_1}\|f_\varphi\|_{s_1(\cdot)}^{(1+\zeta)o_1}\sum_{p=\varphi+2}^{-1} 2^{b_1(p-\varphi)\frac{(1+\zeta)o_1}{2}}\right]^{\frac{1}{(1+\zeta)o_1}}\\
\leq&\sup_{\zeta>0}\sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda_1}\left[\zeta^\theta\sum_{\varphi=-\infty}^{-1} 2^{\varphi\alpha_1(0)(1+\zeta)o_1}\|f_\varphi\|_{s_1(\cdot)}^{(1+\zeta)o_1}\sum_{p=\varphi+2}^{-1} 2^{b_1(p-\varphi)\frac{o_1}{2}}\right]^{\frac{1}{(1+\zeta)o_1}}\\
\leq&\sup_{\zeta>0}\sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda_1}\left[\zeta^\theta\sum_{\varphi=-\infty}^{-1} 2^{\varphi\alpha_1(0)(1+\zeta)o_1}\|f_\varphi\|_{s_1(\cdot)}^{(1+\zeta)o_1}\right]^{\frac{1}{(1+\zeta)o_1}}\\
\leq&\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),o_1),\theta}_{\lambda_1,s_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

The estimate for $I_{112}\leq\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),o_2),\theta}_{\lambda_2,s_2(\cdot)}(\mathbb{R}^n)}$ can be obtained in similar manner, so

$$
I_{11}\leq\|f_1\|_{M\dot{K}^{\alpha_1(\cdot),o_1),\theta}_{\lambda_1,s_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),o_2),\theta}_{\lambda_2,s_2(\cdot)}(\mathbb{R}^n)}. \tag{3.13}
$$

For estimating I_{12}, we split as following

$$
\begin{aligned}
I_{12}\leq&\sup_{\zeta>0}\sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda}\left(\zeta^\theta\sum_{p\in\mathbb{N}_{k_0}} 2^{p\alpha(\infty)r(1+\zeta)}\left\|\sum_{\varphi\in\mathbb{Z}_-}\sum_{\sigma\in\mathbb{Z}_-} T(f_\varphi,f_\sigma)\mathbf{1}_p\right\|_{s(\cdot)}^{r(1+\zeta)}\right)^{\frac{1}{r(1+\zeta)}}\\
&+\sup_{\zeta>0}\sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda}\left(\zeta^\theta\sum_{p\in\mathbb{N}_{k_0}} 2^{p\alpha(\infty)r(1+\zeta)}\left\|\sum_{\varphi=0}^{p-2}\sum_{\sigma=0}^{p-2} T(f_\varphi,f_\sigma)\mathbf{1}_p\right\|_{s(\cdot)}^{r(1+\zeta)}\right)^{\frac{1}{r(1+\zeta)}}\\
=:&I_{121}+I_{122}.
\end{aligned}
$$

The estimate I_{122} can be obtained similarly to I_{11} by simply replacing $\alpha_i(0)$ by $\alpha_i(\infty)$ and using the fact that $s_i(0)\leq s_i(\infty)$.
For I_{121}, by Hölder's inequality, Lemma 2.2, and $s_i(0)\leq s_i(\infty)$ we get

$$
\begin{aligned}
v_p(\mathbb{Z}_-,\mathbb{Z}_-)\leq&\sum_{\varphi\in\mathbb{Z}_-}\sum_{\sigma\in\mathbb{Z}_-} 2^{-2pn}\|f_\varphi\|_{s_1(\cdot)}\|\mathbf{1}_\varphi\|_{s_1'(\cdot)}\|f_\sigma\|_{s_2(\cdot)}\|\mathbf{1}_\sigma\|_{s_2'(\cdot)}\|\mathbf{1}_p\|_{s(\cdot)}\\
\leq&\sum_{\varphi\in\mathbb{Z}_-}\sum_{\sigma\in\mathbb{Z}_-} 2^{-2pn}\|f_\varphi\|_{s_1(\cdot)}2^{\varphi n/s_1'(0)}\|f_\sigma\|_{s_2(\cdot)}2^{\varphi n/s_2'(0)}2^{kn/s(\infty)}\\
\leq&\left(\sum_{\varphi\in\mathbb{Z}_-} 2^{(p-\varphi)(n/s_1(0)-n)}\|f_\varphi\|_{s_1(\cdot)}\right)\left(\sum_{\sigma\in\mathbb{Z}_-} 2^{(p-\sigma)(n/s_2(0)-n)}\|f_\sigma\|_{s_2(\cdot)}\right).
\end{aligned}
$$

By the estimate of $v_p(\mathbb{Z}_-,\mathbb{Z}_-)$, the equalities $\lambda=\lambda_1+\lambda_2$, $1/r=1/o_1+1/o_2$, and Hölder's inequality yield

$$
I_{121}\leq\sup_{\zeta>0}\sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda_1}\left[\zeta^\theta\sum_{p\in\mathbb{N}_{k_0}}\left(\sum_{\varphi\in\mathbb{Z}_-} 2^{(p-\varphi)(n/s_1(0)-n)+p\alpha_1(\infty)}\|f_\varphi\|_{s_1(\cdot)}\right)^{(1+\zeta)o_1}\right]^{\frac{1}{(1+\zeta)o_1}}
$$

$$\times \sup_{\zeta>0} \sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda_2}\left[\zeta^\theta \sum_{p\in\mathbb{N}_{k_0}}\left(\sum_{\sigma\in\mathbb{Z}_-} 2^{(p-\sigma)(n/s_2(0)-n)+p\alpha_2(\infty)}\|f_\sigma\|_{s_2(\cdot)}\right)^{o_2(1+\zeta)}\right]^{\frac{1}{o_2(1+\zeta)}}$$

$$=:A_1A_2.$$

Applying Hölder's inequality, and defining $\omega_1 := n/s_1(0) - n + \alpha_1(\infty) < 0$ we get,

$$\begin{aligned}
A_1 \leq & \sup_{\zeta>0} \sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda_1}\Bigg[\zeta^\theta \sum_{p\in\mathbb{N}_{k_0}}\left(\sum_{\varphi\in\mathbb{Z}_-} 2^{\omega_1(p-\varphi)\frac{(1+\zeta)o_1}{2}} 2^{\varphi\alpha_1(\infty)(1+\zeta)o_1}\|f_\varphi\|_{s_1(\cdot)}^{(1+\zeta)o_1}\right)^{(1+\zeta)o_1} \\
& \times \left(\sum_{\varphi\in\mathbb{Z}_-} 2^{\omega_1(p-\varphi)\frac{((1+\zeta)o_1)'}{2}}\right)^{\frac{(1+\zeta)o_1}{((1+\zeta)o_1)'}}\Bigg]^{\frac{1}{(1+\zeta)o_1}} \\
\leq & \sup_{\zeta>0} \sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda_1}\Bigg[\zeta^\theta \sum_{\varphi\in\mathbb{Z}_-} 2^{\varphi\alpha_1(\infty)(1+\zeta)o_1} 2^{\varphi\omega_1\frac{(1+\zeta)o_1}{2}}\|f_\varphi\|_{s_1(\cdot)}^{(1+\zeta)o_1} \sum_{k\in\mathbb{N}_{k_0}} 2^{\omega_1 k\frac{(1+\zeta)o_1}{2}}\Bigg]^{\frac{1}{(1+\zeta)o_1}} \\
\leq & \sup_{\zeta>0} \sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda_1}\Bigg[\zeta^\theta \sum_{\varphi\in\mathbb{Z}_-} 2^{\varphi\alpha_1(\infty)(1+\zeta)o_1} 2^{\varphi\omega_1\frac{(1+\zeta)o_1}{2}}\|f_\varphi\|_{s_1(\cdot)}^{(1+\zeta)o_1} \sum_{k\in\mathbb{N}_{k_0}} 2^{\omega_1 k\frac{o_1}{2}}\Bigg]^{\frac{1}{(1+\zeta)o_1}} \\
\leq & \sup_{\zeta>0} \sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda_1}\Bigg[\zeta^\theta \sum_{\varphi=-\infty}^{-1} 2^{\varphi\alpha_1(\infty)(1+\zeta)o_1}\|f_\varphi\|_{s_1(\cdot)}^{(1+\zeta)o_1}\Bigg]^{\frac{1}{(1+\zeta)o_1}} \\
\leq & \|f_1\|_{M\dot{K}^{\alpha_1(\cdot),o_1),\theta}_{\lambda_1,s_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

The estimate for $A_2 \leq \|f_2\|_{M\dot{K}^{\alpha_2(\cdot),o_2),\theta}_{\lambda_2,s_2(\cdot)}(\mathbb{R}^n)}$ can be obtained in similar manner, so $I_{12} \leq \|f_1\|_{M\dot{K}^{\alpha_1(\cdot),o_1),\theta}_{\lambda_1,s_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),o_2),\theta}_{\lambda_2,s_2(\cdot)}(\mathbb{R}^n)}$. The estimate of I'_{11} can be estimated similarly to I_{11}. Hence

$$I_1 \leq \|f_1\|_{M\dot{K}^{\alpha_1(\cdot),o_1),\theta}_{\lambda_1,s_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),o_2),\theta}_{\lambda_2,s_2(\cdot)}(\mathbb{R}^n)}. \tag{3.14}$$

Estimation for I_2:

$$\begin{aligned}
I_2 \leq & \sup_{\zeta>0} \sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda}\left(\zeta^\theta \sum_{p\in\mathbb{Z}_-} 2^{p\alpha(0)r(1+\zeta)} v_p(E_p,F_p)^{r(1+\zeta)}\right)^{\frac{1}{r(1+\zeta)}} \\
& + \sup_{\zeta>0} \sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda}\left(\zeta^\theta \sum_{p\in\mathbb{N}_{k_0}} 2^{p\alpha(\infty)r(1+\zeta)} v_p(E_p,F_p)^{r(1+\zeta)}\right)^{\frac{1}{r(1+\zeta)}} \\
& + \sup_{\zeta>0} \sup_{k_0\in\mathbb{Z}^-\cup\{0\}} 2^{-k_0\lambda}\left(\zeta^\theta \sum_{p=-\infty}^{k_0} 2^{p\alpha(0)r(1+\zeta)} v_p(E_p,F_p)^{r(1+\zeta)}\right)^{\frac{1}{r(1+\zeta)}} \\
=: & I_{21} + I_{22} + I'_{21}.
\end{aligned}$$

Note that, for $z \in R_p$, $t_1 \in R_\varphi$, $t_2 \in R_\sigma$, $\varphi \in E_p$, and $\sigma \in F_p$, we have

$$|z - t_1| + |z - t_2| \geq |z - t_1| \geq ||z| - |t_1|| \geq 2^p, \tag{3.15}$$

by Lemma 2.2, we have

$$v_p(E_p,F_p)\le\left(\sum_{\varphi\in E_p}2^{(p-\varphi)(n/s_1(0)-n)}\|f_\varphi\|_{s_1(\cdot)}\right)\left(\sum_{\sigma\in F_p}2^{(\sigma-p)n}2^{(p-\sigma)n/s_2(0))}\|f_\sigma\|_{s_2(\cdot)}\right).$$

Taking into account the estimate of $v_p(E_p,F_p)$, and the equalities $\lambda=\lambda_1+\lambda_2$, $1/r=1/o_1+1/o_2$, and Hölder's inequality yields

$$\begin{aligned}I_{21}\le&\sup_{\zeta>0}\sup_{k_0\in\mathbb{Z}^+}2^{-k_0\lambda_1}\left[\zeta^\theta\sum_{p\in\mathbb{Z}_-}\left(\sum_{\varphi\in E_p}2^{(p-\varphi)(n/s_1(0)-n)+p\alpha_1(0))}\|f_\varphi\|_{s_1(\cdot)}\right)^{(1+\zeta)o_1}\right]^{\frac{1}{(1+\zeta)o_1}}\\&\times\sup_{\zeta>0}\sup_{k_0\in\mathbb{Z}^+}2^{-k_0\lambda_2}\left[\zeta^\theta\sum_{p\in\mathbb{Z}_-}\left(\sum_{\sigma\in F_p}2^{(\sigma-p)n+p\alpha_2(0)}\|f_\sigma\|_{s_2(\cdot)}\right)^{o_2(1+\zeta)}\right]^{\frac{1}{o_2(1+\zeta)}}\\&=:I_{211}I_{212}.\end{aligned}$$

As $I_{211}=I_{111}$, to estimate I_{212},

$$\begin{aligned}I_{212}\le&\sup_{\zeta>0}\sup_{k_0\in\mathbb{Z}^+}2^{-k_0\lambda_2}\left[\zeta^\theta\sum_{p\in\mathbb{Z}_-}2^{p\alpha_2(0)o_2(1+\zeta)}\left(\sum_{\sigma\in F_p}2^{(\sigma-p)n}\|f_\sigma\|_{s_2(\cdot)}\right)^{o_2(1+\zeta)}\right]^{\frac{1}{o_2(1+\zeta)}}\\\le&\sup_{\zeta>0}\sup_{k_0\in\mathbb{Z}^+}2^{-k_0\lambda_2}\left[\zeta^\theta\sum_{p\in\mathbb{Z}_-}2^{p\alpha_2(0)o_2(1+\zeta)}\|f_\sigma\|^{o_2(1+\zeta)}_{s_2(\cdot)}\right]^{\frac{1}{o_2(1+\zeta)}}\\\le&\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),o_2),\theta}_{\lambda_2,s_2(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

The term I_{22} is estimated by

$$\begin{aligned}I_{22}\le&\sup_{\zeta>0}\sup_{k_0\in\mathbb{Z}^+}2^{-k_0\lambda}\left(\zeta^\theta\sum_{p\in\mathbb{N}_{k_0}}2^{p\alpha(\infty)r(1+\zeta)}v_p(\mathbb{Z}_-,M_p)^{r(1+\zeta)}\right)^{\frac{1}{r(1+\zeta)}}\\&+\sup_{\zeta>0}\sup_{k_0\in\mathbb{Z}^+}2^{-k_0\lambda}\left(\zeta^\theta\sum_{p\in\mathbb{N}_{k_0}}2^{p\alpha(\infty)r(1+\zeta)}v_p(0,^-k2,F_p)^{r(1+\zeta)}\right)^{\frac{1}{r(1+\zeta)}}\\&=:I_{221}+I_{222}.\end{aligned}$$

For I_{221}, by Hölder's inequality, Lemma 2.2, and the inequality $s_1(0)\le s_1(\infty)$, yield

$$v_p(\mathbb{Z}_-,M_p)\le\left(\sum_{\varphi\in\mathbb{Z}}2^{(p-\varphi)(n/s_1(0)-n)}\|f_\varphi\|_{s_1(\cdot)}\right)\left(\sum_{\sigma\in\mathbb{M}_p}2^{(p-\sigma)n/s_2(\infty)}2^{(\sigma-k)n}\|f_\sigma\|_{s_2(\cdot)}\right).$$

Thus, by using the above estimate and Hölder's inequality, we have

$$I_{221}\le\sup_{\zeta>0}\sup_{k_0\in\mathbb{Z}^+}2^{-k_0\lambda_1}\left[\zeta^\theta\sum_{p\in\mathbb{N}_{k_0}}\left(\sum_{\varphi\in\mathbb{Z}_-}2^{(p-\varphi)(n/s_1(0)-n)+p\alpha_1(\infty)}\|f_\varphi\|_{s_1(\cdot)}\right)^{(1+\zeta)o_1}\right]^{\frac{1}{(1+\zeta)o_1}}$$

$$\times \sup_{\zeta>0} \sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda_2}\left[\zeta^\theta \sum_{p\in\mathbb{N}_{k_0}}\left(\sum_{\sigma\in\mathbb{M}_p} 2^{(\sigma-p)n+p\alpha_2(\infty)}\|f_\sigma\|_{s_2(\cdot)}\right)^{o_2(1+\zeta)}\right]^{\frac{1}{o_2(1+\zeta)}}$$
$$=: B_1B_2.$$

The term B_1 is equal to A_1.
For the term I_{222}, similarly, we get

$$v_p(0,^- k2, F_p) \le \left(\sum_{\varphi=0}^{p-2} 2^{(p-\varphi)(n/s_1(\infty)-n)}\|f_\varphi\|_{s_1(\cdot)}\right)\left(\sum_{\sigma\in F_p} 2^{(\sigma-p)n}\|f_\sigma\|_{s_2(\cdot)}\right).$$

As a result, we get

$$I_{222} \le \sup_{\zeta>0} \sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda_1}\left[\zeta^\theta \sum_{p\in\mathbb{N}_{k_0}}\left(\sum_{\varphi=0}^{p-2} 2^{(p-\varphi)(n/s_1(\infty)-n)+p\alpha_1(\infty)}\|f_\varphi\|_{s_1(\cdot)}\right)^{(1+\zeta)o_1}\right]^{\frac{1}{(1+\zeta)o_1}}$$
$$\times \sup_{\zeta>0} \sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda_2}\left[\zeta^\theta \sum_{p\in\mathbb{N}_{k_0}}\left(\sum_{\sigma\in F_p} 2^{(\sigma-p)n+p\alpha_2(\infty)}\|f_\sigma\|_{s_2(\cdot)}\right)^{o_2(1+\zeta)}\right]^{\frac{1}{o_2(1+\zeta)}}$$
$$=: B_1'B_2'.$$

As we can see $B_2' = B_2$ and estimate for B_1' is similar to I_{212}, we obtain $I_{22} \le \|f_1\|_{M\dot{K}^{\alpha_1(\cdot),o_1),\theta}_{\lambda_1,s_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),o_2),\theta}_{\lambda_2,s_2(\cdot)}(\mathbb{R}^n)}$. For B_2 we use similar arguments as we used for I_{222}, by replacing $\alpha_2(0)$ with $\alpha_2(\infty)$. The estimate of I_{21}' is similar to the estimate of I_{21} from which we get

$$I_2 \le \|f_1\|_{M\dot{K}^{\alpha_1(\cdot),o_1),\theta}_{\lambda_1,s_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),o_2),\theta}_{\lambda_2,s_2(\cdot)}(\mathbb{R}^n)}. \tag{3.16}$$

For the estimation of I_3 we have

$$I_3 \le \sup_{\zeta>0} \sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda}\left(\zeta^\theta \sum_{p\in\mathbb{Z}_-} 2^{p\alpha(0)r(1+\zeta)} v_p(E_p, G_p)^{r(1+\zeta)}\right)^{\frac{1}{r(1+\zeta)}}$$
$$+ \sup_{\zeta>0} \sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda}\left(\zeta^\theta \sum_{p\in\mathbb{N}_{k_0}} 2^{p\alpha(\infty)r(1+\zeta)} v_p(E_p, G_p)^{r(1+\zeta)}\right)^{\frac{1}{r(1+\zeta)}}$$
$$+ \sup_{\zeta>0} \sup_{k_0\in\mathbb{Z}^-\cup\{0\}} 2^{-k_0\lambda}\left(\zeta^\theta \sum_{p=-\infty}^{k_0} 2^{p\alpha(0)r(1+\zeta)} v_p(E_p, G_p)^{r(1+\zeta)}\right)^{\frac{1}{r(1+\zeta)}}$$
$$=: I_{31} + I_{32} + I_{31}'.$$

Note that, for $z \in R_p$, $t_1 \in R_\varphi$, $t_2 \in R_\sigma$, $\varphi \in E_p$, and $\sigma \in G_p$, we have

$$|z - t_1| \ge |z| - |t_1| \ge 2^p, |z - t_2| \ge |t_2| - |z| \ge 2^\sigma, \tag{3.17}$$

by considering $(2^p+2^\sigma)^{2n} \geq 2^{pn+\sigma n}$, we have

$$\left|T(f_\varphi, f_\sigma)(z)\right| \leq 2^{-pn}\int_{\mathbb{R}^n}|f_\varphi(t_1)|dt_1 2^{-\sigma n}\int_{\mathbb{R}^n}|f_\sigma(t_2)|dt_2.$$

By Hölder's inequality, Lemma 2.2, $s_2(0) \leq s_2(\infty)$, and $p \in \mathbb{Z}_-$, we have

$$v_p(E_p, E_p) \leq \left(\sum_{\varphi\in E_p} 2^{(p-\varphi)(n/s_1(0)-n)}\|f_\varphi\|_{s_1(\cdot)}\right)\left(\sum_{\sigma\in G_p} 2^{(p-\sigma)n/s_2(\infty)}\|f_\sigma\|_{s_2(\cdot)}\right).$$

To estimate I_{31} we have

$$\begin{aligned} I_{31} \leq& \sup_{\zeta>0}\sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda_1}\left[\zeta^\theta\sum_{p\in\mathbb{Z}_-}\left(\sum_{\varphi\in E_p} 2^{(p-\varphi)(n/s_1(0)-n+\alpha_1(0))}2^{\varphi\alpha_1(0)}\|f_\varphi\|_{s_1(\cdot)}\right)^{(1+\zeta)o_1}\right]^{\frac{1}{(1+\zeta)o_1}} \\ &\times\sup_{\zeta>0}\sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda_2}\left[\zeta^\theta\sum_{p\in\mathbb{Z}_-}\left(\sum_{\sigma\in G_p} 2^{(p-\sigma)n/s_2(\infty)+\alpha_2(0))}2^{\sigma\alpha_2(0)}\|f_\sigma\|_{s_2(\cdot)}\right)^{o_2(1+\zeta)}\right]^{\frac{1}{o_2(1+\zeta)}} \\ =:& I_{311}I_{312}. \end{aligned}$$

We have $I_{311} \leq \|f_1\|_{M\dot{K}^{\alpha_1(\cdot),o_1),\theta}_{\lambda,s_1(\cdot)}(\mathbb{R}^n)}$, since $I_{311} = I_{111}$. For I_{312}, we split as follows

$$\begin{aligned} I_{312} \leq& \sup_{\zeta>0}\sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda_2}\left[\zeta^\theta\sum_{p\in\mathbb{Z}_-}\left(\sum_{\sigma=p+2}^{-1} 2^{(p-\sigma)(n/s_2(\infty)+\alpha_2(0))}2^{\sigma\alpha_2(0)}\|f_\sigma\|_{s_2(\cdot)}\right)^{o_2(1+\zeta)}\right]^{\frac{1}{o_2(1+\zeta)}} \\ &+\sup_{\zeta>0}\sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda_2}\left[\zeta^\theta\sum_{p\in\mathbb{Z}_-}\left(\sum_{\sigma\in\mathbb{N}_0} 2^{(p-\sigma)(n/s_2(\infty)+\alpha_2(0))}2^{\sigma\alpha_2(0)}\|f_\sigma\|_{s_2(\cdot)}\right)^{o_2(1+\zeta)}\right]^{\frac{1}{o_2(1+\zeta)}} \\ =:& D_1 + D_2. \end{aligned}$$

By applying Fubini's theorem and Hölder's inequality, and the inequality $n/s_2(\infty) + \alpha_2(0) > 0$ and $2^{-o_2(1+\zeta)} < 2^{-o_2}$, we have

$$\begin{aligned} D_1 \leq& \sup_{\zeta>0}\sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda_2}\Bigg[\zeta^\theta\sum_{p\in\mathbb{Z}_-}\left(\sum_{\sigma=p+2}^{-1} 2^{\sigma\alpha_2(0)o_2(1+\zeta)}\|f_\sigma\|^{o_2(1+\zeta)}_{s_2(\cdot)}2^{[(p-\sigma)n/s_2(\infty)+\alpha_2(0)o_2(1+\zeta)]/2}\right) \\ &\times\left(\sum_{\sigma=p+2}^{-1} 2^{[(p-\sigma)n/s_2(\infty)+\alpha_2(0)(o_2(1+\zeta))']/2}\right)^{\frac{o_2(1+\zeta)}{(o_2(1+\zeta))'}}\Bigg]^{\frac{1}{o_2(1+\zeta)}} \\ \leq& \sup_{\zeta>0}\sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda_2}\left[\zeta^\theta\sum_{p\in\mathbb{Z}_-}\left(\sum_{\sigma=p+2}^{-1} 2^{\sigma\alpha_2(0)o_2(1+\zeta)}\|f_\sigma\|^{o_2(1+\zeta)}_{s_2(\cdot)}2^{[(p-\sigma)n/s_2(\infty)+\alpha_2(0)o_2(1+\zeta)]/2}\right)\right]^{\frac{1}{o_2(1+\zeta)}} \\ \leq& \sup_{\zeta>0}\sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda_2}\left[\zeta^\theta\sum_{p\in\mathbb{Z}_-}\left(\sum_{\sigma=\infty}^{-1} 2^{\sigma\alpha_2(0)o_2(1+\zeta)}\|f_\sigma\|^{o_2(1+\zeta)}_{s_2(\cdot)}\sum_{p=-\infty}^{\sigma-2} 2^{[(p-\sigma)n/s_2(\infty)+\alpha_2(0)]/2}\right)\right]^{\frac{1}{o_2(1+\zeta)}} \end{aligned}$$

$$\leq \|f_2\|_{M\dot{K}^{\alpha_2(\cdot),o_2),\theta}_{\lambda_2,s_2(\cdot)}(\mathbb{R}^n)}.$$

For D_2, by using the same arguments and the inequality $\alpha_2(0) \leq \alpha_2(\infty)$, we obtain $D_2 \leq \|f_2\|_{M\dot{K}^{\alpha_2(\cdot),o_2),\theta}_{\lambda_2,s_2(\cdot)}(\mathbb{R}^n)}$, so

$$I_{31} \leq \|f_1\|_{M\dot{K}^{\alpha_1(\cdot),o_1),\theta}_{\lambda_1,s_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),o_2),\theta}_{\lambda_2,s_2(\cdot)}(\mathbb{R}^n)}.$$

The term I_{32} can be estimated as

$$\begin{aligned}
I_{32} \leq & \sup_{\zeta>0}\sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda}\left(\zeta^\theta \sum_{p\in\mathbb{N}_{k_0}} 2^{p\alpha(\infty)r(1+\zeta)} v_p(\mathbb{Z}_-, N_p)^{r(1+\zeta)}\right)^{\frac{1}{r(1+\zeta)}} \\
& + \sup_{\zeta>0}\sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda}\left(\zeta^\theta \sum_{p\in\mathbb{N}_{k_0}} 2^{p\alpha(\infty)r(1+\zeta)} v_p(0,^- m2, G_p)^{r(1+\zeta)}\right)^{\frac{1}{r(1+\zeta)}} \\
=: & I_{321} + I_{322}.
\end{aligned}$$

To estimate I_{321}, by Hölder's inequality, Lemma 2.2 and the inequality $s_1(0) \leq s_1(\infty)$, we have

$$v_p(\mathbb{Z}_-, N_p) \leq \left(\sum_{\varphi\in\mathbb{Z}_-} 2^{(p-\varphi)(n/s_1(0)-n)}\|f_\varphi\|_{s_1(\cdot)}\right)\left(\sum_{\sigma\in\mathbb{N}_p} 2^{(p-\sigma)(n/s_2(\infty)-n)}\|f_\sigma\|_{s_2(\cdot)}\right).$$

By Hölder's inequality, we get

$$\begin{aligned}
I_{321} \leq & \sup_{\zeta>0}\sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda_1}\left[\zeta^\theta \sum_{p\in\mathbb{N}_{k_0}}\left(\sum_{\varphi\in\mathbb{Z}_-} 2^{(p-\varphi)(n/s_1(0)-n+\alpha_1(\infty))}2^{\varphi\alpha_1(\infty)}\|f_\varphi\|_{s_1(\cdot)}\right)^{(1+\zeta)o_1}\right]^{\frac{1}{(1+\zeta)o_1}} \\
& \times \sup_{\zeta>0}\sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda_2}\left[\zeta^\theta \sum_{p\in\mathbb{N}_{k_0}}\left(\sum_{\sigma\in\mathbb{N}_p} 2^{(p-\sigma)(n/s_2(\infty)+\alpha_2(\infty))}2^{\sigma\alpha_2(\infty)}\|f_\sigma\|_{s_2(\cdot)}\right)^{o_2(1+\zeta)}\right]^{\frac{1}{o_2(1+\zeta)}} \\
=: & D_1' D_2'.
\end{aligned}$$

Notice that $D_1' = A_1$. The estimate for D_2' can be obtained by similar way as for I_{312}, by replacing $\alpha_2(0)$ with $\alpha_2(\infty)$.

To estimate I_{322}, we use Hölder's inequality and Lemma 2.2, we have

$$v_p(0,^- k2, G_p) \leq \left(\sum_{\varphi=0}^{p-2} 2^{(p-\varphi)(n/s_1(\infty)-n)}\|f_\varphi\|_{s_1(\cdot)}\right)\left(\sum_{\sigma\in G_p} 2^{(p-\sigma)/s_2(\infty)}\|f_\sigma\|_{s_2(\cdot)}\right).$$

By using above estimate and Hölder's inequality,

$$I_{322} \leq \sup_{\zeta>0}\sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda_1}\left[\zeta^\theta \sum_{p\in\mathbb{N}_{k_0}}\left(\sum_{\varphi=0}^{p-2} 2^{(p-\varphi)(n/s_1(\infty)-n+\alpha_1(\infty)}2^{\varphi\alpha_1(\infty)}\|f_\varphi\|_{s_1(\cdot)}\right)^{(1+\zeta)o_1}\right]^{\frac{1}{(1+\zeta)o_1}}$$

$$\times \sup_{\zeta>0} \sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda_2} \left[\zeta^{\theta} \sum_{p\in\mathbb{N}_{k_0}} \left(\sum_{\sigma\in G_p} 2^{(p-\sigma)n/s_2(\infty)+\alpha_2(\infty)} 2^{\sigma\alpha_2(\infty)} \|f_\sigma\|_{s_2(\cdot)} \right)^{o_2(1+\zeta)} \right]^{\frac{1}{o_2(1+\zeta)}}$$

$$=: D_1'' D_2''.$$

As $D_2'' = D_2'$ and the estimate of D_1'' is similar to D_1'. Remaining estimate of I_{31}' can be estimated similarly. By taking all the estimates into consideration we get

$$I_3 \le \|f_1\|_{M\dot{K}^{\alpha_1(\cdot),o_1),\theta}_{\lambda_1,s_1(\cdot)}(\mathbb{R}^n)} \|f_2\|_{M\dot{K}^{\alpha_2(\cdot),o_2),\theta}_{\lambda_2,s_2(\cdot)}(\mathbb{R}^n)}. \tag{3.18}$$

Estimate of I_5: We have

$$\begin{aligned} I_5 \le & \sup_{\zeta>0} \sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda} \left(\zeta^{\theta} \sum_{p\in\mathbb{Z}_-} 2^{p\alpha(0)r(1+\zeta)} v_p(F_p, F_p)^{r(1+\zeta)} \right)^{\frac{1}{r(1+\zeta)}} \\ & + \sup_{\zeta>0} \sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda} \left(\zeta^{\theta} \sum_{p\in\mathbb{N}_{k_0}} 2^{p\alpha(\infty)r(1+\zeta)} v_p(F_p, F_p)^{r(1+\zeta)} \right)^{\frac{1}{r(1+\zeta)}} \\ & + \sup_{\zeta>0} \sup_{k_0\in\mathbb{Z}^-\cup\{0\}} 2^{-k_0\lambda} \left(\zeta^{\theta} \sum_{p=-\infty}^{k_0} 2^{p\alpha(0)r(1+\zeta)} v_p(F_p, F_p)^{r(1+\zeta)} \right)^{\frac{1}{r(1+\zeta)}} \\ =: & I_{51} + I_{52} + I_{51}'. \end{aligned}$$

By the boundedness of T from the product space to $L^{s(\cdot)}$, we obtain that

$$v_p(F_p, F_p) \le \left(\sum_{\varphi\in F_p} \|f_\varphi\|_{s_1(\cdot)} \right) \left(\sum_{\sigma\in F_p} \|f_\sigma\|_{s_2(\cdot)} \right).$$

By applying Hölder's inequality we obtain

$$\begin{aligned} I_{51} \le & \sup_{\zeta>0} \sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda_1} \left(\zeta^{\theta} \sum_{p\in\mathbb{Z}_-} 2^{p\alpha_1(0)(1+\zeta)o_1} \sum_{\varphi\in F_p} \|f_\varphi\|_{s_1(\cdot)}^{(1+\zeta)o_1} \right)^{\frac{1}{(1+\zeta)o_1}} \\ & \times \sup_{\zeta>0} \sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda_2} \left(\zeta^{\theta} \sum_{p\in\mathbb{Z}_-} 2^{p\alpha_2(0)o_2(1+\zeta)} \sum_{\sigma\in F_p} \|f_\sigma\|_{s_2(\cdot)}^{o_2(1+\zeta)} \right)^{\frac{1}{o_2(1+\zeta)}} \\ \le & \|f_1\|_{M\dot{K}^{\alpha_1(\cdot),o_1),\theta}_{\lambda_1,s_1(\cdot)}(\mathbb{R}^n)} \|f_2\|_{M\dot{K}^{\alpha_2(\cdot),o_2),\theta}_{\lambda_2,s_2(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

Similarly, we can estimate for I_{52} and I_{51}' by replacing $\alpha_i(0)$ by $\alpha_i(\infty)$ with $p \in \mathbb{N}_0$. Therefore

$$I_5 \le \|f_1\|_{M\dot{K}^{\alpha_1(\cdot),o_1),\theta}_{\lambda_1,s_1(\cdot)}(\mathbb{R}^n)} \|f_2\|_{M\dot{K}^{\alpha_2(\cdot),o_2),\theta}_{\lambda_2,s_2(\cdot)}(\mathbb{R}^n)}. \tag{3.19}$$

Estimate of I_6, we have

$$I_6 \le \sup_{\zeta>0} \sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda} \left(\zeta^{\theta} \sum_{p\in\mathbb{Z}_-} 2^{p\alpha(0)r(1+\zeta)} v_p(F_p, G_p)^{r(1+\zeta)} \right)^{\frac{1}{r(1+\zeta)}}$$

$$+\sup_{\zeta>0}\sup_{k_0\in\mathbb{Z}^+}2^{-k_0\lambda}\left(\zeta^\theta\sum_{p\in\mathbb{N}_{k_0}}2^{p\alpha(\infty)r(1+\zeta)}v_p(F_p,G_p)^{r(1+\zeta)}\right)^{\frac{1}{r(1+\zeta)}}$$
$$+\sup_{\zeta>0}\sup_{k_0\in\mathbb{Z}^-\cup\{0\}}2^{-k_0\lambda}\left(\zeta^\theta\sum_{p=-\infty}^{k_0}2^{p\alpha(0)r(1+\zeta)}v_p(F_p,G_p)^{r(1+\zeta)}\right)^{\frac{1}{r(1+\zeta)}}$$
$$=:I_{61}+I_{62}+I'_{61}.$$

For $z\in R_p$, $\varphi\in F_p$, and $\sigma\in G_p$, $p\in\mathbb{Z}_-$, $t_1\in R_\varphi$, $t_2\in R_\sigma$, we have

$$|z-t_2|\le|t_2|-|z|2^\sigma,$$

$$\left|T(f_\varphi,f_\sigma)(z)\right|\le 2^{-pn}\int_{\mathbb{R}^n}|f_\varphi(t_1)|dt_1 2^{-\sigma n}\int_{\mathbb{R}^n}|f_\sigma(t_2)|dt_2.$$

If $p\in\mathbb{Z}_-$, and $s_2(0)\le s_2(\infty)$, then Hölder's inequality and Lemma 2.2 yield

$$v_p(F_p,G_p)\le\left(\sum_{\varphi\in F_p}2^{(\varphi-k)n}\|f_\varphi\|_{s_1(\cdot)}\right)\left(\sum_{\sigma\in G_p}2^{(p-\sigma)n/s_2(\infty)}\|f_\sigma\|_{s_2(\cdot)}\right).$$

Consequently, we get

$$I_{61}\le\sup_{\zeta>0}\sup_{k_0\in\mathbb{Z}^+}2^{-k_0\lambda_1}\left[\zeta^\theta\sum_{p\in\mathbb{Z}_-}\left(\sum_{\varphi\in F_p}2^{(\varphi-k)n+k\alpha_1(0)}\|f_\varphi\|_{s_1(\cdot)}\right)^{(1+\zeta)o_1}\right]^{\frac{1}{(1+\zeta)o_1}}$$
$$\times\sup_{\zeta>0}\sup_{k_0\in\mathbb{Z}^+}2^{-k_0\lambda_2}\left[\zeta^\theta\sum_{p\in\mathbb{Z}_-}\left(\sum_{\sigma\in G_p}2^{(p-\sigma)(n/s_2(\infty)+\alpha_2(0))}2^{\sigma\alpha_2(0)}\|f_\sigma\|_{s_2(\cdot)}\right)^{o_2(1+\zeta)}\right]^{\frac{1}{o_2(1+\zeta)}}$$
$$=:I_{611}I_{612}.$$

Notice that $I_{611}=I_{51}$ and estimate of I_{612} can be obtained similarly to I_{312}.
For $p\in\mathbb{N}_0$, we have

$$v_p(F_p,G_p)\le\left(\sum_{\varphi\in F_p}2^{(\varphi-k)n}\|f_\varphi\|_{s_1(\cdot)}\right)\left(\sum_{\sigma\in G_p}2^{(p-\sigma)n/s_2(\infty)}\|f_\sigma\|_{s_2(\cdot)}\right).$$

By Hölder's inequality, we get

$$I_{62}\le\sup_{\zeta>0}\sup_{k_0\in\mathbb{Z}^+}2^{-k_0\lambda_1}\left[\zeta^\theta\sum_{p\in\mathbb{Z}_-}\left(\sum_{\varphi\in F_p}2^{(\varphi-k)n+k\alpha_1(\infty)}\|f_\varphi\|_{s_1(\cdot)}\right)^{(1+\zeta)o_1}\right]^{\frac{1}{(1+\zeta)o_1}}$$
$$\times\sup_{\zeta>0}\sup_{k_0\in\mathbb{Z}^+}2^{-k_0\lambda_2}\left[\zeta^\theta\sum_{p\in\mathbb{Z}_-}\left(\sum_{\sigma\in G_p}2^{(p-\sigma)(n/s_2(\infty)+\alpha_2(\infty))}2^{\sigma\alpha_2(\infty)}\|f_\sigma\|_{s_2(\cdot)}\right)^{o_2(1+\zeta)}\right]^{\frac{1}{o_2(1+\zeta)}}$$

$$=: I_{621} I_{622}.$$

The estimate of I_{621} can be obtained similar to I_{52} and we notice that $I_{622} = D_2''$. The estimate of I_{61}' can be obtained similarly. So we have

$$I_6 \leq \|f_1\|_{M\dot{K}^{\alpha_1(\cdot),o_1),\theta}_{\lambda_1,s_1(\cdot)}(\mathbb{R}^n)} \|f_2\|_{M\dot{K}^{\alpha_2(\cdot),o_2),\theta}_{\lambda_2,s_2(\cdot)}(\mathbb{R}^n)}. \tag{3.20}$$

Estimation of I_9,

$$\begin{aligned}
I_9 \leq & \sup_{\zeta>0} \sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda} \left(\zeta^\theta \sum_{p\in\mathbb{Z}_-} 2^{p\alpha(0)r(1+\zeta)} v_p(G_p, G_p)^{r(1+\zeta)} \right)^{\frac{1}{r(1+\zeta)}} \\
& + \sup_{\zeta>0} \sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda} \left(\zeta^\theta \sum_{p\in\mathbb{N}_{k_0}} 2^{p\alpha(\infty)r(1+\zeta)} v_p(G_p, G_p)^{r(1+\zeta)} \right)^{\frac{1}{r(1+\zeta)}} \\
& + \sup_{\zeta>0} \sup_{k_0\in\mathbb{Z}^-\cup\{0\}} 2^{-k_0\lambda} \left(\zeta^\theta \sum_{p=-\infty}^{k_0} 2^{p\alpha(0)r(1+\zeta)} v_p(G_p, G_p)^{r(1+\zeta)} \right)^{\frac{1}{r(1+\zeta)}} \\
=: & I_{91} + I_{92} + I_{91}'.
\end{aligned}$$

For $\varphi \in G_p$, $t_1 \in R_\varphi$ and $z \in R_p$, we have $|z - t_1| \leq 2^\varphi$. Whereas for $\sigma \in G_p$, $t_2 \in R_\sigma$ and $z \in R_p$, we have $|z - t_2| \leq 2^\sigma$.

$$\left|T(f_\varphi, f_\sigma)(z)\right| \leq 2^{-\varphi n} \int_{\mathbb{R}^n} |f_\varphi(t_1)| dt_1 2^{-\sigma n} \int_{\mathbb{R}^n} |f_\sigma(t_2)| dt_2.$$

By using Hölder's inequality, by Lemma 2.2 and $q_j(0) \leq q_j(\infty)$, we obtain

$$v_p(G_p, G_p) \leq \left(\sum_{\varphi\in G_p} 2^{(p-\varphi)n/s_1(\infty)} \|f_\varphi\|_{s_1(\cdot)} \right) \left(\sum_{\sigma\in G_p} 2^{(p-\sigma)n/s_2(\infty)} \|f_\sigma\|_{s_2(\cdot)} \right).$$

Thus we get

$$\begin{aligned}
I_{91} \leq & \sup_{\zeta>0} \sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda_1} \left[\zeta^\theta \sum_{p\in\mathbb{Z}_-} \left(\sum_{\varphi\in G_p} 2^{(p-\varphi)(n/s_1(\infty)+\alpha_1(0))+\varphi\alpha_1(0)} \|f_\varphi\|_{s_1(\cdot)} \right)^{(1+\zeta)o_1} \right]^{\frac{1}{(1+\zeta)o_1}} \\
& \times \sup_{\zeta>0} \sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda_2} \left[\zeta^\theta \sum_{p\in\mathbb{Z}_-} \left(\sum_{\sigma\in G_p} 2^{(p-\sigma)(n/s_2(\infty)+\alpha_2(0))} 2^{\sigma\alpha_2(0)} \|f_\sigma\|_{s_2(\cdot)} \right)^{o_2(1+\zeta)} \right]^{\frac{1}{o_2(1+\zeta)}} \\
=: & I_{911} I_{912}.
\end{aligned}$$

Estimations of I_{911} and I_{912} are obtained similarly to I_{312}. Therefore

$$I_{91} \leq \|f_1\|_{M\dot{K}^{\alpha_1(\cdot),o_1),\theta}_{\lambda_1,s_1(\cdot)}(\mathbb{R}^n)} \|f_2\|_{M\dot{K}^{\alpha_2(\cdot),o_2),\theta}_{\lambda_1,s_2(\cdot)}(\mathbb{R}^n)}. \tag{3.21}$$

$$
\begin{aligned}
I_{92} \leq & \sup_{\zeta>0} \sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda_1}\left[\zeta^{\theta}\sum_{p\in\mathbb{N}_0}\left(\sum_{\varphi\in G_p} 2^{(p-\varphi)(n/s_1(\infty)+\alpha_1(\infty))+\varphi\alpha_1(\infty)}\|f_\varphi\|_{s_1(\cdot)}\right)^{(1+\zeta)o_1}\right]^{\frac{1}{(1+\zeta)o_1}} \\
& \times \sup_{\zeta>0} \sup_{k_0\in\mathbb{Z}^+} 2^{-k_0\lambda_2}\left[\zeta^{\theta}\sum_{p\in\mathbb{N}_0}\left(\sum_{\sigma\in G_p} 2^{(p-\sigma)(n/s_2(\infty)+\alpha_2(\infty))}2^{\sigma\alpha_2(\infty)}\|f_\sigma\|_{s_2(\cdot)}\right)^{o_2(1+\zeta)}\right]^{\frac{1}{o_2(1+\zeta)}} \\
& =: I_{921}I_{922}.
\end{aligned}
$$

The estimates for I_{921}, I_{922} can be obtained in a similar way to I_{322}. The estimate for I'_{91} can be obtained similarly. Thus

$$
I_9 \leq \|f_1\|_{M\dot{K}^{\alpha_1(\cdot),o_1),\theta}_{\lambda_1,s_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),o_2),\theta}_{\lambda_2,s_2(\cdot)}(\mathbb{R}^n)}. \tag{3.22}
$$

Taking into account all the estimates (3.14), (3.16), (3.18)–(3.20), and (3.22), we get

$$
\|T(f_1, f_2)\|_{M\dot{K}^{\alpha(\cdot),r),\theta}_{\lambda,s(\cdot)}(\mathbb{R}^n)} \leq \|f_1\|_{M\dot{K}^{\alpha_1(\cdot),o_1),\theta}_{\lambda_1,s_1(\cdot)}(\mathbb{R}^n)}\|f_2\|_{M\dot{K}^{\alpha_2(\cdot),o_2),\theta}_{\lambda_2,s_2(\cdot)}(\mathbb{R}^n)},
$$

which completes the proof. □

4. Conclusions

In this paper, we proved the boundedness of the multilinear Calderón-Zygmund operators on grand vaiable Herz-Morrey spaces under some proper assumptions on the exponents. These results generalized some previous results on variable Herz spaces and grand variable Herz spaces.

Author contributions

All authors of this article have been contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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