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Research article

On the commuting problem of Toeplitz operators on the harmonic Bergman space

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Abstract: In this paper, we provide a complete characterization of bounded Toeplitz operators T_f on the harmonic Bergman space of the unit disk, where the symbol f has a polar decomposition truncated above, that commute with $T_{z+\bar{g}}$ for a bounded analytic function g.

Keywords: harmonic Bergman space; Toeplitz operators; Mellin transform
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1. Introduction

Let $dA = r\,dr\,\frac{d\theta}{\pi}$, where (r, θ) are the polar coordinates, denote the normalized Lebesgue area measure on the unit disk $\mathbb{D}$ so that $\mathbb{D}$ has the measure 1. The space $L^2(\mathbb{D}, dA)$ consists of all Lebesgue square-integrable functions on $\mathbb{D}$ and forms a Hilbert space with the inner product

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}\,dA(z).$$

The harmonic Bergman space, denoted $L^2_h(\mathbb{D})$, is the closed subspace of $L^2(\mathbb{D}, dA)$ comprising all complex-valued L^2-harmonic functions on $\mathbb{D}$. Let Q represent the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $L^2_h(\mathbb{D})$. This projection is given by the integral operator

$$Qf(z) = \int_{\mathbb{D}} \left(\frac{1}{(1-\bar{z}w)^2} + \frac{1}{(1-z\bar{w})^2} - 1\right) f(w)\,dA(w), \quad z \in \mathbb{D},$$

for any $f \in L^2(\mathbb{D}, dA)$. It is well known that Q is bounded from $L^2(\mathbb{D}, dA)$ onto $L^2_h(\mathbb{D})$.

For a function $u \in L^1(\mathbb{D}, dA)$, we define the Toeplitz operator T_u with the symbol u on L^2_h by

$$T_u f = Q(uf), \tag{1.1}$$

for $f \in L_h^2(\mathbb{D})$, provided that the product uf is in $L^2(\mathbb{D}, dA)$. This operator is densely defined on the polynomials and is not bounded in general. However, if u is bounded on $\mathbb{D}$, then T_u is bounded and $\|T_u\| \le \|u\|_\infty$.

A symbol u is called quasihomogeneous of degree p, where p is an integer, if it can be expressed in the form $u(re^{i\theta}) = e^{ip\theta}\phi(r)$, where ϕ is a radial function. In this case, the associated Toeplitz operator T_u is called a quasihomogeneous Toeplitz operator of degree p. The study of these operators is motivated by the structural decomposition of $L^2(\mathbb{D}, dA)$, which can be written as $L^2(\mathbb{D}, dA) = \bigoplus_{k\in\mathbb{Z}} e^{ik\theta}\mathcal{R}$, where $\mathcal{R}$ denotes the space of square-integrable radial functions on $[0, 1)$ with respect to the measure rdr. This decomposition implies that any function $f \in L^2(\mathbb{D}, dA)$ admits a polar decomposition $f(z) = f(re^{i\theta}) = \sum_{k\in\mathbb{Z}} e^{ik\theta} f_k(r)$, where each $f_k(r)$ is a radial function. Moreover, we say that f is truncated above if its polar decomposition is of the form $f(re^{i\theta}) = \sum_{k=-\infty}^{N} e^{ik\theta} f_k(r)$, for some integer N.

Our focus is on identifying the conditions that characterize the symbols of commuting Toeplitz operators on $L_h^2(\mathbb{D})$. This problem has been extensively explored in the contexts of the classical Hardy space and the analytic Bergman space over the years. The study of Toeplitz operators on $L_h^2(\mathbb{D})$ exhibits significant differences compared with their counterparts on the analytic Bergman space and remains less understood. However, there has been growing interest in investigating this issue within the framework of the harmonic Bergman space. For instance, Choe and Lee [1] established that two analytic Toeplitz operators on L_h^2 (i.e., Toeplitz operators with analytic symbols) commute if and only if their symbols, along with the constant function 1, are linearly dependent. Subsequent works such as [2,3] demonstrated that an analytic Toeplitz operator and a co-analytic Toeplitz operator on $L_h^2(\mathbb{D})$ can commute only if at least one of their symbols is a constant function. In [4], the conditions under which the product of two quasihomogeneous Toeplitz operators remains a Toeplitz operator were investigated. Building upon this, the work in [5] delved into the commuting problem for quasihomogeneous Toeplitz operators on $L_h^2(\mathbb{D})$, where the authors characterized the commuting Toeplitz operators with quasihomogeneous symbols. In addition, they showed that a Toeplitz operator with an analytic or co-analytic monomial symbol commutes with another Toeplitz operator only in the trivial case. For further results on commuting Toeplitz operators in harmonic Bergman spaces, the reader may consult [1,2,6] and the references therein.

The primary goal of our study is to characterize a special class of commuting Toeplitz operators acting on $L_h^2(\mathbb{D})$. More specifically, we characterize all Toeplitz operators with truncated-above symbols that commute with the Toeplitz operator T_u, whose symbol is the harmonic function $u(z) = z + \overline{g(z)}$, where $g(z) = \sum_{n=0}^{\infty} a_n z^n$ is a bounded analytic function on $\mathbb{D}$.

One of the main challenges in this problem arises from the interplay between the multiplication operators induced by the symbols and the projection onto the harmonic Bergman space. Unlike the analytic Bergman space, where the Bergman projection has an explicit integral representation, the harmonic Bergman projection introduces additional complexities that make computing Toeplitz operator products more difficult. Consequently, many classical results from the analytic setting do not directly extend to the harmonic case, necessitating the development of new techniques and approaches. To emphasize the difference between analytic and harmonic Bergman spaces, the authors in [7] proved, that in the analytic Bergman space, if T_f (with f having a polar decomposition truncated above) commutes with $T_{z+\bar{z}}$, then T_f must be a polynomial in $T_{z+\bar{z}}$ of degree at most 3; similarly, if T_f commutes with $T_{z+\bar{z}^2}$, then T_f is necessarily a polynomial in $T_{z+\bar{z}}$ of degree at most 2. However, as we will show in our main theorem, this property does not hold in the harmonic Bergman space.

To structure our analysis effectively, this paper is organized as follows: Section 2 presents the key preliminary results that are essential for proving the main theorem. Section 3 formally states the main result. Finally, Section 4 is devoted to its proof, which is divided into several lemmas to enhance clarity and systematically manage the technical details.

2. Tools

The Mellin transform $\widehat{\phi}$ of a radial function $\phi \in L^1([0,1), r\,dr)$ is given by

$$\widehat{\phi}(z) = \int_0^1 \phi(r) r^{z-1}\, dr.$$

It is well known that for such functions, the Mellin transform is bounded in the right half-plane $\{z \in \mathbb{C} : \Re z \geq 2\}$ and is analytic in $\{z \in \mathbb{C} : \Re z > 2\}$.

The following lemma describes the action of quasihomogeneous Toeplitz operators on elements of the orthogonal basis of $L^2_h(\mathbb{D})$. See [4, Lemma 2.1].

Lemma 2.1. *Let $k \in \mathbb{Z}$ and let ϕ be a radial in $L^1([0,1), rdr)$. Then, for each $n \in \mathbb{N}$, the Toeplitz operator $T_{e^{ik\theta}\phi}$ satisfies*

$$T_{e^{ik\theta}\phi}(z^n) = \begin{cases} 2(n+k+1)\widehat{\phi}(2n+k+2) z^{n+k}, & \text{if } n \geq -k, \\ 2(-n-k+1)\widehat{\phi}(-k+2)\bar{z}^{-n-k}, & \text{if } n < -k. \end{cases}$$

Similarly,

$$T_{e^{ik\theta}\phi}(\bar{z}^n) = \begin{cases} 2(n-k+1)\widehat{\phi}(2n-k+2)\bar{z}^{n-k}, & \text{if } n \geq k, \\ 2(k-n+1)\widehat{\phi}(k+2) z^{k-n}, & \text{if } n < k. \end{cases}$$

A fundamental result states that the Mellin transform of a function is uniquely determined by its values on an arithmetic sequence of integers. This is formalized in the following classical theorem [8, p. 102].

Theorem 2.1. *Let f be a bounded analytic function in the right half-plane $\{z \in \mathbb{C} : \Re z > 0\}$ that vanishes at an infinite sequence of distinct points $d_1, d_2, \ldots$ satisfying the following:*

(i) $\inf\{|d_n|\} > 0$, *and*
(ii) $\sum_{n\geq 1} \Re\left(\frac{1}{d_n}\right) = \infty$.

Then f must be identically zero on $\{z \in \mathbb{C} : \Re z > 0\}$.

Another important result we frequently use is the following classical lemma (see [9, Lemma 7]):

Lemma 2.2. *A bounded, periodic, meromorphic function defined on a right half-plane must be constant.*

The following lemma is crucial for the proof of the main result and can be deduced from [5, Theorem 3.8]:

Lemma 2.3. *Let $f(re^{i\theta}) = e^{ip\theta}\phi(r)$ be a quasihomogenous symbol, where $p \in \mathbb{Z}_+$ and $\phi(r) \in L^1([0,1), rdr)$. If $T_f T_{z^n} = T_{z^n} T_f$ for $n \geq 1$ integer, then $\phi(r) = Cr^p$. In other words, f must be analytic of the form $f(z) = Cz^p$.*

Remark 2.1. *The following observations will be useful in our proofs.*

1) *A straightforward calculation shows that*

$$\widehat{r^n}(z) = \frac{1}{z+n}, \quad \text{for } n \in \mathbb{Z},$$

and

$$r^a \widehat{\ln(r)^b}(z) = \frac{(-1)^b b!}{(a+z)^{b+1}},$$

where $a > 0$ and b is a non-negative integer.

2) *Regarding Theorem 2.1, we apply it in the following setting. Suppose $(n_k)_k$ is an arithmetic sequence of positive integers and that, for some radial function ϕ, we have $\widehat{\phi}(n_k) = 0$ for all k. By Theorem 2.1, this forces $\widehat{\phi}$ to be identically zero in the right half-plane, implying that ϕ itself must vanish there as well.*
3) *Lemma 2.2 is a key tool in our arguments. In our proofs, we encounter functional equations of the form*

$$F(z+p) - F(z) = G(z+p) - G(z),$$

where $\Re(z) > 0$, p is an integer, and F and G are bounded analytic functions in the right half-plane. Applying Lemma 2.2, we conclude that $F(z) = C + G(z)$ for some constant C.

3. Main result

Given a symbol $u(z) = z + \overline{g(z)}$, where $g(z) = \sum_{n=1}^{\infty} a_n z^n$ is a bounded analytic function on $\mathbb{D}$, we aim to characterize all symbols of the form (i.e., symbols whose polar decomposition is truncated above)

$$f(re^{i\theta}) = \sum_{n=-\infty}^{N} e^{in\theta} f_n(r) \; N \geq 1,$$

in $L^1(\mathbb{D}, dA)$ for which the associated Toeplitz operators T_f are bounded and commute with T_u. It is understood here that $f_N \neq 0$. We recall that T_f commutes with T_u if and only if

$$T_f T_u(z^k) = T_u T_f(z^k), \tag{3.1}$$

and

$$T_f T_u(\bar{z}^k) = T_u T_f(\bar{z}^k), \tag{3.2}$$

for all vectors z^k and $\bar{z}^k$ in the orthogonal basis of $L^2_h(\mathbb{D})$.

Our main theorem can be stated as follows.

Theorem 3.1. *Let $u(z) = z + \sum_{l=1}^{\infty} \bar{a}_l \bar{z}^l$. If a nonzero function f of the form $f(re^{i\theta}) = \sum_{k=-\infty}^{N} e^{ik\theta} f_k(r)$, with $N \geq 1$, such that T_f commutes with T_u, then T_f is a polynomial of degree at most one in T_u. In other words, the constants C_1, C_0 exist such that $T_f = C_1 T_u + C_0 I$, where I denotes the identity operator.*

4. Key lemmas for the proof of the main theorem

The proof of our main result is quite lengthy and involves intricate computations. To enhance clarity and readability, we have structured the proof into several lemmas. The first lemma establishes that the highest degree N of f in Theorem 3.1 cannot exceed 3. However, we will later demonstrate that N must, in fact, be equal to 1.

Lemma 4.1. *Under the hypothesis of Theorem 3.1, we have $N \leq 3$.*

Proof. The term in z of degree $n+N+1$ in Eq (3.1) appears on both sides, originating from $T_{e^{iN\theta}f_N}T_z(z^n)$ and $T_zT_{e^{iN\theta}f_N}(z^n)$. Therefore, we have

$$T_{e^{iN\theta}f_N}T_z(z^n) = T_zT_{e^{iN\theta}f_N}(z^n),$$

for every n. By Lemma 2.3, this implies that $e^{iN\theta}f_N = C_Nz^N$ for some constant C_N. Similarly, the term in z of degree $n+N$ appears on both sides only from $T_{e^{i(N-1)\theta}f_{N-1}}T_z(z^n)$ and $T_zT_{e^{i(N-1)\theta}f_{N-1}}(z^n)$. Applying Lemma 2.3 again, we conclude that $e^{i(N-1)\theta}f_{N-1}$ is analytic and satisfies $e^{i(N-1)\theta}f_{N-1} = C_{N-1}z^{N-1}$ for some constant C_{N-1}.

Next, we turn our attention to the radial function f_{N-2}. The term in z of degree $n+N-1$ comes from

$$\left(T_{C_Nz^N}T_{\bar{a}_1\bar{z}} + T_{e^{i(N-2)\theta}f_{N-2}}T_z\right)(z^n) \quad = \quad \left(T_{\bar{a}_1\bar{z}}T_{C_Nz^N} + T_zT_{e^{i(N-2)\theta}f_{N-2}}\right)(z^n).$$

Using Lemma 2.1, the previous equation implies

$$C_N\bar{a}_1\frac{2n}{2n+2} + (2n+2N)\widehat{f_{N-2}}(2n+N+2) = C_N\bar{a}_1\frac{2n+2N}{2n+2N+2} + (2n+2N-2)\widehat{f_{N-2}}(2n+N),$$

which is equivalent to

$$2(n+N)\widehat{f_{N-2}}(2n+N+2) - 2(n+N-1)\widehat{f_{N-2}}(2n+N) = C_N\bar{a}_1\frac{2n+2N}{2n+2N+2} - C_N\bar{a}_1\frac{2n}{2n+2}.$$

We complexify the equation above by letting $z = 2n$ and we use point (2) of Remark 2.1 to obtain

$$(z+2N)\widehat{f_{N-2}}(z+N+2) - (z+2N-2)\widehat{f_{N-2}}(z+N) = C_N\bar{a}_1\frac{z+2N}{z+2N+2} - C_N\bar{a}_1\frac{z}{z+2}. \tag{4.1}$$

Define $F(z) = (z+2N-2)\widehat{f_{N-2}}(z+N)$ and $G(z) = C_N\bar{a}_1\sum_{i=0}^{N-1}\frac{z+2i}{z+2i+2}$. Then Eq (4.1) becomes

$$F(z+2) - F(z) = G(z+2) - G(z).$$

Thus, point (3) of Remark 2.1 implies the existence of a constant C_{N-2} such that $F(z) - G(z) = C_{N-2}$. Equivalently

$$(z+2N-2)\widehat{f_{N-2}}(z+N) = C_{N-2} + C_N\bar{a}_1\sum_{i=0}^{N-1}\frac{z+2i}{z+2i+2}. \tag{4.2}$$

Using partial fraction decomposition and observing that $\widehat{f_{N-2}}(z+N) = \widehat{r^N f_{N-2}}(z)$, we obtain

$$\begin{aligned}
\widehat{r^N f_{N-2}}(z) &= \frac{C_{N-2}}{z+2N-2} + C_N \bar{a}_1 \sum_{i=0}^{N-1} \frac{z+2i}{(z+2N-2)(z+2i+2)} \\
&= \frac{C_{N-2}}{z+2N-2} + C_N \bar{a}_1 \left[\frac{1}{z+2N} + \sum_{i=0}^{N-2} \frac{z+2i}{(z+2N-2)(z+2i+2)} \right] \\
&= \frac{C_{N-2}}{z+2N-2} + C_N \bar{a}_1 \left[\frac{1}{z+2N} + \frac{z+2N-4}{(z+2N-2)^2} + \sum_{i=0}^{N-3} \frac{z+2i}{(z+2N-2)(z+2i+2)} \right] \\
&= \frac{C_{N-2}}{z+2N-2} + C_N \bar{a}_1 \left[\frac{1}{z+2N} + \frac{1}{z+2N-2} - \frac{2}{(z+2N-2)^2} \right. \\
&\quad \left. + \sum_{i=0}^{N-3} \left(\frac{2-2N+2i}{(4-2N+2i)(z+2N-2)} - \frac{2}{(2N-2i-4)(z+2i+2)} \right) \right].
\end{aligned}$$

Using point (1) of Remark 2.1, it follows that

$$\begin{aligned}
\widehat{r^N f_{N-2}}(z) &= C_{N-2} \widehat{r^{2N-2}}(z) + C_N \bar{a}_1 \left[\widehat{r^{2N}}(z) + \widehat{r^{2N-2}}(z) + 2\widehat{r^{2N-2} \ln r}(z) \right. \\
&\quad \left. + \sum_{i=0}^{N-3} \left(\frac{2-2N+2i}{4-2N+2i} \widehat{r^{2N-2}}(z) - \frac{2}{2N-2i-4} \widehat{r^{2i+2}}(z) \right) \right].
\end{aligned}$$

Therefore

$$f_{N-2}(r) = C_{N-2} r^{N-2} + C_N \bar{a}_1 \left[r^N + r^{N-2} + 2r^{N-2} \ln r + \sum_{i=0}^{N-3} \left(\frac{2-2N+2i}{4-2N+2i} r^{N-2} - \frac{2}{2N-2i-4} r^{2i+2-N} \right) \right].$$

Observe that f_{N-2} belongs $L^1([0,1), rdr)$ if and only if $2i+2-N+1 \geq 0$, which simplifies to $N \leq 2i+3$ for all $i = 0, 1, 2, \ldots, N-3$. Consequently, this condition must hold for $i = 0$, which leads to $N \leq 3$. □

Note. With respect to the notation used in the previous proof, we would like to remind the reader and draw their attention to the fact that the functions $f_k(r)$ represent the radial component in the polar decomposition

$$f(re^{i\theta}) = \sum_{k=-\infty}^{N} e^{ik\theta} f_k(r).$$

The constants C_k appear naturally in the process of determining each corresponding radial function f_k. The subscript k in C_k is intentionally chosen to indicate the association between the constant and the corresponding radial term f_k. This notation is consistently used throughout the paper to maintain the clarity and traceability of the decomposition components.

Remark 4.1. *Lemma 4.1 implies the following:*

(1) $f_3(r) = C_3 r^3$.
(2) $f_2(r) = C_2 r^2$.

(3) To find $f_1(r)$, we plug $N = 3$ in Eq (4.2) to obtain

$$\begin{aligned}\widehat{r^3 f_1}(z) &= \frac{C_1}{z+4} + C_3\bar{a}_1 \sum_{i=0}^{2} \frac{z+2i}{(z+4)(z+2i+2)} \\ &= \frac{C_1}{z+4} + C_3\bar{a}_1 \left[\frac{z}{(z+4)(z+2)} + \frac{z+2}{(z+4)^2} + \frac{z+4}{(z+6)(z+4)}\right] \\ &= \frac{C_1}{z+4} + C_3\bar{a}_1 \left[\frac{-1}{z+2} + \frac{3}{z+4} + \frac{-2}{(z+4)^2} + \frac{1}{z+6}\right] \\ &= C_1\widehat{r^4}(z) + C_3\bar{a}_1\left[\widehat{r^6}(z) + 3\widehat{r^4}(z) + 2\widehat{r^4 \ln r}(z) - \widehat{r^2}(z)\right].\end{aligned}$$

Hence, point (2) of Remark 2.1 yields

$$f_1(r) = C_1 r + C_3\bar{a}_1\left[r^3 + 3r + 2r\ln r - \frac{1}{r}\right].$$

So far, using Lemma 4.1 and Remark 4.1, we have established that any Toeplitz operator with the symbol $f(re^{i\theta}) = \sum_{k=-\infty}^{N} e^{ik\theta} f_k(r)$ that commutes with $T_{z+\overline{g}}$, where $g(z) = \sum_{l=1}^{\infty} a_l z^l$ is a bounded analytic function on $\mathbb{D}$, must take the form

$$f(re^{i\theta}) = C_3 z^3 + C_2 z^2 + e^{i\theta}\left(C_1 r + C_3\bar{a}_1\left[r^3 + 3r + 2r\ln r - \frac{1}{r}\right]\right) + \sum_{k=-\infty}^{0} e^{ik\theta} f_k(r).$$

In the following lemmas, we compute the exact expressions of $f_0(r)$, $f_{-1}(r)$, and $f_{-2}(r)$.

Lemma 4.2. *Under the hypothesis of Theorem 3.1, we have*

$$f_0(r) = C_0 + C_2\bar{a}_1\left[1 + 2\ln r + r^2\right] + C_3\bar{a}_2\left[4\ln r + 2r^2 + r^4\right].$$

Proof. For $n \geq 1$, the term z^{n+1} in $T_f T_{z+\overline{g}}(z^n) = T_{z+\overline{g}} T_f(z^n)$ appears both sides only from the expressions

$$\left(T_{f_0}T_z + T_{e^{2i\theta}f_2}T_{\bar{a}_1\bar{z}} + T_{e^{3i\theta}f_3}T_{\bar{a}_2\bar{z}^2}\right)(z^n),$$

and

$$\left(T_z T_{f_0} + T_{\bar{a}_1\bar{z}}T_{e^{2i\theta}f_2} + T_{\bar{a}_2\bar{z}^2}T_{e^{3i\theta}f_3}\right)(z^n).$$

Thus, applying Lemma 2.1, we obtain

$$(2n+4)\widehat{f_0}(2n+4) + C_2\bar{a}_1\frac{2n}{2n+2} + C_3\bar{a}_2\frac{2n-2}{2n+2} = (2n+2)\widehat{f_0}(2n+2) + C_2\bar{a}_1\frac{2n+4}{2n+6} + C_3\bar{a}_2\frac{2n+4}{2n+8},$$

which can be written as

$$(2n+4)\widehat{f_0}(2n+4) - (2n+2)\widehat{f_0}(2n+2) = C_2\bar{a}_1\left[\frac{2n+4}{2n+6} - \frac{2n}{2n+2}\right] + C_3\bar{a}_2\left[\frac{2n+4}{2n+8} - \frac{2n-2}{2n+2}\right].$$

We complexify the equation above by considering $z = 2n - 2$ and we use point (2) of Remark 2.1 to obtain

$$(z+6)\widehat{f_0}(z+6)-(z+4)\widehat{f_0}(z+4)=C_2\bar{a}_1\left[\frac{z+6}{z+8}-\frac{z+2}{z+4}\right]+C_3\bar{a}_2\left[\frac{z+6}{z+10}-\frac{z}{z+4}\right].$$

This equation can be expressed in the form

$$F(z+2)-F(z)=G(z+2)-G(z),$$

where $F(z)=(z+4)\widehat{f_0}(z+4)$ and $G(z)=C_2\bar{a}_1\sum_{i=0}^{1}\frac{z+2i+2}{z+2i+4}+C_3\bar{a}_2\sum_{i=0}^{2}\frac{z+2i}{z+2i+4}$. By point (3) of Remark 2.1, it follows that a constant C_0 exists such that $F(z)-G(z)=C_0$. Hence, since $\widehat{f_0}(z+4)=\widehat{r^4f_0}(z)$, we have

$$\begin{aligned}\widehat{r^4f_0}(z)&=\frac{C_0}{z+4}+C_2\bar{a}_1\left[\frac{z+2}{(z+4)^2}+\frac{1}{z+6}\right]+C_3\bar{a}_2\left[\frac{z}{(z+2)^2}+\frac{z+2}{(z+4)(z+6)}+\frac{1}{z+8}\right]\\&=\frac{C_0}{z+4}+C_2\bar{a}_1\left[\frac{1}{z+4}-\frac{2}{(z+4)^2}+\frac{1}{z+6}\right]+C_3\bar{a}_2\left[\frac{1}{z+4}-\frac{4}{(z+4)^2}+\frac{2}{z+6}-\frac{1}{z+4}+\frac{1}{z+8}\right].\end{aligned}$$

Thus point (1) of Remark 2.1 implies

$$\widehat{r^4f_0}(z)=C_0\widehat{r^4}(z)+C_2\bar{a}_1\left[\widehat{r^4}(z)+2\widehat{r^4\ln r}(z)+\widehat{r^6}(z)\right]+C_3\bar{a}_2\left[4\widehat{r^4\ln r}(z)+2\widehat{r^6}(z)+\widehat{r^8}(z)\right].$$

Therefore

$$f_0(r)=C_0+C_2\bar{a}_1\left[1+2\ln r+r^2\right]+C_3\bar{a}_2\left[4\ln r+2r^2+r^4\right].$$

□

Next, we proceed to compute the radial function f_{-1}.

Lemma 4.3. *Under the hypothesis of Theorem 3.1, we have*

$$f_{-1}(r)=\frac{C_{-1}}{r}+C_1\bar{a}_1r+C_3\bar{a}_1^2\left[3r+2r\ln r+r^3\right]+C_2\bar{a}_2\left[2r-\frac{1}{r}+r^3\right]+C_3\bar{a}_3\left[3r-\frac{5}{2r}+\frac{3r^3}{2}+r^5\right].$$

Proof. The term z^n in $T_fT_{z+\bar{g}}(z^n)=T_{z+\bar{g}}T_f(z^n)$ appears both sides only from the expressions

$$\left(T_{e^{-i\theta}f_{-1}}T_z+T_{e^{i\theta}f_1}T_{\bar{a}_1\bar{z}}+T_{e^{2i\theta}f_2}T_{\bar{a}_2\bar{z}^2}+T_{e^{3i\theta}f_3}T_{\bar{a}_3\bar{z}^3}\right)(z^n),$$

and

$$\left(T_zT_{e^{-i\theta}f_{-1}}+T_{\bar{a}_1\bar{z}}T_{e^{i\theta}f_1}+T_{\bar{a}_2\bar{z}^2}T_{e^{2i\theta}f_2}+T_{\bar{a}_3\bar{z}^3}T_{e^{3i\theta}f_3}\right)(z^n).$$

So both sides must be equal. Using the results of the previous lemmas and evaluating each term on both sides yields the following:

$$\begin{aligned}&(2n+2)\widehat{f_{-1}}(2n+3)-2n\widehat{f_{-1}}(2n+1)\\=&C_1\bar{a}_1\left(\frac{2n+2}{2n+4}-\frac{2n}{2n+2}\right)+C_3\bar{a}_1^2\left[3\left(\frac{2n+2}{2n+4}-\frac{2n}{2n+2}\right)\right.\\&\left.-2\left(\frac{2n+2}{(2n+4)^2}-\frac{2n}{(2n+2)^2}\right)+\frac{2n+2}{2n+6}-\frac{2n}{2n+4}\right]\\&+C_2\bar{a}_2\left(\frac{2n+2}{2n+6}-\frac{2n-2}{2n+2}\right)+C_3\bar{a}_3\left(\frac{2n+2}{2n+8}-\frac{2n-4}{2n+2}\right).\end{aligned}\tag{4.3}$$

We complexify the above equation by considering $z = 2n - 4$ and we use point (2) of Remark 2.1 to obtain

$$\begin{aligned}
&(z+6)\widehat{f_{-1}}(z+7) - (z+4)\widehat{f_{-1}}(z+5)\\
=&C_1\bar{a}_1\left(\frac{z+6}{z+8} - \frac{z+4}{z+6}\right) + C_3\bar{a}_1^2\Bigg[3\left(\frac{z+6}{z+8} - \frac{z+4}{z+6}\right)\\
&-2\left(\frac{z+6}{(z+8)^2} - \frac{z+4}{(z+6)^2}\right) + \frac{z+6}{z+10} - \frac{z+4}{z+8}\Bigg]\\
&+C_2\bar{a}_2\left(\frac{z+6}{z+10} - \frac{z+2}{z+6}\right) + C_3\bar{a}_3\left(\frac{z+6}{z+12} - \frac{z}{z+6}\right).
\end{aligned} \tag{4.4}$$

We let

$$F(z) = (z+4)\widehat{f_{-1}}(z+5) = (z+4)\widehat{r^5 f_{-1}}(z),$$

and

$$G(z) = C_1\bar{a}_1\frac{z+4}{z+6} + C_3\bar{a}_1^2\left[3\frac{z+4}{z+6} - 2\frac{z+4}{(z+6)^2} + \frac{z+4}{z+8}\right] + C_2\bar{a}_2\sum_{i=0}^{1}\frac{z+2i+2}{z+2i+6} + C_3\bar{a}_3\sum_{i=0}^{2}\frac{z+2i}{z+2i+6}.$$

Then Eq (4.4) can be written as $F(z+2) - F(z) = G(z+2) - G(z)$. Thus, by point (3) of Remark 2.1, a constant C_{-1} exists such that $F(z) = C_{-1} + G(z)$. Applying partial fraction decomposition to the terms of G and using point (1) of Remark 2.1, we find that

$$\begin{aligned}
\widehat{r^5 f_{-1}}(z) &= C_{-1}\widehat{r^4}(z) + C_1\bar{a}_1\widehat{r^6}(z) + C_3\bar{a}_1^2\left[3\widehat{r^6}(z) + 2\widehat{r^6\ln r}(z) + \widehat{r^8}(z)\right]\\
&\quad+C_2\bar{a}_2\left[2\widehat{r^6}(z) - \widehat{r^4}(z) + \widehat{r^8}(z)\right] + C_3\bar{a}_3\Big[3\widehat{r^6}(z) - 2\widehat{r^4}(z) - \frac{1}{2}\widehat{r^4}(z) + \frac{3}{2}\widehat{r^8}(z) + \widehat{r^{10}}(z)\Big].
\end{aligned}$$

Hence

$$f_{-1}(r) = \frac{C_{-1}}{r} + C_1\bar{a}_1 r + C_3\bar{a}_1^2\left[3r + 2r\ln r + r^3\right] + C_2\bar{a}_2\left[2r - \frac{1}{r} + r^3\right] + C_3\bar{a}_3\left[3r - \frac{5}{2r} + \frac{3r^3}{2} + r^5\right].$$

□

The main purpose of the following lemma is to evaluate the radial function f_{-2}. However, we will omit some of the lengthy calculations, as they are similar to those in the previous lemmas.

Lemma 4.4. *Under the hypothesis of Theorem 3.1, we have*

$$\begin{aligned}
f_{-2}(r) &= \frac{C_{-2}}{r^2} - C_2\bar{a}_1^2\left(\frac{1}{r^2} - r^2\right) - C_3\bar{a}_1\bar{a}_2\left(\frac{31}{4r^2} - 6r^2 - 2r^4 - 2r^2\ln r + \frac{1}{4r^6} + \frac{1}{2r^4} - \frac{\ln r}{r^2} - \frac{1}{2}\right)\\
&\quad+C_1\bar{a}_2 r^2 - C_2\bar{a}_3\left(\frac{3}{2r^2} - \frac{3}{2}r^2 + \frac{1}{r^2} - r^4\right) - C_3\bar{a}_4\left(\frac{13}{3r^2} - 2r^2 - \frac{4}{3}r^4 - r^6\right).
\end{aligned}$$

Proof. The term z^{n-1} appears on the left-hand side and right-hand side of Eq (3.1) only from

$$\left(T_{e^{-2i\theta}f_{-2}}T_z + T_{f_0}T_{\bar{a}_1\bar{z}} + T_{e^{i\theta}f_1}T_{\bar{a}_2\bar{z}^2} + T_{e^{2i\theta}f_2}T_{\bar{a}_3\bar{z}^3} + T_{e^{3i\theta}f_3}T_{\bar{a}_4\bar{z}^4}\right)(z^n),$$

and

$$\Big(T_zT_{e^{-2i\theta}f_{-2}} + T_{\bar{a}_1\bar{z}}T_{f_0} + T_{\bar{a}_2\bar{z}^2}T_{e^{i\theta}f_1} + T_{\bar{a}_3\bar{z}^3}T_{e^{2i\theta}f_2} + T_{\bar{a}_4\bar{z}^4}T_{e^{3i\theta}f_3}\Big)(z^n).$$

So both sides must be equal. Next, using Lemma 2.1, we evaluate each term on both sides and we obtain the following:

(1) $T_{e^{-2i\theta}f_{-2}}T_z(z^n) = 2n\widehat{f_{-2}}(2n+2)z^{n-1}$,

(2)

$$\begin{aligned} T_{f_0}T_{\bar{a}_1\bar{z}}(z^n) &= \bar{a}_1\frac{(2n)^2}{2n+2}\widehat{f_0}(2n)z^{n-1} \\ &= \Bigg[C_0\bar{a}_1\frac{2n}{2n+2} + C_2\bar{a}_1^2\Big(\frac{2n}{2n+2} - \frac{2}{2n+2} + \frac{(2n)^2}{(2n+2)^2}\Big) \\ &\quad + C_3\bar{a}_1\bar{a}_2\left(-\frac{4}{2n+2} + 2\frac{(2n)^2}{(2n+2)^2} + \frac{(2n)^2}{(2n+2)(2n+4)}\right)\Bigg]z^{n-1}, \end{aligned}$$

(3)

$$\begin{aligned} T_{e^{i\theta}f_1}T_{\bar{a}_2\bar{z}^2}(z^n) &= \bar{a}_2\frac{2n(2n-2)}{2n+2}\widehat{f_1}(2n-1)z^{n-1} \\ &= \Bigg[C_1\bar{a}_2\frac{2n-2}{2n+2} + C_3\bar{a}_1\bar{a}_2\Big(\frac{2n(2n-2)}{(2n+2)^2} + 3\frac{2n-2}{2n+2} - 2\frac{2n-2}{2n(2n+2)} - \frac{2n}{2n+2}\Big)\Bigg]z^{n-1}, \end{aligned}$$

(4) $T_{e^{2i\theta}f_2}T_{\bar{a}_3\bar{z}^3}(z^n) = C_2\bar{a}_3\dfrac{2n-4}{2n+2}z^{n-1}$,

(5) $T_{e^{3i\theta}f_3}T_{\bar{a}_4\bar{z}^4}(z^n) = C_3\bar{a}_4\dfrac{2n-6}{2n+2}z^{n-1}$,

(6) $T_zT_{e^{-2i\theta}f_{-2}}(z^n) = (2n-2)\widehat{f_{-2}}(2n)z^{n-1}$,

(7)

$$\begin{aligned} T_{\bar{a}_1\bar{z}}T_{f_0}(z^n) &= 2n\bar{a}_1(2n+2)\widehat{f_0}(2n+2)\widehat{r}(2n+1)z^{n-1} \\ &= \Bigg[C_0\bar{a}_1\frac{2n}{2n+2} + C_2\bar{a}_1^2\left(\frac{2n}{2n+2} - \frac{4n}{(2n+2)^2} + \frac{2n}{2n+4}\right) \\ &\quad + C_3\bar{a}_1\bar{a}_2\left(-\frac{8n}{(2n+2)^2} + \frac{4n}{2n+4} + \frac{2n}{2n+6}\right)\Bigg]z^{n-1}, \end{aligned}$$

(8) $T_{\bar{a}_2\bar{z}^2}T_{e^{i\theta}f_1}(z^n) = \Bigg[C_1\bar{a}_2\dfrac{2n}{2n+4} + C_3\bar{a}_1\bar{a}_2\Big(\dfrac{2n}{2n+6} + \dfrac{6n}{2n+4} - \dfrac{4n}{(2n+4)^2} - \dfrac{2n}{2n+2}\Big)\Bigg]z^{n-1}$,

(9) $T_{\bar{a}_3\bar{z}^3}T_{e^{2i\theta}f_2}(z^n) = C_2\bar{a}_3\dfrac{2n}{2n+6}z^{n-1}$,

(10) $T_{\bar{a}_4\bar{z}^4}T_{e^{3i\theta}f_3}(z^n) = C_3\bar{a}_4\dfrac{2n}{2n+8}z^{n-1}$.

We substitute the 10 quantities above into both sides, equate them, complexify the expression by setting $z = 2n-6$, and then use point (2) of Remark 2.1 to obtain the following:

$$
\begin{aligned}
&(z+6)\widehat{f_{-2}}(z+8)-(z+4)\widehat{f_{-2}}(z+6)\\
=&C_2\bar{a}_1^2\left(\frac{-2(z+6)}{(z+8)^2}+\frac{z+6}{z+10}\right)-C_2\bar{a}_1^2\left(-\frac{2}{z+8}+\frac{(z+6)^2}{(z+10)^2}\right)\\
&+C_3\bar{a}_1\bar{a}_2\left(\frac{-4(z+6)}{(z+8)^2}+\frac{2(z+6)}{z+10}+\frac{z+6}{z+12}\right)-C_3\bar{a}_1\bar{a}_2\left(\frac{-4}{z+8}+\frac{2(z+6)^2}{(z+8)^2}+\frac{(z+6)^2}{(z+8)(z+10)}\right)\\
&+C_1\bar{a}_2\frac{z+6}{z+10}-C_1\bar{a}_2\frac{z+4}{z+8}+C_3\bar{a}_1\bar{a}_2\left(\frac{z+6}{z+12}+\frac{3(z+6)}{z+10}-\frac{2(z+6)}{(z+10)^2}\right)\\
&-C_3\bar{a}_1\bar{a}_2\left(\frac{(z+4)(z+6)}{(z+8)^2}+\frac{3(z+4)}{z+8}-\frac{2(z+4)}{z(z+8)}\right)+C_2\bar{a}_3\frac{z+6}{z+12}-C_2\bar{a}_3\frac{z+2}{z+8}+C_3\bar{a}_4\frac{z+6}{z+8}-C_3\bar{a}_4\frac{z}{z+8}.
\end{aligned} \tag{4.5}
$$

Now, let $F(z) = (z+4)\widehat{f_{-2}}(z+6) = (z+4)\widehat{r^6 f_{-2}}(z)$ and define the function $G(z)$ as $G(z) = \sum_{i=1}^{9} G_i(z)$, where the values of G are given by

(1) $G_1(z) = -4C_2\bar{a}_1^2\frac{1}{z+8}$,

(2) $G_2(z) = -6C_3\bar{a}_1\bar{a}_2\sum_{i=0}^{1}\frac{1}{z+2i+8}$,

(3) $G_3(z) = C_1\bar{a}_2\frac{z+4}{z+8}$,

(4) $G_4(z) = -6C_3\bar{a}_1\bar{a}_2\sum_{i=0}^{1}\frac{1}{z+2i+8}$,

(5) $G_5(z) = -14C_3\bar{a}_1\bar{a}_2\frac{1}{z+8}$,

(6) $G_6(z) = 8C_3\bar{a}_1\bar{a}_2\frac{1}{(z+8)^2}$,

(7) $G_7(z) = -C_3\bar{a}_1\bar{a}_2\sum_{i=0}^{3}\frac{1}{z+2i}$,

(8) $G_8(z) = -6C_2\bar{a}_3\sum_{i=0}^{1}\frac{1}{z+2i+8}$,

(9) $G_9(z) = -6C_3\bar{a}_4\sum_{i=0}^{2}\frac{1}{z+2i+8}$.

Thus, Eq (4.5) simplifies to $F(z+2)-F(z) = G(z+2)-G(z)$. Therefore, by point (3) of Remark 2.1, a constant C_{-2} exits such that $F(z) = C_{-2} + G(z)$, which is equivalent to

$$(z+4)\widehat{r^6 f_{-2}}(z) = C_{-2} + G(z).$$

Now, dividing both sides by $(z+4)$ and expanding $G(z)$ using the sums of partial fractions, we obtain

$$\widehat{r^6 f_{-2}}(z) = \frac{C_{-2}}{z+4} - C_2\bar{a}_1^2\left(\frac{1}{z+4}-\frac{1}{z+8}\right) - \frac{3}{2}C_3\bar{a}_1\bar{a}_2\left(\frac{1}{z+4}-\frac{1}{z+8}\right)$$

$$
\begin{aligned}
&-C_3\bar{a}_1\bar{a}_2\left(\frac{1}{z+4}-\frac{1}{z+10}\right)+C_1\bar{a}_2\frac{1}{z+8}-\frac{3}{2}C_3\bar{a}_1\bar{a}_2\left(\frac{1}{z+4}-\frac{1}{z+8}\right)\\
&-C_3\bar{a}_1\bar{a}_2\left(\frac{1}{z+4}-\frac{1}{z+10}\right)-\frac{7}{2}C_3\bar{a}_1\bar{a}_2\left(\frac{1}{z+4}-\frac{1}{z+8}\right)\\
&+\frac{1}{2}C_3\bar{a}_1\bar{a}_2\frac{1}{z+4}-\frac{1}{2}C_3\bar{a}_1\bar{a}_2\frac{1}{z+8}-2C_3\bar{a}_1\bar{a}_2\frac{1}{(z+8)^2}\\
&-\frac{1}{4}C_3\bar{a}_1\bar{a}_2\left(\frac{1}{z}-\frac{1}{z+4}\right)-\frac{1}{2}C_3\bar{a}_1\bar{a}_2\left(\frac{1}{z+2}-\frac{1}{z+4}\right)\\
&-C_3\bar{a}_1\bar{a}_2\frac{1}{(z+4)^2}-\frac{1}{2}C_3\bar{a}_1\bar{a}_2\left(\frac{1}{z+4}-\frac{1}{z+6}\right)-\frac{3}{2}C_2\bar{a}_3\left(\frac{1}{z+4}-\frac{1}{z+8}\right)\\
&-C_2\bar{a}_3\left(\frac{1}{z+4}-\frac{1}{z+10}\right)-C_3\bar{a}_4\left(\frac{1}{z+4}-\frac{1}{z+8}\right)\\
&-\frac{4}{3}C_3\bar{a}_4\left(\frac{1}{z+4}-\frac{1}{z+10}\right)-C_3\bar{a}_4\left(\frac{1}{z+4}-\frac{1}{z+12}\right).
\end{aligned}
$$

Finally, if we apply point (1) of Remark 2.1, the equation above implies

$$
\begin{aligned}
f_{-2}(r) \;=\; &\frac{C_{-2}}{r^2}-C_2\bar{a}_1^2\left(\frac{1}{r^2}-r^2\right)-C_3\bar{a}_1\bar{a}_2\left(\frac{31}{4r^2}-6r^2-2r^4-2r^2\ln r+\frac{1}{4r^6}\right.\\
&\left.+\frac{1}{2r^4}-\frac{\ln r}{r^2}-\frac{1}{2}\right)+C_1\bar{a}_2r^2-C_2\bar{a}_3\left(\frac{3}{2r^2}-\frac{3}{2}r^2+\frac{1}{r^2}-r^4\right)\\
&-C_3\bar{a}_4\left(\frac{13}{3r^2}-2r^2-\frac{4}{3}r^4-r^6\right).
\end{aligned}
$$

□

We observe that f_{-2}, obtained in the previous lemma, belongs to $L^1([0,1), rdr)$ if and only if $C_{-2}=0$, $C_2=0$, and $C_3=0$. Thus, by Remark 4.1, and Lemmas 4.2 and 4.5, we establish the following:

(1) $T_{e^{3i\theta}f_3}=0$,
(2) $T_{e^{2i\theta}f_2}=0$,
(3) $T_{e^{i\theta}f_1}=C_1T_z$,
(4) $T_{f_0}=C_0$,
(5) $f_{-1}=\frac{C_{-1}}{r}+C_1\bar{a}_1r$,
(6) $T_{e^{-2i\theta}f_{-2}}=C_1T_{\bar{a}_2\bar{z}^2}$.

This implies that $N=1$ in the polar decomposition of the symbol f in Theorem 3.1, and that $f(re^{i\theta})=\sum_{k=-\infty}^{1}e^{ik\theta}f_k(r)$.

Lemma 4.5. *Under the hypothesis of Theorem 3.1, we have*

$$f_{-1}(r)=C_1\bar{a}_1r \;\; \textit{and} \;\; f_{-3}(r)=C_1\bar{a}_3r^3.$$

Proof. In Eq (3.2), the term $\bar{z}^{n+2}$ arises from

$$\left(T_{e^{-i\theta}f_{-1}}T_{\bar{a}_1\bar{z}}+T_{f_0}T_{\bar{a}_2\bar{z}^2}+T_{e^{i\theta}f_1}T_{\bar{a}_3\bar{z}^3}+T_{e^{-3i\theta}f_{-3}}T_z\right)(\bar{z}^n)$$

$$= \left(T_{\bar{a}_1\bar{z}}T_{e^{-i\theta}f_{-1}} + T_{\bar{a}_2\bar{z}^2}T_{f_0} + T_{\bar{a}_3\bar{z}^3}T_{e^{i\theta}f_1} + T_zT_{e^{-3i\theta}f_{-3}}\right)(\bar{z}^n) \tag{4.6}$$

By using Lemma 2.1, we evaluate each term in Eq (4.6) and we obtain

(1) $T_{e^{-i\theta}f_{-1}}T_{\bar{a}_1\bar{z}}(\bar{z}^n) = \bar{a}_1 2(n+3)\widehat{f_{-1}}(2n+5)\bar{z}^{n+2} = \bar{a}_1(2n+6)\left[\frac{C_{-1}}{2n+4} + \frac{C_1\bar{a}_1}{2n+6}\right]\bar{z}^{n+2}$,
(2) Since $f_0(r) = C_0$, $T_{f_0}T_{\bar{a}_2\bar{z}^2}(\bar{z}^n) = T_{\bar{a}_2\bar{z}^2}T_{f_0}(\bar{z}^n) = C_0\bar{z}^{n+2}$,
(3) $T_{e^{i\theta}f_1}T_{\bar{a}_3\bar{z}^3}(\bar{z}^n) = C_1\bar{a}_3 2(n+3)\widehat{r}(2n+7)\bar{z}^{n+2} = C_1\bar{a}_3\frac{2n+6}{2n+8}\bar{z}^{n+2}$,
(4) $T_{e^{-3i\theta}f_{-3}}T_z(\bar{z}^n) = \frac{2n(2n+6)}{2n+2}\widehat{f_{-3}}(2n+3)\bar{z}^{n+2}$,
(5) $T_{\bar{a}_1\bar{z}}T_{e^{-i\theta}f_{-1}}(\bar{z}^n) = \bar{a}_1(2n+4)\widehat{f_{-1}}(2n+3)\bar{z}^{n+2} = \bar{a}_1(2n+4)\left[\frac{C_{-1}}{2n+2} + \frac{C_1\bar{a}_1}{2n+4}\right]\bar{z}^{n+2}$,
(6) $T_{\bar{a}_3\bar{z}^3}T_{e^{i\theta}f_1}(\bar{z}^n) = C_1\bar{a}_3\frac{2n}{2n+2}\bar{z}^{n+2}$,
(7) $T_zT_{e^{-3i\theta}f_{-3}}(\bar{z}^n) = (2n+6)\widehat{f_{-3}}(2n+5)\bar{z}^{n+2}$.

Substituting these quantities into Eq (4.6) and rearranging them yields

$$(2n+6)\widehat{f_{-3}}(2n+5) - \frac{2n(2n+6)}{2n+2}\widehat{f_{-3}}(2n+3) = C_{-1}\bar{a}_1\frac{2n+6}{2n+4} - C_{-1}\bar{a}_1\frac{2n+4}{2n+2} + C_1\bar{a}_3\frac{2n+6}{2n+8} - C_1\bar{a}_3\frac{2n}{2n+2},$$

which is equivalent to

$$(2n+2)\widehat{f_{-3}}(2n+5) - 2n\widehat{f_{-3}}(2n+3) = C_{-1}\bar{a}_1\frac{2n+2}{2n+4} - C_{-1}\bar{a}_1\frac{2n+4}{2n+6} + C_1\bar{a}_3\frac{2n+2}{2n+8} - C_1\bar{a}_3\frac{2n}{2n+6}. \tag{4.7}$$

We set $z = 2n$ to complexify Eq (4.7) and we obtain

$$(z+2)\widehat{f_{-3}}(z+5) - z\widehat{f_{-3}}(z+3) = C_{-1}\bar{a}_1\frac{z+2}{z+4} - C_{-1}\bar{a}_1\frac{z+4}{z+6} + C_1\bar{a}_3\frac{z+2}{z+8} - C_1\bar{a}_3\frac{z}{z+6}.$$

We let $F(z) = z\widehat{f_{-3}}(z+3) = z\widehat{r^3 f_{-3}}(z)$ and the function G be defined as $G(z) = C_1\bar{a}_3\frac{z}{z+6} - C_{-1}\bar{a}_1\frac{z+2}{z+4}$. Then the equation above simplifies to

$$F(z+2) - F(z) = G(z+2) - G(z).$$

Therefore, by point (3) of Remark 2.1, a constant C_{-3} exists such that $F(z) = C_{-3} + G(z)$. Hence we have

$$\widehat{r^3 f_{-3}}(z) = \frac{C_{-3}}{z} + C_1\bar{a}_3\frac{1}{z+6} - C_{-1}\bar{a}_1\frac{z+2}{z(z+4)} = \frac{C_{-3}}{z} + C_1\bar{a}_3\frac{1}{z+6} - C_{-1}\bar{a}_1\left(\frac{1}{2z} + \frac{1}{2(z+4)}\right).$$

Using point (1) of Remark 2.1, we deduce that

$$\widehat{r^3 f_{-3}}(z) = C_{-3}\widehat{1}(z) + C_1\bar{a}_3\widehat{r^6}(z) - C_{-1}\bar{a}_1\left(\frac{1}{2}\widehat{1}(z) + \frac{1}{2}\widehat{r^4}(z)\right).$$

Therefore,

$$f_{-3}(r) = \frac{C_{-3}}{r^3} + C_1\bar{a}_3 r^3 - \frac{1}{2}C_{-1}\bar{a}_1\left(\frac{1}{r^3} + r\right).$$

Clearly, $f_{-3}(r)$ belongs to $L^1([0,1), rdr)$ if and only if $C_{-3} = 0$ and $C_{-1} = 0$. In this case, we have $f_{-1}(r) = C_1\bar{a}_1 r$ and $f_{-3}(r) = C_1\bar{a}_3 r^3$. □

We now proceed with the computation of f_{-4}.

Lemma 4.6. *Under the hypothesis of Theorem 3.1, we have* $f_{-4}(r) = C_1\bar{a}_4 r^4$.

Proof. In Eq (3.2), the term $\bar{z}^{n+3}$ arises from the following expression:

$$\begin{aligned} & \left(T_{e^{-4i\theta}f_{-4}}T_z + T_{f_0}T_{\bar{a}_3\bar{z}^3} + T_{e^{i\theta}f_1}T_{\bar{a}_4\bar{z}^4} + T_{e^{-i\theta}f_{-1}}T_{\bar{a}_2\bar{z}^2}\right)(\bar{z}^n) \\ = & \left(T_zT_{e^{-4i\theta}f_{-4}} + T_{\bar{a}_3\bar{z}^3}T_{f_0} + T_{\bar{a}_4\bar{z}^4}T_{e^{i\theta}f_1} + T_{\bar{a}_2\bar{z}^2}T_{e^{-i\theta}f_{-1}}\right)(\bar{z}^n). \end{aligned} \tag{4.8}$$

We use Lemma 2.1 to evaluate each term appearing in Eq (4.8), and we obtain

(1) $T_{e^{-4i\theta}f_{-4}}T_z(\bar{z}^n) = \dfrac{2n(2n+8)}{2n+2}\widehat{f_{-4}}(2n+4)\bar{z}^{n+3}$,
(2) since $f_0(r) = C_0$, $T_{f_0}T_{\bar{a}_3\bar{z}^3}(\bar{z}^n) = T_{\bar{a}_3\bar{z}^3}T_{f_0}(\bar{z}^n) = C_0\bar{z}^{n+3}$,
(3) $T_{e^{i\theta}f_1}T_{\bar{a}_4\bar{z}^4}(\bar{z}^n) = C_1\bar{a}_4 2(n+4)\widehat{r}(2n+9)\bar{z}^{n+3} = C_1\bar{a}_4\dfrac{2n+8}{2n+10}\bar{z}^{n+3}$,
(4) $T_{e^{-i\theta}f_{-1}}T_{\bar{a}_2\bar{z}^2}(\bar{z}^n) = C_1\bar{a}_1\bar{a}_2\bar{z}^{n+3}$,
(5) $T_zT_{e^{-4i\theta}f_{-4}}(\bar{z}^n) = (2n+8)\widehat{f_{-4}}(2n+6)\bar{z}^{n+3}$,
(6) $T_{\bar{a}_4\bar{z}^4}T_{e^{i\theta}f_1}(\bar{z}^n) = C_1\bar{a}_4\dfrac{2n}{2n+2}\bar{z}^{n+3}$.

After substituting these terms in Eq (4.8) and rearranging them, we obtain

$$(2n+2)\widehat{f_{-4}}(2n+6) - 2n\widehat{f_{-4}}(2n+4) = C_1\bar{a}_4\frac{2n+2}{2n+10} - C_1\bar{a}_4\frac{2n}{2n+8}.$$

Thus, by setting $z = 2n$, the equation above becomes

$$F(z+2) - F(z) = G(z+2) - G(z),$$

where $F(z) = z\widehat{f_{-4}}(z+4) = z\widehat{r^4 f_{-4}}(z)$ and $G(z) = C_1\bar{a}_4\frac{z}{z+8}$. Point (3) of Remark 2.1 implies the existence of a constant C_{-4} such that $F(z) = C_{-4} + G(z)$. Hence

$$\widehat{r^4 f_{-4}} = C_{-4}\widehat{1}(z) + C_1\bar{a}_4\widehat{r^8}(z).$$

Therefore, we deduce that $f_{-4}(r) = \dfrac{C_{-4}}{r^4} + C_1\bar{a}_4 r^4$. Clearly, f_4 belongs to $L^1([0,1), rdr)$ if and only if $C_{-4} = 0$. Finally, we must have $f_{-4}(r) = C_1\bar{a}_4 r^4$. □

Using the same technique as in the previous lemmas, we establish the following by induction.

Lemma 4.7. *If Eqs* (3.1) *and* (3.2) *are satisfied, then for all* $k \geq 1$, *we have* $f_{-k}(r) = C_1\bar{a}_k r^k$.

Proof. By Lemma 4.5, we have $f_{-1}(r) = C_1\bar{a}_1 r$, which establishes the base case. Now, assume that the formula holds for some $k \geq 1$; that is

$$f_{-k}(r) = C_1\bar{a}_k r^k.$$

Following a similar argument as in the proof of Lemma 4.6, we obtain

$$f_{-(k+1)}(r) = \frac{C_{-(k+1)}}{r^{k+1}} + C_1\bar{a}_{k+1}r^{k+1}.$$

For $f_{-(k+1)}$ to belong to $L^1([0,1), rdr)$, it must satisfy $C_{-(k+1)} = 0$. Thus, we conclude that

$$f_{-(k+1)}(r) = C_1\bar{a}_{k+1}r^{k+1},$$

which completes the induction. □

5. Proof of the main theorem

Combining all the lemmas from the previous section, we conclude that the symbol f in Theorem 3.1 can be written as

$$f(re^{i\theta}) = e^{i\theta} f_1(r) + f_0(r) + \sum_{k=-\infty}^{-1} e^{ik\theta} f_k(r),$$

which, by the results established in the lemmas, becomes

$$f(re^{i\theta}) = C_1 e^{i\theta} r + C_0 + \sum_{k=-\infty}^{-1} e^{-ik\theta} C_1 \bar{a}_k r^{-k}.$$

This is equivalent to

$$f(z) = C_1 z + C_0 + C_1 \sum_{l=1}^{\infty} \bar{a}_l \bar{z}^l.$$

Using the linearity of Toeplitz operators with respect to the symbol, this implies that the Toeplitz operator T_f takes the form

$$T_f = C_1 T_u + C_0 I,$$

where u is as in Theorem 3.1 and I denotes the identity operator. This completes the proof.

Final Remark. The results in this paper describe bounded Toeplitz operators with truncated symbols that commute with $T_{z+\bar{g}}$, where g is an analytic function. It is worth noting that the analytic part z in the symbol $z + \bar{g}$ can be replaced by z^n or, more generally, by a polynomial in z, and the same proof techniques can still be applied. However, this generalization comes at the cost of significantly more involved calculations, which can quickly become tedious and lengthy.

Indeed, in our main result, where the analytic polynomial is simply z, we already needed to compute the radial components f_k explicitly for $k = N$ down to $k = -4$. It is natural to expect that replacing z by a higher-degree polynomial in z would require computing even more radial components f_k.

Nevertheless, we are confident that our proof strategy can be adjusted to accommodate analytic polynomials of an arbitrary degree. In particular, we believe that the main result can be extended to the following more general statement: If T_f, as in Theorem 3.1, commutes with T_u, where $u(z) = p(z) + \overline{g(z)}$, with $p(z)$ being an analytic polynomial and $g(z) = \sum_{l=1}^{\infty} a_l z^l$ being a bounded analytic function, then T_f must be a polynomial in T_u of degree at most one.

Author contributions

Hasan Iqtaish: Formal analysis, investigation, resources; Issam Louhichi: Conceptualization, investigation, project administration, methodology; Abdelrahman Yousef: Methodology, supervision, writing-original draft, writing-review and editing. All authors have read and agreed to the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

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