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Research article

Inclusion of Bessel functions in a subclass of spiral functions

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Abstract: In this article, we derived conditions on the order ν of the classical Bessel functions J_ν that guarantee the inclusion of three distinct normalized forms of J_ν in a subclass of α-spirallike functions. The primary goal was to determine the subintervals within $(-\frac{\pi}{2}, \frac{\pi}{2})$ where these inclusion conditions are satisfied. A key component in establishing our results was the upper bound of the ratio $J_{\nu+1}(1)/J_\nu(1)$. The theoretical findings were validated through numerical experiments and accompanying graphical demonstrations.

Keywords: spiral functions; Bessel function; zero of Bessel functions; starlike
Mathematics Subject Classification: 30C45, 30C80, 40G05

1. Introduction

The study of analytic and univalent functions within the open unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

has led to the emergence of various geometric subclasses [1], each encapsulating distinct mapping behaviors. Of these, the class $\mathcal{A}$—consisting of functions analytic in $\mathbb{D}$ with normalization $f(0) = 0$ and $f'(0) = 1$—serves as the foundational framework. Within this class, numerous subclasses have been introduced to capture specific geometric and functional properties relevant in both theoretical investigations and applications. A central subclass for this study is the family of *starlike functions of order* η, denoted as $\mathrm{S}^*(\eta)$, defined by the inequality

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \eta, \quad 0 \le \eta < 1, \quad z \in \mathbb{D}.$$

This analytic condition geometrically implies that the image of $\mathbb{D}$ under such mappings is star-shaped with respect to the origin.

In the pursuit of generalizing classical starlikeness, Rønning [2] proposed the *parabolic starlike class* S_p^*, characterized by:

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right|, \quad z \in \mathbb{D}.$$

This condition defines a parabolic region in the complex plane symmetric about the real axis. Ali [3] introduced a unifying formulation using the set

$$\Omega_\rho = \{w = u + iv \in \mathbb{C} : v^2 < 4(1-\rho)(u-\rho)\}, \quad 0 \le \rho < 1,$$

leading to the class $\mathrm{PS}^*(\rho)$ such that

$$\frac{zf'(z)}{f(z)} \in \Omega_\rho.$$

This yields a broader understanding of parabolic-type domains under parameter variation.

The classical notion of starlike domains and functions can be extended by replacing straight line segments with logarithmic spirals. For a real parameter α satisfying $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$, define the curve

$$\gamma_\alpha(t) = \exp(te^{i\alpha}), \quad t \in \mathbb{R},$$

known as an *α-spiral*. More generally, its rotations $e^{i\theta}\gamma_\alpha(t)$, with $\theta \in \mathbb{R}$, also form α-spirals.

These curves are characterized (up to reparametrization) by having a constant radial angle α, i.e.,

$$\arg\left(\frac{\gamma_\alpha'(t)}{\gamma_\alpha(t)}\right) = \alpha.$$

Notably, this family of curves remains invariant under non-zero complex dilations $z \mapsto cz$ for $c \in \mathbb{C}\backslash\{0\}$.

Given $w \in \mathbb{C}$, the associated *α-spiral segment* from the origin to w is defined as

$$[0, w]_\alpha = \{w \exp(te^{i\alpha}) : t \le 0\} \cup \{0\}.$$

When $\alpha = 0$, this reduces to the standard line segment $[0, w]$.

A domain $\Omega \subset \mathbb{C}$ containing the origin is called *α-spirallike with respect to the origin* if for every $w \in \Omega$, the spiral segment $[0, w]_\alpha$ is entirely contained in Ω.

Analogously, an analytic function f on the unit disk $\mathbb{D}$ with $f(0) = 0$ is called *α-spirallike* if it maps $\mathbb{D}$ univalently onto an α-spirallike domain. It is known that such a function f is α-spirallike if and only if it satisfies the condition

$$\operatorname{Re}\left(e^{-i\alpha}\frac{zf'(z)}{f(z)}\right) > 0, \quad |z| < 1. \tag{1.1}$$

For a detailed proof and geometric interpretation, see §2.7 of Duren's book [1]. Some authors adopt a slightly different convention, using $i\alpha$ instead of $-i\alpha$ in condition (1.1) (see, for example, [4, §9.3]). The notion of the above spiralikeness also holds for $f \in \mathcal{A}$. We define the class $\mathcal{F}_\alpha \subset \mathcal{A}$ as those functions satisfying condition (1.1) for a fixed $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$. In particular, $\mathcal{F}_0$ coincides with the classical class $\mathcal{S}^*$ of starlike functions. Define the union

$$\mathcal{F} = \bigcup_{|\alpha|<\frac{\pi}{2}} \mathcal{F}_\alpha,$$

whose elements are simply referred to as *spirallike functions*. It is evident that $\mathcal{F} \subset \mathcal{S}$, but this class is more general and exhibits more intricate behavior. For more correspondence between $\mathcal{F}$ and $\mathcal{S}^*$, see [5]. A function f is convex spirallike if $zf'(z)$ is spirallike, combining the properties of convexity and spirallike behavior. The notion of α-spirallikeness of order δ was introduced by Libra in [6] through the analytic characterization

$$\operatorname{Re}\left(e^{-i\alpha}\frac{zf'(z)}{f(z)}\right) > \delta\cos(\alpha), \quad z \in \mathbb{D}, \quad 0 \le \delta < 1. \tag{1.2}$$

Ravichandran et al. [7] proposed a hybrid class combining spirallikeness with parabolic boundaries. A function f belongs to $SP_p(\alpha)$ if:

$$\operatorname{Re}\left(e^{-i\alpha}\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right|, \quad z \in \mathbb{D}, \quad |\alpha| < \frac{\pi}{2}. \tag{1.3}$$

From a geometric perspective, the inequality in (1.3) implies that the quantity $\frac{zf'(z)}{f(z)}$ lies within the parabolic region

$$P_\alpha = \left\{\omega \in \mathbb{C} : \operatorname{Re}\left(e^{-i\alpha}\omega\right) > |\omega - 1|\right\}.$$

The region P_α, which depends on the parameter α, is illustrated in Figure 1. We note here that the class P_α and Figure 1 ar also discussed in [8]. As discussed in [7], if $\omega \in P_\alpha$, then it follows that

$$\operatorname{Re}\left(e^{-i\alpha}\omega\right) > \frac{\cos(\alpha)}{2},$$

a fact that can also be observed from Figure 1. According to (1.2), this condition corresponds, from a geometric viewpoint, to α-spirallikeness of order $1/2$.

Inclusion properties of Bessel functions in the class $SP_p(\alpha)$ are the main topic for this study.

Let

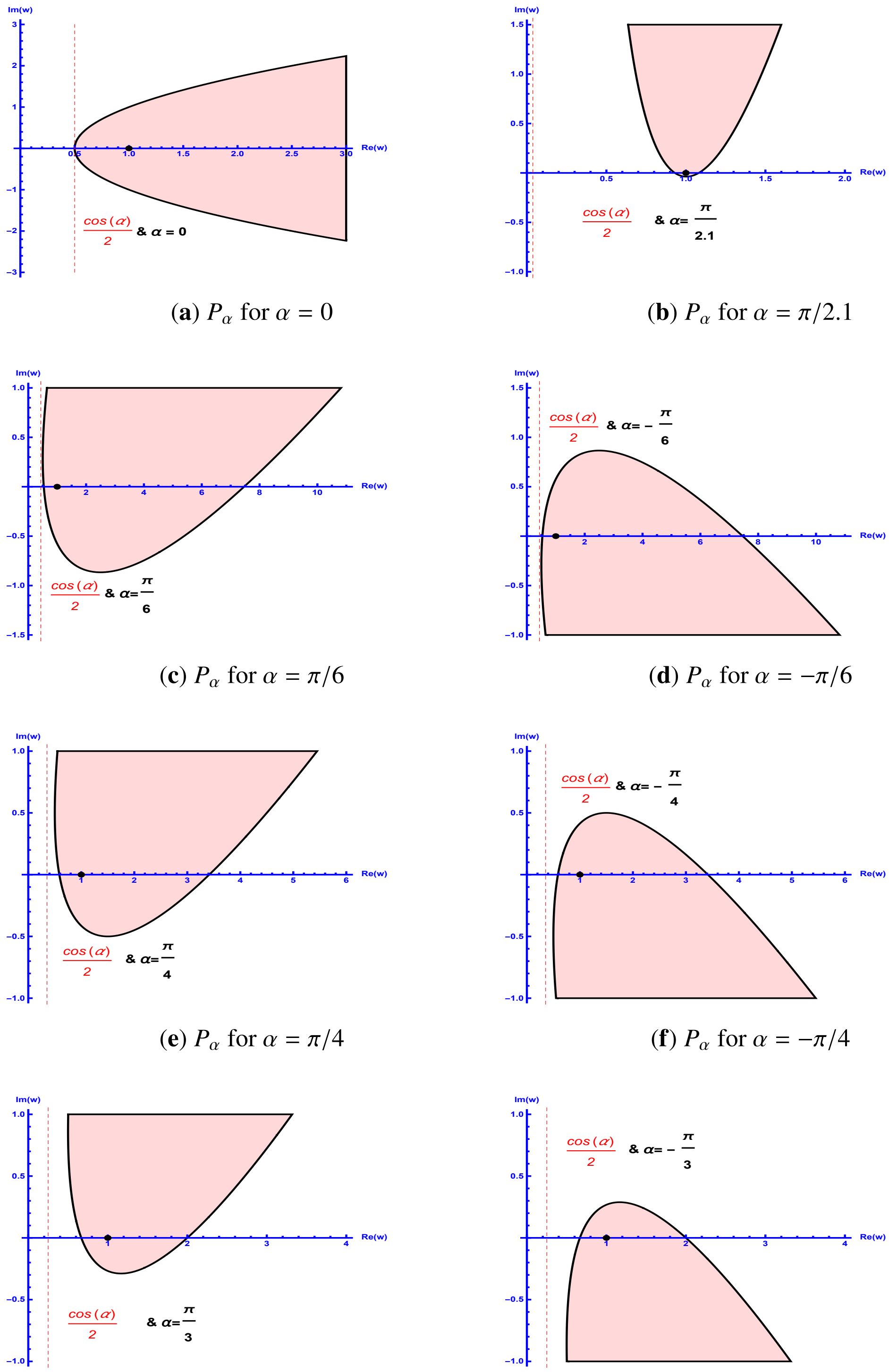
$$R_a = \begin{cases} a - \frac{1}{2}, & \frac{1}{2} < a \le \frac{3}{2}, \\ \sqrt{2a-2}, & \frac{3}{2} \le a < 3. \end{cases} \tag{1.4}$$

Then the following sufficient condition for a function to be in $SP_p(\alpha)$ is the main tool for this study.

Lemma 1.1 ([7]). *Let $\frac{1}{2} < a < 3$ and R_a be defined as above. If $f \in A$ satisfies*

$$\left|\frac{zf'(z)}{f(z)} - (a\cos\alpha - i\sin\alpha)\, e^{i\alpha}\right| \le R_a \cos\alpha, \tag{1.5}$$

then $f \in SP_p(\alpha)$.



(**a**) P_α for $\alpha = 0$ (**b**) P_α for $\alpha = \pi/2.1$

(**c**) P_α for $\alpha = \pi/6$ (**d**) P_α for $\alpha = -\pi/6$

(**e**) P_α for $\alpha = \pi/4$ (**f**) P_α for $\alpha = -\pi/4$

(**g**) P_α for $\alpha = \pi/3$ (**h**) P_α for $\alpha = -\pi/3$

Figure 1. Representation of parabolic regions P_α for different values of α [8].

Using Lemma 1.1, we are going to find sufficient conditions on the parameter of special functions f by which $f \in SP_p(\alpha)$. After a computation, one can see that the inequality (1.5) is equivalent to show

$$\left|\frac{zf'(z)}{f(z)} - 1 + (1-a)e^{i\alpha}\cos\alpha\right| \leq R_a \cos\alpha. \tag{1.6}$$

For $a = 1$, the inequality (1.6) is reduced to

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\cos\alpha}{2}, \tag{1.7}$$

which implies that $f \in S^*\left(1 - \frac{\cos\alpha}{2}\right)$.

This study relies on Lemma 1.1 as a foundational tool, primarily because of the geometric interpretation and the well-established inclusion properties associated with the starlike class.

The positive zeros $j_n(\nu)$ of the well-known classical Bessel function J_ν follow the increasing order $j_1(\nu) < j_2(\nu) < \ldots$ for $\nu \geq 0$. Furthermore, the Bessel function can also be represented by

$$J_\nu(z) = \frac{z^\nu}{2^\nu \Gamma(\nu+1)} \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{j_n^2(\nu)}\right). \tag{1.8}$$

For more details on the classical Bessel functions, see [9].

A logarithmic differentiation of (1.8) yields

$$\frac{zJ'_\nu(z)}{J_\nu(z)} = \nu - \sum_{n=1}^{\infty} \frac{2z^2}{j_n^2(\nu) - z^2}. \tag{1.9}$$

Before proceeding further, we recall the following inequality from [10, p. 670]

$$\frac{J_{\nu+1}(x)}{J_\nu(x)} < \frac{x}{2\nu+1}, \quad \nu > 0, \quad 0 < x \leq \frac{\pi}{2}. \tag{1.10}$$

A more stringent inequality than (1.10) is given in [11] as follwos:

$$\frac{J_{\nu+1}(x)}{J_\nu(x)} < \frac{x}{2(\nu+1)} + \frac{x^3 j_{\nu,1}^2}{8(\nu+1)^2(2+\nu)(j_{\nu,1}^2 - x^2)}, \quad \nu > -1, \quad 0 < x < j_{\nu,1}. \tag{1.11}$$

For our study, we need the best possible upper bound for $J_{\nu+1}(1)/J_\nu(1)$. We can consider the inequality (1.11) for $x = 1$ if $j_{\nu,1} > 1$ which is true for $\nu > \nu_0 = -0.774565$. Thus, for our purpose, we consider the following inequality at $x = 1$:

$$\frac{J_{\nu+1}(1)}{J_\nu(1)} < \frac{1}{2(\nu+1)} + \frac{j_{\nu,1}^2}{8(\nu+1)^2(2+\nu)(j_{\nu,1}^2 - 1)}, \quad \nu > \nu_0 = -0.774565. \tag{1.12}$$

The following three normalizations can be obtained from (1.8):

(i) $\mathcal{B}_1(\nu, z) := 2^\nu \Gamma(\nu+1) z^{1-\frac{\nu}{2}} J_\nu(\sqrt{z})$;
(ii) $\mathcal{B}_2(\nu, z) = 2^\nu \Gamma(\nu+1) z^{1-\nu} J_\nu(z)$;
(iii) $\mathcal{B}_3(\nu, z) = \left(2^\nu \Gamma(\nu+1) J_\nu(z)\right)^{\frac{1}{\nu}}$, $\nu > 0$.

The logarithmic differentiation of $\mathcal{B}_i(\nu, z)$, $i = 1, 2, 3$ yields

$$\frac{z\mathcal{B}_1'(\nu, z)}{\mathcal{B}_1(\nu, z)} = 1 - \frac{\nu}{2} + \frac{\sqrt{z}\, J_\nu'(\sqrt{z})}{2\, J_\nu(\sqrt{z})}, \tag{1.13}$$

$$\frac{z\mathcal{B}_2'(\nu, z)}{\mathcal{B}_2(\nu, z)} = 1 - \nu + \frac{z\, J_\nu'(z)}{J_\nu(z)}, \tag{1.14}$$

$$\frac{z\mathcal{B}_3'(\nu, z)}{\mathcal{B}_3(\nu, z)} = \frac{z\, J_\nu'(z)}{\nu\, J_\nu(z)}. \tag{1.15}$$

Next, we are going to study the spirallikeness of each of the above normalizations by using condition (1.6) and Lemma 1.1.

Our aim here is to answer the following problem:

Problem 1.1. *Determine the values of the parameter pair (a, ν) for which there exists a nontrivial interval $I_{a,\nu} \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that, for every $\alpha \in I_{a,\nu}$, the functions $\mathcal{B}_i(\nu, z)$, for $i = 1, 2, 3$, belong to the class $SP_p(\alpha)$.*

Before going further, it is helpful to review the important research that connects Bessel functions and their generalizations with geometric function theory (GFT). This connection began in the 1960s with early work by Brown [12–14], and Kreyszig and Todd [15]. Since then, many researchers have studied the geometric properties of Bessel functions—such as starlikeness, convexity (including convexity of order α), and close-to-convexity. Baricz and his co-authors have made major contributions in this area by finding clear conditions under which these properties hold [16, 17]. Their work also covers related topics like integral transforms and higher-order derivatives. Other studies on generalized Bessel functions can be found in [18, 19].

Researchers have also looked at more specific types of geometric functions, like exponential and lemniscate starlike or convex functions, in relation to Bessel-type functions [20, 21]. A key problem in this area is finding the largest radius in which these functions keep their geometric properties. For important work on this radius problem, we refer: [22] for radius of convexity of normalized Bessel functions; [23] in relation of bounds for radii of starlikeness and convexity of some special functions; [24] for radii of uniform convexity of some special functions; [25] for radius of uniform convexity of Bessel functions. However, there are fewer results on the spirallikeness of Bessel functions [8, 26], which encourages us to study Problem 1.1 to enhance the existing research.

The article is outlined as follows: Section 2 presents the theoretical results about the spiriallikeness of functions $\mathcal{B}_i(\nu, z)$ and solves Problem 1.1. Section 3 provides numerical and graphical validation of the main theorems.

2. Spirallikeness of Bessel functions

2.1. Spirallikeness of $\mathcal{B}_1(\nu, z)$

Theorem 2.1. *Suppose that for a fixed $a \in (1/2, 3)$ and $\nu > \nu_0$, there exists an interval $I_{a,\nu} \subseteq (-\pi/2, \pi/2)$ such that, for $\alpha \in I_{a,\nu}$,*

$$(R_a - |a - 1|)\cos(\alpha) > \frac{1}{4(\nu + 1)} + \frac{j_{\nu,1}^2}{16(\nu + 1)^2(2 + \nu)(j_{\nu,1}^2 - 1)}. \tag{2.1}$$

Then $\mathcal{B}_1(\nu, z) \in SP_p(\alpha)$.

Proof. Replace $f(z) = \mathcal{B}_1(\nu, z)$ in (1.6). Then, we can rewrite (1.6) as

$$\left|\frac{z\mathcal{B}_1'(\nu, z)}{\mathcal{B}_1(\nu, z)} - 1 + (1-a)e^{i\alpha}\cos\alpha\right| \leq R_a \cos\alpha. \tag{2.2}$$

For $z \in \mathbb{D}$, denote

$$F_{\mathcal{B}_1}(a, \alpha, \nu) := \left|\frac{z\mathcal{B}_1'(\nu, z)}{\mathcal{B}_1(\nu, z)} - 1 + (1-a)e^{i\alpha}\cos\alpha\right| - R_a \cos\alpha. \tag{2.3}$$

It follows from (1.8) that

$$\mathcal{B}_1(\nu, z) := 2^\nu \Gamma(\nu+1) z^{1-\frac{\nu}{2}} J_\nu(\sqrt{z}) = z \prod_{n=1}^{\infty}\left(1 - \frac{z}{j_n^2(\nu)}\right). \tag{2.4}$$

The logarithmic derivative of both sides of (2.4) gives

$$\frac{z\mathcal{B}_1'(\nu, z)}{\mathcal{B}_1(\nu, z)} = 1 - \sum_{n=1}^{\infty} \frac{z}{j_n^2(\nu) - z}. \tag{2.5}$$

Now we need the following inequality from [25]: If $|z| \leq r < a < b$, and $\alpha \in [0, 1]$, then

$$\left|\frac{z}{b-z} - \lambda\frac{z}{a-z}\right| \leq \frac{r}{b-r} - \lambda\frac{r}{a-r}. \tag{2.6}$$

Let $\lambda = 0$ and $b = j_n^2(\nu) > 1$. Since $|z| < 1$, we have

$$\left|\frac{z\mathcal{B}_1'(\nu, z)}{\mathcal{B}_1(\nu, z)} - 1\right| = \sum_{n=1}^{\infty}\left|\frac{z}{j_n^2(\nu) - z}\right| < \sum_{n=1}^{\infty} \frac{1}{j_n^2(\nu) - 1} = 1 - \frac{\mathcal{B}_1'(\nu, 1)}{\mathcal{B}_1(\nu, 1)}. \tag{2.7}$$

To prove (2.2), it is enough to show $F_{\mathcal{B}_1}(a, \alpha, \nu) < 0$. By applying the inequality (2.7), we have

$$\begin{aligned} F_{\mathcal{B}_1}(a, \alpha, \nu) &\leq \left|\frac{z\mathcal{B}_1'(\nu, z)}{\mathcal{B}_1(\nu, z)} - 1\right| - (R_a - |1-a|)\cos\alpha \\ &< 1 - \frac{\mathcal{B}_1'(\nu, 1)}{\mathcal{B}_1(\nu, 1)} - (R_a - |1-a|)\cos\alpha. \end{aligned} \tag{2.8}$$

Using well-known recurrence relations of Bessel functions and (1.13), we have

$$\begin{aligned} F_{\mathcal{B}_1}(a, \alpha, \nu) &< 1 - \frac{\mathcal{B}_1'(\nu, 1)}{\mathcal{B}_1(\nu, 1)} - (R_a - |1-a|)\cos\alpha \\ &= 1 - \left(1 - \frac{\nu}{2} + \frac{J_\nu'(1)}{2\,J_\nu(1)}\right) - (R_a - |1-a|)\cos\alpha \\ &= \frac{J_{\nu+1}(1)}{2\,J_\nu(1)} - (R_a - |1-a|)\cos\alpha \\ &< \frac{1}{4(\nu+1)} + \frac{j_{\nu,1}^2}{16(\nu+1)^2(2+\nu)(j_{\nu,1}^2 - 1)} - (R_a - |1-a|)\cos\alpha, \qquad \nu > \nu_0. \end{aligned}$$

Finally, $F_{\mathcal{B}_1}(a, \alpha, \nu) < 0$ follows from the given hypothesis (2.1). □

Next, we will refine Theorem 2.1 after conducting some numerical investigations. For further simplification, we denote

$$\mathtt{M}_1(\nu) := \frac{1}{4(\nu+1)} + \frac{j_{\nu,1}^2}{16(\nu+1)^2(2+\nu)(j_{\nu,1}^2-1)}, \nu > \nu_0. \tag{2.9}$$

Figure 2 represents $\mathtt{M}_1(\nu)$ in $(-1, 0]$. Clearly, $\mathtt{M}_1(\nu)$ has an asymptote, $\nu_0 = -0.774565$, which is the root of $j_{\nu,1}^2 - 1 = 0$ in $(-1, 0]$.

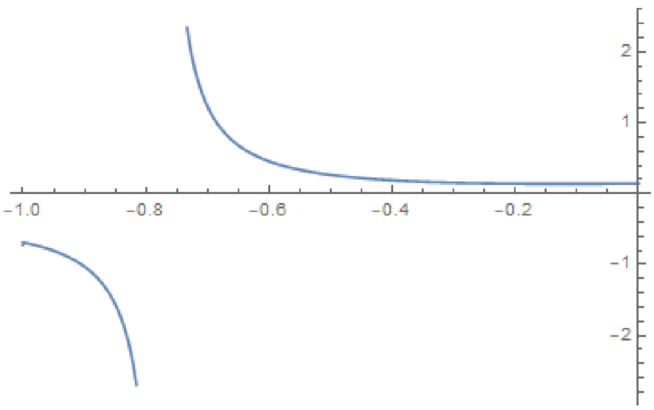


Figure 2. Graph of $\mathtt{M}_1(\nu)$ for $(-1, 0]$.

For the special case $a = 1$, we have the following result:

Corollary 2.1. *For a fixed $\nu \geq \nu_1 = -0.339429$, there exists an interval*

$$_{\mathcal{B}_1}I_\nu = \left(-\cos^{-1}(2\mathtt{M}_1(\nu)), \cos^{-1}(2\mathtt{M}_1(\nu))\right)$$

such that $\mathcal{B}_1(\nu, z) \in \mathtt{S}^*\left(1 - \frac{\cos\alpha}{2}\right)$ for $\alpha \in I_\nu$.

Here, ν_1 is the solution of $2\mathtt{M}_1(\nu) = 1$.

2.2. *Spirallikeness of $\mathcal{B}_2(\nu, z)$*

Now we are going to discuss the spirallikeness of $\mathcal{B}_2(\nu, z)$. For $z \in \mathbb{D}$, denote

$$F_{\mathcal{B}_2}(a, \alpha, \nu) := \left| \frac{z\mathcal{B}_2'(\nu, z)}{\mathcal{B}_2(\nu, z)} - 1 + (1-a)e^{i\alpha}\cos\alpha \right| - R_a \cos\alpha. \tag{2.10}$$

Following the proof of Theorem 2.1, we can prove that

$$F_{\mathcal{B}_2}(a, \alpha, \nu) = \frac{J_{\nu+1}(1)}{J_\nu(1)} - (R_a - |1-a|)\cos\alpha.$$

Thus, we have the following theorem.

Theorem 2.2. *Suppose that for a fixed* $a \in (1/2, 3)$ *and* $\nu > \nu_0$*, there exists an interval* $I_{a,\nu} \subseteq (-\pi/2, \pi/2)$ *such that, for* $\alpha \in I_{a,\nu}$*,*

$$(R_a - |a-1|)\cos(\alpha) > \frac{1}{2(\nu+1)} + \frac{j_{\nu,1}^2}{8(\nu+1)^2(2+\nu)(j_{\nu,1}^2-1)}, \quad \nu > \nu_0. \tag{2.11}$$

Then, $\mathcal{B}_2(\nu, z) \in SP_p(\alpha)$*.*

Now, it is interesting to observe and investigate the validity of (2.11) for $\nu > -1$ and $a \in (1/2, 3)$. We define

$$\mathtt{M}_2(\nu) := \frac{1}{2(\nu+1)} + \frac{j_{\nu,1}^2}{8(\nu+1)^2(2+\nu)(j_{\nu,1}^2-1)}, \quad \nu > \nu_0. \tag{2.12}$$

For the special case $a = 1$, we have the following result:

Corollary 2.2. *For a fixed* $\nu \geq \nu_2 = 0.123164$*, there exists an interval*

$${}_{\mathcal{B}_2}I_\nu = \left(-\cos^{-1}(2\mathtt{M}_2(\nu)), \cos^{-1}(2\mathtt{M}_2(\nu))\right)$$

such that $\mathcal{B}_2(\nu, z) \in \mathtt{S}^*\left(1 - \frac{\cos\alpha}{2}\right)$ *for* $\alpha \in {}_{\mathcal{B}_2}I_\nu$*.*

Here, ν_2 is the solution of $2\mathtt{M}_2(\nu) = 1$.

2.3. Spirallikeness of $\mathcal{B}_3(\nu, z)$

Following the approach of Theorem 2.1, we now investigate the spirallikeness of $\mathcal{B}_3(\nu, z)$.

Theorem 2.3. *Suppose that for a fixed* $a \in (1/2, 3)$ *and* $\nu > 0$*, there exists an interval* $I_{a,\nu} \subseteq (-\pi/2, \pi/2)$ *such that, for* $\alpha \in I_{a,\nu}$*,*

$$(R_a - |a-1|)\cos(\alpha) > \frac{1}{2\nu(\nu+1)} + \frac{j_{\nu,1}^2}{8\nu(\nu+1)^2(2+\nu)(j_{\nu,1}^2-1)}, \quad \nu > 0. \tag{2.13}$$

Then $\mathcal{B}_3(\nu, z) \in SP_p(\alpha)$*.*

Denote

$$\mathtt{M}_3(\nu) := \frac{1}{2\nu(\nu+1)} + \frac{j_{\nu,1}^2}{8\nu(\nu+1)^2(2+\nu)(j_{\nu,1}^2-1)}.$$

For the special case $a = 1$, we have the following result:

Corollary 2.3. *For a fixed* $\nu \geq \nu_3 = 0.645878$*, there exists an interval*

$${}_{\mathcal{B}_3}I_\nu = \left(-\cos^{-1}(2\mathtt{M}_3(\nu)), \cos^{-1}(2\mathtt{M}_3(\nu))\right)$$

such that $\mathcal{B}_3(\nu, z) \in \mathtt{S}^*\left(1 - \frac{\cos\alpha}{2}\right)$ *for* $\alpha \in {}_{\mathcal{B}_3}I_\nu$*.*

Here, ν_3 is the solution of $2\mathtt{M}_3(\nu) = 1$.

3. Numerical and graphical validations of main results

In this section, we are going to present the numerical and graphical validation of our main results presented in Section 2. For this, first we simplify the expression $R_a - |a-1|$.

$$R_a - |a-1| = \begin{cases} 2a - \frac{3}{2} < 0, & \frac{1}{2} < a < \frac{3}{4}, \\ 2a - \frac{3}{2} \geq 0, & \frac{3}{4} \leq a < 1, \\ \frac{1}{2}, & 1 \leq a \leq \frac{3}{2}, \\ \sqrt{2a-2} - a + 1, & \frac{3}{2} \leq a < 3. \end{cases}$$

Clealry, $R_a - |a-1| < 0$ for $a \in (1/2, 3/4)$ and $R_a - |a-1| \geq 0$ for $a \in [3/4, 3]$. Now we discuss several cases based on ν.

3.1. Spirallikeness of $\mathcal{B}_1(\nu, z)$

Target 1. *For $a \in (1/2, 3)$ and $\nu \in (\nu_0, 0]$, find $I_{a,\nu} \subseteq (-\pi/2, \pi/2)$ such that, for $\alpha \in I_{a,\nu}$,*

$$(R_a - |a-1|)\cos(\alpha) - M_1(\nu) > 0.$$

We discuss the prospective solutions of Target 1 by considering different specific values of a. The main aim is to see how the interval $I_{a,\nu}$ changes based on a and ν. We mainly calculate the value $\cos^{-1}(M_1(\nu)/(R_a - |a-1|))$ when it is defined in $(0, 1]$, and otherwise we state that the interval $I_{a,\nu}$ does not exist. Basically, our goal is to find the values of ν and a such that $(M_1(\nu)/(R_a - |a-1|)) < 1$.

(i) $(1/2, 3/4]$: In this case, $R_a - |a-1| \leq 0$. Thus, there is no $I_{a,\nu}$ such that

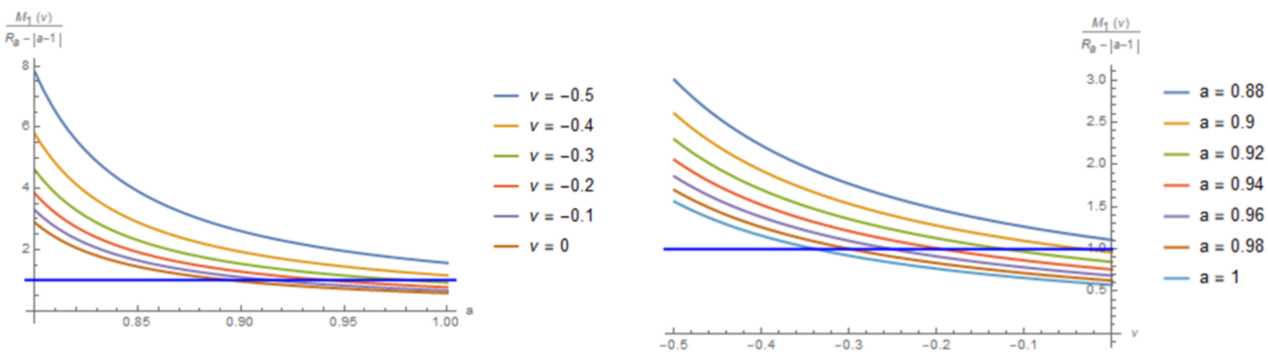
$$(R_a - |a-1|)\cos(\alpha) - M_1(\nu) > 0$$

holds.

(ii) $(3/4, 1]$: The possible cases when $I_{a,\nu}$ can exist are highlighted in Table 1. Similar validation is also done graphically in Figure 3a. To have a clear graphical presentation, we consider only $\nu = -0.5, -0.4, -0.3, -0.2, -0.1, 0$ and $-0.8 < a \leq 1$. For validation purposes, we have to observe the graph of $M_1(\nu)/(R_a - |a-1|)$, which lies between 0 and 1, that is below the bold blue line in Figure 3a. It is evident that for $\nu = -0.5, -0.4$, the curve for $M_1(\nu)/(R_a - |a-1|)$ never crosses that blue line, while $\nu = -0.3, -0.2, -0.1, 0$ does. It is interesting to find the best range of a for a fixed ν such that $(M_1(\nu))/(R_a - |a-1|) < 1$. We present these values in Table 2. A similar graphical validation can also be done by taking $\nu \in (\nu_0, 0]$ and a few fixed $a \in (3/4, 1]$ as presented in Figure 3b. The best range of ν for fixed a are given in Table 3.

Table 1. Solution of Target 1 for $a \in (3/4, 1]$.

$v \setminus a$	0.76	0.77	0.79	0.8	0.9	0.95	[1, 3/2]
− 0.7	140.117	70.0583	35.0291	28.0233	9.3411	7.00583	5.60466
− 0.6	60.6576	30.3288	15.1644	12.1315	4.04384	3.03288	2.42631
− 0.5	39.0123	19.5062	9.75308	7.80246	2.60082	1.95062	1.56049
− 0.4	28.8848	14.4424	7.2212	5.77696	1.92565	1.44424	1.15539
− 0.3	22.9994	11.4997	5.74986	4.59989	1.5333	1.14997	**0.919978**
− 0.2	19.1448	9.57238	4.78619	3.82895	1.27632	**0.957238**	**0.76579**
− 0.1	16.42	8.20999	4.105	3.284	1.09467	**0.820999**	**0.656799**
0	14.3892	7.19458	3.59729	2.87783	**0.959278**	**0.719458**	**0.575567**



(**a**) Target 1 for $a \in (3/4, 1]$ and some fixed $v_0 < v \leq 0$. (**b**) Target 1 for $v \in (v_0, 0]$ and some fixed $3/4 < a \leq 1$.

Figure 3. Graphical validation of Target 1 for $a \in (3/4, 1]$.

Table 2. Best range of a for a fixed v.

v	Best range of a
− 0.3	(0.979994, 1]
− 0.2	(0.941448, 1]
− 0.1	(0.9142, 1]
0	(0.893892, 1]

Table 3. Best range of v for a fixed a.

a	Best range of v
0.9	(− 0.0330101, 0]
0.92	(− 0.124054, 0]
0.94	(− 0.195429, 0]
0.96	(− 0.252847, 0]
0.98	(− 0.300012, 0]
1	(− 0.339429, 0]

(iii) $[1, 3/2]$: The last column in Table 1 presents the possible cases. In this case, $R_a - |a-1| = 1/2$ is a constant, and $(M_1(\nu))/(R_a - |a-1|) < 1$ is valid for $\nu \in (-0.339429, 0]$ as presented in Figure 4.

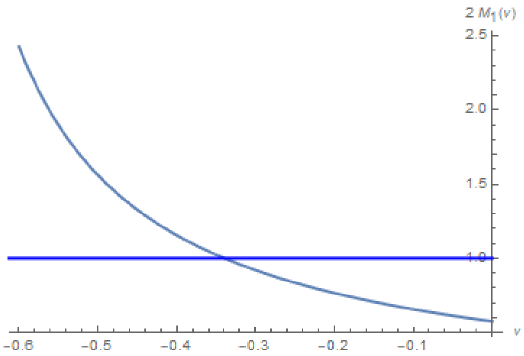
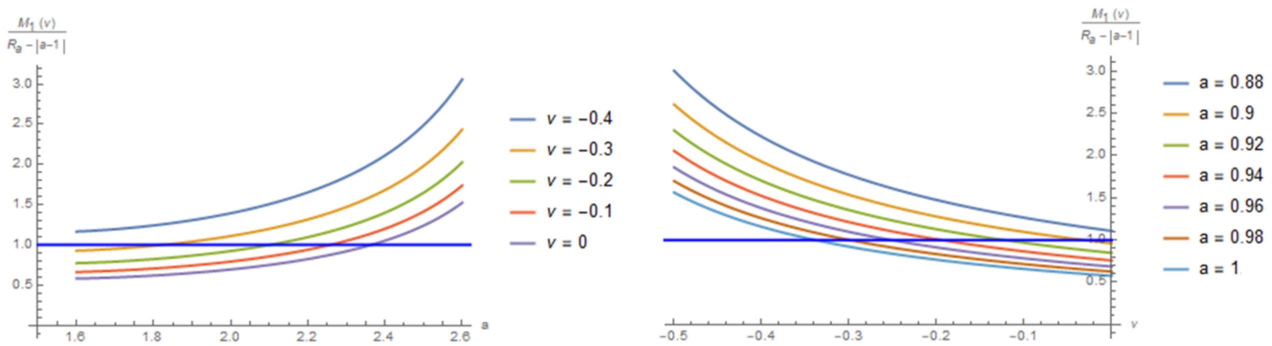


Figure 4. Best range of ν for a fixed $a \in [1, 3/2]$.

(iv) $(3/2, 3)$: The possible cases when $I_{a,\nu}$ can exist are highlighted in Table 4. Figure 5a present the case for $a \in (3/2, 3)$ and some fixed $\nu_0 < \nu \le 0$, while Figure 5b represents graphically the case for $\nu \in (\nu_0, 0]$ and some fixed $3/2 < a \le 3$.



(**a**) Target 1 for $a \in (3/2, 3)$ and some fixed $\nu_0 < \nu \le 0$.

(**b**) Target 1 for $\nu \in (\nu_0, 0]$ and some fixed $3/2 < a \le 3$.

Figure 5. Graphical validation of Target 1 for $a \in (3/2, 3)$.

Table 4. Solution of Target 1 for $a \in (3/2, 3)$.

$\nu \setminus a$	1.6	1.7	1.9	2	2.2	2.5	2.8	2.9	2.99
− 0.7	5.66	5.8	6.35	6.77	8.03	12.08	28.78	56.78	561.17
− 0.6	2.45	2.51	2.75	2.93	3.47	5.23	12.46	24.58	242.94
− 0.5	1.57	1.61	1.77	1.88	2.23	3.36	8.01	15.81	156.25
− 0.4	1.17	1.19	1.31	1.39	1.65	2.49	5.93	11.70	115.68
− 0.3	**0.93**	**0.95**	1.04	1.11	1.32	1.98	4.72	9.32	92.11
− 0.2	**0.77**	**0.79**	**0.87**	**0.92**	1.1	1.65	3.93	7.76	76.68
− 0.1	**0.66**	**0.68**	**0.74**	**0.79**	**0.94**	1.41	3.37	6.65	65.76
0	**0.58**	**0.6**	**0.65**	**0.69**	**0.82**	1.24	2.96	5.83	57.63

We note that Target 1 covers the discussion for $\nu \in (\nu_0, 0]$. Next, we will discuss the cases where $\nu \in (0, \infty)$.

Target 2. *For $a \in (1/2, 3)$ and $\nu \in (0, \infty)$, find $I_{a,\nu} \subseteq (-\pi/2, \pi/2)$ such that, for $\alpha \in I_{a,\nu}$, we have* $(R_a - |a-1|)\cos(\alpha) - \mathtt{M}_1(\nu) > 0$.

As earlier, we will separate it into four parts as follows:

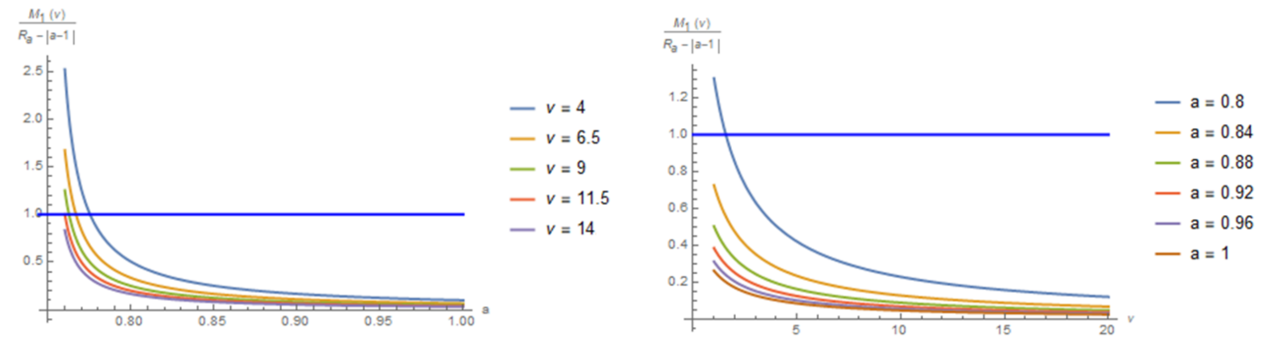
(i) $(1/2, 3/4]$: In this case, $R_a - |a-1| \leq 0$. Thus, there is no $I_{a,\nu}$ such that $(R_a - |a-1|)\cos(\alpha) - M_1(\nu) > 0$ holds.

(ii) $(3/4, 1]$: The possible cases when $I_{a,\nu}$ can exist are presented in Table 5. It can be observed that for $a = 0.9$ and $a = 0.95$, the existence of α is granted for all $\nu > 0$. On the other hand, for $\nu \geq 11.5$, the value of α can be obtained for all $a \in (3/4, 1]$. The graphical validation Figure 6a and Figure 6b provides a clearer view of this aspect.

(iii) $[1, 3/2]$: The last column in Table 5 presents the possible cases. It can be observed that α exists for all $\nu > 0$ and $a \in [1, 3/2]$. Graphically, this case is presented in Figure 7.

Table 5. Solution of Target 2 for $a \in (3/4, 3/2]$.

	0.76	0.77	0.79	0.8	0.9	0.95	$[1, 3/2]$
0	14.3892	7.19458	3.59729	2.87783	0.959278	0.719458	0.575567
1	6.52945	3.26473	1.63236	1.30589	0.435297	0.326473	0.261178
2	4.25689	2.12845	1.06422	0.851379	0.283793	0.212845	0.170276
3	3.16505	1.58252	0.791262	0.633009	0.211003	0.158252	0.126602
4	2.5212	1.2606	0.6303	0.50424	0.16808	0.12606	0.100848
5	2.0959	1.04795	0.523974	0.419179	0.139726	0.104795	0.0838359
6	1.79377	0.896884	0.448442	0.358754	0.119585	0.0896884	0.0717507
7	1.56797	0.783985	0.391992	0.313594	0.104531	0.0783985	0.0627188
8	1.39277	0.696386	0.348193	0.278555	0.0928515	0.0696386	0.0557109
9	1.25286	0.626428	0.313214	0.250571	0.0835238	0.0626428	0.0501143
10	1.13853	0.569263	0.284632	0.227705	0.0759017	0.0569263	0.045541
11	1.04334	0.521671	0.260836	0.208669	0.0695562	0.0521671	0.0417337
11.2	1.02619	0.513094	0.256547	0.205237	0.0684125	0.0513094	0.0410475
11.3	1.01782	0.50891	0.254455	0.203564	0.0678546	0.050891	0.0407128
11.4	1.00959	0.504794	0.252397	0.201917	0.0673058	0.0504794	0.0403835
11.5	0.999963	0.499981	0.249991	0.199993	0.0666642	0.0499981	0.0399985
11.6	0.993516	0.496758	0.248379	0.198703	0.0662344	0.0496758	0.0397407
12	0.962864	0.481432	0.240716	0.192573	0.0641909	0.0481432	0.0385146
13	0.893923	0.446962	0.223481	0.178785	0.0595949	0.0446962	0.0357569
14	0.834204	0.417102	0.208551	0.166841	0.0556136	0.0417102	0.0333682
15	0.78197	0.390985	0.195492	0.156394	0.0521313	0.0390985	0.0312788
16	0.735896	0.367948	0.183974	0.147179	0.0490597	0.0367948	0.0294358
17	0.694953	0.347477	0.173738	0.138991	0.0463302	0.0347477	0.0277981
18	0.658328	0.329164	0.164582	0.131666	0.0438886	0.0329164	0.0263331
19	0.625373	0.312686	0.156343	0.125075	0.0416915	0.0312686	0.0250149



(**a**) Target 2 for $a \in (3/4, 1]$ and some fixed $v \geq 0$.

(**b**) Target 2 for $v > 0$ and some fixed $a \in (3/4, 1]$.

Figure 6. Graphical validation of Target 2 for $a \in (3/4, 1]$.

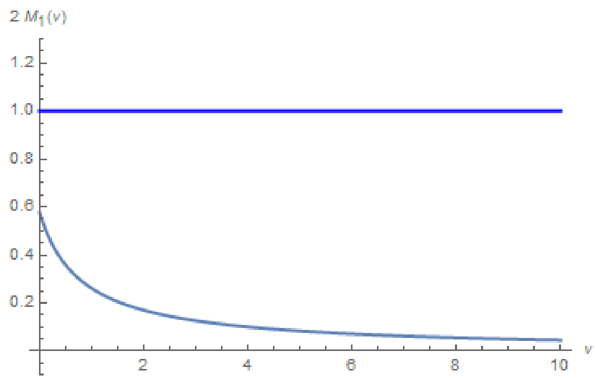
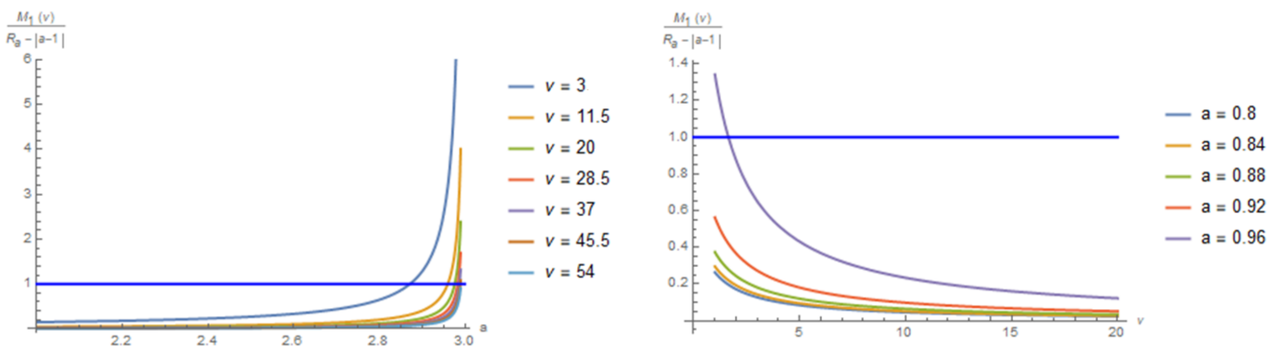


Figure 7. Best range of v for a fixed $a \in [1, 3/2]$ (Target 2).

(iv) $(3/2, 3)$: The possible cases when $I_{a,v}$ can exist are highlighted in Table 6. Graphically, these cases are illustrated in Figure 8a,8b.



(**a**) Target 2 for $a \in (3/2, 3)$ and some fixed $v \geq 0$.

(**b**) Target 2 for $v > 0$ and some fixed $a \in (3/2, 3)$.

Figure 8. Graphical validation of Target 2 for $a \in (3/2, 3)$.

Table 6. Solution of Target 2 for $a \in (3/2, 3)$.

	1.6	1.7	1.9	2	2.3	2.8	2.9	2.99
0	0.580858	0.595558	0.651623	0.69477	0.921049	2.95567	5.83043	57.6289
1	0.263579	0.27025	0.295691	0.31527	0.41795	1.34121	2.6457	26.1506
2	0.171841	0.17619	0.192776	0.205541	0.272483	0.874405	1.72487	17.0489
3	0.127766	0.130999	0.143331	0.152822	0.202594	0.65013	1.28246	12.6761
4	0.101775	0.104351	0.114174	0.121734	0.161382	0.517878	1.02158	10.0975
5	0.0846066	0.0867479	0.0949141	0.101199	0.134158	0.430517	0.849249	8.39411
6	0.0724104	0.0742429	0.081232	0.0866108	0.114819	0.368457	0.726827	7.18407
7	0.0632954	0.0648973	0.0710066	0.0757083	0.100366	0.322076	0.635335	6.27975
8	0.0562231	0.057646	0.0630727	0.067249	0.0891513	0.286088	0.564346	5.57808
9	0.050575	0.0518549	0.0567365	0.0604933	0.0801953	0.257348	0.507652	5.01772
10	0.0459597	0.0471229	0.0515589	0.0549729	0.072877	0.233864	0.461326	4.55982
11	0.0421174	0.0431833	0.0472485	0.0503771	0.0667843	0.214312	0.422758	4.17861
12	0.0388686	0.0398523	0.0436039	0.0464912	0.0616329	0.197781	0.390148	3.85629
13	0.0360857	0.0369989	0.0404819	0.0431624	0.05722	0.18362	0.362214	3.58018
14	0.0336749	0.0345272	0.0377775	0.0402789	0.0533973	0.171353	0.338016	3.341
15	0.0315664	0.0323652	0.035412	0.0377568	0.0500538	0.160624	0.316851	3.1318
20	0.0240414	0.0246499	0.0269704	0.0287562	0.0381218	0.122334	0.241319	2.38523
30	0.0162814	0.0166935	0.018265	0.0194744	0.025817	0.0828472	0.163427	1.61533
40	0.012309	0.0126205	0.0138086	0.014723	0.0195181	0.0626339	0.123553	1.22122
49.5	0.009993	0.010246	0.011210	0.011953	0.015846	0.050849	0.100306	0.991437

3.2. Spirallikeness of $\mathcal{B}_2(\nu, z)$

Note that $\mathtt{M}_2(\nu) = 2\mathtt{M}_1(\nu)$. Thus, we can reciprocate Targets 1 and 2 in association with $\mathtt{M}_2(\nu)$. Following the discussion of the solution of Target 1, it can be shown that: $\forall a \in (1/2, 3/4)$ and $\nu > \nu_0$, there is no $\alpha \in (-\pi/2, \pi/2)$, such that (2.11) holds. Next, consider $\nu \in (\nu_0, 0)$. Then, due to the relation $\mathtt{M}_2(\nu) = 2\mathtt{M}_1(\nu)$, multiplying each entry of Tables 1 and 4, we observe that all values are greater than 1, and therefore there does not exist any $\alpha \in (-\pi/2, \pi/2)$ such that (2.11) holds. Based on observations from Tables 1 and 4, we propose the following open problem:

Open Problem 1. *For a fixed $a \in (3/4, 3)$ and for a fixed $\nu \in (\nu_0, 0]$, there exist no $I_{a,\nu}$ such that inequality* (2.11) *is true for $\alpha \in I_{a,\nu}$.*

However, from Table 7, it is clear that for a fixed $a \in (3/4, 3/2]$, the required interval $I_{a,\nu}$ exists when ν is increasing, and for $\nu \geq 24.1$, the interval $I_{a,\nu}$ exists for all $a \in (3/4, 3/2]$. Similarly, from Table 8, one can see that the interval $I_{a,\nu}$ exists for all $a \in (3/2, 3)$ when $\nu \geq 49.07$.

Table 7. Existence of $I_{a,\nu}$ for fixed $\nu > 0$ and $a \in (3/4, 3/2]$ in relation to $\mathtt{M}_2$.

ν/a	0.76	0.77	0.79	0.8	0.9	0.95	[1, 3/2]
0	28.7783	14.3892	7.19458	5.75567	1.91856	1.43892	1.15113
1	13.0589	6.52945	3.26473	2.61178	0.870593	0.652945	0.522356
2	8.51379	4.25689	2.12845	1.70276	0.567586	0.425689	0.340551
3	6.33009	3.16505	1.58252	1.26602	0.422006	0.316505	0.253204
4	5.0424	2.5212	1.2606	1.00848	0.33616	0.25212	0.201696
5	4.19179	2.0959	1.04795	0.838359	0.279453	0.20959	0.167672
6	3.58754	1.79377	0.896884	0.717507	0.239169	0.179377	0.143501
7	3.13594	1.56797	0.783985	0.627188	0.209063	0.156797	0.125438
8	2.78555	1.39277	0.696386	0.557109	0.185703	0.139277	0.111422
9	2.50571	1.25286	0.626428	0.501143	0.167048	0.125286	0.100229
10	2.27705	1.13853	0.569263	0.45541	0.151803	0.113853	0.0910821
15	1.56394	0.78197	0.390985	0.312788	0.104263	0.078197	0.0625576
20	1.19112	0.595561	0.29778	0.238224	0.0794081	0.0595561	0.0476449
24	1.00039	0.500193	0.250096	0.200077	0.0666923	0.0500193	0.0400154
24.1	0.996396	0.498198	0.249099	0.199279	0.0664264	0.0498198	0.0398559

Table 8. Existence of $I_{a,\nu}$ for fixed $\nu > 0$ and $a \in (3/2, 3)$ in relation to $\mathtt{M}_2$.

ν/a	1.6	1.7	1.9	2	2.3	2.8	2.9	2.99
0	0.580858	0.595558	0.651623	0.69477	0.921049	2.95567	5.83043	57.6289
1	0.263579	0.27025	0.295691	0.31527	0.41795	1.34121	2.6457	26.1506
2	0.171841	0.17619	0.192776	0.205541	0.272483	0.874405	1.72487	17.0489
3	0.127766	0.130999	0.143331	0.152822	0.202594	0.65013	1.28246	12.6761
4	0.101775	0.104351	0.114174	0.121734	0.161382	0.517878	1.02158	10.0975
5	0.0846066	0.0867479	0.0949141	0.101199	0.134158	0.430517	0.849249	8.39411
6	0.0724104	0.0742429	0.081232	0.0866108	0.114819	0.368457	0.726827	7.18407
7	0.0632954	0.0648973	0.0710066	0.0757083	0.100366	0.322076	0.635335	6.27975
8	0.0562231	0.057646	0.0630727	0.067249	0.0891513	0.286088	0.564346	5.57808
9	0.050575	0.0518549	0.0567365	0.0604933	0.0801953	0.257348	0.507652	5.01772
10	0.0459597	0.0471229	0.0515589	0.0549729	0.072877	0.233864	0.461326	4.55982
20	0.0240414	0.0246499	0.0269704	0.0287562	0.0381218	0.122334	0.241319	2.38523
49	0.0100929	0.0103484	0.0113225	0.0120723	0.0160041	0.0513573	0.101309	1.00135
49.07	0.0100788	0.0103339	0.0113067	0.0120554	0.0159817	0.0512855	0.101167	0.999953

3.3. Spirallikeness of $\mathcal{B}_3(\nu, z)$

As earlier, we also validate (2.13) in Tables 9 and 10. It is evident from Table 9 that for $\nu \gtrapprox 4.5424$, the interval $I_{a,\nu}$ exists for all $a \in (3/4, 1]$, while the last column of Table 9 indicates that for all $a \in [1, 3/2]$, the interval $I_{a,\nu}$ exists for $\nu \gtrapprox 0.646$. Similarly, Table 10 represents the case when $a \in (3/2, 3)$.

Remark 3.1. *Spirallikeness of $\mathcal{B}_2(\nu, z)$ and $\mathcal{B}_3(\nu, z)$ can also be graphically validated, as demonstrated* *earlier for $\mathcal{B}_1(\nu, z)$. However, due to the similarity in their visual patterns, the corresponding plots are omitted.*

Table 9. Existence of $I_{a,\nu}$ for fixed $\nu > 0$ and $a \in (3/4, 3/2]$ in relation to $\mathtt{M}_3$.

ν/a	0.76	0.77	0.78	0.79	0.8	0.9	0.95	[1, 3/2]
0.1	256.309	128.154	85.4362	64.0772	51.2617	17.0872	12.8154	10.2523
0.5	35.8061	17.903	11.9354	8.95152	7.16122	2.38707	1.7903	1.43224
0.646	24.9932	12.4966	8.33108	6.24831	4.99865	1.66622	1.24966	0.999729
1	13.0589	6.52945	4.35297	3.26473	2.61178	0.870593	0.652945	0.522356
2	4.25689	2.12845	1.41896	1.06422	0.851379	0.283793	0.212845	0.170276
3	2.11003	1.05502	0.703344	0.527508	0.422006	0.140669	0.105502	0.0844012
4	1.2606	0.6303	0.4202	0.31515	0.25212	0.0840401	0.06303	0.050424
4.5424	0.999966	0.499983	0.333322	0.249992	0.199993	0.0666644	0.0499983	0.0399986
5	0.838359	0.419179	0.279453	0.20959	0.167672	0.0558906	0.0419179	0.0335344
6	0.597923	0.298961	0.199308	0.149481	0.119585	0.0398615	0.0298961	0.0239169
7	0.447991	0.223996	0.14933	0.111998	0.0895983	0.0298661	0.0223996	0.0179197
8	0.348193	0.174097	0.116064	0.0870483	0.0696386	0.0232129	0.0174097	0.0139277

Table 10. Existence of $I_{a,\nu}$ for fixed $\nu > 0$ and $a \in (3/2, 3)$ in relation to $\mathtt{M}_3$.

ν/a	1.6	1.7	1.9	2.3	2.8	2.9	2.99
0.5	1.44541	1.48199	1.6215	2.29195	7.3549	14.5085	143.404
1	0.527158	0.5405	0.591381	0.835899	2.68242	5.29141	52.3011
2	0.171841	0.17619	0.192776	0.272483	0.874405	1.72487	17.0489
3	0.0851772	0.0873328	0.0955542	0.135063	0.43342	0.854975	8.45071
4	0.0508876	0.0521755	0.0570872	0.080691	0.258939	0.51079	5.04873
5	0.0338427	0.0346991	0.0379656	0.0536633	0.172207	0.339699	3.35764
6	0.0241368	0.0247476	0.0270773	0.038273	0.122819	0.242276	2.39469
7	0.0180844	0.0185421	0.0202876	0.0286759	0.0920216	0.181524	1.79421
8	0.0140558	0.0144115	0.0157682	0.0222878	0.0715221	0.141086	1.39452
9	0.0112389	0.0115233	0.0126081	0.0178212	0.0571885	0.112812	1.11505
9.5292	0.0100791	0.0103342	0.011307	0.0159821	0.0512869	0.10117	0.999979
10	0.00919195	0.00942457	0.0103118	0.0145754	0.0467728	0.0922652	0.911964

4. Conclusions

In this study, we have successfully employed Lemma 1.1 to establish relationships among the triplet (a, ν, α), which ensure that three distinct normalizations of the Bessel function belong to a subclass of spirallike functions. In a special case, we derived a connection to the class of starlike functions of order $1 - \cos(\alpha)/2$.

Given that the zeros of Bessel functions and the upper bound of the ratio $J_{\nu+1}/J_\nu$ are central to our analysis, similar investigations can be extended to other special functions. In particular, analogous results may be obtained for the Struve, Lommel, and confluent hypergeometric functions by determining or using (if already existing in the literature) appropriate upper bounds for the respective function ratios.

Author contributions

Both authors of this article have contributed equally. Both authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. P. L. Duren, *Univalent functions*, Vol. 259, New York: Springer-Verlag, 1983.
2. F. Rønning, Uniformly convex functions and a corresponding class of starlike functions, *Proc. Amer. Math. Soc.*, **118** (1993), 189–196.
3. R. M. Ali, Starlikeness associated with parabolic regions, *Int. J. Math. Math. Sci.*, **4** (2005), 561–570. https://doi.org/10.1155/IJMMS.2005.561
4. A. W. Goodman, *Univalent functions*, Vol. I, Mariner Publishing Co., Inc., 1983.
5. Y. C. Kim, T. Sugawa, Correspondence between spirallike functions and starlike functions, *Math. Nachr.*, **285** (2012), 322–331. https://doi.org/10.1002/mana.201010020
6. R. J. Libera, Univalent α-spiral functions, *Can. J. Math.*, **19** (1967), 449–456. https://doi.org/10.4153/CJM-1967-038-0
7. V. Ravichandran, C. Selvaraj, R. Rajagopal, On uniformly convex spiral functions and uniformly spirallike functions, *Soochow J. Math.*, **29** (2003), 393–405.
8. N. Alabkary, S. R. Mondal, On spirallikeness of entire functions, *Mathematics*, **13** (2025), 1566. https://doi.org/10.3390/math13101566
9. G. N. Watson, *A treatise on the theory of Bessel functions*, 2 Eds., Cambridge University Press, 1944.

10. D. K. Ross, Inequalities for special functions, *SIAM Rev.*, **15** (1973), 665–670.
11. E. K. Ifantis, P. D. Siafarikas, Inequalities involving Bessel and modified Bessel functions, *J. Math. Anal. Appl.*, **147** (1990), 214–227. https://doi.org/10.1016/0022-247X(90)90394-U
12. R. K. Brown, Univalence of Bessel functions, *Proc. Amer. Math. Soc.*, **11** (1960), 278–283. https://doi.org/10.2307/2032969
13. R. K. Brown, Univalent solutions of $W'' + pW = 0$, *Can. J. Math.*, **14** (1962), 69–78.
14. R. K. Brown, Univalence of normalized solutions of $W''(z) + p(z)W(z) = 0$, *Int. J. Math. Math. Sci.*, **5** (1982), 459–483. https://doi.org/10.1155/S0161171282000441
15. E. Kreyszig, J. Todd, The radius of univalence of Bessel functions I, *Illinois J. Math.*, **4** (1960), 143–149.
16. A. Baricz, Geometric properties of generalized Bessel functions, *Publ. Math. Debrecen*, **73** (2008), 155–178.
17. A. Baricz, Geometric properties of generalized Bessel functions of complex order, *Mathematica*, **48** (2006), 13–18.
18. S. Kanas, S. R. Mondal, A. D. Mohammed, Relations between the generalized Bessel functions and the Janowski class, *Math. Inequal. Appl.*, **21** (2018), 165–178. https://doi.org/10.7153/mia-2018-21-14
19. S. R. Mondal, A. Swaminathan, Geometric properties of generalized Bessel functions, *Bull. Malays. Math. Sci. Soc.*, **35** (2012), 179–194.
20. N. Bohra, V. Ravichandran, Radii problems for normalized Bessel functions of first kind, *Comput. Methods Funct. Theory*, **18** (2018), 99–123. https://doi.org/10.1007/s40315-017-0216-0
21. V. Madaan, A. Kumar, V. Ravichandran, Starlikeness associated with lemniscate of Bernoulli, *Filomat*, **33** (2019), 1937–1955. https://doi.org/10.2298/FIL1907937M
22. A. Baricz, R. Szász, The radius of convexity of normalized Bessel functions of the first kind, *Anal. Appl.*, **12** (2014), 485–509. https://doi.org/10.1142/S0219530514500316
23. I. Aktaş, A. Baricz, H. Orhan, Bounds for radii of starlikeness and convexity of some special functions, *Turk. J. Math.*, **42** (2018), 211–226. https://doi.org/10.3906/mat-1610-41
24. I. Aktaş, E. Toklu, H. Orhan, Radii of uniform convexity of some special functions, *Turk. J. Math.*, **42** (2018), 3010–3024. https://doi.org/10.3906/mat-1806-43
25. E. Deniz, R. Szász, The radius of uniform convexity of Bessel functions, *J. Math. Anal. Appl.*, **453** (2017), 572–588. https://doi.org/10.1016/j.jmaa.2017.03.079
26. S. Kanas, K. Gangania, Radius of uniformly convex γ-spirallikeness of combination of derivatives of Bessel functions, *Axioms*, **12** (2023), 468. https://doi.org/10.3390/axioms12050468