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Research article

Nyström method for the system of multi-dimensional nonlinear Fredholm integral equations of the second kind by the variable transformation

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Abstract: An efficient numerical scheme is developed to solve the system of multi-dimensional nonlinear Fredholm integral equations (MDNFIEs) of the second kind in terms of the variable transformation technique of Sidi type in conjunction with the trapezoidal quadrature rule. Using the product Nyström method, the considered integral system is discretized into a set of nonlinear algebraic equations. Additionally, a rigorous convergence analysis of the proposed method is provided, thus demonstrating that compared with the classical trapezoidal quadrature rule approach, this scheme achieves a significantly improved convergence rate. Furthermore, numerical examples are presented to validate the efficiency and accuracy of the described method.

Keywords: system of multi-dimensional nonlinear Fredholm integral equations; Sidi's transformation; trapezoidal quadrature rule; Nyström method
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1. Introduction

Integral equations have become indispensable in modeling a wide range of scientific and engineering applications. They are widely encountered in disciplines such as elasticity [1, 2], heat and mass transfer [3], solid and fluid dynamics [4], and electrodynamics [5, 6]. Notably, Fredholm integral equations (FIEs) play a pivotal role in various fields, including plasma physics [7], electromagnetic analyses [8], and so on.

A considerable number of studies have been carried out to solve one-dimensional FIEs, for example collocation method [9, 10], Haar wavelets method[11], modified mthod [12], iteration method [13] and weighted optimal quadrature formula method [14]. However, the numerical solutions for the system of multi-dimensional FIEs (MDFIEs), particularly in the nonlinear case, have not been extensively explored. In [15], the fast matrix-vector multiplication algorithms were presented to solve linear two dimensional FIEs. Bazm et al. [16] presented a Gauss product quadrature technique for the single integral equation with a two dimensional case. Pouria Assari et al. [17] and Hedayat Fatahi et al. [18] proposed meshless approaches using the moving least squares and radial point interpolation techniques, respectively, to find the approximate solution for two dimensional nonlinear FIEs (2DNFIEs). Moreover, the meshfree scheme was employed to solve MDFIEs on the hypercube domain by means of radial basis functions and barycentric Lagrange basis functions separately in [19, 20]. A method was proposed in [21] that applied the integral mean value theorem to solve the linear system of MDFIEs. In [22], a numerical iterative method was proposed based on a Picard iteration and newton-cotes rules to solve NMDFIEs. In [23], a unified spectral collocation method was developed to solve a system of multi-dimensional nonlinear Volterra-Fredholm integral equations.

This paper consider the system of MDNFIEs of the second kind that encompasses n undetermined functions, namely,

$$u_i(\boldsymbol{x}) - \sum_{j=1}^{n} \int_{\Omega} K_{ij}(\boldsymbol{x}, \boldsymbol{t}, u_j(\boldsymbol{t}))\mathrm{d}^m\boldsymbol{t} = g_i(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \tag{1.1}$$

where $\boldsymbol{x} = (x_1, \dots, x_m)$, $\boldsymbol{t} = (t_1, \dots, t_m)$, $\mathrm{d}^m\boldsymbol{t} = \mathrm{d}t_1\mathrm{d}t_2 \cdots \mathrm{d}t_m$ and the notation

$$\int_{\Omega} \cdot \mathrm{d}^m\boldsymbol{t} = \int_{\boldsymbol{a}}^{\boldsymbol{b}} \cdot \mathrm{d}^m\boldsymbol{t} = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \cdot \mathrm{d}t_1 \cdots \mathrm{d}t_m,$$

where $\Omega = \prod_{k=1}^{m}[a_k, b_k]$ is a bounded domain in $\mathbb{R}^m$ with $m \geq 2$ and $m \in \mathbb{N}$, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{N}$ denotes the set of positive integers, where $\boldsymbol{a} < \boldsymbol{b}$ if and only if $a_k < b_k$. Let $D = \Omega \times \Omega \times (-\infty, \infty)$. The functions $g_i(\boldsymbol{x})$ and $K_{ij}(\boldsymbol{x}, \boldsymbol{t}, u_j(\boldsymbol{t}))$ are presumed to be predefined smooth real valued functions over Ω and D, and $u_i(\boldsymbol{x})$ represents the i-th solution that needs to be ascertained. In addition, $K_{ij}(\boldsymbol{x}, \boldsymbol{t}, u_j(\boldsymbol{t}))$ is nonlinear concerning $u_j(\boldsymbol{t})$, where $i, j = 1, \dots, n$, $n \geq 1$ and $n \in \mathbb{N}$. When $n = 1$, the system (1.1) can conveniently yield a single MDNFIE:

$$u(\boldsymbol{x}) - \int_{\Omega} K(\boldsymbol{x}, \boldsymbol{t}, u(\boldsymbol{t}))\mathrm{d}^m\boldsymbol{t} = g(\boldsymbol{x}). \tag{1.2}$$

This paper introduces the following n-colunm vectors:

$$\boldsymbol{U}(\boldsymbol{x}) = \begin{bmatrix} u_1(\boldsymbol{x}) \\ u_2(\boldsymbol{x}) \\ \vdots \\ u_n(\boldsymbol{x}) \end{bmatrix}_{n\times 1}, \quad \boldsymbol{G}(\boldsymbol{x}) = \begin{bmatrix} g_1(\boldsymbol{x}) \\ g_2(\boldsymbol{x}) \\ \vdots \\ g_n(\boldsymbol{x}) \end{bmatrix}_{n\times 1},$$

and

$$\boldsymbol{K}(\boldsymbol{x}, \boldsymbol{t}, u(\boldsymbol{t})) = \begin{bmatrix} \sum_{j=1}^{n} K_{1j}(\boldsymbol{x}, \boldsymbol{t}, u_j(\boldsymbol{t})) \\ \sum_{j=1}^{n} K_{2j}(\boldsymbol{x}, \boldsymbol{t}, u_j(\boldsymbol{t})) \\ \vdots \\ \sum_{j=1}^{n} K_{nj}(\boldsymbol{x}, \boldsymbol{t}, u_j(\boldsymbol{t})) \end{bmatrix}_{n\times 1};$$

then, system (1.1) is reformulated into the matrix-vector format as follows:

$$\boldsymbol{U}(\boldsymbol{x}) - \int_{\Omega} \boldsymbol{K}(\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{U}(\boldsymbol{t}))\mathrm{d}^m\boldsymbol{t} = \boldsymbol{G}(\boldsymbol{x}). \tag{1.3}$$

In the following, we assume that $\Omega = \prod_{k=1}^{m}[0, 1]$, because $[a_k.b_k]$ can be converted to $[0, 1]$ using the change of variables $y_k = (x_k - a_k)/(b_k - a_k)$, $k = 1, \cdots, m$.

In recent years, Sidi's transformation technique has been widely applied in numerical integrations [24, 25] and the one-dimensional integral equations [26, 27, 28]. Furthermore, a numerical quadrature rule for multi-dimensional integrals was derived by combining Sidi's transformation with the trapezoidal quadrature rule in [25]. The constructed quadrature formula achieves a more favorable accuracy than the classical trapezoidal quadrature rule. This was due to the fact that several of the derivatives contained in the Euler-Maclaurin sum of the integration error formula equated to zero. The present study aims to develop an efficient scheme to solve Equation (1.2). This approach is established based on the transformation of a Sidi type [24] combined with the trapezoidal quadrature rule. System (1.3) is converted into a system of nonlinear algebraic equations. Based on the Nyström interpolant technique, the approximate solution for system (1.3) is obtained. Error estimate and convergence analyses are explored for the presented approach. Notably, the method avoids using computationally expensive product integration rules during integral evaluations, thus offering advantages of low computational cost, a simple structure, and an ease of implementation. Moreover, the described scheme can further enhance the accuracy through the judicious adjustment of the value of the transformation parameter.

This study is structured as follows: Section 2 introduces some relevant background materials associated with the application of Sidi's transformation in numerical integrations; Section 3 is dedicated to formulating an Nyström approach to solve the system of MDNFIEs by applying Sidi's transformation; Section 4 explores a convergence analysis of the established approach; in Section 5, numerical examples are exhibited; and finally, some conclusions are drawn in Section 6.

2. Preliminaries

For the sake of brevity, we first introduce some notations as follows:

- Let $\mathbb{R}$ denote the set of all real numbers, $\mathbb{N}$ denote the set of all nonnegative integers, and $\mathbb{N}_0 = \mathbb{N} \cup 0$.
- Given $m \in \mathbb{N}$ and a scalar $h \in \mathbb{R}$, we denote m-dimensional multi-indexes and vectors by boldface lowercase letters. For instance, $\boldsymbol{i} = (i_1, \dots, i_m) \in \mathbb{N}_0^m$, $\boldsymbol{\mu} = (\mu_1, \dots, \mu_m) \in \mathbb{N}^m$,

$$\boldsymbol{i}h = (i_1 h, \dots, i_m h), \quad \sum_{\boldsymbol{i}=\boldsymbol{0}}^{\boldsymbol{N}}(\cdot) := \sum_{i_1=0}^{N_1} \cdots \sum_{i_m=0}^{N_m}(\cdot),$$

$$|\boldsymbol{\mu}|_1 = \sum_{i=1}^{m} \mu_i, \quad |\boldsymbol{\mu}|_\infty = \max_{1 \le i \le m} \mu_i.$$

- Let $f(\boldsymbol{t})$ be a multivariate function; and the $|\boldsymbol{\mu}|_1$-th (mixed) partial derivative is represented as

$$\partial_{\boldsymbol{t}}^{\boldsymbol{\mu}} f := \partial_{t_1}^{\mu_1} \cdots \partial_{t_m}^{\mu_m} f := \frac{\partial^{|\boldsymbol{\mu}|_1} f}{\partial t_1^{\mu_1} \cdots \partial t_m^{\mu_m}},$$

We begin by reviewing Sidi's trigonometric transformation and the relevant properties [24]. The $\sin^m$-transformation is a representative class of Sidi's transformation, is defined via the integral representation

$$\psi_q(t) = \frac{\theta_q(t)}{\theta_q(1)}, \tag{2.1}$$

where

$$\theta_q(t) = \int_0^t (\sin \pi u)^q \mathrm{d}u, \quad q = 1, 2, \cdots . \tag{2.2}$$

and $\theta_q(t)$ satisfies the following recursion relations

$$\begin{aligned} \theta_q(t) &= -\frac{1}{\pi q}(\sin \pi t)^{q-1} \cos \pi t + \frac{q-1}{q}\theta_{p-2}(t), \\ \theta_q(1) &= \frac{q-1}{q}\theta_{q-2}(1), \quad q = 2, 3, \cdots , \end{aligned} \tag{2.3}$$

according to the initial conditions

$$\begin{aligned} &\theta_0(t) = t \quad \text{and} \quad \theta_1(t) = \frac{1}{\pi}(1 - \cos \pi t), \\ &\theta_0(1) = 1 \quad \text{and} \quad \theta_1(1) = \frac{2}{\pi}, \end{aligned} \tag{2.4}$$

respectively. In addition, the derivatives of $\psi_q(t)$ is formulated as follows:

$$\psi_q'(t) = \kappa_q(\sin \pi t)^q, \tag{2.5}$$

where $\kappa_q = \frac{1}{\theta_q(1)}$. $\psi_q(t)$ has the following properties:

$$\begin{aligned} &\psi_q(0) = 0,\ \psi_q(1) = 1,\ \psi_q'(t) > 0, \quad \forall t \in (0, 1), \\ &\psi_q^{(i)}(0) = 0,\ \psi_q^{(i)}(1) = 0, \quad \forall i \in [1, q],\ q \geq 1, \\ &\psi_q'(t) = \psi_q'(1 - t),\ \psi_q(t) \in C^\infty[0, 1]. \end{aligned} \tag{2.6}$$

Consider the integral

$$I(f) = \int_{[0,1]^m} f(\boldsymbol{x})\mathrm{d}^m\boldsymbol{x}, \quad m \geq 2. \tag{2.7}$$

Its approximation can be formulated by means of the trapezoidal rule as follows:

$$\begin{aligned} Q_N(f) &= h^m \sum_{i_1=0}^{N} \cdots \sum_{i_m=0}^{N} \Big(\prod_{k=1}^{m} w_{i_k}^{(k)} f(i_1 h, \cdots , i_m h) \Big), \\ &= h^m \sum_{\boldsymbol{i}=\boldsymbol{0}}^{\boldsymbol{N}} \boldsymbol{w}_{\boldsymbol{i}}^{(k)} f(\boldsymbol{i}\boldsymbol{h}) \end{aligned} \tag{2.8}$$

where $\boldsymbol{w}_{\boldsymbol{i}}^{(k)} = \prod_{k=1}^{m} w_{i_k}^{(k)},\ k = 1, \ldots, m$, with

$$w_{i_k}^{(k)} = \begin{cases} \frac{1}{2}, & i_k = 0 \text{ or } N, \\ 1, & i_k = 1, \ldots, N - 1. \end{cases}$$

and $h = 1/N$. It should be emphasized that in the case where $0 < i_k < N$, $k = 1, \cdots, m$, then $\boldsymbol{w}_{\boldsymbol{i}}^{(k)} = 1$. Due to the above mentioned properties of Sidi's transformation, [25] introduced a numerical integration over $[0,1]^m$ using Sidi's transformation for each of the m components, and the corresponding quadrature error expansion of the Euler-Maclaurin type was presented. By applying a variable transformation $\boldsymbol{x} = (\psi_q(t_1), \cdots, \psi_q(t_m)) = \boldsymbol{\psi_q}(\boldsymbol{t})$ to (2.7), then the integral (2.7) can subsequently be converted into the following form:

$$\begin{aligned} I(f) &= \int_{[0,1]^m} f(\boldsymbol{\psi_q}(\boldsymbol{t})) \prod_{i=1}^{m} \psi_q'(t_i) \mathrm{d}^m \boldsymbol{t} \\ &= \int_{[0,1]^m} f(\boldsymbol{\psi_q}(\boldsymbol{t})) \boldsymbol{\psi_q'}(\boldsymbol{t}) \mathrm{d}^m \boldsymbol{t}, \quad m = 2, 3 \ldots. \end{aligned} \tag{2.9}$$

Therefore, the trapezoidal quadrature rule (2.8) is written as follows:

$$Q_N(f) = h^m \sum_{\boldsymbol{i}=\boldsymbol{0}}^{\boldsymbol{N}} \boldsymbol{w}_{\boldsymbol{i}}^{(k)} f(\boldsymbol{ih}) \boldsymbol{\psi_q'}(\boldsymbol{ih}). \tag{2.10}$$

where $\boldsymbol{\psi_q'}(\boldsymbol{ih}) = \prod_{k=1}^{m} \psi_q'(i_k h)$. Given that $\psi_q'(0) = \psi_q'(1) = 0$, $q \geq 1$, the above quadrature rule (2.10) is capable of being expressed in an alternative form as follows:

$$\begin{aligned} Q_N(f) &= h^m \sum_{i_1=1}^{N-1} \cdots \sum_{i_m=1}^{N-1} \left(\prod_{k=1}^{m} w_{i_k}^{(k)} f(\psi_q(i_1 h), \cdots, \psi_q(i_m h)) \prod_{k=1}^{m} \psi_q'(i_k h) \right) \\ &= h^m \sum_{i_1=1}^{N-1} \cdots \sum_{i_m=1}^{N-1} f(\psi_q(i_1 h), \cdots, \psi_q(i_m h)) \prod_{k=1}^{m} \psi_q'(i_k h) \\ &= h^m \sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{N-1}} f(\boldsymbol{ih}) \boldsymbol{\psi_q'}(\boldsymbol{ih}). \end{aligned} \tag{2.11}$$

Based on [25], we present the error expansion of quadrature rule (2.10) or (2.11) as follows. To simplify this discussion, we assume that $f_{\boldsymbol{q}}(\boldsymbol{t}) = f(\boldsymbol{\psi_q}(\boldsymbol{t})) \prod_{i=1}^{m} \psi_q'(t_i)$ in the following discussion.

Theorem 2.1. *([25]) Assume that $f(\boldsymbol{x})$ and all partial derivatives of total order p are integrable over $[0,1]^m$, and $Q_N(f)$ is given by* (2.10) *or* (2.11)*; then, we have*

$$Q_N(f) - I(f) = \sum_{\substack{\boldsymbol{\mu} \text{ is even}, \\ \boldsymbol{\mu} \in [\bar{q}+1, p]}} \frac{B_{\boldsymbol{\mu}}(Q, f_{\boldsymbol{q}})}{N^{\boldsymbol{\mu}}} + o(N^{-p}), \tag{2.12}$$

where $\bar{q} = 2q + 1$ or q depending on whether q is even or odd, and

$$B_{\boldsymbol{\mu}}(Q, f_{\boldsymbol{q}}) = \sum_{\sum_{i=1}^{m} \mu_i = \boldsymbol{\mu}} B_{\mu_1, \mu_2, \cdots, \mu_m}(Q, f_{\boldsymbol{q}}),$$

with

$$B_{\mu_1,\mu_2,\cdots,\mu_m}(Q,f_{\boldsymbol{q}})=\prod_{i=1}^{m}c_{\mu_i}\int_{[0,1]^m}\frac{\partial^{\mu}}{\partial t_1^{\mu_1}\partial t_2^{\mu_2}\cdots\partial t_m^{\mu_m}}(f_{\boldsymbol{q}}(\boldsymbol{t}))\mathrm{d}^m\boldsymbol{t},$$

where c_{μ_i} *are constants independent of* $f_{\boldsymbol{q}}(\boldsymbol{t})$ *and* N*, but are related to the Bernoulli numbers and polynomials.*

3. Nyström method and variable transformation to solve (1.3)

In this section, by integrating Sidi's transformation with the trapezoidal quadrature rule, a numerical approach is developed to solve the system of MDNFIEs with a continuous kernel.

Using the transformations $\boldsymbol{t}=(t_1,t_2,\cdots,t_m)=\boldsymbol{\psi_q}(\boldsymbol{s})=(\psi_q(s_1),\cdots,\psi_q(s_m))$ and $\boldsymbol{x}=(x_1,x_2,\cdots,x_m)=\boldsymbol{\psi_q}(\boldsymbol{y})=(\psi_q(y_1),\cdots,\psi_q(y_m))$ in Equation (1.3), we obtain the following:

$$\boldsymbol{U}(\boldsymbol{\psi_q}(\boldsymbol{y}))-\int_{\Omega}\boldsymbol{K}(\boldsymbol{\psi_q}(\boldsymbol{y}),\boldsymbol{\psi_q}(\boldsymbol{s}),\boldsymbol{U}(\boldsymbol{\psi_q}(\boldsymbol{s})))\prod_{k=1}^{m}\psi_q'(s_k)\mathrm{d}^m\boldsymbol{s}=\boldsymbol{G}(\boldsymbol{\psi_q}(\boldsymbol{y})).\tag{3.1}$$

Setting $\boldsymbol{V}(\boldsymbol{y})=\boldsymbol{U}(\boldsymbol{\psi_q}(\boldsymbol{y}))$, $\boldsymbol{F}(\boldsymbol{y})=\boldsymbol{G}(\boldsymbol{\psi_q}(\boldsymbol{y}))$, $\boldsymbol{\psi_q'}(\boldsymbol{s})=\prod_{k=1}^{m}\psi_q'(s_k)$, then (3.1) can be written as follows:

$$\boldsymbol{V}(\boldsymbol{y})-\int_{\Omega}\boldsymbol{K}(\boldsymbol{y},\boldsymbol{s},\boldsymbol{V}(\boldsymbol{s}))\boldsymbol{\psi_q'}(\boldsymbol{s})\mathrm{d}^m\boldsymbol{s}=\boldsymbol{F}(\boldsymbol{y}),\tag{3.2}$$

where

$$\boldsymbol{V}(\boldsymbol{y})=\begin{bmatrix}v_1(\boldsymbol{y})\\v_2(\boldsymbol{y})\\\vdots\\v_n(\boldsymbol{y})\end{bmatrix}_{n\times1}=\begin{bmatrix}u_1(\boldsymbol{\psi_q}(\boldsymbol{y}))\\u_2(\boldsymbol{\psi_q}(\boldsymbol{y}))\\\vdots\\u_n(\boldsymbol{\psi_q}(\boldsymbol{y}))\end{bmatrix}_{n\times1},\quad \boldsymbol{F}(\boldsymbol{y})=\begin{bmatrix}f_1(\boldsymbol{y})\\f_2(\boldsymbol{y})\\\vdots\\f_n(\boldsymbol{y})\end{bmatrix}_{n\times1}=\begin{bmatrix}g_1(\boldsymbol{\psi_q}(\boldsymbol{y}))\\g_2(\boldsymbol{\psi_q}(\boldsymbol{y}))\\\vdots\\g_n(\boldsymbol{\psi_q}(\boldsymbol{y}))\end{bmatrix}_{n\times1},$$

$$\boldsymbol{K}(\boldsymbol{y},\boldsymbol{s},\boldsymbol{V}(\boldsymbol{s}))=\begin{bmatrix}\sum_{j=1}^{n}K_{1j}(\boldsymbol{y},\boldsymbol{s},v_j(\boldsymbol{s}))\\\sum_{j=1}^{n}K_{2j}(\boldsymbol{y},\boldsymbol{s},v_j(\boldsymbol{s}))\\\vdots\\\sum_{j=1}^{n}K_{nj}(\boldsymbol{y},\boldsymbol{s},v_j(\boldsymbol{s}))\end{bmatrix}_{n\times1}=\begin{bmatrix}\sum_{j=1}^{n}K_{1j}(\boldsymbol{\psi_q}(\boldsymbol{y}),\boldsymbol{\psi_q}(\boldsymbol{s}),u_j(\boldsymbol{\psi_q}(\boldsymbol{s})))\\\sum_{j=1}^{n}K_{2j}(\boldsymbol{\psi_q}(\boldsymbol{y}),\boldsymbol{\psi_q}(\boldsymbol{s}),u_j(\boldsymbol{\psi_q}(\boldsymbol{s})))\\\vdots\\\sum_{j=1}^{n}K_{nj}(\boldsymbol{\psi_q}(\boldsymbol{y}),\boldsymbol{\psi_q}(\boldsymbol{s}),u_j(\boldsymbol{\psi_q}(\boldsymbol{s})))\end{bmatrix}_{n\times1}.$$

We define a completely continuous operator $\mathcal{K}$ from $C(\Omega)$ to $C(\Omega)$ by the following

$$\mathcal{K}\boldsymbol{V}(\boldsymbol{y})=\int_{\Omega}\boldsymbol{K}(\boldsymbol{y},\boldsymbol{s},\boldsymbol{V}(\boldsymbol{s}))\boldsymbol{\psi_q'}(\boldsymbol{s})\mathrm{d}^m\boldsymbol{s}.\tag{3.3}$$

Therefore, (3.2) is represented as the following operator form:

$$(\mathcal{I}-\mathcal{K})\boldsymbol{V}=\boldsymbol{F}.\tag{3.4}$$

By applying the trapezoidal quadrature rule to the integrals in (3.2), it follows that

$$\boldsymbol{V}(\boldsymbol{y}) - h^m \sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{N-1}} \boldsymbol{K}(\boldsymbol{y}, \boldsymbol{s_i}, \boldsymbol{V}(\boldsymbol{s_i}))\psi_{\boldsymbol{q}}'(\boldsymbol{s_i}) = \boldsymbol{F}(\boldsymbol{y}), \tag{3.5}$$

where $h = 1/N$, and $\boldsymbol{s_i} = \boldsymbol{ih} = (i_1 h, \cdots, i_m h)$. Subsequently, we proceed to define the approximation operator $\mathcal{K}_{N^m}$ on $C([0,1]^m)$ by the following:

$$\mathcal{K}_{N^m}\boldsymbol{V}(\boldsymbol{y}) = \begin{bmatrix} \sum_{j=1}^{n} \mathcal{K}_{1j,N^m} v_j(\boldsymbol{y}) \\ \sum_{j=1}^{n} \mathcal{K}_{2j,N^m} v_j(\boldsymbol{y}) \\ \vdots \\ \sum_{j=1}^{n} \mathcal{K}_{nj,N^m} v_j(\boldsymbol{y}) \end{bmatrix}_{n\times 1},$$

with $\mathcal{K}_{ij,N^m} v_j(\boldsymbol{y}) = h^m \sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{N-1}} K_{ij}(\boldsymbol{y}, \boldsymbol{s_i}, v_j(\boldsymbol{s_i}))\psi_{\boldsymbol{q}}'(\boldsymbol{s_i})$. Thus, (3.5) can be written as follows:

$$(\mathcal{I} - \mathcal{K}_{N^m})\boldsymbol{V} = \boldsymbol{F}. \tag{3.6}$$

Substituting $\boldsymbol{y_k} = \boldsymbol{kh} = (k_1 h, \cdots, k_m h) = \boldsymbol{y}$ into (3.5), where $k_j = 1, \cdots, N-1$ and $j = 1, \cdots, n$, the values $v_j(\boldsymbol{s_i})$ for $j = 1, \cdots, n$ are determined via the following:

$$\boldsymbol{V_k} - h^m \sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{N-1}} \boldsymbol{K}(\boldsymbol{y_k}, \boldsymbol{s_i}, \boldsymbol{V_i})\psi_{\boldsymbol{q}}'(\boldsymbol{s_i}) = \boldsymbol{F}(\boldsymbol{y_k}). \tag{3.7}$$

where $\boldsymbol{V_k} = [v_{l,\boldsymbol{k}}]_{(N-1)n\times 1}$, and $v_{l,\boldsymbol{k}}$, $l = 1, \cdots, n$, $k = 1, \cdots, N-1$ is the approximation value of $v_l(\boldsymbol{s})$ at the mesh point $\boldsymbol{s_k}$. A set of nonlinear algebraic equations (3.7) is addressed by applying Newton's iteration technique. Simultaneously, (3.7) is expressed as the following operator form:

$$(\mathcal{I} - \mathcal{K}_{N^m})\boldsymbol{V}_{N^m} = \boldsymbol{F}_{N^m}. \tag{3.8}$$

Based on the Nyström interpolant method, we derive the approximation solution of Equation (3.2) as follows:

$$\boldsymbol{V}_{N^m}(\boldsymbol{y}) = h^m \sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{N-1}} \boldsymbol{K}(\boldsymbol{y}, \boldsymbol{s_i}, \boldsymbol{V_i})\psi_{\boldsymbol{q}}'(\boldsymbol{s_i}) + \boldsymbol{F}(\boldsymbol{y}), \quad \boldsymbol{y} \in \Omega, \tag{3.9}$$

where

$$\boldsymbol{V}_{N^m}(\boldsymbol{y}) = \left[v_{l,N^m}(\boldsymbol{y})\right]_{n\times 1} = \begin{bmatrix} h^m \sum_{j=1}^{n} \sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{N-1}} K_{1j}(\boldsymbol{y}, \boldsymbol{s_i}, v_{j,i})\psi_{\boldsymbol{q}}'(\boldsymbol{s_i}) + f_1(\boldsymbol{y}) \\ h^m \sum_{j=1}^{n} \sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{N-1}} K_{2j}(\boldsymbol{y}, \boldsymbol{s_i}, v_{j,i})\psi_{\boldsymbol{q}}'(\boldsymbol{s_i}) + f_2(\boldsymbol{y}) \\ \vdots \\ h^m \sum_{j=1}^{n} \sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{N-1}} K_{nj}(\boldsymbol{y}, \boldsymbol{s_i}, v_{j,i})\psi_{\boldsymbol{q}}'(\boldsymbol{s_i}) + f_n(\boldsymbol{y}) \end{bmatrix}_{n\times 1}.$$

Consequently, by setting $\boldsymbol{y} = \psi_q^{-1}(\boldsymbol{x})$, we obtain the approximate solution for Equation (1.2) as follows:

$$\boldsymbol{U}_{N^m}(\boldsymbol{x}) = h^m \sum_{i=1}^{N-1} \boldsymbol{K}(\boldsymbol{x}, \boldsymbol{s}_i, \boldsymbol{V}_i) + \boldsymbol{G}(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \tag{3.10}$$

with

$$\boldsymbol{U}_{N^m}(\boldsymbol{x}) = \left[u_{l,N^m}(\boldsymbol{x})\right]_{n\times 1} = \begin{bmatrix} h^m \sum_{j=1}^{n} \sum_{i=1}^{N-1} K_{1j}(\boldsymbol{x}, \boldsymbol{t}_i, v_{j,i})\psi_q'(\boldsymbol{s}_i) + g_1(\boldsymbol{x}) \\ h^m \sum_{j=1}^{n} \sum_{i=1}^{N-1} K_{2j}(\boldsymbol{x}, \boldsymbol{t}_i, v_{j,i})\psi_q'(\boldsymbol{s}_i) + g_2(\boldsymbol{x}) \\ \vdots \\ h^m \sum_{j=1}^{n} \sum_{i=1}^{N-1} K_{nj}(\boldsymbol{x}, \boldsymbol{t}_i, v_{j,i})\psi_q'(\boldsymbol{s}_i) + g_n(\boldsymbol{x}) \end{bmatrix}_{n\times 1}.$$

4. Convergence analysis

In what follows, the existence, uniqueness, and convergence of the solution for Equation (3.6) are demonstrated. Assume that functions g_i, K_{ij}, and u_i satisfy the following assumptions:

(i) System (1.3) has a unique solution $u_j^*(\boldsymbol{x}) \in C(\Omega)$;
(ii) $\frac{\partial^2 K_{lj}(\boldsymbol{x},\boldsymbol{t},u_j(\boldsymbol{t}))}{\partial {u_j}^2}$ is continuous at $(\boldsymbol{x}, \boldsymbol{t}, u_j) \in D$;
(iii) $[\mathcal{I} - \mathcal{K}'(\boldsymbol{U}^*)]$ is nonsingular with $\boldsymbol{U}^*(\boldsymbol{x}) = (u_1^*(\boldsymbol{x}) \cdots, u_n^*(\boldsymbol{x}))^{\mathrm{T}}$.

Condition (i) indicates that (3.2) has only one solution: $\boldsymbol{V}^*(\boldsymbol{x}) \in C(\Omega)$. Based on the postulated condition (ii), it can be observed that $K_{lj}(\boldsymbol{y}, \boldsymbol{s}, v_j(\boldsymbol{s}))$ has a continuous first derivative and a bounded second derivative with respect to v_j on $B(v_j^*; r_j) = \{v_j : \|v_j - v_j^*\| \le r_0; r_0 > 0\}, j = 1 \cdots, n$. Additionally, $\mathcal{K}_{lj}'(v_j)$ is a linear operator which is defined by

$$\mathcal{K}_{lj}'(v_j z_j)(\boldsymbol{y}) = \int_\Omega \frac{\partial K_{lj}(\boldsymbol{y}, \boldsymbol{s}, v_j(\boldsymbol{s}))\psi_q'(\boldsymbol{s})}{\partial v_j} z_j(\boldsymbol{s})\mathrm{d}^{\boldsymbol{s}}, \tag{4.1}$$

and

$$(\mathcal{K}_{lj}''(v_j)w_j z_j)(\boldsymbol{y}) = \int_\Omega \frac{\partial^2}{\partial v_j^2} K_{lj}(\boldsymbol{y}, \boldsymbol{s}, v_j(\boldsymbol{s}))\psi_q'(\boldsymbol{s}) w_j(\boldsymbol{s}) z_j(\boldsymbol{s})\mathrm{d}^m \boldsymbol{s} \tag{4.2}$$

is bilinear operator, where $w_j(\boldsymbol{s}), z_j(\boldsymbol{s}) \in C(\Omega)$, $i, j = 1 \cdots, n$.

Lemma 4.1. *For* $\psi_q(\boldsymbol{s}) = \prod_{k=1}^{m} \psi_q(s_k)$, *we obtain* $\max_{\boldsymbol{s}\in\Omega} |\psi_q(\boldsymbol{s})| \le 1$.

Proof. *By* $\psi_q(s_k) = \kappa_q(\sin \pi s_k)^q$, *we have the following:*

$$\begin{aligned} |\psi_q'(\boldsymbol{s})| &= \left| \prod_{k=1}^{m} \kappa_q(\sin(\pi s_k))^q \right| \\ &= \left| \theta_q^m(1) \prod_{k=1}^{m} (\sin(\pi s_k))^q \right| \end{aligned}$$

$$\leq \begin{cases} \left[\dfrac{(q-1)!!}{q!!}\right]^m, & q \text{ is even,} \\ \left[\dfrac{(q-1)!!}{q!!}\right]^m \dfrac{1}{\pi^m}, & q \text{ is odd.} \end{cases}$$

Hence, the following inequality holds:

$$|\psi'_{\boldsymbol{q}}(\boldsymbol{s})| \leq 1.$$

Regarding Lemma 4.1, it can be easily verified that $\mathcal{K}'_{lj}(v_j)$ is a bounded operator from $C(\Omega)$ to $C(\Omega)$. Furthermore, the operator $\mathcal{K}''_{lj}(v_j)$ is bounded due to the following:

$$\begin{aligned} \|\mathcal{K}''_{lj}(v_j)\|_\infty &= \sup_{\|w_j\|,\|z_j\|\leq 1} \|\mathcal{K}''_{lj}(v_j)w_j z_j\|_\infty \\ &= \sup_{\|w_j\|,\|z_j\|\leq 1} \max_{\boldsymbol{y}\in\Omega, 1\leq l\leq n} \int_\Omega \left| \frac{\partial^2}{\partial v_j^2} K_{lj}(\boldsymbol{y}, \boldsymbol{s}, v_j(\boldsymbol{s}))\psi'_{\boldsymbol{q}}(\boldsymbol{s}) w_j(\boldsymbol{s}) z_j(\boldsymbol{s}) \right| \mathrm{d}^m \boldsymbol{s} \\ &\leq \text{const.} \end{aligned}$$

Based on Theorem 2.1, it can be observed that $\mathcal{K}_{lj,N^m}$ is pointwise convergent to $\mathcal{K}_{lj}$. Moreover, in light of [29, 30] and the assumptions, we conclude that $\{\mathcal{K}_{lj,N^m} | N, m \geq 1\}$ is a family of collectively compacts. The definitions of $\mathcal{K}'_{lj,N^m}(v_j)$ and $\mathcal{K}''_{lj,N^m}(v_j)$ are the same as those of $\mathcal{K}'_{lj}(v_j)$ and $\mathcal{K}''_{lj}(v_j)$, respectively, with the only difference being that the integrals are replaced by numerical approximations. Thus, we have the discrete operators as follows:

$$\mathcal{K}'_{N^m} \boldsymbol{V} \boldsymbol{z}(\boldsymbol{y}) = \begin{bmatrix} \sum_{j=1}^n (\mathcal{K}'_{1j,N^m}(v_j) z_j)(\boldsymbol{y}) \\ \sum_{j=1}^n (\mathcal{K}'_{2j,N^m}(v_j) z_j)(\boldsymbol{y}) \\ \vdots \\ \sum_{j=1}^n (\mathcal{K}'_{nj,N^m}(v_j) z_j)(\boldsymbol{y}) \end{bmatrix}_{n\times 1}, \tag{4.3}$$

and

$$\mathcal{K}''_{N^m} \boldsymbol{V} \boldsymbol{z}(\boldsymbol{y}) = \begin{bmatrix} \sum_{j=1}^n (\mathcal{K}''_{1j,N^m}(v_j) w_j z_j)(\boldsymbol{y}) \\ \sum_{j=1}^n (\mathcal{K}''_{2j,N^m}(v_j) w_j z_j)(\boldsymbol{y}) \\ \vdots \\ \sum_{j=1}^n (\mathcal{K}''_{nj,N^m}(v_j) w_j z_j)(\boldsymbol{y}) \end{bmatrix}_{n\times 1}, \tag{4.4}$$

where

$$\begin{aligned} (\mathcal{K}'_{lj,N^m}(v_j) z_j)(\boldsymbol{y}) &= h^m \sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{N}-\boldsymbol{1}} \frac{\partial}{\partial v_j} K_{lj}(\boldsymbol{y}, \boldsymbol{s_i}, v_j(\boldsymbol{s_i}))\psi'_{\boldsymbol{q}}(\boldsymbol{s_i}) z_j(\boldsymbol{s_i}), \\ (\mathcal{K}''_{lj,N^m}(v_j) w_j z_j)(\boldsymbol{y}) &= h^m \sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{N}-\boldsymbol{1}} \frac{\partial^2}{\partial v_j^2} K_{lj}(\boldsymbol{y}, \boldsymbol{s_i}, v_j(\boldsymbol{s_i}))\psi'_{\boldsymbol{q}}(\boldsymbol{s_i}) w_j(\boldsymbol{s_i}) z_j(\boldsymbol{s_i}), \end{aligned}$$

with $w_j(\boldsymbol{s}), z_j(\boldsymbol{s}) \in C(\Omega)$, $i, j = 1, \cdots, n$. From (4.3) and (4.4), we have the following:

$$\begin{aligned}
\|\mathcal{K}'_{N^m}(\boldsymbol{V})\|_\infty &= \sup_{\|\boldsymbol{W}\|\le 1} \|\mathcal{K}'_{N^m}(\boldsymbol{V})\boldsymbol{W}\|_\infty \\
&= \sup_{\|\boldsymbol{W}\|\le 1} \max_{\boldsymbol{y}\in\Omega, 1\le l\le n} \left\| h^m \sum_{j=1}^{n} \sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{N}-\boldsymbol{1}} \frac{\partial}{\partial v_j} K_{lj}(\boldsymbol{y}, \boldsymbol{s_i}, v_j(\boldsymbol{s_i}))\boldsymbol{\psi}'_{\boldsymbol{q}}(\boldsymbol{s_i}) w_j(\boldsymbol{s_i}) \right\|_\infty \\
&\le n \max_{\boldsymbol{y}\in\Omega, 1\le l\le n} \left| \frac{\partial}{\partial v_j} K_{lj}(\boldsymbol{y}, \boldsymbol{s_i}, v_j(\boldsymbol{s_i})) w_j(\boldsymbol{s_i}) \right|, \\
\|\mathcal{K}''_{N^m}(\boldsymbol{V})\|_\infty &= \sup_{\|\boldsymbol{W}\|\le 1, \|\boldsymbol{Z}\|\le 1} \|\mathcal{K}''_{N^m}(\boldsymbol{V})\boldsymbol{W}\boldsymbol{Z}\|_\infty \\
&= \sup_{\|\boldsymbol{W}\|\le 1, \|\boldsymbol{Z}\|\le 1} \max_{\boldsymbol{y}\in\Omega, 1\le l\le n} \left\| h^m \sum_{j=1}^{n} \sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{N}-\boldsymbol{1}} \frac{\partial^2}{\partial v_j^2} K_{lj}(\boldsymbol{y}, \boldsymbol{s_i}, v_j(\boldsymbol{s_i}))\boldsymbol{\psi}'_{\boldsymbol{q}}(\boldsymbol{s_i}) w_j(\boldsymbol{s_i}) z_j(\boldsymbol{s_i}) \right\|_\infty \\
&\le n \max_{\boldsymbol{y}\in\Omega, 1\le l\le n} \left| \frac{\partial^2}{\partial v_j^2} K_{lj}(\boldsymbol{y}, \boldsymbol{s_i}, v_j(\boldsymbol{s_i})) w_j(\boldsymbol{s_i}) z_j(\boldsymbol{s_i}) \right|,
\end{aligned}$$

where final term within the above mentioned inequalities is bounded due to the hypothesis imposed on K_{lj}. It is demonstrated that $\mathcal{K}'_{N^m}(\boldsymbol{V})$ and $\mathcal{K}''_{N^m}(\boldsymbol{V})$ are bounded operators for any $v_j \in B(v_j^*, r_0)$. Therefore, regarding the existence, uniqueness and convergence for the solution to (3.6), the following are inferred.

Lemma 4.2. *Given the assumptions (i)-(iii), when N is large enough, namely, the operator* $\mathcal{I} - \mathcal{K}'_{N^m}(\boldsymbol{V}^*)$ *are nonsingular,*

$$\|[\mathcal{I} - \mathcal{K}'_{N^m}(\boldsymbol{V}^*)]^{-1}\|_\infty \le C_0 < \infty, \quad N \ge N_1.$$

Proof. *This lemma can be proved from [30].*

The primary outcomes for this study are elaborated in the subsequent theorem.

Theorem 4.1. *Given the assumptions (i)-(iii), and* $\boldsymbol{V}^* = (v_1^*, v_2^*, \cdots, v_n^*)^T$ *represents an isolated solution for Equation* (3.4)*, then there exist constants* ρ_0*,* $0 \le \rho_0 \le r_0$*, and C such that Equation* (3.6) *has a unique solution* $\boldsymbol{V}_{N^m} = (v_{1,N^m}, v_{2,N^m}, \cdots, v_{n,N^m})^T \in \boldsymbol{B}(\boldsymbol{V^*}; \rho_0) = [B(v_1^*, \rho_0), \cdots, B(v_n^*, \rho_0)]^T$ *for an arbitrary sufficiently large N, and*

$$\|\boldsymbol{V}_{N^m} - \boldsymbol{V}^*\|_\infty \le C\|\mathcal{K}_{N^m}\boldsymbol{V}^* - \mathcal{K}\boldsymbol{V}^*\|_\infty. \tag{4.5}$$

Proof. *In accordance with Lemma 4.2 and Equations* (3.4) *and* (3.6)*, we obtain the following:*

$$\boldsymbol{V} = \boldsymbol{V} - [\mathcal{I} - \mathcal{K}'_{N^m}(\boldsymbol{V}^*)]^{-1}[\boldsymbol{V} - \mathcal{K}_{N^m}\boldsymbol{V} - \boldsymbol{G}] = \mathcal{T}\boldsymbol{V}. \tag{4.6}$$

For any $\boldsymbol{V}_1, \boldsymbol{V}_2 \in \boldsymbol{B}(\boldsymbol{V^*}; \rho_0)$*, it follows that*

$$\begin{aligned}
\mathcal{T}\boldsymbol{V}_1 - \mathcal{T}\boldsymbol{V}_2 =& \boldsymbol{V}_1 - \boldsymbol{V}_2 - [\mathcal{I} - \mathcal{K}'_{N^m}(\boldsymbol{V}^*)]^{-1}[\boldsymbol{V}_1 - \boldsymbol{V}_2 - (\mathcal{K}_{N^m}\boldsymbol{V}_1 - \mathcal{K}_{N^m}\boldsymbol{V}_2)] \\
=& [\mathcal{I} - \mathcal{K}'_{N^m}(\boldsymbol{V}^*)]^{-1}(\mathcal{K}_{N^m}\boldsymbol{V}_1 - \mathcal{K}_{N^m}\boldsymbol{V}_2 - \mathcal{K}'_{N^m}(\boldsymbol{V}^*)) \\
=& [\mathcal{I} - \mathcal{K}'_{N^m}(\boldsymbol{V}^*)]^{-1}[\overline{\mathcal{K}}_{N^m} - \mathcal{K}'_{N^m}(\boldsymbol{V}^*)][\boldsymbol{V}_1 - \boldsymbol{V}_2],
\end{aligned} \tag{4.7}$$

where

$$\overline{\mathcal{K}}_{N^m} = \int_0^1 \mathcal{K}'_{N^m}(r\boldsymbol{V}_1 + (1-r)\boldsymbol{V}_2)\mathrm{d}r, \quad r \in (0,1), \tag{4.8}$$

in light of the fact that $\mathcal{K}_{N^m}$ *is differentiable and* $\boldsymbol{B}(\boldsymbol{V}^*;\rho_0)$ *exhibits convexity. Owing to the boundedness of* $\mathcal{K}'_{N^m}$*, we obtain the following:*

$$\|\overline{\mathcal{K}}_{N^m} - \mathcal{K}'_{N^m}(\boldsymbol{V}^*)\|_\infty = \left\| \int_0^1 [\mathcal{K}'_{N^m}(r\boldsymbol{V}_1 + (1-r)\boldsymbol{V}_2) - \mathcal{K}'_{N^m}(\boldsymbol{V}^*)]\mathrm{d}r \right\|_\infty \le M\rho_0, \tag{4.9}$$

where $\|\mathcal{K}''_{N^m}(\boldsymbol{V})\|_\infty \le M$ *for any* $\boldsymbol{V} \in \boldsymbol{B}(\boldsymbol{V}^*;\rho_0)$*. Therefore, by utilizing* (4.7) *and* (4.9)*, we deduce the following:*

$$\|\mathcal{T}\boldsymbol{V}_1 - \mathcal{T}\boldsymbol{V}_2\|_\infty \le \alpha\|\boldsymbol{V}_1 - \boldsymbol{V}_2\|_\infty, \quad \boldsymbol{V}_1, \boldsymbol{V}_2 \in \boldsymbol{B}(\boldsymbol{V}^*;\rho_0), \tag{4.10}$$

where $\alpha = MC_0\rho_0$*, and we select* ρ_0 *to be so small that* $\alpha < 1$*. According to* (4.10)*, T is a contractive mapping. Additionally, we obtain the following:*

$$\begin{aligned}\mathcal{T}\boldsymbol{V}^* - \boldsymbol{V}^* &= [\mathcal{I} - \mathcal{K}'_{N^m}(\boldsymbol{V}^*)]^{-1}][\mathcal{K}_{N^m}(\boldsymbol{V}^*) - \boldsymbol{V}^* + G] \\ &= [\mathcal{I} - \mathcal{K}'_{N^m}(\boldsymbol{V}^*)]^{-1}][\mathcal{K}_{N^m}(\boldsymbol{V}^*) - \mathcal{K}(\boldsymbol{V}^*)].\end{aligned}$$

Given that $\mathcal{K}_{N^m}(\boldsymbol{V}^*)$ *pointwise convergence to* $\mathcal{K}(\boldsymbol{V}^*)$*, when N is large enough, more precisely, for* $N \ge N_2$*, we can derive the following:*

$$\|\mathcal{T}\boldsymbol{V}^* - \boldsymbol{V}^*\|_\infty \le (1-\alpha)\rho_0.$$

On the contrary, in the case where $N \ge N_3 = \max\{N_1, N_2\}$*, for an arbitrary* $\boldsymbol{V} \in \boldsymbol{B}(\boldsymbol{V}^*, \rho_0)$*,*

$$\begin{aligned}\|\mathcal{T}\boldsymbol{V} - \boldsymbol{V}^*\|_\infty &\le \|\mathcal{T}\boldsymbol{V} - \boldsymbol{V}^*\|_\infty + \|\mathcal{T}\boldsymbol{V}^* - \boldsymbol{V}^*\|_\infty \\ &\le \alpha\rho_0 + (1-\alpha)\rho_0 = \rho_0.\end{aligned}$$

Hence, for $N \ge N_3$*,* $\mathcal{T}$ *maps* $\boldsymbol{B}(\boldsymbol{V}^*;\rho_0)$ *into itself; by* (4.10)*, this leads to a contradiction. Therefore, the existence and uniqueness for* (3.6) *hold. Then, we aim to prove the convergence for the current numerical approach. By virtue of Equations* (3.4) *and* (3.8)*, we obtain the following:*

$$\begin{aligned}&[\mathcal{I} - \mathcal{K}'_{N^m}(\boldsymbol{V}^*)](\boldsymbol{V}_{N^m} - \boldsymbol{V}^*) \\ &= \mathcal{K}_{N^m}(\boldsymbol{V}^*) - \mathcal{K}(\boldsymbol{V}^*) - [\mathcal{K}_{N^m}(\boldsymbol{V}^*) - \mathcal{K}_{N^m}(\boldsymbol{V}_{N^m}) - \mathcal{K}'_{N^m}(\boldsymbol{V}^*)(\boldsymbol{V}^* - \boldsymbol{V}_{N^m})].\end{aligned}$$

Using the properties of $\mathcal{K}'_{N^m}$*,* $\mathcal{K}''_{N^m}$*, and Lemma 4.2, we obtain the following:*

$$\|\boldsymbol{V}_{N^m} - \boldsymbol{V}^*\|_\infty \le C_0\left\{\|\mathcal{K}_{N^m}(\boldsymbol{V}^*) - \mathcal{K}(\boldsymbol{V}^*)\|_\infty + \frac{1}{2}M\|\boldsymbol{V}_{N^m} - \boldsymbol{V}^*\|_\infty^2\right\};$$

therefore, the following results are concluded

$$\begin{aligned}\|\boldsymbol{V}_{N^m} - \boldsymbol{V}^*\|_\infty &\le \frac{C_0\|\mathcal{K}_{N^m}(\boldsymbol{V}^*) - \mathcal{K}(\boldsymbol{V}^*)\|_\infty}{1 - (MC_0/2)\|\boldsymbol{V}_{N^m} - \boldsymbol{V}^*\|_\infty} \\ &\le \frac{C_0}{1-\alpha/2}\|\mathcal{K}_{N^m}(\boldsymbol{V}^*) - \mathcal{K}(\boldsymbol{V}^*)\|_\infty.\end{aligned}$$

By the Theorem 4.1, the convergence of $\boldsymbol{V}_{N^m}$ *is proven.* □

Lemma 4.3. *Assume that $\boldsymbol{K_q V^*}(\boldsymbol{s}) = \boldsymbol{K}(\boldsymbol{x}, \boldsymbol{s}, \boldsymbol{V}^*(\boldsymbol{s}))\psi_q'(\boldsymbol{s})$, and all partial derivatives of total order q are integrable with $\mathbf{s}$ over $\Omega = [0,1]^m$. Then, we have the following:*

$$\|\mathcal{K}_{N^m} V^* - \mathcal{K} V^*\|_\infty \le \max_{\boldsymbol{y}\in\Omega} \left| \sum_{\boldsymbol{\mu}\ is\ even,\ \boldsymbol{\mu}\in[\bar{q}+1,p]} \frac{B_{\boldsymbol{\mu}}(Q, \boldsymbol{K_q V^*}(\boldsymbol{s}))}{N^{\boldsymbol{\mu}}} \right| + o(N^{-p}), \tag{4.11}$$

where $\bar{q} = 2q+1$ or q depending on whether q is even or odd, and

$$B_{\boldsymbol{\mu}}(Q, \boldsymbol{K V^*}(\boldsymbol{s})) = \sum_{\sum_{i=1}^{m} \mu_i = \boldsymbol{\mu}} B_{\mu_1,\mu_2,\cdots,\mu_m}(Q, \boldsymbol{K_q V^*}),$$

with

$$B_{\mu_1,\mu_2,\cdots,\mu_m}(Q, \boldsymbol{K_q V^*}) = \prod_{i=1}^{m} c_{\mu_i} \int_{[0,1]^m} \frac{\partial^{\mu}}{\partial s_1^{\mu_1} \partial s_2^{\mu_2} \cdots \partial s_m^{\mu_m}} (\boldsymbol{K_q V^*}) d^m \boldsymbol{s},$$

where c_{μ_i} are constants independent of $\boldsymbol{K V^*}(\boldsymbol{s})$ and N, but are related to the Bernoulli numbers and polynomials.

Corollary 4.1. *Under the assumptions (i)-(iii) and supposing that $\mathcal{I} - \mathcal{K}'(\boldsymbol{U}^*)$ is nonsingular (i.e., $\boldsymbol{U}^* = (u_1^*, u_2^*, \cdots, u_n^*)^T$ represents an isolated solution for* (1.3)*), then there exist two constants ρ_0' with $0 \le \rho_0' \le r_0'$ and C' such that* (3.10) *has a unique solution $\boldsymbol{U}_{N^m} = (u_{1,N^m}, u_{2,N^m}, \cdots, u_{n,N^m})^T \in \boldsymbol{B}(\boldsymbol{U}^*; \rho_0') = [B(u_1^*; \rho_0'), B(u_2^*; \rho_0'), \cdots, B(u_n^*; \rho_0')]^T$ for arbitrary large enough N, and*

$$\begin{aligned} \|\boldsymbol{U}_{N^m} - \boldsymbol{U}^*\|_\infty &\le C' \|\mathcal{K}_{N^m} \boldsymbol{U}^* - \mathcal{K} \boldsymbol{U}^*\|_\infty \\ &\le C' \max_{\boldsymbol{x}\in\Omega} \left| \sum_{\boldsymbol{\mu}\ even,\ \boldsymbol{\mu}\in[\bar{q}+1,p]} \frac{B_{\boldsymbol{\mu}}(Q, \boldsymbol{K U^*}(\boldsymbol{t}))}{N^{\boldsymbol{\mu}}} \right| + o(N^{-p}), \end{aligned} \tag{4.12}$$

Proof. *The conclusions are straightforwardly obtained by applying Theorem 2.1 and Lemma 4.2.* □

Remark. *Based on Theorem 4.1 and Corollary 4.1, it can be observed that the numerical error for the given scheme depends on the presented formula* (2.11)*. Consequently, the numerical error is determined by the properties of $K_{ij}(\boldsymbol{x}, \boldsymbol{t}, u_j(\boldsymbol{t}))$ and the transformation parameter value of q. To enhance the accuracy order for the described approach, we can adjust the value of parameter q.*

5. Numerical examples

In this section, a series of numerical examples are presented to display the accuracy for the established scheme. All numerical experiments are performed on a personal computer equipped with MATLAB. To analyze the error characteristics of the proposed approach, the following notations are defined:

$$e_i(\boldsymbol{x}) = |u_i(\boldsymbol{x}) - u_{i,N^m}(\boldsymbol{x})|, \quad \boldsymbol{x} \in \Omega,$$
$$\text{Max Error} = \max_{\boldsymbol{x}\in\Omega} |u_i(\boldsymbol{x}) - u_{i,N^m}(\boldsymbol{x})|,$$

where $u_i(\boldsymbol{x})$ denotes the exact solutions, and $u_{i,N^m}(\boldsymbol{x})$ denotes the approximation solutions.

Example 1. *([31]) Take the subsequent 2DNFIEs into account:*

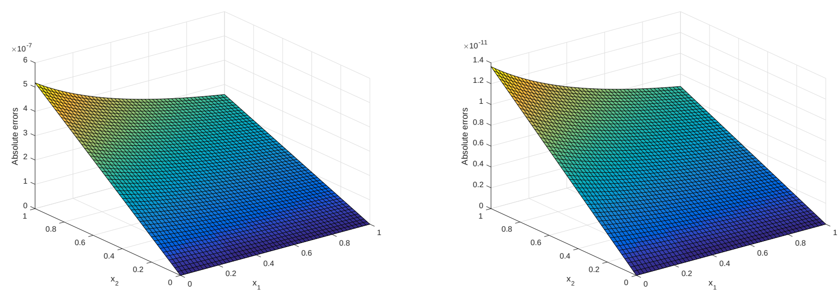
$$u(x_1, x_2) = g(x_1, x_2) + \int_0^1 \int_0^1 \frac{x_1(1 - t_1^2)}{(1 + x_2)(1 + t_2^2)}(1 - e^{-u(t_1,t_2)})\mathrm{d}t_1\mathrm{d}t_2,$$

where

$$g(x_1, x_2) = -\log\left(1 + \frac{x_1 x_2}{1 + x_2^2}\right) + \frac{x_1}{16(1 + x_2)}.$$

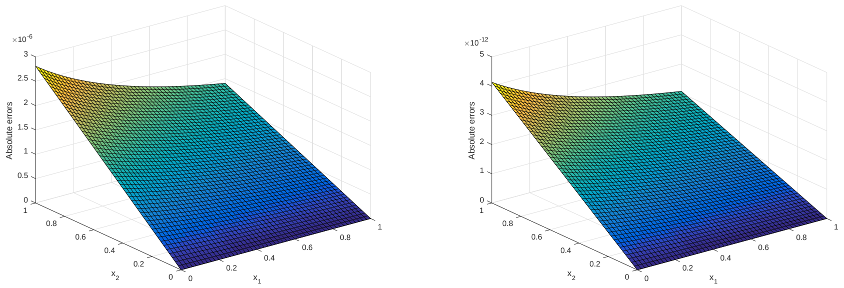
The exact solution is $u(x_1, x_2) = -\log(1 + \frac{x_1 x_2}{1+x_2^2})$*, with* $(x_1, x_2) \in [0, 1] \times [0, 1]$.

The numerical absolute errors for the given approach are plotted for different values of N in Figures 1 and 2, which correspond to $q = 4$ and $q = 6$, respectively. For a fixed $N = 10$, we can derive more accurate results for $q = 4$ than that for $q = 6$. However, for a fixed $N = 20$, the results of Sidi's transformation with $q = 6$ are more accurate than the case with $q = 4$. Figure 3 presents the maximum errors of the proposed scheme for distinct q and N at quadrature mesh points. It should be noted that the trapezoidal rule is directly applied to the proposed method for $q = 0$ without any transformation. The convergence speeds of the given approach for various p can be observed in Figure 3, thus demonstrating a faster convergence compared to the case without transformation. In Table 1, by comparing the established approach with the approach of [31], the presented scheme obtains more accurate numerical results than the approach of [31], both in the cases without extrapolation and with h^2-extrapolation. Moreover, the degrees of accuracy for the established scheme associated with $q = 2$ and $q = 4$ are higher than those of the method in [31] using h^2-extrapolation. Within maximum absolute errors, the computational times of both methods are comparable and exhibit the same order of magnitude.



(a) $q = 4, N = 10$ (b) $q = 4, N = 20$

Figure 1. Absolute errors for various values of N with $q = 4$ in Example 1.



(a) $q = 6, N = 10$ (b) $q = 6, N = 20$

Figure 2. Absolute errors for various values of N with $q = 6$ in Example 1.

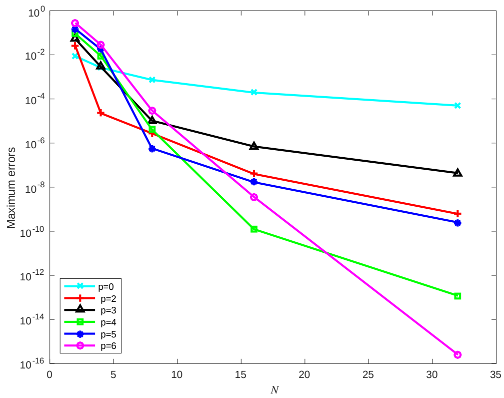


Figure 3. Maximum errors for various values of q and N at quadrature points in Example 1.

Table 1. Comparison of absolute errors with $N = 16$ in Example 1.

(x_1, x_2)	Present method		Method in [31]	
	$q = 2, N = 16$	$q = 4, N = 16$	$N = 16$	h^2-Extrapolation
$(\frac{1}{2^1}, \frac{1}{2^1})$	1.33e-08	4.15e-11	6.90e-05	3.72e-07
$(\frac{1}{2^2}, \frac{1}{2^2})$	7.97e-09	2.49e-11	4.14e-05	2.23e-07
$(\frac{1}{2^3}, \frac{1}{2^3})$	4.43e-09	1.38e-11	2.30e-05	1.24e-07
$(\frac{1}{2^4}, \frac{1}{2^4})$	2.35e-09	7.33e-12	1.22e-05	6.56e-08
$(\frac{1}{2^5}, \frac{1}{2^5})$	1.21e-09	3.77e-12	6.27e-06	3.38e-08
$(\frac{1}{2^6}, \frac{1}{2^6})$	6.13e-10	1.92e-12	3.18e-06	1.72e-08
Max Error	1.93e-08	6.21e-11	2.07e-04	6.74e-08
Cost Time	0.049s	0.077s	0.033s	0.062s

Example 2. *Take the subsequent system of 2DNFIEs into account:*

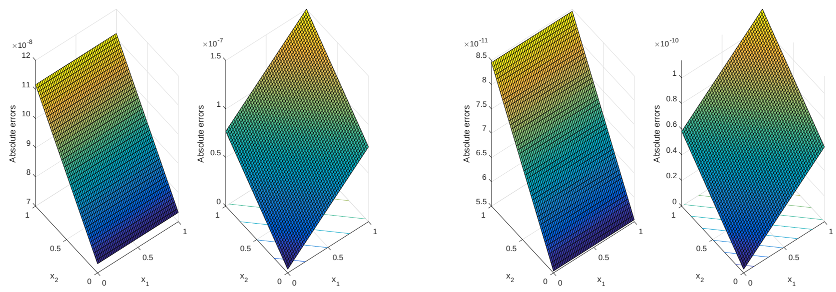
$$\begin{cases} u_1(x_1,x_2) = g_1(x_1,x_2) + \int_0^1\int_0^1 x_1(t_1+t_2)u_1(t_1,t_2)\mathrm{d}t_1\mathrm{d}t_2 - \int_0^1\int_0^1 t_2u_2(t_1,t_2)\mathrm{d}t_1\mathrm{d}t_2, \\ u_2(x_1,x_2) = g_2(x_1,x_2) + \int_0^1\int_0^1 t_1t_2u_1(t_1,t_2)\mathrm{d}t_1\mathrm{d}t_2 - \int_0^1\int_0^1 (x_1+x_2)t_1u_2(t_1,t_2)\mathrm{d}t_1\mathrm{d}t_2, \end{cases}$$

where

$$(g_1(x_1,x_2), g_2(x_1,x_2)) = (-\frac{1}{6}x_1 + x_2 + \frac{1}{3}, \frac{2}{3}x_1 - \frac{1}{3}x_2 - \frac{1}{3}).$$

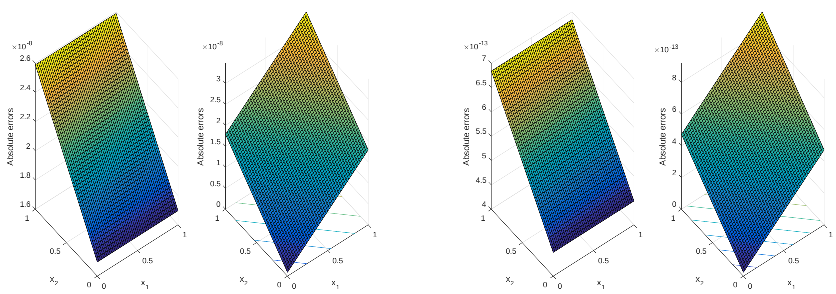
The exact solution is $(u_1(x_1,x_2), u_2(x_1,x_2)) = (x_1+x_2, x_1)$*, with* $(x_1,x_2) \in [0,1]\times[0,1]$.

The absolute errors that correspond to the transformations $\psi_4(t)$ and $\psi_6(t)$ are plotted in Figures 4 and 5, respectively. It can be clearly demonstrated that the approximate solution exhibits a high degree of agreement with the exact solution. Figure 6 demonstrates that the approximate solution converges rapidly to the exact solution as N increases for different values of q. Meanwhile, the convergence rate of the proposed scheme is enhanced by appropriately adjusting the transformation parameter q.



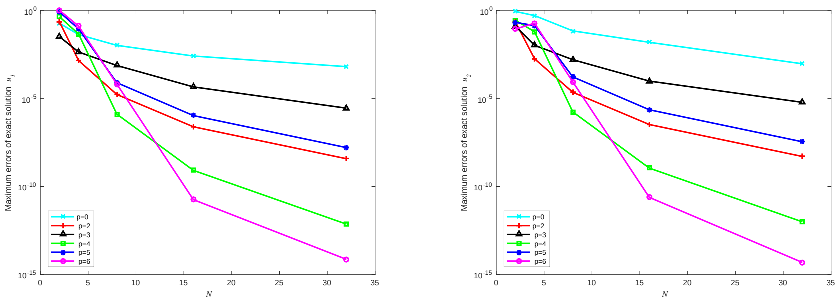
(a) $q = 4, N = 10$ (b) $q = 4, N = 20$

Figure 4. Absolute errors for various values of N with $q = 4$ in Example 2



(a) $q = 6, N = 10$ (b) $q = 6, N = 20$

Figure 5. Absolute errors for various values of N with $q = 6$ in Example 2.



(a) u_1 (b) u_2

Figure 6. Maximum errors for various values of q and N at quadrature points in Example 2.

Example 3. *([20]) Take the subsequent system of 2DNFIEs into account:*

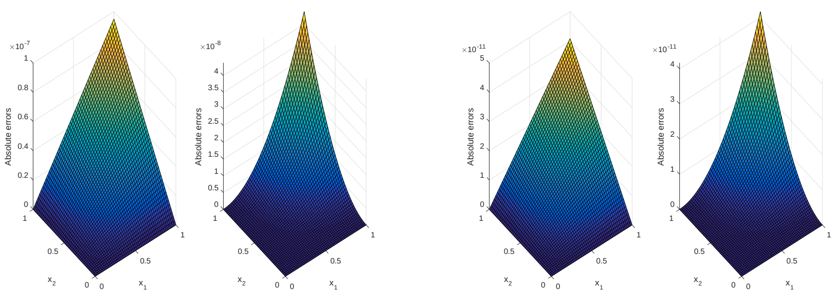
$$\begin{cases} u_1(x_1, x_2) = g_1(x_1, x_2) + \int_0^1 \int_0^1 x_1 x_2 t_1^2 t_2^2 [u_1^2(t_1, t_2) + u_2^2(t_1, t_2)] \mathrm{d}t_1 \mathrm{d}t_2, \\ u_2(x_1, x_2) = g_2(x_1, x_2) + \int_0^1 \int_0^1 x_1^2 x_2^2 t_1 t_2 [u_1^2(t_1, t_2) + u_2^2(t_1, t_2)] \mathrm{d}t_1 \mathrm{d}t_2, \end{cases}$$

where

$$(g_1(x_1, x_2), g_2(x_1, x_2)) = (\frac{1151}{1225} x_1 x_2, \frac{131}{144} x_1^2 x_2^2).$$

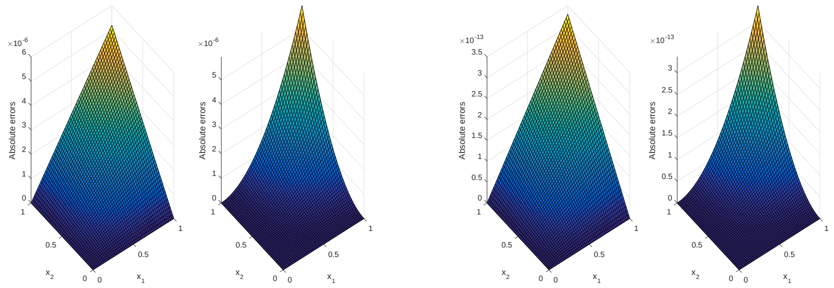
The exact solution is $(u_1(x_1, x_2), u_2(x_1, x_2)) = (x_1 x_2, x_1^2 x_2^2)$, *with* $(x_1, x_2) \in [0, 1] \times [0, 1]$.

In Figures 7 and 8, the absolute errors are calculated toward different values of N with regard to $q = 4$ and $q = 6$, respectively. In Figure 9, the maximum absolute errors are plotted at quadrature points; it can be seen that errors reduce as the quadrature points increases for several selected values of q, which display the accuracy orders of the proposed scheme. The simulation results demonstrate that the presented approach that incorporates Sidi's transformation can enhance the convergence order when q is even. Table 2 compares the absolute errors and cost time of the proposed approach with the Lagrange interpolation method in [20]. Evidently, the given scheme achieves a higher accuracy with fewer number of mesh points and a lower computational cost compared to the approach described in [20].



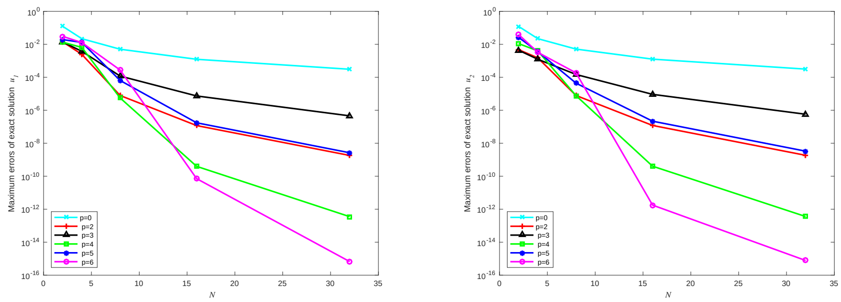
(a) $q = 4, N = 10$ (b) $q = 4, N = 20$

Figure 7. Absolute errors for various values of N with $q = 4$ in Example 3.



(a) $q = 6, N = 10$ (b) $q = 6, N = 20$

Figure 8. Absolute errors for various values of N with $q = 6$ in Example 3.



(a) u_1 (b) u_2

Figure 9. Maximum errors for various values of q and N at quadrature points in Example 3.

Table 2. Comparison of absolute errors in Example 3.

(x_1, x_2)	Present method $q = 4, N = 12$		Method in [20] $N = 16$	
	$e_1(x_1, x_2)$	$e_2(x_1, x_2)$	$e_1(x_1, x_2)$	$e_2(x_1, x_2)$
$(\frac{1}{2^1}, \frac{1}{2^1})$	1.74e-09	4.92e-10	8.49e-08	1.10e-08
$(\frac{1}{2^2}, \frac{1}{2^2})$	4.36e-10	3.07e-11	4.14e-09	7.36e-10
$(\frac{1}{2^3}, \frac{1}{2^3})$	1.09e-10	1.92e-12	2.52e-08	4.60e-10
$(\frac{1}{2^4}, \frac{1}{2^4})$	2.72e-11	1.20e-13	1.34e-09	2.88e-12
$(\frac{1}{2^5}, \frac{1}{2^5})$	6.81e-12	7.50e-15	3.53e-10	1.80e-12
$(\frac{1}{2^6}, \frac{1}{2^6})$	1.70e-12	4.69e-16	8.38e-11	1.12e-14
Max Error	3.20e-08	1.92e-08	9.50e-06	7.34e-06
Cost Time	0.0533	0.0533s	1.99s	1.99s

Example 4. *([20]) Take the subsequent system of 3DNFIEs into account:*

$$\begin{cases} u_1(\boldsymbol{x}) = g_1(\boldsymbol{x}) + \int_0^1 \int_0^1 \int_0^1 x_1x_2x_3u_1(\boldsymbol{t}) + (x_1 + x_2 + x_2)(t_1 + t_2 + t_3)[u_2(\boldsymbol{t}) + u_3(\boldsymbol{t})]\mathrm{d}^3\boldsymbol{t}, \\ u_2(\boldsymbol{x}) = g_2(\boldsymbol{x}) + \int_0^1 \int_0^1 \int_0^1 x_1e^{x_2+x_3}[u_1(\boldsymbol{t}) + u_2^2(\boldsymbol{t}) + u_3(\boldsymbol{t})]\mathrm{d}^3\boldsymbol{t}, \\ u_3(\boldsymbol{x}) = g_3(\boldsymbol{x}) + x_1x_2x_3 \int_0^1 \int_0^1 \int_0^1 t_1t_2t_3[u_1(\boldsymbol{t}) + u_2(\boldsymbol{t})] + t_1^2t_2t_3u_3^3(\boldsymbol{t})\mathrm{d}^3\boldsymbol{t}, \end{cases}$$

where $\boldsymbol{x} = (x_1, x_2, x_3), \boldsymbol{t} = (t_1, t_2, t_3)$*, and* $g_i(\boldsymbol{x})$*,* $i = 1, 2, 3$ *are known functions. The exact solution is* $(u_1(\boldsymbol{x}), u_2(\boldsymbol{x}), u_3(\boldsymbol{x})) = (x_1x_2x_3, x_1e^{x_2}x_3, x_1x_2^2x_3)$*, with* $(x_1, x_2, x_3) \in [0, 1] \times [0, 1] \times [0, 1]$.

Tables 3 and 4 display the absolute errors and convergence rates for the proposed method on the given mesh quadrature points from $q = 2$ and $q = 4$, respectively. The results show that the approximate order of convergence with $q = 4$ is higher than $q = 2$. In Table 5, the errors and computational time of the constructed scheme are compared with the barycentric Lagrange interpolation approach in [20] for $N = 4$. As this table illustrates, the proposed approach proves to be more efficient than the method elaborated in [20].

Table 3. Absolute errors for various values of N with $q = 2$ in Example 4.

$\boldsymbol{x} = (x_1, x_2, x_3)$	$N = 4, q = 2$			$N = 8, q = 2$		
	$e_1(\boldsymbol{x})$	$e_2(\boldsymbol{x})$	$e_3(\boldsymbol{x})$	$e_1(\boldsymbol{x})$	$e_2(\boldsymbol{x})$	$e_3(\boldsymbol{x})$
(0.2, 0.2, 0.2)	2.33e-04	2.19e-04	4.42e-06	1.97e-06	2.14e-06	2.39e-08
(0.4, 0.4, 0.4)	4.97e-04	6.53e-04	3.54e-05	4.22e-06	4.22e-06	1.88e-07
(0.6, 0.6, 0.6)	8.28e-04	1.46e-03	1.19e-04	6.99e-06	6.99e-05	6.34e-06
(0.8, 0.8, 0.8)	1.24e-03	2.91e-03	2.83e-04	1.05e-05	2.85e-05	1.50e-06
(1.0, 1.0, 1.0)	1.79e-03	5.42e-03	5.53e-04	1.52e-05	5.31e-05	2.94e-06
C.O.	–	–	–	6.68	6.33	6.33

Table 4. Absolute errors for various values of N with $q = 4$ in Example 4.

$\boldsymbol{x} = (x_1, x_2, x_3)$	$N = 4, q = 4$			$N = 8, q = 4$		
	$e_1(\boldsymbol{x})$	$e_2(\boldsymbol{x})$	$e_3(\boldsymbol{x})$	$e_1(\boldsymbol{x})$	$e_2(\boldsymbol{x})$	$e_3(\boldsymbol{x})$
(0.2, 0.2, 0.2)	5.25e-03	6.01e-03	4.78e-05	3.19e-07	2.38e-07	5.45e-09
(0.4, 0.4, 0.4)	1.12e-02	1.79e-03	3.82e-04	6.80e-07	7.10e-07	4.36e-08
(0.6, 0.6, 0.6)	1.86e-02	4.01e-02	1.29e-03	1.12e-06	1.59e-06	1.47e-07
(0.8, 0.8, 0.8)	2.81e-02	7.98e-02	3.06e-03	1.70e-06	3.16e-06	3.49e-07
(1.0, 1.0, 1.0)	4.03e-02	1.49e-01	5.98e-03	2.45e-06	5.89e-06	6.81e-07
C.O	–	–	–	13.92	14.46	14.46

Table 5. Comparison of absolute errors in Example 4.

$\boldsymbol{x} = (x_1, x_2, x_3)$	Present method, $q = 2, N = 4$			Method in [20], $N = 4$		
	$e_1(\boldsymbol{x})$	$e_2(\boldsymbol{x})$	$e_3(\boldsymbol{x})$	$e_1(\boldsymbol{x})$	$e_2(\boldsymbol{x})$	$e_3(\boldsymbol{x})$
$(\frac{1}{2^1}, \frac{1}{2^1}, \frac{1}{2^1})$	6.51e-04	9.97e-04	6.91e-05	9.37e-02	1.82e-01	3.43e-03
$(\frac{1}{2^2}, \frac{1}{2^2}, \frac{1}{2^2})$	2.95e-04	3.02e-04	8.64e-05	4.25e-02	5.51e-02	4.35e-04
$(\frac{1}{2^3}, \frac{1}{2^3}, \frac{1}{2^3})$	1.43e-04	1.78e-04	1.08e-06	2.07e-02	2.15e-02	5.44e-05
$(\frac{1}{2^4}, \frac{1}{2^4}, \frac{1}{2^4})$	7.13e-05	5.20e-05	1.35e-07	1.03e-02	9.47e-03	6.80e-06
$(\frac{1}{2^5}, \frac{1}{2^5}, \frac{1}{2^5})$	3.56e-05	2.44e-05	1.69e-08	5.13e-03	4.45e-03	8.50e-07
$(\frac{1}{2^6}, \frac{1}{2^6}, \frac{1}{2^6})$	1.78e-05	1.18e-05	2.11e-09	2.56e-03	2.16e-03	1.06e-07
Max Error	1.79e-03	5.42e-03	5.53e-04	1.11e-01	5.11e-03	2.20e-03
Cost Time	0.0528s	0.0528s	0.0528s	0.212s	0.212s	0.212s

6. Conclusions

In this study, a variable transformation technique was presented to solve the system of MDNFIEs of a second kind by leveraging Sidi's transformation associated with the trapezoidal quadrature rule. An error analysis revealed that the described approach can improve the convergence performance by adjusting the transformation parameter q. Simultaneously, the approach exhibited a high accuracy, low cost, and ease of implementation to solve (1.3). The presented scheme is not limited by the dimensionality of the target equations. Moreover, the proposed approach can effectively compete with the other advanced methods to solve (1.3). The given approach can be effortlessly extended to multi-dimensional Volterra Fredholm integral equations with singular kernels.

Author contributions

Yanying Ma: conceptualization, formal analysis, funding acquisition, investigation, methodology, project administration, software, validation, writing-original draft, writing-review & editing; Zhenxing Hao: software, validation, formal analysis, methodology, writing-original draft; Hongyan Liu: formal analysis, methodology, writing-review & editing; Guodong Wang: conceptualization, formal analysis, methodology, writing-original draft; Changqing Wang: software, formal analysis, methodology, writing-original draft. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this article.

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