Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(7): 17274–17290.
DOI: 10.3934/math.2025772
Received: 09 December 2024
Revised: 29 April 2025
Accepted: 04 June 2025
Published: 31 July 2025

Research article

Analysis of coefficient functionals for analytic functions with bounded turning linked with q-Sine function

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Abstract: This research study was focused on investigating a newly defined subclass of analytic functions, denoted as $R_{\sin_q}$ that was affiliated with the q-analogue of sine function that was defined in the open unit disk. The study examined the q-analogue of the sine function, with a focus on analyzing the upper bound of second and third order Hankel determinants, addressing coefficient problems, investigating the Krushkal inequality and estimating certain sharp bounds for coefficient problems for the corresponding subclass $R_{\sin_q}$. All computed bounds were sharp.

Keywords: analytic function; univalent function; $q-$derivative operator; factorial functional; sin function; Krushkal inequality.
Mathematics Subject Classification: 30C45; 30C80

1. Introduction

The class A comprises analytic functions f defined in the open unit disk

$$\bar{\delta} = \{z \in \mathbb{C} : |z| < 1\},$$

with the normalization properties $f(0) = 0$ and $f'(0) = 1$, has the following Taylor series

$$f(z) = z + \sum_{n=2}^{\infty} \xi_n z^n, \ z \in \bar{\delta}. \tag{1.1}$$

All univalent functions f that are analytic in the open unit disk and are normalized belong to the class S. This can be stated as given below:

$$S = \left\{ f \in A : f \text{ is univalent in } \bar{\delta} \right\}.$$

This class S established the foundation for current studies in this field. The class of starlike functions in $\bar{\delta}$ is represented by S^*, consisting of such analytic functions f which satisfy the $\Re\left(\frac{zf'(z)}{f(z)}\right) > 0$, $z \in \bar{\delta}$, condition; see [1]. One of the most significant topics in geometric function theory is the class of starlike functions. Let $\mathbf{P}$ [2] denote the class of analytic functions p in $\bar{\delta}$ with positive real part on $\bar{\delta}$ given by

$$p(z) = 1 + \sum_{n=1}^{\infty} \wp_n z^n, \; z \in \bar{\delta}. \tag{1.2}$$

Next, we recall the concept of subordination. Let ω be an analytic function in $\bar{\delta}$; it is called a Schwarz function if it satisfies the conditions $\omega(0) = 0$ and $|\omega(z)| < 1$ for all z in $\bar{\delta}$. Let $\hbar(z)$ and $\Upsilon(z)$ be analytic functions in $\bar{\delta}$, and if there exists a Schwarz function ω in $\bar{\delta}$ such that

$$\hbar(z) = \Upsilon(\omega(z)), \; z \in \bar{\delta}, \tag{1.3}$$

then $\hbar$ is said to be subordinate to the function Υ, which is denoted by $\hbar \prec \Upsilon$. Now, we define the well-known class of bounded turning function R_φ by using the subordination relation as follows:

$$R_\varphi = \left\{ f \in S : f'(z) \prec \varphi(z), \; z \in \bar{\delta} \right\},$$

where $\varphi(z)$ can be any suitable function. The R denotes the class of functions with bounded turnings; which are analytic and satisfy $\Re(f'(z)) > 0$, $z \in \bar{\delta}$ and are normalized by $f(0) = 0$ and $f'(0) = 1$. MacGregor [3] conducted a comprehensive study of the subclasses of R. The class R was initially introduced by Janteng et.al [4,5], marking a significant contribution to the study of analytic functions. The class R has several interesting properties that are utilized in complex analysis and related fields. For example, it has been used to study the properties of extremal functions, with intention to derive sharp estimates for the growth of Taylor coefficients and to analyze the convergence of numerical methods for solving certain differential equations. Moreover, the R class provides a framework for understanding the geometric properties of analytic functions.

Babalola [6] calculated the upper bounds of Hankel determinants of third order for the subfamilies of R. Zaprawa improved the results of Babalola; see [7]. Hankel determinants play a valuable role in calculating coefficient problems in the study of analytic and univalent functions. Scholars continue to explore the characteristics and applications of Hankel determinants, providing them with a rich and stimulating field of study in current mathematics. Because of their connections to a wide range of mathematical fields and applications, the Hankel determinant is an important tool for both pure and applied mathematics.

For the given parameters $\iota, m \in \mathbb{N}$, Pommerenke [8,9] defined the Hankel determinant $H_{\iota,m}(f)$ for a function $f \in A$ of the form (1.1) as follows:

$$H_{\iota,m}(f) = \begin{vmatrix} \xi_m & \xi_{m+1} & \cdots & \xi_{m+\iota-1} \\ \xi_{m+1} & \xi_{m+2} & \cdots & \xi_{m+\iota} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{m+\iota-1} & \xi_{m+\iota-2} & \cdots & \xi_{m+2\iota-2} \end{vmatrix} \quad \iota, m \in \mathbb{N}.$$

The Hankel determinants for various orders are calculated by taking multiple values of ι and m. For $\iota = 2$ and $m = 1$, the Hankel determinant $H_{\iota,m}(f)$ takes the following form:

$$H_{2,1}(f) = \begin{vmatrix} \xi_1 & \xi_2 \\ \xi_2 & \xi_3 \end{vmatrix} = \xi_1\xi_3 - \xi_2^2$$

The Hankel determinant $H_{2,1}(f)$ is known as the Fekete-Szegö functional [10]. For $\iota = 2$ and $m = 2$,

$$H_{2,2}(f) = \begin{vmatrix} \xi_2 & \xi_3 \\ \xi_3 & \xi_4 \end{vmatrix} = \xi_2\xi_4 - \xi_3^2.$$

For $\iota = 3$ and $m = 1$,

$$\begin{aligned} H_{3,1}(f) &= \begin{vmatrix} \xi_1 & \xi_2 & \xi_3 \\ \xi_2 & \xi_3 & \xi_4 \\ \xi_3 & \xi_4 & \xi_5 \end{vmatrix} \\ &= \xi_5(\xi_3 - \xi_2^2) - \xi_4(\xi_4 - \xi_2\xi_3) + \xi_3(\xi_2\xi_4 - \xi_3^2). \end{aligned}$$

This implies that

$$\left|H_{3,1}(f)\right| \leq |\xi_5| \left|\xi_3 - \xi_2^2\right| + |\xi_4| \left|\xi_4 - \xi_2\xi_3\right| + |\xi_3| \left|H_{2,2}(f)\right|. \tag{1.4}$$

Extensive research has been carried out to study the upper bounds of $\left|H_{3,1}(f)\right|$ for various subfamilies of analytic and univalent functions which includes the bound for Bazilevic functions by Altinkaya and Yalcin [11], for functions related with bounded variations by Arif et al. [12], for certain locally univalent functions by Bansal et al. [13], for strongly starlike functions by Cho et al. [14], for functions related with lemniscate of Bernoulli by Raza and Malik [15], for functions related to booth Lemniscate by Raza et al. [16], for functions associated with generalized Lemniscate of Bernoulli by Nawaz et al. [17] and many others.

In recent years, researchers have increasingly turned their attention to q-calculus, a mathematical framework that has garnered recognition for its numerous applications and significance in various fields. This renewed focus has stimulated the exploration and definition of important and intriguing subclasses of analytic functions, specifically within the context of q-calculus. In a short period, multiple noteworthy achievements in this field have been presented. Engineers as well as pure and applied mathematicians want to pursue this extended form because of its higher efficiency and wide range of applications. The q-analogue of differential and integral operators were initially introduced by Jackson [18, 19]. His innovative approach has significantly improved our understanding of these mathematical concepts, paving the way for further exploration in q-calculus. In [20], Srivastava presented a comprehensive study of the q-calculus for the sake of developing a mathematical understanding to introduce new ideas in Geometric Function Theory. Mahmood et al. [21, 22] studied the q-analogue of starlike functions by computing certain coefficient problems. Taj et al. [23, 24] introduced the q-versions of sine and cosine functions using the q-exponential function. In [25–27], q-starlike functions associated with q-exponential functions and with the general conic domain are discussed. Zhang et al. [28] studied the q-Hermite polynomials. The q-analogue of close-to-convex functions was studied by Shi et al. [29]. Raza et al. [30] studied the q-analogue of differential subordination associated with Lemniscate of Bernoulli. In [31–33], q-starlike functions associated with k -Fibonacci numbers, conic domains, and Janowski functions, respectively, are discussed.

Quantum calculus, also known as q -calculus, is a mathematical discipline that introduces a new type of derivative, known as the q-derivative. The q-derivative of the complex-valued function f in q-calculus is precisely defined within the domain $\bar{\delta}$, as given below:

$$(D_q f)(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z}, & z \neq 0, \\ f'(0), & z = 0, \end{cases} \tag{1.5}$$

where $q \in (0, 1)$. Additionally,

$$\lim_{q\to 1^-} (D_q f)(z) = \lim_{q\to 1^-} \frac{f(z) - f(qz)}{(1 - q)z} = f'(z),$$

provided f is differentiable in $\bar{\delta}$. The Maclaurin series expansion for the function $(D_q f)$ in equation (1.1) is given by:

$$(D_q f)(z) = \sum_{\tau=0}^{\infty} [\tau]_q \, \xi_\tau z^{\tau-1},$$

where

$$[\tau]_q = \begin{cases} \frac{1-q^\tau}{1-q}, & \text{if } \tau \in \mathbb{C}, \\ \sum_{\tau=0}^{\tau-1} q^\tau, & \text{if } \tau \in \mathbb{N}. \end{cases} \tag{1.6}$$

For more details, see [18, 19]. There are q-analogues of the exponential functional presented in the following form:

$$\varrho_q(z) = \sum_{\tau=0}^{\infty} \frac{z^\tau}{[\tau]_q!}, \; q \in (0, 1); \; |z| < \frac{1}{1-q}.$$

Now, we define $1 + \sin_q(z)$ as given below.

Definition 1.1. The q-version of sine functions is derived by utilizing the q-exponential function, which can be expressed as follows:

$$\sin_q(z) = \frac{\varrho_q^{iz} - \varrho_q^{-iz}}{2i}$$

and the series of $1 + \sin_q(z)$ is presented as follows:

$$1 + \sin_q(z) = 1 + \frac{1}{[1]_q!} z - \frac{1}{[3]_q!} z^3 + \frac{1}{[5]_q!} z^5 - \frac{1}{[7]_q!} z^7 \cdots . \tag{1.7}$$

After briefly studying the applications of q-calculus, we come to know that the q-derivative operator D_q plays a central role in defining and discovering abundant subclasses of analytic functions. Motivated by the above work, we now define the class $R_{\sin_q}$ of bounded turning functions associated with the q-analogue of $1 + \sin_q(z)$, as given below.

Definition 1.2. A function f is said to be in the class $R_{\sin_q}$ if it fulfills the following condition:

$$D_q f(z) \prec 1 + \sin_q(z), \;\; z \in \bar{\delta}, \tag{1.8}$$

that is,

$$R_{\sin_q} = \left\{ f \in A : D_q f(z) \prec 1 + \sin_q(z), \;\; z \in \bar{\delta} \right\}. \tag{1.9}$$

The class $R_{\sin_q}$ extends the class $R_{\sin}$ associated with the function $1 + \sin(z)$ and $\mathrm{Lim}_{q\to 1^-} R_{\sin_q} \cong R_{\sin}$. The class $R_{\sin}$ studied by [34].

The main objective of this research is to study the class $R_{\sin_q}$ by utilizing the following mentioned lemmas, which will serve as key analytical tools to support our findings and conclusions. We are intended to explore this class by investigating the following coefficient problems for the said class.

- The sharp coefficient bounds:

$$|\xi_n| \quad \text{for } n = 2, 3, 4.$$

- The Krushkal inequality:

$$\left|\xi_{\varepsilon}^{\eta} - \xi_2^{\eta(\varepsilon-1)}\right| \leq 2^{\eta(\varepsilon-1)} - \varepsilon^{\eta}, \quad \text{for } \varepsilon = 4 \text{ and } \eta = 1.$$

- The sharp upper bounds of the second-order Hankel determinant and third-order Hankel determinant.

2. A set of lemmas

We will require the following lemmas for finding the main results.

Lemma 2.1. *[2, 35] If $p \in \mathbf{P}$ is of the form* (1.2) *, then*

$$2\wp_2 = \wp_1^2 + \alpha(4 - \wp_1^2),$$

$$4\wp_3 = \wp_1^3 + 2\wp_1(4 - \wp_1^2)\alpha - \wp_1(4 - \wp_1^2)\alpha^2 + 2(4 - \wp_1^2)\left(1 - |\alpha|^2\right)\beta,$$

with $|\alpha| \leq 1$ and $|\beta| \leq 1$.

Lemma 2.2. *[36] If $p \in \mathbf{P}$ is of the form* (1.2) *where $\lambda, \alpha \in (0, 1)$ and*

$$8\lambda(1-\lambda)\left[(\alpha\beta - 2\gamma)^2 + (\alpha(\lambda+\alpha) - \beta)^2\right] + \alpha(1-\alpha)(\beta - 2\lambda\alpha)^2 \leq 4\alpha^2(1-\alpha)^2\lambda(1-\lambda),$$

then

$$\left|\gamma\wp_1^4 + \lambda\wp_2^2 + 2\alpha\wp_1\wp_3 - \frac{3}{2}\beta\wp_1^2\wp_2 - \wp_4\right| \leq 2.$$

Lemma 2.3. *[2] Let $p \in \mathbf{P}$ be of the form* (1.2) *. Then,*

$$|\wp_n| \leq 2 \qquad (n \in \mathbf{N}), \tag{2.1}$$

$$\left|\wp_2 - \frac{\nu}{2}\wp_1^2\right| \leq \begin{cases} 2, & 0 \leq \nu \leq 2, \\ 2|\nu - 1|, & \textit{elsewhere}. \end{cases} \tag{2.2}$$

Lemma 2.4. *[2] Let $p \in \mathbf{P}$, $0 \leq M \leq 1$, and $M(2M-1) \leq N \leq M$. Then,*

$$\left|\wp_3 - 2M\wp_1\wp_2 + N\wp_1^3\right| \leq 2.$$

3. Main results

In this section, we calculate the coefficient bounds for the considered class $R_{\sin_q}$. Coefficient bounds are essential for studying the growth, distortion, stability, and geometric and structural characteristics of analytic and univalent functions.

Theorem 3.1. *If the function f belongs to $R_{\sin_q}$ and has the form* (1.1)*, then*

$$|\xi_n| \leq \frac{1}{\sum\limits_{i=0}^{n-1} q^i}, \tag{3.1}$$

where $n = 2, 3, 4$ for $q \in (0, 1)$ and $n = 5$ for $q \in (0.56, 1)$. All these results are sharp for the function defined by

$$D_q F(z) = 1 + \sin_q(z^n) = 1 + z + \cdots, \text{ for } n = 1, 2, 3, 4. \tag{3.2}$$

Proof. From (1.5) and (1.9), we can write

$$D_q F(z) = 1 + \sin_q(\omega(z)), \quad z \in \overline{\delta},$$

where $\omega(z) = \frac{p(z)-1}{1+p(z)}$. If the function $p(z)$ is in the form described by (1.2), then

$$\omega(z) = \frac{\wp_1 z + \wp_2 z^2 + \wp_3 z^3 + \ldots}{2 + \wp_1 z + \wp_2 z^2 + \wp_3 z^3 + \ldots}.$$

Now consider

$$1 + \sin_q(\omega(z)) = 1 + \sin_q\left(\frac{\wp_1 z + \wp_2 z^2 + \wp_3 z^3 + \ldots}{2 + \wp_1 z + \wp_2 z^2 + \wp_3 z^3 + \ldots}\right), \tag{3.3}$$

we have

$$1 + \sin_q(\omega(z)) = 1 + \frac{\wp_1}{2\,[1]_q!} z + \left(\frac{2\wp_2 - \wp_1^2}{4\,[1]_q!}\right) z^2 + \cdots. \tag{3.4}$$

Now, consider

$$D_q f(z) = 1 + \frac{-\xi_2 q^2 + \xi_2}{1-q} z + \frac{-\xi_3 q^3 + \xi_3}{1-q} z^2 + \frac{-\xi_4 q^4 + \xi_4}{1-q} z^3 + \cdots. \tag{3.5}$$

Using (3.4) and (3.5), we can compute the coefficients of z, z^2, z^3, and z^4 and obtain

$$\xi_2 = \frac{\wp_1}{2(1+q)}, \tag{3.6}$$

$$\xi_3 = \frac{2\wp_2 - \wp_1^2}{4\sum_{i=0}^{2} q^i}, \tag{3.7}$$

$$\xi_4 = \frac{\left(4\,[3]_q!\wp_3 - 4\,[3]_q!\wp_1\wp_2 + \left(-1 + [3]_q!\right)\wp_1^3\right)}{8\,[3]_q! \sum_{i=0}^{3} q^i}, \tag{3.8}$$

and

$$\xi_5 = \frac{6\left(-1 + [3]_q!\right)\wp_1^2\wp_2 + \left(3 - [3]_q!\right)\wp_1^4 - 8\,[3]_q!\wp_1\wp_3 + 8\,[3]_q!\wp_4 - 4\,[3]_q!\wp_2^2}{16\,[3]_q! \sum\limits_{k=0}^{4} q^k}. \tag{3.9}$$

Applying Lemma 2.3 in (3.6) , we get

$$|\xi_2| \leq \frac{1}{1+q},$$

and we can write

$$|\xi_2| \leq \frac{1}{\sum\limits_{k=0}^{1} q^k}.$$

Now, consider

$$\xi_3 = \frac{2\wp_2 - \wp_1^2}{4\sum_{i=0}^{3} q^i},$$

and we can write

$$\xi_3 = \frac{1}{2\sum_{i=0}^{3} q^i}\left[\wp_2 - \frac{\wp_1^2}{2}\right].$$

Using Lemma 2.3 in (3.7) , we get

$$|\xi_3| \leq \frac{1}{\sum_{i=0}^{2} q^i}. \tag{3.10}$$

Now, consider

$$\xi_4 = \frac{\left(4\,[3]_q!\wp_3 - 4\,[3]_q!\wp_1\wp_2 + \left(-1 + [3]_q!\right)\wp_1^3\right)}{8\,[3]_q!\,(1+q+q^2+q^3)}.$$

As $[3]_q! = (1+q)\left(1+q+q^2\right)$, we may write

$$\begin{aligned}
\xi_4 &= \frac{\left(4q^3+8q^2+8q+4\right)}{8\,[3]_q!\,(1+q+q^2+q^3)}\left(\wp_3 - \wp_1\wp_2 + \frac{\left(q^3+2q^2+2q\right)}{(4q^3+8q^2+8q+4)}\wp_1^3\right)\\
&= \frac{4\,(q+1)\left(q+q^2+1\right)}{8\,(1+q)\,(1+q+q^2)\,(1+q+q^2+q^3)}\left(\wp_3 - \wp_1\wp_2 + \frac{\left(q^3+2q^2+2q\right)}{(4q^3+8q^2+8q+4)}\wp_1^3\right)\\
&= \frac{1}{2\,(1+q+q^2+q^3)}\left(\wp_3 - \wp_1\wp_2 + \frac{\left(q^3+2q^2+2q\right)}{(4q^3+8q^2+8q+4)}\wp_1^3\right).
\end{aligned}$$

Now applying Lemma 2.4, we get $M = \frac{1}{2}$ and $N = \frac{\left(q^3+2q^2+2q\right)}{\left(4q^3+8q^2+8q+4\right)}$. It is clearly seen that $0 < M < 1$ and $N < M$ for $q \in (0,1)$. Further, $M\,(2M-1) - N < 0$ for $q \in (0,1)$ .Therefore, by Lemma 2.4, we have

$$|\xi_4| \leq \frac{1}{(1+q+q^2+q^3)}. \tag{3.11}$$

Now, take

$$\xi_5 = \frac{6\left(-1+[3]_q!\right)\wp_1^2\wp_2 + \left(3-[3]_q!\right)\wp_1^4 - 8\,[3]_q!\wp_1\wp_3 + 8\,[3]_q!\wp_4 - 4\,[3]_q!\wp_2^2}{16\,(q^4+q^3+q^2+q+1)\,[3]_q!}.$$

Putting the value of $[3]_q! = (1+q)\left(1+q+q^2\right)$, we have

$$\xi_5 = \frac{1}{2\,(q^4+q^3+q^2+q+1)}\left[\frac{-q^3-2q^2-2q+2}{8\,(1+q)\,(1+q+q^2)}\wp_1^4 + \frac{1}{2}\wp_2^2 + \wp_1\wp_3\right.$$

$$- \frac{3}{2} \frac{q^3 + 2q^2 + 2q}{2(1+q)(1+q+q^2)} \wp_1^2 \wp_2 - \wp_4 \Bigg].$$

After comparing with Lemma 2.2, we get

$$\gamma = \frac{-q^3 - 2q^2 - 2q + 2}{8(1+q)(1+q+q^2)},\ \lambda = \frac{1}{2},\ \alpha = \frac{1}{2},\ \beta = \frac{q^3 + 2q^2 + 2q}{2(1+q)(1+q+q^2)}.$$

Clearly, $0 < \lambda < 1$ and $0 < \alpha < 1$. Also, consider

$$\begin{aligned} &8\lambda(1-\lambda)\left[(\alpha\beta - 2\gamma)^2 + (\alpha(\lambda+\alpha) - n_1)^2\right] \\ &+ \alpha(1-\alpha)(\beta - 2\lambda\alpha)^2 - 4\alpha^2(1-\alpha)^2\lambda(1-\lambda) = \Psi(q), \end{aligned}$$

where

$$\Psi(q) = -\frac{2q^6 + 8q^5 + 16q^4 + 20q^3 + 16q^2 + 8q - 15}{16(q^3 + 2q^2 + 2q + 1)^2}.$$

After some simple calculations, we can see that $\Psi(q) \leq 0$, for all $q \in (0.56, 1)$. By using Lemma 2.2, we obtain

$$|\xi_5| \leq \frac{1}{(q^4 + q^3 + q^2 + q + 1)},\ \text{for } q \in (0.56, 1). \tag{3.12}$$

□

Upon letting $q \to 1^-$, the upper bounds $|\xi_2|$, $|\xi_3|$, and $|\xi_4|$ reduce to the following, proved in [34]. Moreover, the upper bound for $|\xi_5|$ is significantly improved compared to the one proved in [34].

Corollary 3.2. *If the function f belongs to $R_{\sin}$ and has the form* (1.1)*, then*

$$\begin{aligned} |\xi_2| &\leq \frac{1}{2}, \\ |\xi_3| &\leq \frac{1}{3}, \\ |\xi_4| &\leq \frac{1}{4}, \end{aligned}$$

and

$$|\xi_5| \leq \frac{1}{5}.$$

4. Krushkal inequality

The Krushkal inequality comes up when studying Teichmüller spaces and quasi-conformal mappings. It frequently appears in the analysis of extremal problems regarding geometric function theory, function theory, and conformal invariants. Krushkal proposed and verified this inequality for every class of univalent functions in [37].

According to the Krushkal inequality, each $f \in S$ having the form (1.1) satisfies the following sharp inequality.

$$\left|\xi_\varepsilon^\eta - \xi_2^{\eta(\varepsilon-1)}\right| \leq 2^{\eta(\varepsilon-1)} - \varepsilon^\eta, \varepsilon > 3, \eta \geq 1. \tag{4.1}$$

The next result examines the inequality (4.1) for $\varepsilon = 4$ and $\eta = 1$, reducing it to $\left|\xi_4 - \xi_2^3\right| \leq 4$. We encourage viewers to review [38, 39] for a few current studies on the Kruskal inequality.

Theorem 4.1. *If the function f belongs to $R_{\sin_q}$ and has the form* (1.1)*, then*

$$\left|\xi_4 - \xi_2^3\right| \leq \frac{1}{(1+q+q^2+q^3)},\ q \in (0.44, 1).$$

The result is sharp for the function defined in (3.2) *, for $n = 4$.*

Proof. From (3.6) and (3.8) , we have

$$\xi_4 - \xi_2^3 = \frac{\left(4\,[3]_q!\wp_3 - 4\,[3]_q!\wp_1\wp_2 + \left(-1 + [3]_q!\right)\wp_1^3\right)}{8\left([3]_q!\right)\sum_{i=0}^{3} q^i} - \left(\frac{\wp_1}{2\,(1+q)}\right)^3.$$

Putting the value of $[3]_q! = (1+q)\left(1+q+q^2\right)$, after simplification we can write

$$\xi_4 - \xi_2^3 = \frac{1}{2\,(1+q+q^2+q^3)}\left(\wp_3 - \wp_1\wp_2 + \frac{q+2q^2+2q^3-1}{4\,(q+1)^2\,(q+q^2+1)}\wp_1^3\right).$$

Now applying Lemma 2.4, we get $M = \frac{1}{2}$ and $N = -\frac{q+2q^2+2q^3-1}{4(q+1)^2\left(q+q^2+1\right)}$. It is clearly seen that $0 < M < 1$ and $N < M$ for $q \in (0, 1)$. Further, $M\,(2M-1) - N < 0$ for $q \in (0.44, 1)$. So, we can write

$$\left|\xi_4 - \xi_2^3\right| \leq \frac{1}{(1+q+q^2+q^3)},\ q \in (0.44, 1).$$

□

Theorem 4.2. *If the function f belongs to $R_{\sin_q}$ and has the form* (1.1)*, then*

$$|\xi_4 - \xi_2\xi_3| \leq \frac{1}{\sum_{j=0}^{3} q^j}. \tag{4.2}$$

The result is sharp for the function defined in (3.2) *, for $n = 4$.*

Proof. From (3.6) – (3.8), we get

$$\begin{aligned}\xi_4 - \xi_2\xi_3 = {} & \frac{1}{8\left([3]_q!\right)^2(1+q^2)}\left[4\,[3]_q!\sum_{j=0}^{2} q^j\wp_3 - 2\,[3]_q!\left(3q^2+2q+3\right)\wp_1\wp_2\right. \\ & \left. + \left([3]_q!\left(2+q+2q^2\right) - \sum_{j=0}^{2} q^j\right)\wp_1^3\right] \\ & := F_q(\wp, x).\end{aligned}$$

Utilizing Lemma 2.1, we get

$$F_q(\wp, x) = \frac{1}{8\left([3]_q!\right)^2(1+q^2)}\left[-\wp_1\,|x|^2\left(4-\wp_1^2\right)[3]_q!\sum_{j=0}^{2} q^j - \left(q^2+1\right)[3]_q!\wp_1\,|x|\left(4-\wp_1^2\right)\right.$$

$$- \sum_{j=0}^{2} q^j \wp_1^3 + 2\,[3]_q!\left(4-\wp_1^2\right)\left(1-|x|^2\right)|\beta| \sum_{j=0}^{2} q^j \Bigg],$$

Apply the triangular inequality and assume that $|x| = x \in (0,1)$, $c \in (0,2)$, and $|\beta| \leq 1$. After reducing, we obtain

$$F_q(\wp, x) = \frac{1}{8\left([3]_q!\right)^2 (1+q^2)} \Bigg[\wp_1 x^2 \left(4-\wp_1^2\right)[3]_q! \sum_{j=0}^{2} q^j \wp_3 + \left(q^2+1\right)[3]_q! \wp_1 x \left(4-\wp_1^2\right)$$
$$+ \sum_{j=0}^{2} q^j \wp_1^3 + 2\,[3]_q!\left(4-\wp_1^2\right)\sum_{j=0}^{2} q^j \Bigg].$$

Partially differentiating with respect to x, we get

$$\frac{\partial F_q(\wp, x)}{\partial x} = \frac{1}{8\left([3]_q!\right)^2 (1+q^2)} \left[2\wp_1 x \left(4-\wp_1^2\right)[3]_q! \sum_{j=0}^{2} q^j + \left(q^2+1\right)[3]_q! \wp_1 \left(4-\wp_1^2\right)\right].$$

We can observe that $\frac{\partial F_q}{\partial x} > 0$ and $x \in [0,1]$. Therefore, when $x = 1$, the function $F_q(\wp, x)$ reaches its maximum value, as shown by

$$F_q(\wp, 1) = \frac{1}{8\left([3]_q!\right)^2 (1+q^2)} \Bigg[\wp_1 \left(4-\wp_1^2\right)[3]_q! \sum_{j=0}^{2} q^j + \left(q^2+1\right)[3]_q! \wp_1 \left(4-\wp_1^2\right)$$
$$+ \sum_{j=0}^{2} q^j \wp_1^3 + 2\,[3]_q!\left(4-\wp_1^2\right)\sum_{j=0}^{2} q^j \Bigg] := G(\wp).$$

Partial differentiating w.r.t $\wp$, after reducing, we get

$$G'(\wp) = \frac{1}{8\left([3]_q!\right)^2 (1+q^2)} \Bigg[\wp^2 \left(3\sum_{j=0}^{2} q^j - 3\,[3]_q\left(2+q+2q^2\right)\right) + 2\,[3]_q!\left(2+q+2q^2\right)$$
$$- 4\,[3]_q! \wp \sum_{j=0}^{2} q^j \Bigg].$$

We concluded that $G'(\wp) < 0$, and then $G(\wp)$ achieved its maximum value at $\wp = 0$, where we have

$$G(0) = \frac{1}{(1+q)(1+q^2)}.$$

After reducing, we get

$$|\xi_4 - \xi_2\xi_3| \leq \frac{1}{\sum_{j=0}^{3} q^j}.$$

□

We determined that the following bounds are significantly better than the one demonstrated in [34] by taking $q \to 1^-$ in the previous expression.

Corollary 4.3. *If the function f belongs to $R_{\sin}$ and has the series form* (1.1) , *then*

$$|\xi_4 - \xi_2\xi_3| \leq \frac{1}{4}.$$

5. Hankel determinants

Theorem 5.1. *If the function f belongs to $R_{\sin_q}$ and has the series form as it is appeared in* (1.1) , *then*

$$|H_{2,1}(f)| = \left|\xi_3 - \xi_2^2\right| \leq \frac{1}{(q^2+q+1)}, \text{ for } q \in (0,1). \tag{5.1}$$

The outcome is sharp for the function defined in (3.2) , *for $n = 3$.*

Proof. Using (3.6) and (3.7), we have

$$\xi_3 - \xi_2^2 = \frac{1}{2(q^2+q+1)}\left(\wp_2 - \frac{2q^2+3q+2}{2(q+1)^2}\wp_1^2\right).$$

Clearly, $0 < \frac{2q^2+3q+2}{2(q+1)^2} \leq 1$, for $q \in (0,1)$, so applying Lemma 2.3, we have

$$\left|\xi_3 - \xi_2^2\right| \leq \frac{1}{(q^2+q+1)}.$$

□

We concluded the following result showed in [34] by taking $q \to 1^-$ in the previous expression.

Corollary 5.2. *If the function f belongs to $R_{\sin}$ and has form* (1.1)*, then*

$$\left|\xi_3 - \xi_2^2\right| \leq \frac{1}{3}.$$

Theorem 5.3. *If the function f belongs to $R_{\sin_q}$ and has form* (1.1)*, then*

$$|H_{2,2}(f)| \leq \frac{1}{\left(\sum_{j=0}^{2} q^j\right)^2}, \tag{5.2}$$

where $H_{2,2}(f) = \xi_2\xi_4 - \xi_3^2$. The outcome is sharp for the function defined in (3.2) , *for $n = 3$.*

Proof. From (3.6) – (3.8) , we have

$$H_{2,2}(f) = \frac{1}{16\left([3]_q!\right)\left([4]_q!\right)\left(\sum_{j=0}^{3} q^j\right)^2}\left[\left(-1 - 2q + [3]_q!q^2 - 3q^2 - 2q^3 - q^4\right)\wp_1^4 - 4[3]_q!q^2\wp_1^2\wp_2\right.$$
$$\left. +4[3]_q!\wp_1\wp_3\left(1 + 2q + 3q^2 + 2q^3 + q^4\right) - 4[3]_q!\wp_2^2\left(1 + 2q + 2q^2 + 2q^3 + q^4\right)\right].$$

For the convenience of notation, $\wp_1 := \wp \in [0,2]$. By utilizing Lemma 2.1 and simplifying the expression, we get

$$H_{2,2}(f) = \frac{1}{16\,[3]_q!\,[4]_q!\left(\sum_{j=0}^{2l} q^j\right)^2}\Big[-\left(1+2q+2q^2+2q^3+q^4\right)[3]_q!t^2x^2$$
$$-\left(1+2q+3q^2+2q^3+q^4\right)\wp^4 - [3]_q!\wp^2 t x^2\left(1+2q+3q^2+2q^3+q^4\right)$$
$$+2\,[3]_q!\wp\left(1-|x|^2\right)|\beta|\,t\left(1+2q+3q^2+2q^3+q^4\right)\Big].$$

Let t be defined as $(4-\wp^2)$. To apply the modulus, we can utilize the triangle inequality along with the conditions where $|x| = x$ and $|\beta| \leq 1$.

$$\begin{aligned}
\alpha_1 &= \left|-\left(1+2q+2q^2+2q^3+q^4\right)[3]_q!\right| = \left(1+2q+2q^2+2q^3+q^4\right)[3]_q! > 0,\\
\alpha_2 &= \left|-\left(1+2q+3q^2+2q^3+q^4\right)\right| = \left(1+2q+3q^2+2q^3+q^4\right) > 0,\\
\alpha_3 &= \left|-[3]_q!\left(1+2q+3q^2+2q^3+q^4\right)\right| = [3]_q!\left(1+2q+3q^2+2q^3+q^4\right) > 0,\\
\alpha_4 &= \left|2\,[3]_q!\left(1+2q+3q^2+2q^3+q^4\right)\right| = 2\,[3]_q!\left(1+2q+3q^2+2q^3+q^4\right) > 0.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
H_{2,2}(f) &= \frac{1}{16\,[3]_q!\,[4]_q!\left(\sum_{j=0}^{2} q^j\right)^2}\left[\alpha_1 t^2x^2 + \alpha_2\wp^4 + \alpha_3\wp^2 t x^2 + \alpha_4\wp t\left(1-x^2\right)\right]\\
&:= L(\wp, x). \qquad (5.3)
\end{aligned}$$

Assume that the upper bound for the function $L(\wp, x)$ is defined within the interior of the rectangle $[0,2]\times[0,1]$. Differentiating (5.3) with respect to x, we have

$$\begin{aligned}
\frac{\partial L}{\partial x} &= \frac{1}{16\,[3]_q!\,[4]_q!\left(\sum_{j=0}^{2} q^j\right)^2}\left[2\alpha_1 t^2 x + 2\alpha_3\wp^2 t x - 2\alpha_4\wp t x\right]\\
&= \frac{t}{16\,[3]_q!\,[4]_q!\left(\sum_{j=0}^{2} q^j\right)^2}\left[2\alpha_1 t x + 2\alpha_3\wp^2 x - 2\alpha_4\wp x\right],
\end{aligned}$$

where $t := (4-\wp^2)$. Setting $\frac{\partial L}{\partial x} = 0$ implies either $\wp = 2$ or $2\alpha_1(4-\wp^2)x + 2\alpha_3\wp^2 x - 2\alpha_4\wp x = 0$. The points $(\wp, x)$ that satisfy these conditions are not interior points of the rectangle $[0,2]\times[0,1]$. The function $L(\wp, x)$ cannot attain its maximum value within the interior of the rectangle. Therefore, the maximum value must occur at the boundary of the rectangle. For this, We will examine the following cases:

When $\wp = 0$, we have

$$L(0,x) = \frac{\left(1+2q+2q^2+2q^3+q^4\right)x^2}{[4]_q!\left(\sum_{j=0}^{2} q^j\right)^2},$$

$$\max L(0,x) = L(0,1) = \frac{1}{\left(\sum_{j=0}^{2} q^j\right)^2}.$$

Suppose $\wp = 2$, and we get

$$L(2, x) = \frac{1}{[3]_q!\,[4]_q!}.$$

Suppose $x = 0$, and we get

$$L(\wp, 0) = \frac{1}{16\,[3]_q!\,[4]_q!\left(\sum_{j=0}^{2} q^j\right)^2}\left[\alpha_2\wp^4 + \alpha_4\wp t\right] := g(\wp).$$

Taking partial differentiating w.r.t. $\wp$, after simplifying, we get

$$g'(\wp) = \frac{1}{16\,[3]_q!\,[4]_q!\left(\sum_{j=0}^{2} q^j\right)^2}\left[4\alpha_2\wp^3 + 2\alpha_4\wp^2\right].$$

As we can see, $g'(\wp) > 0$, $\wp \in [0, 2]$ and g is an increasing function that reaches its maximum value at $\wp = 2$. After simplifying, and get

$$g(\wp) \leq g(2) = \frac{1}{[3]_q!\,[4]_q!}.$$

Suppose $x = 1$, and we get

$$\begin{aligned} L(\wp, 1) &= \frac{1}{16\,[3]_q!\,[4]_q!\left(\sum_{j=0}^{2} q^j\right)^2}\left[\alpha_1\left(4 - \wp^2\right)^2 + \alpha_2\wp^4 + \alpha_3\wp^2\left(4 - \wp^2\right)\right] \\ &:= g_1(\wp), \end{aligned}$$

as $g_1'(0) = 0$, and

$$g_1''(0) = \frac{-1}{2\left(\sum_{j=0}^{2} q^j\right)^2} < 0,$$

so

$$\begin{aligned} \max L(\wp, 1) &= g_1(0) = \frac{16\alpha_1}{16\,[3]_q!\,[4]_q!\left(\sum_{j=0}^{2} q^j\right)^2} \\ &= \frac{\left(1 + 2q + 2q^2 + 2q^3 + q^4\right)}{[4]_q!\left(\sum_{j=0}^{2} q^j\right)^2}. \end{aligned}$$

Consequently, we obtain

$$|H_{2,2}(f)| \leq \frac{\left(1 + 2q + 2q^2 + 2q^3 + q^4\right)}{[4]_q!\left(\sum_{j=0}^{2} q^j\right)^2}.$$

As $[4]_q! = (1 + q)\left(1 + q + q^2 + q^3\right)$, after simplification, we have

$$|H_{2,2}(f)| \leq \frac{1}{\left(\sum_{j=0}^{2} q^j\right)^2}.$$

□

We determined that the following bounds are significantly better than the one calculated in [34] by taking $q \to 1^-$ in the previous expression.

Corollary 5.4. *If the function f belongs to $R_{\sin}$ and has form* (1.1)*, then*

$$|H_{2,2}(f)| \leq \frac{1}{9}.$$

Theorem 5.5. *If $f \in R_{\sin}$ has the form as* (1.1) *, then*

$$\left|H_{3,1}(f)\right| \leq \frac{Q}{\left(1+q+q^2\right)^3\left(1+q+q^2+q^3\right)^2\left(1+q+q^2+q^3+q^4\right)}. \tag{5.4}$$

with $Q = 4q^{10} + 15q^9 + 36q^8 + 63q^7 + 86q^6 + 95q^5 + 86q^4 + 63q^3 + 36q^2 + 15q + 4.$

Proof. Thus, we use the fact that $a_1 = 1$, together with (3.10), (3.11), (3.12), (4.2), (5.1), and (5.2) in (1.4). □

We arrive at the conclusion that the following bound is significantly better than the one demonstrated in [34] by taking $q \to 1^{-1}$ in (5.4).

Corollary 5.6. *If $f \in R_{\sin}$ has the form as* (1.1) *, then*

$$|H_{3,1}(f)| \leq 0.23287.$$

6. Conclusions

Recently, the basic concepts of q-calculus have attracted the attention of numerous mathematicians due to its extensive applications in both mathematics and physics. We were primarily motivated to conduct the current investigations by the existing research in this field of study, in this contex we introduced a new subclass denoted by $R_{\sin_q}$ of analytic functions. This subclass is associated with the q-analogue of the *sine* function through a subordination relation. We computed the coefficient bounds and solved the Fekete-Szegö problem for this subclass. Furthermore, we derived the Krushkal inequality and determined the third Hankel function. This study presents sharp results regarding the coefficients, Hankel determinants, and the Kruskal inequality, specifically for the defined class $R_{\sin_q}$. Additionally, the specified class $R_{\sin_q}$ can be further examined for future study in order to determine the upper bounds of higher-order Hankel determinants. New directions for research in GFT and related domains can be valuable for future study, possibly as a consequence of our work.

Author contributions

Conceptualization, R.N. and S.N.M.; Methodology, R.N. and S.N.M.; Validation, L-I. C.; Formal analysis, S.N.M.; Investigation, R.N. and S.N.M.; Resources, D.B.; Data curation, D.B.; Visualization, S.N.M.; Supervision, S.N.M.; Project administration, L-I. C.; Funding acquisition, D.B. and L-I. C.. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was carried out for the requirement of a degree program under the synopsis notification No. CUI-Reg/Notif-216/24/226, dated 10 January 2024. The research work of the second author is supported by Project number (Ref No. 20-16231/NRPU/ RD/HEC/2021-2020), Higher Education Commission of Pakistan.

Conflict of interest

The authors declare no conflicts of interest.

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