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Theory article

Robust stability of switched Boolean networks with external disturbances and function perturbation

Zhiqiao Tian and Haitao Li*

School of Mathematics and Statistics, Shandong Normal University, Jinan 250014, China

* **Correspondence:** Email: lihaitao_sdu@163.com.

Abstract: This paper focused on the robust stability of switched Boolean networks (SBNs) subject to external disturbances and one-bit function perturbation. The reachable sets of a perturbed state before and after function perturbation were constructed, and some basic properties of these reachable sets were presented. Based on the reachable sets of a perturbed state, a necessary and sufficient condition was proposed to verify the robust stability of SBNs with one-bit function perturbation.

Keywords: switched Boolean network; external disturbance; function perturbation; robust stability; semi-tensor product of matrices
Mathematics Subject Classification: 93D09, 94C11

1. Introduction

Robust stability of dynamic systems is a significant problem in control theory, which has been extensively studied for several kinds of dynamic systems in the past few decades [1–3]. Robust stability has wide applications in various fields such as the power system and coordinating networks [4, 5]. Particularly, in gene regulation, mutations in certain key genes may lead to the occurrence of cancer [6]. Therefore, it is meaningful to investigate the robust stability of gene regulatory networks with mutations.

Boolean networks (BNs), a kind of dynamic system, were first put forward in [7] to depict gene regulatory networks. In recent years, several new methods have been proposed to reduce the computational complexity of studying BNs. The deep reinforcement learning method was used to deal with the stabilization of large-scale probabilistic BNs [8]. Based on the dependence digraph and feedback arc set, a pinning control scheme was proposed to explore the non-oscillation of large-scale asynchronous BNs [9]. In reality, however, many dynamic evolutionary processes in gene regulation are controlled by different modes [10]. Therefore, the model of switched Boolean networks (SBNs) was proposed in [11] to describe different modes in gene regulatory networks. In order to depict the gene mutations, the concept of function perturbation was established in the model of BNs [12], including one-bit function perturbation and multi-bit function perturbation. The robust stability of BNs subject to multi-bit stochastic function perturbations was studied in [13], and the robust set stability of implicit BNs with a time delay and one-bit function perturbation was investigated in [14].

The semi-tensor product (STP) of matrices [15] is a useful tool for the study of BNs and SBNs. Under the framework of STP, the robust stability of BNs with external disturbances was explored in [16–19]. Using STP, some basic issues of SBNs were well addressed, including stability and global stability [11,20], synchronization [21,22], stabilization [23], and output tracking [24]. The STP method is also used to discuss the impact of function perturbations on BNs such as robust stability [25], stabilization [26], set controllability [27], cluster synchronization [28], and topological structure [29]. In [25], pointwise stabilizability and consistent stabilizability of SBNs with function perturbation were investigated. Besides, the influence of one-bit function perturbation on SBNs was considered in [30].

Intracellular molecular activities are considerably influenced by thermal fluctuations and noisy process, but cellular functions are generally robust to these external disturbances [31]. Moreover, gene mutations described as function perturbations can lead to some diseases such as cancer and diabetes [12]. When both external disturbances and function perturbation affect a BN, the robust stability becomes more challenging [32]. The set stability of SBNs (BNs with disturbances can be regarded as SBNs) [33] and the disturbance decoupling controller design of SBNs with disturbances [34] were studied, and the robust stability of SBNs subject to perturbation was investigated in [25, 30]. However, there exist few studies on SBNs subject to both external disturbances and function perturbations. Based on the above motivations, we investigate the influence of one-bit function perturbation on SBNs with external disturbances via constructing the reachable sets of a perturbed state. Compared with [25], which used the reachable matrix of SBNs after perturbation, we just employ the information of a perturbed state to derive a necessary and sufficient condition for the robust stability of disturbed SBNs with function perturbation, which needs less information about the perturbed system. Furthermore, the new criterion can degenerate into the results of [30, 32] when the considered SBN is just affected by function perturbation or only has one mode.

The rest of this article is structured as follows. Section 2 presents the problem formulation. The robust stability criterion of disturbed SBNs subject to one-bit function perturbation is given in Section 3. A concluding summary is provided in Section 4.

Notations: $\mathbb{N}$ and $\mathbb{Z}_+$ denote the sets of natural numbers and positive numbers, respectively. Given $\alpha, \beta \in \mathbb{N}$ and $\alpha < \beta$, denote $[\alpha : \beta] := \{\alpha, \alpha+1, \cdots, \beta\}$. $\mathcal{D} := \{0, 1\}$ and $\mathcal{D}^n := \underbrace{\mathcal{D} \times \cdots \times \mathcal{D}}_{n}$. $A_{i,j}$ represents the element in the i-th row and the j-th column of matrix A. $Blk_i(A)$ denotes the i-th equal block of an $m \times mn$ matrix A, where $i \in [1 : n]$. $Col_i(A)$ denotes the i-th column of matrix A. $Row_j(A)$ represents the j-th row of matrix A. $\Delta_n := \{\delta_n^1, \cdots, \delta_n^n\}$ and $\Delta_2 := \Delta$, where $\delta_n^i = Col_i(I_n)$, $\forall\, i \in [1 : n]$, and I_n represents the n-dimensional identity matrix. If $L = [\delta_m^{i_1}\ \delta_m^{i_2}\ \cdots\ \delta_m^{i_n}]$, then the $m \times n$ matrix L is called a logical matrix. For simplicity, $[\delta_m^{i_1}\ \delta_m^{i_2}\ \cdots\ \delta_m^{i_n}]$ is briefly expressed as $\delta_m[i_1\ i_2\ \cdots\ i_n]$. "$\ltimes$" denotes the STP of matrices, which is omitted without causing confusion. For $s \times t$ matrix A and $p \times q$ matrix B, denote $A \ltimes B = (A \otimes I_{r/t})(B \otimes I_{r/p})$, where r is the least common multiple of t and p.

2. Problem formulation

Normally, an SBN with n nodes, ω switching signals, and q disturbance inputs is described as

$$\begin{cases} x_1(t+1) = f_1^{\sigma(t)}(X(t), \Xi(t)), \\ x_2(t+1) = f_2^{\sigma(t)}(X(t), \Xi(t)), \\ \quad \vdots \\ x_n(t+1) = f_n^{\sigma(t)}(X(t), \Xi(t)), \end{cases} \tag{2.1}$$

where $X(t) = (x_1(t), x_2(t), \cdots, x_n(t)) \in \mathcal{D}^n$ is the state vector, $\Xi(t) = (\xi_1(t), \xi_2(t), \cdots, \xi_q(t)) \in \mathcal{D}^q$ is the disturbance input vector, $\sigma : \mathbb{N} \to [1 : \omega]$ is the switching signal, and $f_i^l : \mathcal{D}^{n+q} \to \mathcal{D}$, $i \in [1 : n]$, $l \in [1 : \omega]$ are Boolean functions. $\{\Xi(t) : t \in \mathbb{N}\} \subseteq \mathcal{D}^q$ and $\{\Sigma(t) : t \in \mathbb{N}\} \subseteq [1 : \omega]$ are a sequence of disturbance inputs and a switching sequence, respectively, under which we denote the state trajectory of system (2.1) starting from an initial state $X(0) \in \mathcal{D}^n$ by $X(t; X(0), \Sigma, \Xi)$.

The definition of robust stability for system (2.1) is given below.

Definition 2.1. *System (2.1) is said to be robustly stable at $X_e = (x_e, x_e, \cdots, x_e) \in \mathcal{D}^n$ under an arbitrary switching signal, if there exists a positive integer T such that $X(t; X(0), \Sigma, \Xi) = X_e$ holds for any $X(0) \in \mathcal{D}^n$, any $\{\Sigma(t) : t \in \mathbb{N}\} \subseteq [1 : \omega]$, any $\{\Xi(t) : t \in \mathbb{N}\} \subseteq \mathcal{D}^q$, and any integer $t \geq T$.*

The STP of matrices is the main research tool in this paper and its definition is given in notations. For the specific properties of the STP of matrices, please refer to [15].

Using the STP, we can convert $X(t)$, $\sigma(t) = i$, and $\Xi(t)$ into the equivalent vectors $x(t) = \ltimes_{i=1}^n x_i(t)$, $\sigma(t) = \delta_\omega^i$, $i \in [1 : \omega]$, and $\xi(t) = \ltimes_{j=1}^q \xi_j(t)$, respectively. Then, system (2.1) is expressed as the following equivalent algebraic form:

$$x(t+1) = L\sigma(t)\xi(t)x(t), \tag{2.2}$$

where $L = \delta_{2^n}[\nu_1\ \nu_2 \cdots \nu_{\omega 2^{n+q}}] \in \mathcal{L}_{2^n \times \omega 2^{n+q}}$ is the state transition matrix. L can be divided into $\omega 2^q$ equal parts by column. We denote $Blk_i(L) = L_i$, where $L_i = \delta_{2^n}[\beta_1^i\ \beta_2^i \cdots \beta_{2^n}^i]$ and $i \in [1 : \omega 2^q]$. Additionally, $M = \sum_{i=1}^{\omega 2^q} L_i$ is a one-step reachable matrix.

When converting an SBN into the equivalent algebraic form (2.2), the function perturbation is depicted by the change of some columns in the state transition matrix [26]. Thus, we present the following assumptions about the considered system and the type of function perturbation throughout the article.

Assumption 2.1. *System (2.2) is robustly stable at $x_e = \delta_{2^n}^\theta$ under an arbitrary switching signal.*

Assumption 2.2. *After one-bit function perturbation, the ζ-th column of L is changed and the other columns do not change. Specifically, assume that L changes to $\widetilde{L} = \delta_{2^n}[\rho_1\ \rho_2 \cdots \rho_{\omega 2^{n+q}}]$ after one-bit function perturbation. Then, $Col_\zeta(\widetilde{L}) = \delta_{2^n}^{\rho_\zeta} \neq \delta_{2^n}^{\nu_\zeta} = Col_\zeta(L)$ and $Col_i(\widetilde{L}) = Col_i(L)$, where $i \in [1 : \omega 2^{n+q}] \setminus \{\zeta\}$.*

Remark 2.1. *Similar to L, $\widetilde{L}$ can be divided into $\omega 2^q$ equal parts by column. We denote $\widetilde{L}_i = \delta_{2^n}[\gamma_1^i\ \gamma_2^i \cdots \gamma_{2^n}^i]$, where $i \in [1 : \omega 2^q]$. For the above-mentioned integer ζ, there exist unique integers $\iota \in [1 : \omega]$, $\kappa \in [1 : 2^q]$, and $\mu \in [1 : 2^n]$ satisfying $\zeta = (\iota-1)2^{q+n} + (\kappa-1)2^n + \mu = [(\iota-1)2^q + (\kappa-1)]2^n + \mu$, which means that there exists unique integer $p = (\iota - 1)2^q + \kappa \in [1 : \omega 2^q]$ such that $\zeta = (p-1)2^n + \mu$. Thus, only the μ-th column of the p-th block in state transition matrix L changes.*

We summarize the above analysis as the following assumption.

Assumption 2.3. *For system (2.2), $Col_\mu(\widetilde{L}_p) = \delta_{2^n}^{\gamma_\mu^p} \neq \delta_{2^n}^{\beta_\mu^p} = Col_\mu(L_p)$, $Col_i(\widetilde{L}_p) = Col_i(L_p)$ is satisfied for any integer $i \in [1 : 2^n] \setminus \{\mu\}$, and $\widetilde{L}_j = L_j$ holds for any integer $j \in [1 : \omega 2^q] \setminus \{p\}$.*

Under Assumption 2.2, system (2.2) becomes

$$\widetilde{x}(t+1) = \widetilde{L}\widetilde{\sigma}(t)\widetilde{\xi}(t)\widetilde{x}(t). \tag{2.3}$$

Here, $\widetilde{x}(t) = \ltimes_{i=1}^n \widetilde{x}_i(t) \in \Delta_{2^n}$, $\widetilde{L} \in \mathcal{L}_{2^n \times \omega 2^{n+q}}$, $\widetilde{\sigma}(t) \in \Delta_\omega$, and $\widetilde{\xi}(t) \in \Delta_{2^q}$ are the state variable, state transition matrix, switching signal, and disturbance input of system (2.2) subject to perturbation in Assumption 2.2, respectively. Note that $\widetilde{\sigma}(t) = \sigma(t)$ and $\widetilde{\xi}(t) = \xi(t)$ hold for any $t \in \mathbb{N}$. Correspondingly, $\widetilde{M} = \sum_{i=1}^{\omega 2^q} L_i - L_p + \widetilde{L}_p$ is the one-step reachable matrix of system (2.3).

The purpose of this article is to study the impact of one-bit function perturbation in Assumption 2.2 on the robust stability of SBNs.

3. Main results

In this section, we propose some criteria to determine the robust stability of SBNs subject to external disturbances and one-bit function perturbation. We first construct a series of reachable sets of $\delta_{2^n}^\mu$. Then, we explore the relation of reachable sets of $\delta_{2^n}^\mu$ before and after one-bit function perturbation.

First of all, we present two necessary conditions for the robust stability of SBNs.

Proposition 3.1. *Under Assumptions 2.1–2.3, if system (2.3) is still robustly stable at $x_e = \delta_{2^n}^\theta$, then $\mu \neq \theta$ holds.*

Proof. Suppose that $\mu = \theta$. According to Assumption 2.1, we know $\delta_{2^n}^{\beta_\theta^p} = \delta_{2^n}^\theta$. Then we have $\widetilde{L}\delta_\omega^\iota \delta_{2^n}^\kappa x_e = \widetilde{L}_p x_e = \widetilde{L}_p \delta_{2^n}^\theta = \delta_{2^n}^{\gamma_\theta^p} \neq \delta_{2^n}^{\beta_\theta^p} = \delta_{2^n}^\theta$. It indicates that x_e is not the fixed point of $\widetilde{L}_p$. Therefore, $x(t+1) = \widetilde{L}_p x(t)$ is not stable at x_e, which contradicts the condition that system (2.3) is robustly stable at x_e. Thus, $\mu \neq \theta$ holds.

Proposition 3.2. *Under Assumptions 2.1–2.3, if system (2.3) is still robustly stable at $x_e = \delta_{2^n}^\theta$, then $\gamma_\mu^p \neq \mu$ holds.*

Proof. Suppose that $\gamma_\mu^p = \mu$. For any integer $k \in \mathbb{Z}_+$, one derives $(\widetilde{L}\delta_\omega^\iota \delta_{2^n}^\kappa)^k \delta_{2^n}^\mu = (\widetilde{L}_p)^k \delta_{2^n}^\mu = \delta_{2^n}^{\gamma_\mu^p} = \delta_{2^n}^\mu$ for system (2.3). However, we conclude $\mu \neq \theta$ from Proposition 3.1. Hence, $(\widetilde{L}\delta_\omega^\iota \delta_{2^n}^\kappa)^k \delta_{2^n}^\mu = \delta_{2^n}^\mu$, which contradicts the condition that system (2.3) is robustly stable at x_e. Therefore, we obtain $\gamma_\mu^p \neq \mu$.

It follows that $\gamma_\mu^p \neq \mu$ and $\mu \neq \theta$ are necessary conditions for the robust stability of system (2.2) with external disturbances and one-bit function perturbation. As a result, we naturally assume that $\gamma_\mu^p \neq \mu$ and $\mu \neq \theta$ hold in this context.

We use $m_1, m_2, \cdots, m_{2^n}$ and $\widetilde{m}_1, \widetilde{m}_2, \cdots, \widetilde{m}_{2^n}$ to represent the column vectors of M and $\widetilde{M}$, respectively. Then, $M := [m_1\ m_2 \cdots m_{2^n}]$ and $\widetilde{M} := [\widetilde{m}_1\ \widetilde{m}_2 \cdots \widetilde{m}_{2^n}]$. The following lemma reveals the relation between m_j and $\widetilde{m}_j$, $j \in [1 : 2^n]$.

Lemma 3.1. *Under Assumptions 2.2 and 2.3, it holds that*

$$\widetilde{m}_j = \begin{cases} m_j, \ j \in [1 : 2^n] \setminus \{\mu\}, \\ m_j - \delta_{2^n}^{\beta_\mu^p} + \delta_{2^n}^{\gamma_\mu^p}, \ j = \mu. \end{cases} \tag{3.1}$$

Proof. According to Assumptions 2.2 and 2.3, $\widetilde{m}_j = \sum_{i=1}^{\omega 2^q} Col_j(\widetilde{L}_i) = \sum_{i=1}^{\omega 2^q} Col_j(L_i) = m_j, \ \forall j \in [1 : 2^n] \setminus \{\mu\}$ is apparently established. Because

$$\begin{aligned} M &= \sum_{i=1}^{\omega 2^q} L_i = \sum_{i=1}^{\omega 2^q} \delta_{2^n}[\beta_1^i \, \beta_2^i \cdots \beta_{2^n}^i] \\ &= [\sum_{i=1}^{\omega 2^q} \delta_{2^n}^{\beta_1^i} \, \sum_{i=1}^{\omega 2^q} \delta_{2^n}^{\beta_2^i} \cdots \sum_{i=1}^{\omega 2^q} \delta_{2^n}^{\beta_{2^n}^i}] \\ &= [m_1 \, m_2 \cdots m_{2^n}] \end{aligned}$$

and

$$\begin{aligned} \widetilde{M} &= \sum_{i=1}^{\omega 2^q} L_i - L_p + \widetilde{L}_p \\ &= [\sum_{i=1}^{\omega 2^q} \delta_{2^n}^{\beta_1^i} - \delta_{2^n}^{\beta_1^p} + \delta_{2^n}^{\gamma_1^p} \, \sum_{i=1}^{\omega 2^q} \delta_{2^n}^{\beta_2^i} - \delta_{2^n}^{\beta_2^p} + \delta_{2^n}^{\gamma_2^p} \cdots \sum_{i=1}^{\omega 2^q} \delta_{2^n}^{\beta_{2^n}^i} - \delta_{2^n}^{\beta_{2^n}^p} + \delta_{2^n}^{\gamma_{2^n}^p}] \\ &= [\widetilde{m}_1 \, \widetilde{m}_2 \cdots \widetilde{m}_{2^n}], \end{aligned}$$

it holds that $\widetilde{m}_\mu = \sum_{i=1}^{\omega 2^q} \delta_{2^n}^{\beta_\mu^i} - \delta_{2^n}^{\beta_\mu^p} + \delta_{2^n}^{\gamma_\mu^p} = m_\mu - \delta_{2^n}^{\beta_\mu^p} + \delta_{2^n}^{\gamma_\mu^p}$.

In order to investigate the robust stability of system (2.3), we construct a sequence of reachable sets of $\delta_{2^n}^{\mu}$ before and after one-bit function perturbation below:

$$E_k(\mu) := \{j : (M^k)_{\mu,j} > 0\}, \ k \in \mathbb{Z}_+, \ E_0(\mu) := \{\mu\}, \tag{3.2}$$

$$\widetilde{E}_k(\mu) := \{j : (\widetilde{M}^k)_{\mu,j} > 0\}, \ k \in \mathbb{Z}_+, \ \widetilde{E}_0(\mu) := \{\mu\}. \tag{3.3}$$

The influence of perturbation on the states which reach $\delta_{2^n}^{\mu}$ can be analyzed by comparing the elements in the above sets. The elements contained in $E_k(\mu)$ and $\widetilde{E}_k(\mu)$ are the states that can reach $\delta_{2^n}^{\mu}$ under an arbitrary switching signal before and after perturbation, respectively. Then, through the above reachable sets, we explore whether the states that can reach $\delta_{2^n}^{\mu}$ are affected by the one-bit function perturbation.

Lemma 3.2. *Under Assumptions 2.1–2.3, if system (2.3) is robustly stable at x_e, then*

$$\bigcup_{k=0}^{2^n} E_k(\mu) = \bigcup_{k=0}^{2^n} \widetilde{E}_k(\mu).$$

Proof. One concludes that $(M^k)_{\mu,\mu} = (\widetilde{M}^k)_{\mu,\mu} = 0$ is satisfied for any integer $k \in \mathbb{Z}_+$ because systems (2.2) and (2.3) are robustly stable at x_e. Then, in the light of Assumption 2.1 and Proposition 3.2, we know that $\beta_\mu^p \neq \mu$ and $\gamma_\mu^p \neq \mu$ hold, which together with Lemma 3.1 imply that

$$(\widetilde{M})_{\mu,j} = M_{\mu,j}, \ \forall j \in [1 : 2^n]. \tag{3.4}$$

We prove that $(\widetilde{M}^k)_{\mu,j} = (M^k)_{\mu,j}$ holds for any integer $k \in \mathbb{Z}_+$ and any integer $j \in [1 : 2^n] \setminus \{\mu\}$ by induction.

According to (3.4), the conclusion is clearly true when $k = 1$. When $k = s \in \mathbb{Z}_+$, assume that $(\widetilde{M}^s)_{\mu,j} = (M^s)_{\mu,j}$ is true for any $j \in [1:2^n] \setminus \{\mu\}$. For any integer $j \in [1:2^n] \setminus \{\mu\}$, one obtains from Lemma 3.1 that $\widetilde{M}_{l,j} = M_{l,j}$, $\forall\, l \in [1:2^n]$. Then,

$$\begin{aligned}(\widetilde{M}^{s+1})_{\mu,j} &= \sum_{i=1}^{2^n}(\widetilde{M}^s)_{\mu,i}\widetilde{M}_{i,j} = \sum_{i\neq\mu}(M^s)_{\mu,i}M_{i,j} + (\widetilde{M}^s)_{\mu,\mu}\widetilde{M}_{\mu,j}\\ &= \sum_{i\neq\mu}(M^s)_{\mu,i}M_{i,j} + 0\times M_{\mu,j}\\ &= \sum_{i\neq\mu}(M^s)_{\mu,i}M_{i,j} + (M^s)_{\mu,\mu}M_{\mu,j} = (M^{s+1})_{\mu,j}.\end{aligned}$$

Therefore, $(\widetilde{M}^k)_{\mu,j} > 0$ is equivalent to $(M^k)_{\mu,j} > 0$, $\forall\, j \in [1:2^n] \setminus \{\mu\}$, $\forall\, k \in \mathbb{Z}_+$. Then, based on (3.2) and (3.3), the conclusion follows.

Instead of calculating $Row_\theta(\widetilde{M}^k)$, $k \in [1:2^n]$, directly, Lemma 3.2 can aid us to explore the robust stability of system (2.3) by employing the information of system (2.2) and function perturbation.

Theorem 3.1. *Under Assumptions 2.2 and 2.3, if $\eta \in [1:2^n]$ satisfies $\eta \notin \bigcup_{k=0}^{2^n} E_k(\mu)$, then $\eta \notin \bigcup_{k=0}^{2^n} \widetilde{E}_k(\mu)$.*

Proof. We prove the conclusion by a reduction to absurdity. Suppose $\eta \in \bigcup_{k=0}^{2^n} \widetilde{E}_k(\mu)$. Then, we have $\eta \in [\bigcup_{k=0}^{2^n} \widetilde{E}_k(\mu)] \setminus \{\mu\}$ or $\eta = \mu$. Note that $\eta = \mu$ contradicts $\eta \notin \bigcup_{k=0}^{2^n} E_k(\mu)$. Hence, we assume that there exists a minimum integer $k^* \in [1:2^n]$ satisfying $(\widetilde{M}^{k^*})_{\mu,\eta} > 0$ based on $\eta \in [\bigcup_{k=0}^{2^n} \widetilde{E}_k(\mu)] \setminus \{\mu\}$. Then, there exists a sequence of switching signals $\widetilde{\sigma}(0) = \sigma(0), \widetilde{\sigma}(1) = \sigma(1), \cdots, \widetilde{\sigma}(k^*-1) = \sigma(k^*-1)$ and a sequence of disturbance inputs $\widetilde{\xi}(0) = \xi(0), \widetilde{\xi}(1) = \xi(1), \cdots, \widetilde{\xi}(k^*-1) = \xi(k^*-1)$ satisfying

$$[\ltimes_{l=k^*-1}^{0}\widetilde{L}\widetilde{\sigma}(l)\widetilde{\xi}(l)]\delta_{2^n}^{\eta} = \delta_{2^n}^{\mu}. \tag{3.5}$$

Next, we prove that

$$[\ltimes_{l=k}^{0}L\sigma(l)\xi(l)]\delta_{2^n}^{\eta} = [\ltimes_{l=k}^{0}\widetilde{L}\widetilde{\sigma}(l)\widetilde{\xi}(l)]\delta_{2^n}^{\eta},\ \forall\, k \in [0:k^*-1] \tag{3.6}$$

holds by induction. When $k = 0$, we derive from Assumptions 2.2 and 2.3, the minimality of k^*, and $\eta \neq \mu$ that $\widetilde{L}\widetilde{\sigma}(0)\widetilde{\xi}(0)\delta_{2^n}^{\eta} = L\sigma(0)\xi(0)\delta_{2^n}^{\eta}$. Then, assume that $[\ltimes_{l=s}^{0}L\sigma(l)\xi(l)]\delta_{2^n}^{\eta} = [\ltimes_{l=s}^{0}\widetilde{L}\widetilde{\sigma}(l)\widetilde{\xi}(l)]\delta_{2^n}^{\eta}$ is true for $k = s \in [0:k^*-2]$. Considering the case of $k = s+1$, we obtain

$$\begin{aligned}[\ltimes_{l=s+1}^{0}L\sigma(l)\xi(l)]\delta_{2^n}^{\eta} &= [L\sigma(s+1)\xi(s+1)][\ltimes_{l=s}^{0}\widetilde{L}\widetilde{\sigma}(l)\widetilde{\xi}(l)]\delta_{2^n}^{\eta}\\ &= [\widetilde{L}\widetilde{\sigma}(s+1)\widetilde{\xi}(s+1)][\ltimes_{l=s}^{0}\widetilde{L}\widetilde{\sigma}(l)\widetilde{\xi}(l)]\delta_{2^n}^{\eta}\\ &= [\ltimes_{l=s+1}^{0}\widetilde{L}\widetilde{\sigma}(l)\widetilde{\xi}(l)]\delta_{2^n}^{\eta}\end{aligned}$$

based on Assumptions 2.2 and 2.3 and $\eta \neq \mu$. Accordingly, (3.6) is satisfied. Therefore, in view of (3.5) and (3.6), we conclude $[\ltimes_{l=k^*-1}^{0}L\sigma(l)\xi(l)]\delta_{2^n}^{\eta} = [\ltimes_{l=k^*-1}^{0}\widetilde{L}\widetilde{\sigma}(l)\widetilde{\xi}(l)]\delta_{2^n}^{\eta} = \delta_{2^n}^{\mu}$, which implies $\eta \in E_{k^*}(\mu)$. Because $\eta \in E_{k^*}(\mu)$ contradicts $\eta \notin \bigcup_{k=0}^{2^n} E_k(\mu)$, the conclusion follows.

Remark 3.1. *Since k^* is the minimum integer that satisfies $(\widetilde{M}^{k^*})_{\mu,\eta} > 0$, one concludes that $\delta_{2^n}^{\eta}$ cannot reach $\delta_{2^n}^{\mu}$ within k^* steps. When time increases from 0 to s step by step, $s+1$ steps are actually taken. Thus, we derive from $s \in [0:k^*-2]$ that $[\ltimes_{l=s}^{0}\widetilde{L}\widetilde{\sigma}(l)\widetilde{\xi}(l)]\delta_{2^n}^{\eta} \neq \delta_{2^n}^{\mu}$, which means that $\delta_{2^n}^{\mu}$ cannot be* *reached from $\delta_{2^n}^{\eta}$ within $s+1$ steps. Suppose that $[\ltimes_{l=s}^{0}\widetilde{L\sigma}(l)\widetilde{\xi}(l)]\delta_{2^n}^{\eta} = \delta_{2^n}^{y}$. Then, we have $\delta_{2^n}^{y} \neq \delta_{2^n}^{\mu}$. Thus, we derive from Assumptions 2.2 and 2.3 that $\widetilde{L\sigma}(s+1)\widetilde{\xi}(s+1)\delta_{2^n}^{y} = L\sigma(s+1)\xi(s+1)\delta_{2^n}^{y}$.*

Now, we apply the above results to explore the robust stability of system (2.3).

Theorem 3.2. *Under Assumptions 2.1–2.3, system (2.3) is robustly stable at $x_e = \delta_{2^n}^{\theta}$, if and only if*

$$\gamma_{\mu}^{p} \notin \bigcup_{k=0}^{2^n} E_k(\mu). \tag{3.7}$$

Proof. (Necessity) In order to prove $\gamma_{\mu}^{p} \notin \bigcup_{k=0}^{2^n} E_k(\mu)$, we only need to demonstrate $\gamma_{\mu}^{p} \notin \bigcup_{k=0}^{2^n} \widetilde{E}_k(\mu)$ in the light of Lemma 3.2. If $\gamma_{\mu}^{p} \in \bigcup_{k=0}^{2^n} \widetilde{E}_k(\mu)$, then there exists $k^* \in [1:2^n]$ satisfying $(\widetilde{M}^{k^*})_{\mu,\gamma_{\mu}^{p}} > 0$. Moreover, we have $\widetilde{M}_{\gamma_{\mu}^{p},\mu} > 0$ on the basis of $Col_{\mu}(\widetilde{L}_p) = \delta_{2^n}^{\gamma_{\mu}^{p}}$. Therefore, we conclude

$$(\widetilde{M}^{k^*+1})_{\mu,\mu} = \sum_{j=1}^{2^n} (\widetilde{M}^{k^*})_{\mu,j} \widetilde{M}_{j,\mu} \geq (\widetilde{M}^{k^*})_{\mu,\gamma_{\mu}^{p}} \widetilde{M}_{\gamma_{\mu}^{p},\mu} > 0,$$

which contradicts the robust stability at x_e of system (2.3).

(Sufficiency) We derive from $\gamma_{\mu}^{p} \notin E_0(\mu)$ that $\gamma_{\mu}^{p} \neq \mu$.

Assume that system (2.3) is not robustly stable to x_e. Then, there exists another attractor C other than x_e. We denote $C := \{\delta_{2^n}^{i_1}, \cdots, \delta_{2^n}^{i_r}\}$, $r \in [2 : 2^n - 1]$. Then, we prove $\delta_{2^n}^{\gamma_{\mu}^{p}} \in C$ by a reduction to absurdity. If $i_1, \cdots, i_r \neq \gamma_{\mu}^{p}$ hold, then there exists a sequence of switching signals $\widetilde{\sigma}^*(0) = \sigma^*(0)$, $\widetilde{\sigma}^*(1) = \sigma^*(1), \cdots, \widetilde{\sigma}^*(r-2) = \sigma^*(r-2)$, $\widetilde{\sigma}^*(r-1) = \sigma^*(r-1)$ and a sequence of disturbance inputs $\widetilde{\xi}^*(0) = \xi^*(0)$, $\widetilde{\xi}^*(1) = \xi^*(1), \cdots, \widetilde{\xi}^*(r-2) = \xi^*(r-2)$, $\widetilde{\xi}^*(r-1) = \xi^*(r-1)$ satisfying $\widetilde{L\sigma}^*(0)\widetilde{\xi}^*(0)\delta_{2^n}^{i_1} = \delta_{2^n}^{i_2} \neq \delta_{2^n}^{\gamma_{\mu}^{p}}$, $\widetilde{L\sigma}^*(1)\widetilde{\xi}^*(1)\delta_{2^n}^{i_2} = \delta_{2^n}^{i_3} \neq \delta_{2^n}^{\gamma_{\mu}^{p}}, \cdots$, $\widetilde{L\sigma}^*(r-2)\widetilde{\xi}^*(r-2)\delta_{2^n}^{i_{r-1}} = \delta_{2^n}^{i_r} \neq \delta_{2^n}^{\gamma_{\mu}^{p}}$, $\widetilde{L\sigma}^*(r-1)\widetilde{\xi}^*(r-1)\delta_{2^n}^{i_r} = \delta_{2^n}^{i_1} \neq \delta_{2^n}^{\gamma_{\mu}^{p}}$. We consider the following two cases:

(i) $\widetilde{L\sigma}^*(j)\widetilde{\xi}^*(j) \neq \widetilde{L}_p$ is satisfied for any integer $j \in [0 : r-1]$.
(ii) There exists an integer $j \in [0 : r-1]$ such that $\widetilde{L\sigma}^*(j)\widetilde{\xi}^*(j) = \widetilde{L}_p$ holds.

For item (i), according to Assumptions 2.2 and 2.3, we know $L\sigma^*(j)\xi^*(j) = \widetilde{L\sigma}^*(j)\widetilde{\xi}^*(j)$, $\forall j \in [0 : r-1]$. Thereby, we derive the following state transitions for system (2.2):

$$\begin{aligned}
L\sigma^*(0)\xi^*(0)\delta_{2^n}^{i_1} &= \widetilde{L\sigma}^*(0)\widetilde{\xi}^*(0)\delta_{2^n}^{i_1} = \delta_{2^n}^{i_2} \neq \delta_{2^n}^{\gamma_{\mu}^{p}}, \\
L\sigma^*(1)\xi^*(1)\delta_{2^n}^{i_2} &= \widetilde{L\sigma}^*(1)\widetilde{\xi}^*(1)\delta_{2^n}^{i_2} = \delta_{2^n}^{i_3} \neq \delta_{2^n}^{\gamma_{\mu}^{p}}, \\
&\vdots \\
L\sigma^*(r-2)\xi^*(r-2)\delta_{2^n}^{i_{r-1}} &= \widetilde{L\sigma}^*(r-2)\widetilde{\xi}^*(r-2)\delta_{2^n}^{i_{r-1}} = \delta_{2^n}^{i_r} \neq \delta_{2^n}^{\gamma_{\mu}^{p}}, \\
L\sigma^*(r-1)\xi^*(r-1)\delta_{2^n}^{i_r} &= \widetilde{L\sigma}^*(r-1)\widetilde{\xi}^*(r-1)\delta_{2^n}^{i_r} = \delta_{2^n}^{i_1} \neq \delta_{2^n}^{\gamma_{\mu}^{p}}.
\end{aligned} \tag{3.8}$$

Hence, C is a cycle of system (2.2), which contradicts Assumption 2.1.

For item (ii), denote all integers $j \in [0 : r-1]$ satisfying $\widetilde{L\sigma}^*(j)\widetilde{\xi}^*(j) = \widetilde{L}_p$ as $j_1, \cdots, j_\alpha$, where $j_1 < \cdots < j_\alpha$. Then, we can get $L\sigma^*(j_l)\xi^*(j_l) = L_p$, $\forall\, l \in [1 : \alpha]$. Noticing that $Col_{\mu}(\widetilde{L}_p) = \delta_{2^n}^{\gamma_{\mu}^{p}}$ and

$$\widetilde{L\sigma}^*(j_l)\widetilde{\xi}^*(j_l)\delta_{2^n}^{i_{j_l+1}} = \widetilde{L}_p\delta_{2^n}^{i_{j_l+1}} = \begin{cases} \delta_{2^n}^{i_{j_l+2}} \neq \delta_{2^n}^{\gamma_{\mu}^{p}}, & j_l \in [0 : r-2], \\ \delta_{2^n}^{i_1} \neq \delta_{2^n}^{\gamma_{\mu}^{p}}, & j_l = r-1, \end{cases}$$

we conclude $i_{j_l+1} \neq \mu$ holds for any integer $j_l \in [0 : r-1]$, where $l \in [1 : \alpha]$. Then, according to (3.8) and Assumption 2.3, one has

$$L\sigma^*(j_l)\xi^*(j_l)\delta_{2^n}^{i_{j_l+1}} = L_p\delta_{2^n}^{i_{j_l+1}} = \widetilde{L}_p\delta_{2^n}^{i_{j_l+1}} = \begin{cases} \delta_{2^n}^{i_{j_l+2}} \neq \delta_{2^n}^{\gamma_\mu^p}, \ j_l \in [0 : r-2], \\ \delta_{2^n}^{i_1} \neq \delta_{2^n}^{\gamma_\mu^p}, \ j_l = r-1. \end{cases}$$

Then, (3.8) is established for system (2.2). Hence, C is a cycle of system (2.2), which contradicts Assumption 2.1.

To sum up, it can be inferred that $\delta_{2^n}^{\gamma_\mu^p} \in C$ in system (2.3). Accordingly, there exists $k^* \in \mathbb{Z}_+$ satisfying

$$(\widetilde{M}^{k^*})_{\gamma_\mu^p,\gamma_\mu^p} > 0. \tag{3.9}$$

Next, we prove

$$(\widetilde{M}^k)_{i,\gamma_\mu^p} = (M^k)_{i,\gamma_\mu^p}, \ \forall\ i \in [1 : 2^n] \setminus \{\mu\}, \ k \in \mathbb{Z}_+ \tag{3.10}$$

by induction. On account of $\gamma_\mu^p \neq \mu$ and Lemma 3.1, we have $\widetilde{M}_{i,\gamma_\mu^p} = M_{i,\gamma_\mu^p}$, $i \in [1 : 2^n]$. Thus, $(\widetilde{M}^k)_{i,\gamma_\mu^p} = (M^k)_{i,\gamma_\mu^p}$ is satisfied for $i \in [1 : 2^n] \setminus \{\mu\}$ and $k = 1$. Assume that $(\widetilde{M}^s)_{i,\gamma_\mu^p} = (M^s)_{i,\gamma_\mu^p}$ is true for $k = s \in \mathbb{Z}_+$ and $i \in [1 : 2^n] \setminus \{\mu\}$. Then we have

$$\begin{aligned} (\widetilde{M}^{s+1})_{i,\gamma_\mu^p} &= \sum_{j\neq\mu} \widetilde{M}_{i,j}(\widetilde{M}^s)_{j,\gamma_\mu^p} + \widetilde{M}_{i,\mu}(\widetilde{M}^s)_{\mu,\gamma_\mu^p} \\ &= \sum_{j\neq\mu} \widetilde{M}_{i,j}(M^s)_{j,\gamma_\mu^p} + \widetilde{M}_{i,\mu}(\widetilde{M}^s)_{\mu,\gamma_\mu^p}. \end{aligned} \tag{3.11}$$

According to $\gamma_\mu^p \notin \bigcup_{k=0}^{2^n} E_k(\mu)$ and Theorem 3.1, we obtain $\gamma_\mu^p \notin \bigcup_{k=0}^{2^n} \widetilde{E}_k(\mu)$. Hence, $(\widetilde{M}^k)_{\mu,\gamma_\mu^p} = (M^k)_{\mu,\gamma_\mu^p} = 0$ holds for any $k \in \mathbb{Z}_+$. Besides, we derive from Lemma 3.1 that $\widetilde{M}_{i,j} = M_{i,j}$, $j \in [1 : 2^n]\setminus\{\mu\}$, $i \in [1 : 2^n]$. Then, (3.11) can be converted into

$$\begin{aligned} (\widetilde{M}^{s+1})_{i,\gamma_\mu^p} &= \sum_{j\neq\mu} M_{i,j}(M^s)_{j,\gamma_\mu^p} + \widetilde{M}_{i,\mu} \times 0 = \sum_{j\neq\mu} M_{i,j}(M^s)_{j,\gamma_\mu^p} + M_{i,\mu} \times 0 \\ &= \sum_{j\neq\mu} M_{i,j}(M^s)_{j,\gamma_\mu^p} + M_{i,\mu}(M^s)_{\mu,\gamma_\mu^p} = (M^{s+1})_{i,\gamma_\mu^p}. \end{aligned}$$

Therefore, it can be concluded that (3.10) is true.

Noting that $x_e \notin C$ and $\delta_{2^n}^{\gamma_\mu^p} \in C$, we conclude that $\gamma_\mu^p \neq \theta$. Based on Assumption 2.1, we know $(M^k)_{\gamma_\mu^p,\gamma_\mu^p} = 0$, $\forall\ k \in \mathbb{Z}_+$. Thus, we know from (3.10) and $\gamma_\mu^p \neq \mu$ that $(\widetilde{M}^k)_{\gamma_\mu^p,\gamma_\mu^p} = (M^k)_{\gamma_\mu^p,\gamma_\mu^p} = 0$, $\forall\ k \in \mathbb{Z}_+$, which contradicts (3.9). Thus, system (2.3) is robustly stable at x_e.

Remark 3.2. *Assume that system (2.3) is not robustly stable at x_e. According to Assumption 2.2 and $\gamma_\mu^p \neq \mu$, we know that no new fixed point is generated in system (2.3) after perturbation. Thereby, a new attractor which is a cycle different from x_e is bound to be produced.*

Finally, we give an example to support the main results.

Example 3.1. Consider the following SBN with external disturbance and three modes:

$$\begin{cases} x_1(t+1) = x_1(t) \vee \{\neg x_1(t) \wedge [(x_2(t)\wedge\neg x_3(t)) \vee (\xi(t)\wedge\neg x_2(t) \wedge x_3(t)) \\ \qquad \vee (\neg\xi(t)\wedge\neg x_2(t))]\}, \\ x_2(t+1) = \{\xi(t) \wedge x_3(t) \wedge [(x_1(t) \wedge x_2(t))\vee\neg x_1(t)]\} \vee \{\neg\xi(t)\wedge\neg x_1(t) \\ \qquad \wedge [(x_2(t)\wedge\neg x_3(t)) \vee (\neg x_2(t) \wedge x_3(t))]\}, \\ x_3(t+1) = (\xi(t)\wedge\neg x_1(t)\wedge\neg x_2(t)\wedge\neg x_3(t)) \vee \{(\neg\xi(t) \wedge x_2(t)) \wedge [(x_1(t) \\ \qquad \wedge x_3(t))\vee\neg x_1(t)]\}, \end{cases}$$
$$\begin{cases} x_1(t+1) = x_1(t) \vee \{\neg x_1(t) \wedge [(\xi(t) \wedge x_2(t)) \vee (\neg\xi(t)\wedge\neg x_2(t)) \vee (\xi(t) \\ \qquad \wedge\neg x_2(t) \wedge x_3(t)) \vee (\neg\xi(t) \wedge x_2(t))\wedge\neg x_3(t)]\}, \\ x_2(t+1) = (\xi(t) \wedge x_1(t) \wedge x_3(t)) \vee \{\neg x_1(t)\wedge\neg x_2(t) \wedge [(\xi(t) \wedge x_3(t)) \\ \qquad \vee (\neg\xi(t)\wedge\neg x_3(t))]\}, \\ x_3(t+1) = \xi(t)\wedge\neg x_1(t)\wedge\neg x_2(t), \end{cases} \tag{3.12}$$
$$\begin{cases} x_1(t+1) = x_1(t) \vee \{\neg x_1(t) \wedge [(\xi(t)\wedge\neg x_3(t)) \vee (\neg\xi(t) \wedge x_2(t)\wedge\neg x_3(t)) \\ \qquad \vee (\neg\xi(t)\wedge\neg x_2(t))]\}, \\ x_2(t+1) = \{x_1(t) \wedge x_3(t) \wedge [(\xi(t)\wedge\neg x_2(t))\vee\neg\xi(t)]\} \vee \{[(\xi(t)\wedge\neg x_1(t)) \\ \qquad \vee (\neg\xi(t)\wedge\neg x_1(t))] \wedge [(x_2(t)\wedge\neg x_3(t)) \vee (\neg x_2(t) \wedge x_3(t))]\}, \\ x_3(t+1) = \{x_2(t) \wedge x_3(t) \wedge [(\xi(t) \wedge x_1(t)) \vee (\neg\xi(t)\wedge\neg x_1(t))]\}\vee \\ \qquad (\neg\xi(t)\wedge\neg x_1(t)\wedge\neg x_2(t)), \end{cases}$$

which has the following algebraic form:

$$x(t+1) = L\sigma(t)\xi(t)x(t), \tag{3.13}$$

where $x(t) \in \Delta_8$, $\sigma : \mathbb{N} \to \{1,2,3\}$, $\xi(t) \in \Delta$, and

$$\begin{aligned} L \ = \ & \delta_8[4\ 4\ 2\ 4\ 6\ 4\ 2\ 7\ 3\ 4\ 4\ 4\ 7\ 1\ 2\ 4 \\ & 2\ 4\ 2\ 4\ 4\ 4\ 1\ 7\ 3\ 4\ 4\ 4\ 8\ 4\ 4\ 2 \\ & 3\ 4\ 2\ 4\ 8\ 2\ 6\ 4\ 2\ 4\ 2\ 4\ 7\ 2\ 1\ 3]. \end{aligned}$$

We partition L into six equal parts $L_1,\ L_2,\ L_3,\ L_4,\ L_5$, and L_6 by column. Note that

$$M = \sum_{i=1}^{6} L_i = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 2 & 0 & 4 & 0 & 0 & 2 & 2 & 1 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 6 & 2 & 6 & 1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \end{bmatrix}.$$

Then, we have

$$M^7 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 279,936 & 279,936 & 279,936 & 279,936 & 279,936 & 279,936 & 279,936 & 279,936 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Based on [11], system (3.13) is robustly stable at δ_8^4.

Assume that the one-bit function perturbation changes the value of $Col_3(L_2)$. After a simple calculation, we obtain $E_0(3) = \{3\}$, $E_1(3) = \{1, 8\}$, $E_2(3) = \{5, 6, 7\}$, $E_3(3) = \{5, 7, 8\}$, $E_4(3) = \{5, 8\}$, $E_5(3) = \{5\}$, and $E_6(3) = E_7(3) = E_8(3) = \emptyset$. Therefore, $\bigcup_{k=0}^{2^3} E_k(3) = \{1, 3, 5, 6, 7, 8\}$. In the light of Theorem 3.2, we conclude that system (3.13) is robustly stable at δ_8^4 after the value of $Col_3(L_2)$ is changed to δ_8^2, while system (3.13) is not robustly stable at δ_8^4 after the value of $Col_3(L_2)$ is changed to δ_8^1, δ_8^3, δ_8^5, δ_8^6, δ_8^7, or δ_8^8. These results can be demonstrated in Figures 1 and 2, where $\sigma(4j) = \delta_3^1$, $\sigma(4j + 1) = \delta_3^1$, $\sigma(4j + 2) = \delta_3^1$, $\sigma(4j + 3) = \delta_3^3$, $\xi(2j) = \delta_2^2$, $\xi(2j + 1) = \delta_2^1$, $\forall j \in \mathbb{N}$.

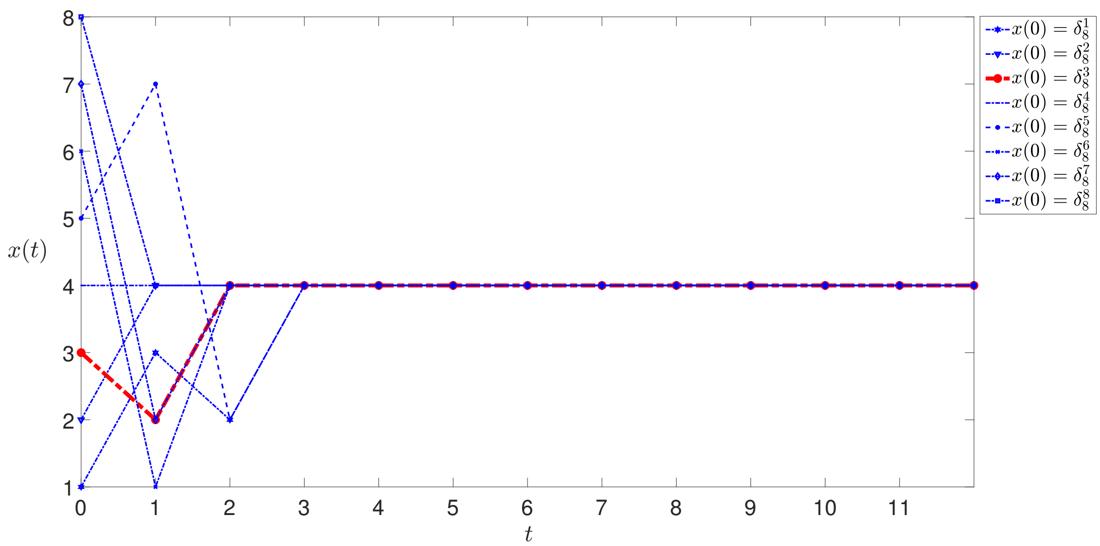

Figure 1. State trajectory of system (3.13) subject to the one-bit function perturbation which changes the value of $Col_3(L_2)$ to δ_8^2.

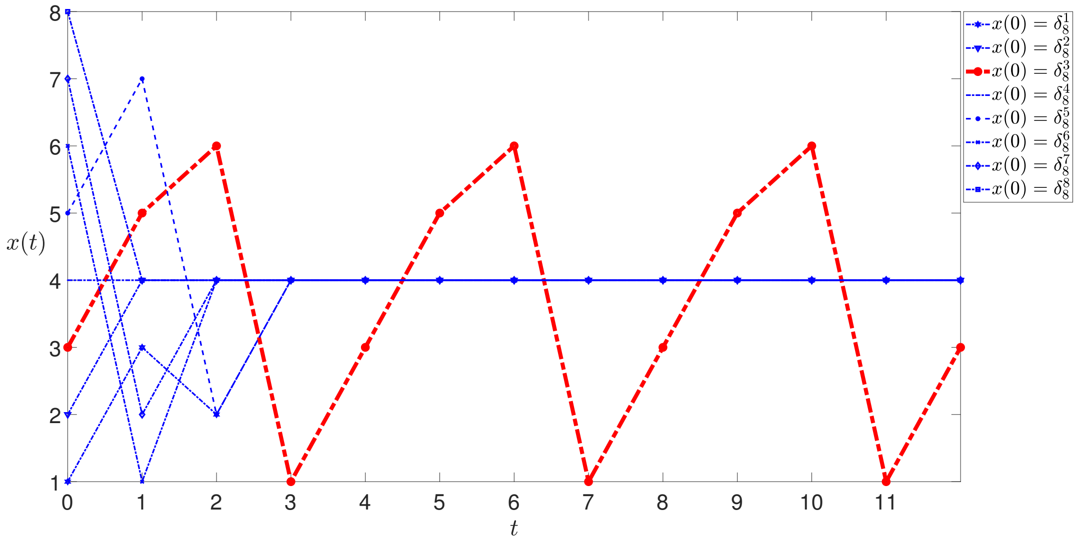

Figure 2. State trajectory of system (3.13) subject to the one-bit function perturbation which changes the value of $Col_3(L_2)$ to δ_8^5.

4. Conclusions

In this paper, we have investigated the robust stability of SBNs subject to external disturbance and one-bit function perturbation. We have constructed the reachable sets of the perturbed state before and after function perturbation. On the basis of the reachable sets, we have further studied the properties of the elements which do not belong to the sets. Then, with the assistance of the reachable sets, we have proposed a new criterion for verifying whether the perturbed SBN is robustly stable or not.

The computational complexity of our results is high. Therefore, we plan to further explore new criteria with lower computational complexity in the future. One can design flipping control or time-variant feedback control [35] to enable the robust stability of perturbed SBNs when the condition of Theorem 3.2 is no longer satisfied. Moreover, further investigation can focus on the robust stability of SBNs subject to Markovian jump disturbances [36] and multi-bit function perturbations.

Author contributions

Zhiqiao Tian: Investigation, Writing original draft; Haitao Li: Conceptualization, Writing–review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Prof. Haitao Li is an editorial board member for AIMS Mathematics and was not involved in the editorial review or the decision to publish this article. All authors declare no conflicts of interest in this paper.

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