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Research article

Several boundary value problems for bi-analytic complex partial differential systems of second order

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Abstract: In this paper, several boundary value problems for second order complex partial differential systems of bi-analytic functions were investigated on the bicylinder. Homogeneous and nonhomogeneous Dirichlet problems for $(\lambda, 1)$ bi-analytic functions were first discussed on the bicylinder. Applying the Cauchy-Pompeiu formula and the properties of the Poisson kernel, the expressions of the solutions to the Dirichlet problems were obtained. Thereafter, the Riemann problems and the inverse problems for $(\lambda, 1)$ bi-analytic functions were explored on the generalized bicylinder in $\mathbb{C}^2$. Applying the Plemelj formula, the solutions to the corresponding problems were obtained.

Keywords: complex partial differential equations; Dirichlet problems; Riemann problems; bi-analytic functions; Cauchy-Pompeiu formula
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1. Introduction

Dirichlet and Riemann problems are classical boundary value problems. The associated research on analytic functions has been very comprehensive. A class of generalized functions, namely bi-analytic functions, has aroused considerable interest.

The concept of bi-analytic functions in the complex plane was introduced by Sander [1] in 1961. He first gave the exact definitions of bi-analytic functions of type $k(\neq -1)$:

$$\begin{cases} u_x - v_y = \theta, \quad u_y + v_x = \omega, \\ (k+1)\theta_x + \omega_y = 0, \quad (k+1)\theta_y - \omega_x = 0, \end{cases} \tag{1.1}$$

in which $u(x, y)$, $v(x, y)$, and $\theta(x, y)$, $\omega(x, y)$ have continuous derivatives of the first order, and $(k+1)\theta - i\omega$ is called the associated function of $u(x, y) + iv(x, y)$. Sander also introduced bi-analytic functions of type -1:

$$\begin{cases} u_x - v_y = \theta, \quad u_y + v_x = 0, \\ \theta_x - \omega_y = 0, \quad \theta_y + \omega_x = 0, \end{cases} \tag{1.2}$$

where u, v, θ, and ω have continuous derivatives of the first order and $\theta + i\omega$ is called the associated function of $u(x, y) + iv(x, y)$. Bi-analytic functions of type -1 are not special cases of bi-analytic functions of type $k(\neq -1)$. However, the system of Eq (1.1) converts to (1.2) provided that k and ω are set equal to -1 and 0, respectively, and $\frac{\omega}{k+1}$ is replaced by $-\omega$. Sander showed the connection between biharmonic functions and bi-analytic functions of type k with $k \neq 0$, and discussed the constructing, the algebra, and the Cauchy theorem of bi-analytic functions. He first proposed that the equations of the flow of viscous fluids can establish a connection with bi-analytic functions. Thus, problems of viscous fluid flow, plane strain, and generalized plane stress can be related to the theory of bi-analytic functions.

Lin and Wu [2] extended bi-analytic functions of type k to type (λ, k):

$$\frac{k+1}{2}\frac{\partial f}{\partial \bar{z}} - \frac{k-1}{2}\frac{\partial f}{\partial z} = \frac{\lambda - k}{4\lambda}\varphi(z) + \frac{\lambda + k}{4\lambda}\overline{\varphi(z)},$$

which are connected with the system of the elliptic equations of second order:

$$\Big[\begin{pmatrix} 1 & 0 \\ 0 & \frac{\lambda}{k^2} \end{pmatrix}\frac{\partial^2}{\partial x^2} + \begin{pmatrix} 0 & \frac{\lambda - k^2}{k} \\ \frac{\lambda - 1}{k} & 0 \end{pmatrix}\frac{\partial^2}{\partial x \partial y} + \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}\frac{\partial^2}{\partial y^2}\Big]\begin{pmatrix} u \\ v \end{pmatrix} = 0,$$

in which λ, k are real constants with $\lambda \neq 0, 1, k^2$, and $0 < k \leq 1$, and the analytic function $\varphi(z)$ is called the associate function of $f(z) = u + iv$. They investigated the properties of (λ, k) bi-analytic functions and extended the Cauchy theory of analytic functions to (λ, k) bi-analytic functions and obtained some approximations.

Applying the function theory of bi-analytic functions, some boundary value problems for the elliptic system

$$\Big[\begin{pmatrix} k & 0 \\ 0 & -\lambda \end{pmatrix}\frac{\partial}{\partial x} + \begin{pmatrix} 0 & \lambda \\ k & 0 \end{pmatrix}\frac{\partial}{\partial y}\Big]\Big[\begin{pmatrix} \frac{1}{k} & 0 \\ 0 & \frac{1}{k} \end{pmatrix}\frac{\partial}{\partial x} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial y}\Big]\begin{pmatrix} u \\ v \end{pmatrix} = 0,$$

were solved in [3, 4]. The theory of (λ, k) bi-analytic functions, which are closely associated with systems of partial differential equations of second order, was developed by Hua, Wu, and Lin [5, 6]. In addition, Gilbert and Lin [7] explored the representation formula for the Dirichlet problem of $(\lambda, 1)$ bi-analytic functions on the unit disc. Then, applying the $(\lambda, 1)$ bi-analytic function theory, they put forward a simplified method to solve the plane strain problems of orthotropic elasticity. It is thus clear that (λ, k) bi-analytic functions provide a favorable tool to deal with isotropic elasticity and orthotropic plane elasticity. Many scholars have been dedicated to studying the boundary value problems of (λ, k) bi-analytic functions. For instance, Xu [8] studied the properties of generalized (λ, k) bi-analytic functions on a bounded domain in the complex plane and explored the solution to a Riemann-Hilbert boundary value problem on the unit disc for a class of nonlinear second order elliptic systems associated with (λ, k) bi-analytic functions.

In [9], Begehr and Kumar developed the theory of complex bi-analytic functions in n variables determined by

$$\frac{k_j+1}{2}\frac{\partial f^k}{\partial \bar{z}_j} - \frac{k_j-1}{2}\frac{\partial f^k}{\partial z_j} = \frac{\lambda_j - k_j}{4\lambda_j}\varphi_j^k + \frac{\lambda_j + k_j}{4\lambda_j}\overline{\varphi_j^k},$$

where λ_j, k_j are real constants with $\lambda_j \neq 0, 1, k_j^2$ and $0 < k_j \leq 1$ $(1 \leq j \leq n)$, $z_j = x_{2j-1} + ix_{2j}$, $f^k = u_k(x_1, \cdots, x_{2n}) + iu_{(2n+1)-k}(x_1, \cdots, x_{2n})$ $(1 \leq k \leq n)$, and $\varphi_j^k(z_1, \cdots, z_n)$ is analytic, and

$$\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_{2j-1}} + i\frac{\partial}{\partial x_{2j}}\Big), \quad \frac{\partial}{\partial z_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_{2j-1}} - i\frac{\partial}{\partial x_{2j}}\Big).$$

They obtained the generalized Cauchy integral formula, Taylor series, and Poisson integral formula on the polydisc. Xu [10] discussed the Riemann problem and inverse Riemann problem of $(\lambda, 1)$ bi-analytic functions in the complex plane. An interesting question related to the generalized classical second-order linear PDEs is the uniqueness of solutions for nonlocal equations. In recent years, there have been many excellent conclusions for second-order linear PDEs. For example, Ma [11] discussed the nonlocal partial differential equations and obtained the solution to the corresponding Cauchy problem via Fourier series. There exist, of course, many conclusions about nonlinear, nonlocal PDE problems. For example, a local and nonlocal second-order boundary value problem with in-homogeneous Cauchy-Neumann boundary conditions were discussed in [12]. The existence of the solutions to a class of nonlinear problems with nonlocal reaction term were investigated via Topological methods in [13]. These conclusions provide the inspiration for studying the corresponding complex partial differential equation problems. There have been some new conclusions regarding the boundary value problems for bi-analytic functions.

The conclusions about bi-analytic functions are almost on the complex plane. However, there are few results available regarding the boundary value problems of $(\lambda, 1)$ bi-analytic functions in spaces of several complex variables. Stimulated by this, in this paper, we mainly investigate several Dirichlet and Riemann boundary value problems for bi-analytic functions on the bicylinder in $\mathbb{C}^2$.

Throughout this paper, let $G = G_1 \times G_2$ be the generalized bicylinder in $\mathbb{C}^2$, where G_1 and G_2 are bounded domains. The boundary L_m of G_m $(m = 1, 2)$ is a finite smooth simple closed curve. Let $L = L_1 \times L_2$, and let G_m^+ and G_m^- denote bounded domains and unbounded domains, respectively, enclosed by L_m. Let $f^{++}(t)$, $f^{+-}(t)$, $f^{-+}(t)$, and $f^{--}(t)$ denote the limit values of $f(z)$ as $z \to t (t \in L)$ along the side of $G_1^+ \times G_2^+$, $G_1^+ \times G_2^-$, $G_1^- \times G_2^+$, and $G_1^- \times G_2^-$, respectively. Let $H_\alpha(\overline{G})$ represent the set of Hölder continuous functions on G.

2. Some definitions and lemmas

To get the major results, we need the following definition and lemmas.

Definition 2.1. *Let G be the generalized bicylinder in $\mathbb{C}^2$, and let $\lambda \in \mathbb{R} \setminus \{0, 1, -1\}$. Then, $f(z)$ is called a bi-analytic function of type $(\lambda, 1)$ on G, if*

$$\partial_{\bar{z}_1}\partial_{\bar{z}_2} f(z) = \frac{\lambda - 1}{4\lambda}\varphi(z) + \frac{\lambda + 1}{4\lambda}\overline{\varphi(z)}, \quad \partial_{\bar{z}_1}\partial_{\bar{z}_2}\varphi(z) = 0,\ z = (z_1, z_2),$$

where $\varphi(z)$ is determined by $\partial_{\bar{z}_1}\partial_{\bar{z}_2}\varphi(z) = 0$, which is called the associate function of $f(z)$.

Lemma 2.2. *[14] Let $w \in C^1(G_0; \mathbb{C}) \bigcap C(\overline{G_0}; \mathbb{C})$, where G_0 is a bounded smooth domain in the complex plane, then*

$$w(z) = \frac{1}{2\pi i}\int_{\partial G_0} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_{G_0} w_{\bar{\zeta}}(\zeta)\frac{d\sigma_\zeta}{\zeta - z},$$
$$\int_{G_0} w_{\bar{z}}(z)d\sigma_z = \frac{1}{2i}\int_{\partial G_0} w(z)dz, \int_{G_0} w_z(z)d\sigma_z = -\frac{1}{2i}\int_{\partial G_0} w(z)d\bar{z}.$$

Lemma 2.3. *[14] Let G_0 be a bounded smooth domain in the complex plane, and let $f \in L_1(G_0; \mathbb{C})$ and*

$$Tf(z) = \frac{-1}{\pi}\int_{G_0}\frac{f(\zeta)}{\zeta - z}d\xi d\eta,\ \ \zeta = (\xi, \eta).$$

Then $\partial_{\bar z} Tf(z) = f(z)$.

Lemma 2.4. *[15] Let G be the generalized bicylinder in* $\mathbb{C}^2$. *Let* $\varphi(t) \in H_\alpha(L)$ *and*

$$\Phi(z_1, z_2) = \frac{1}{(2\pi i)^2} \int_L \frac{\varphi(\tau_1, \tau_2) d\tau_1 d\tau_2}{(\tau_1 - z_1)(\tau_2 - z_2)} \quad (z_m \notin L_m),$$

then

$$\begin{cases} \Phi^{++}(t_1, t_2) = \frac{1}{4}[\varphi + S_1\varphi + S_2\varphi + S_3\varphi], & \Phi^{+-}(t_1, t_2) = \frac{1}{4}[-\varphi - S_1\varphi + S_2\varphi + S_3\varphi], \\ \Phi^{-+}(t_1, t_2) = \frac{1}{4}[-\varphi + S_1\varphi - S_2\varphi + S_3\varphi], & \Phi^{--}(t_1, t_2) = \frac{1}{4}[\varphi - S_1\varphi - S_2\varphi + S_3\varphi], \end{cases}$$

where $t_m \in L_m$, $\Phi^{\pm\pm}(t_1, t_2)$ *represents the limit value of* $\Phi(z)$ *when* $z \to t$ *in* $D_1^\pm \times D_2^\pm$ (D_m^+ *and* D_m^- *represent the interior and exterior domain enclosed by* L_m*, respectively, m=1,2) and*

$$S_1\varphi = \frac{1}{\pi i} \int_{L_1} \frac{\varphi(\tau_1, t_2)}{\tau_1 - t_1} d\tau_1, \quad S_2\varphi = \frac{1}{\pi i} \int_{L_2} \frac{\varphi(t_1, \tau_2)}{\tau_2 - t_2} d\tau_2,$$

$$S_3\varphi = \frac{1}{(\pi i)^2} \int_L \frac{\varphi(\tau_1, \tau_2)}{(\tau_1 - t_1)(\tau_2 - t_2)} d\tau_1 d\tau_2.$$

3. Dirichlet problems

In this section, we discuss several Dirichlet problems for $(\lambda, 1)$ bi-analytic functions on the bicylinder $D = D_1 \times D_2 = \{(z_1, z_2) : |z_1| < 1, |z_2| < 1\}$. Let $\partial_0 D$ be the characteristic boundary of D.

Theorem 3.1. *Let* $h \in C(\partial_0 D)$, $\varphi_1 \in C(D_1 \times \partial D_2)$, *and* $\varphi_2 \in C(\partial D_1 \times D_2)$ *be given functions, and let* $\lambda \in \mathbb{R} \setminus \{-1, 0, 1\}$. *Then, the solution of the problem*

$$\begin{cases} \partial_{\bar z_1}\partial_{\bar z_2}\Phi(z) = \frac{\lambda-1}{4\lambda}\varphi(z) + \frac{\lambda+1}{4\lambda}\overline{\varphi(z)},\ \partial_{\bar z_1}\partial_{\bar z_2}\varphi(z) = 0,\ \Phi(\zeta) = h(\zeta)\ \ (z \in D,\ \zeta \in \partial D), \\ \partial_{\bar z_1}\Phi(z_1, \zeta_2) = \varphi_1(z_1, \zeta_2), \partial_{\bar z_2}\Phi(\zeta_1, z_2) = \varphi_2(\zeta_1, z_2)\ \ (\zeta_1 \in \partial D_1, \zeta_2 \in \partial D_2) \end{cases} \tag{3.1}$$

(where $z = (z_1, z_2)$*, and* $\varphi(z)$ *is the associate function of* $\Phi(z)$*) is given by*

$$\begin{aligned} \Phi(z) = {} & \frac{1}{(2\pi i)^2} \int_{\partial_0 D} \Big(\frac{\zeta_1 + z_1}{\zeta_1 - z_1} + \frac{\bar\zeta_1 + \bar z_1}{\bar\zeta_1 - \bar z_1}\Big)\Big(\frac{\zeta_2 + z_2}{\zeta_2 - z_2} + \frac{\bar\zeta_2 + \bar z_2}{\bar\zeta_2 - \bar z_2}\Big)\frac{h(\zeta)}{4\zeta_1\zeta_2} d\zeta \\ & + \frac{1}{2\pi i} \int_{\partial D_1} \Big(\frac{\zeta_1 + z_1}{\zeta_1 - z_1} + \frac{\bar\zeta_1 + \bar z_1}{\bar\zeta_1 - \bar z_1}\Big)\Big[\frac{1}{\pi} \int_{D_2} \Big(\frac{1}{\zeta_2 - \frac{1}{\bar z_2}} - \frac{1}{\zeta_2 - z_2}\Big)\varphi_2(\zeta) d\sigma_{\zeta_2}\Big]\frac{d\zeta_1}{2\zeta_1} \\ & + \frac{1}{2\pi i} \int_{\partial D_2} \Big(\frac{\zeta_2 + z_2}{\zeta_2 - z_2} + \frac{\bar\zeta_2 + \bar z_2}{\bar\zeta_2 - \bar z_2}\Big)\Big[\frac{1}{\pi} \int_{D_1} \Big(\frac{1}{\zeta_1 - \frac{1}{\bar z_1}} - \frac{1}{\zeta_1 - z_1}\Big)\varphi_1(\zeta) d\sigma_{\zeta_1}\Big]\frac{d\zeta_2}{2\zeta_2} \\ & + \frac{\lambda - 1}{\lambda + 1}\Big(\frac{1}{z_1} - \bar z_1\Big)\Big(\frac{1}{z_2} - \bar z_2\Big)\frac{1}{2\pi i} \int_{\partial D_1} \Big[\overline{\varphi_2(\zeta_1, z_2)} - \overline{\varphi_1(\zeta_1, 0)}\Big]\Big[\frac{1}{(\zeta_1 - z_1)^2} - \frac{1}{\zeta_1^2}\Big] d\zeta_1. \end{aligned} \tag{3.2}$$

Proof. By Lemma 2.2, for $\forall z_1 \in D_1$, applying the Cauchy-Pompeiu formula on $\Phi(z_1, \cdot)$, we get that

$$\Phi(z) = \frac{1}{2\pi i} \int_{\partial D_1} \frac{\Phi(\zeta_1, z_2)}{\zeta_1 - z_1} d\zeta_1 - \frac{1}{\pi} \int_{D_1} \frac{\Phi_{\bar\zeta_1}(\zeta_1, z_2)}{\zeta_1 - z_1} d\sigma_{\zeta_1}. \tag{3.3}$$

Similarly, for $\forall z_2 \in D_2$, applying the Cauchy-Pompeiu formula on $\Phi(\zeta_1, z_2)$, we get that

$$\Phi(\zeta_1, z_2) = \frac{1}{2\pi i}\int_{\partial D_2} \frac{\Phi(\zeta_1, \zeta_2)}{\zeta_2 - z_2} d\zeta_2 - \frac{1}{\pi}\int_{D_2} \frac{\Phi_{\bar{\zeta}_2}(\zeta_1, \zeta_2)}{\zeta_2 - z_2} d\sigma_{\zeta_2}. \tag{3.4}$$

Plugging (3.4) into (3.3), we obtain that

$$\begin{aligned}\Phi(z) = {} & \frac{1}{(2\pi i)^2}\int_{\partial_0 D} \frac{\Phi(\zeta) d\zeta_1 d\zeta_2}{(\zeta_1 - z_1)(\zeta_2 - z_2)} + \frac{1}{\pi^2}\int_D \frac{\partial_{\bar{\zeta}_1}\partial_{\bar{\zeta}_2}\Phi(\zeta) d\sigma_{\zeta_1} d\sigma_{\zeta_2}}{(\zeta_1 - z_1)(\zeta_2 - z_2)} \\ & - \frac{1}{2\pi i}\int_{\partial D_1}\Big[\frac{1}{\pi}\int_{D_2}\frac{\partial_{\bar{\zeta}_2}\Phi(\zeta)}{\zeta_2 - z_2} d\sigma_{\zeta_2}\Big]\frac{d\zeta_1}{\zeta_1 - z_1} - \frac{1}{2\pi i}\int_{\partial D_2}\Big[\frac{1}{\pi}\int_{D_1}\frac{\partial_{\bar{\zeta}_1}\Phi(\zeta)}{\zeta_1 - z_1} d\sigma_{\zeta_1}\Big]\frac{d\zeta_2}{\zeta_2 - z_2}.\end{aligned} \tag{3.5}$$

Let $\widetilde{\varphi}(z) = \int_0^{z_1}\int_0^{z_2} \varphi(\zeta) d\zeta_1 d\zeta_2$, then we have that

$$\begin{aligned}& \frac{1}{\pi^2}\int_D \frac{\partial_{\bar{\zeta}_1}\partial_{\bar{\zeta}_2}\Phi(\zeta) d\sigma_{\zeta_1} d\sigma_{\zeta_2}}{(\zeta_1 - z_1)(\zeta_2 - z_2)} \\ = & \frac{1}{\pi^2}\int_D \Big[\frac{\lambda - 1}{4\lambda}\varphi(\zeta) + \frac{\lambda + 1}{4\lambda}\overline{\varphi(\zeta)}\Big]\frac{d\sigma_{\zeta_1} d\sigma_{\zeta_2}}{(\zeta_1 - z_1)(\zeta_2 - z_2)} \\ = & \frac{\lambda - 1}{4\lambda}\frac{1}{\pi^2}\int_D \frac{\partial_{\bar{\zeta}_1}\partial_{\bar{\zeta}_2}[\bar{\zeta}_1\bar{\zeta}_2\varphi(\zeta)]}{(\zeta_1 - z_1)(\zeta_2 - z_2)} d\sigma_{\zeta_1} d\sigma_{\zeta_2} + \frac{\lambda + 1}{4\lambda}\frac{1}{\pi^2}\int_D \frac{\partial_{\bar{\zeta}_1}\partial_{\bar{\zeta}_2}\overline{\widetilde{\varphi}(\zeta)}}{(\zeta_1 - z_1)(\zeta_2 - z_2)} d\sigma_{\zeta_1} d\sigma_{\zeta_2}.\end{aligned} \tag{3.6}$$

In consideration of $\partial_{\bar{z}_1}\partial_{\bar{z}_2}\varphi(z) = 0$ (i.e., $\varphi(z)$ is analytic with respect to z_1 and z_2), by (3.5) we get that

$$\begin{aligned}& \frac{1}{\pi^2}\int_D \frac{\partial_{\bar{\zeta}_1}\partial_{\bar{\zeta}_2}[\bar{\zeta}_1\bar{\zeta}_2\varphi(\zeta)]}{(\zeta_1 - z_1)(\zeta_2 - z_2)} d\sigma_{\zeta_1} d\sigma_{\zeta_2} \\ = {} & \bar{z}_1\bar{z}_2\varphi(z) - \frac{1}{(2\pi i)^2}\int_{\partial_0 D} \frac{\bar{\zeta}_1\bar{\zeta}_2\varphi(\zeta) d\zeta_1 d\zeta_2}{(\zeta_1 - z_1)(\zeta_2 - z_2)} \\ & + \frac{1}{2\pi i}\int_{\partial D_1}\bar{\zeta}_1\Big[\frac{1}{\pi}\int_{D_2}\frac{\partial_{\bar{\zeta}_2}(\bar{\zeta}_2\varphi(\zeta))}{\zeta_2 - z_2} d\sigma_{\zeta_2}\Big]\frac{d\zeta_1}{\zeta_1 - z_1} + \frac{1}{2\pi i}\int_{\partial D_2}\bar{\zeta}_2\Big[\frac{1}{\pi}\int_{D_1}\frac{\partial_{\bar{\zeta}_1}(\bar{\zeta}_1\varphi(\zeta))}{\zeta_1 - z_1} d\sigma_{\zeta_1}\Big]\frac{d\zeta_2}{\zeta_2 - z_2} \\ = {} & \bar{z}_1\bar{z}_2\varphi(z) - \frac{1}{(2\pi i)^2}\int_{\partial_0 D} \frac{\bar{\zeta}_1\bar{\zeta}_2\varphi(\zeta) d\zeta_1 d\zeta_2}{(\zeta_1 - z_1)(\zeta_2 - z_2)} \\ & + \frac{1}{2\pi i}\int_{\partial D_1}\bar{\zeta}_1\Big[\frac{1}{2\pi i}\int_{\partial D_2}\frac{\bar{\zeta}_2\varphi(\zeta)}{\zeta_2 - z_2} d\zeta_2 - \bar{z}_2\varphi(\zeta_1, z_2)\Big]\frac{d\zeta_1}{\zeta_1 - z_1} \\ & + \frac{1}{2\pi i}\int_{\partial D_2}\bar{\zeta}_2\Big[\frac{1}{2\pi i}\int_{\partial D_1}\frac{\bar{\zeta}_1\varphi(\zeta)}{\zeta_1 - z_1} d\zeta_1 - \bar{z}_1\varphi(z_1, \zeta_2)\Big]\frac{d\zeta_2}{\zeta_2 - z_2} \\ = {} & \bar{z}_1\bar{z}_2\varphi(z) + \frac{1}{(2\pi i)^2}\int_{\partial_0 D} \frac{\bar{\zeta}_1\bar{\zeta}_2\varphi(\zeta) d\zeta_1 d\zeta_2}{(\zeta_1 - z_1)(\zeta_2 - z_2)} \\ & - \frac{1}{(2\pi i)^2}\int_{\partial_0 D} \frac{\bar{\zeta}_1\bar{z}_2\varphi(\zeta) d\zeta_1 d\zeta_2}{(\zeta_1 - z_1)(\zeta_2 - z_2)} - \frac{1}{(2\pi i)^2}\int_{\partial_0 D} \frac{\bar{z}_1\bar{\zeta}_2\varphi(\zeta) d\zeta_1 d\zeta_2}{(\zeta_1 - z_1)(\zeta_2 - z_2)} \\ = {} & \frac{1}{(2\pi i)^2}\int_{\partial_0 D} \frac{(\bar{\zeta}_1 - \bar{z}_1)(\bar{\zeta}_2 - \bar{z}_2)\varphi(\zeta) d\zeta_1 d\zeta_2}{(\zeta_1 - z_1)(\zeta_2 - z_2)}.\end{aligned} \tag{3.7}$$

Similarly, we have that

$$\begin{aligned}
&\frac{1}{\pi^2}\int_D \frac{\partial_{\bar\zeta_1}\partial_{\bar\zeta_2}\overline{\widetilde{\varphi}(\zeta)}}{(\zeta_1-z_1)(\zeta_2-z_2)}d\sigma_{\zeta_1}d\sigma_{\zeta_2}\\
&=\overline{\widetilde{\varphi}(z)}-\frac{1}{(2\pi i)^2}\int_{\partial_0 D}\frac{\overline{\widetilde{\varphi}(\zeta)}d\zeta_1 d\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}\\
&\quad+\frac{1}{2\pi i}\int_{\partial D_1}\Big[\frac{1}{\pi}\int_{D_2}\frac{\partial_{\bar\zeta_2}\overline{\widetilde{\varphi}(\zeta)}}{\zeta_2-z_2}d\sigma_{\zeta_2}\Big]\frac{d\zeta_1}{\zeta_1-z_1}+\frac{1}{2\pi i}\int_{\partial D_2}\Big[\frac{1}{\pi}\int_{D_1}\frac{\partial_{\bar\zeta_1}\overline{\widetilde{\varphi}(\zeta)}}{\zeta_1-z_1}d\sigma_{\zeta_1}\Big]\frac{d\zeta_2}{\zeta_2-z_2}\\
&=\overline{\widetilde{\varphi}(z)}-\frac{1}{(2\pi i)^2}\int_{\partial_0 D}\frac{\overline{\widetilde{\varphi}(\zeta)}d\zeta_1 d\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}+\frac{1}{2\pi i}\int_{\partial D_1}\Big[\frac{1}{2\pi i}\int_{\partial D_2}\frac{\overline{\widetilde{\varphi}(\zeta)}}{\zeta_2-z_2}d\zeta_2-\overline{\widetilde{\varphi}(\zeta_1,z_2)}\Big]\frac{d\zeta_1}{\zeta_1-z_1}\\
&\quad+\frac{1}{2\pi i}\int_{\partial D_2}\Big[\frac{1}{2\pi i}\int_{\partial D_1}\frac{\overline{\widetilde{\varphi}(\zeta)}}{\zeta_1-z_1}d\zeta_1-\overline{\widetilde{\varphi}(z_1,\zeta_2)}\Big]\frac{d\zeta_2}{\zeta_2-z_2}\\
&=\frac{1}{(2\pi i)^2}\int_{\partial_0 D}\overline{\widetilde{\varphi}(\zeta)}\Big(\frac{d\zeta_1}{\zeta_1-z_1}+\frac{d\bar\zeta_1}{\overline{\zeta_1-z_1}}\Big)\Big(\frac{d\zeta_2}{\zeta_2-z_2}+\frac{d\bar\zeta_2}{\overline{\zeta_2-z_2}}\Big).
\end{aligned}\tag{3.8}$$

Plugging (3.7) and (3.8) into (3.6), we get that

$$\begin{aligned}
&\frac{1}{\pi^2}\int_D \frac{\partial_{\bar\zeta_1}\partial_{\bar\zeta_2}\Phi(\zeta)d\sigma_{\zeta_1}d\sigma_{\zeta_2}}{(\zeta_1-z_1)(\zeta_2-z_2)}\\
&=\frac{\lambda-1}{4\lambda}\frac{1}{(2\pi i)^2}\int_{\partial_0 D}\frac{(\bar\zeta_1-\bar z_1)(\bar\zeta_2-\bar z_2)\varphi(\zeta)d\zeta_1 d\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}\\
&\quad+\frac{\lambda+1}{4\lambda}\frac{1}{(2\pi i)^2}\int_{\partial_0 D}\overline{\widetilde{\varphi}(\zeta)}\Big(\frac{d\zeta_1}{\zeta_1-z_1}+\frac{d\bar\zeta_1}{\overline{\zeta_1-z_1}}\Big)\Big(\frac{d\zeta_2}{\zeta_2-z_2}+\frac{d\bar\zeta_2}{\overline{\zeta_2-z_2}}\Big).
\end{aligned}\tag{3.9}$$

As $\Phi(\zeta)=h(\zeta)$, (3.5) and (3.9) follow that

$$\begin{aligned}
\Phi(z)&=\frac{1}{(2\pi i)^2}\int_{\partial_0 D}\frac{h(\zeta)d\zeta_1 d\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}\\
&\quad-\frac{1}{2\pi i}\int_{\partial D_1}\Big[\frac{1}{\pi}\int_{D_2}\frac{\partial_{\bar\zeta_2}\Phi(\zeta)}{\zeta_2-z_2}d\sigma_{\zeta_2}\Big]\frac{d\zeta_1}{\zeta_1-z_1}-\frac{1}{2\pi i}\int_{\partial D_2}\Big[\frac{1}{\pi}\int_{D_1}\frac{\partial_{\bar\zeta_1}\Phi(\zeta)}{\zeta_1-z_1}d\sigma_{\zeta_1}\Big]\frac{d\zeta_2}{\zeta_2-z_2}\\
&\quad+\frac{\lambda-1}{4\lambda}\frac{1}{(2\pi i)^2}\int_{\partial_0 D}\frac{(\bar\zeta_1-\bar z_1)(\bar\zeta_2-\bar z_2)\varphi(\zeta)d\zeta_1 d\zeta_2}{(\zeta_1-z_1)(\zeta_2-z_2)}\\
&\quad+\frac{\lambda+1}{4\lambda}\frac{1}{(2\pi i)^2}\int_{\partial_0 D}\overline{\widetilde{\varphi}(\zeta)}\Big(\frac{d\zeta_1}{\zeta_1-z_1}+\frac{d\bar\zeta_1}{\overline{\zeta_1-z_1}}\Big)\Big(\frac{d\zeta_2}{\zeta_2-z_2}+\frac{d\bar\zeta_2}{\overline{\zeta_2-z_2}}\Big).
\end{aligned}\tag{3.10}$$

As $|\frac{1}{\bar z_1}|>1$, if z_1 is replaced with $\frac{1}{\bar z_1}$, then the left-hand end of Eq (3.3) is zero, so the left end of Eq (3.10) as well as (3.5) is zero, that is

$$\begin{aligned}
0&=\frac{1}{(2\pi i)^2}\int_{\partial_0 D}\frac{h(\zeta)d\zeta_1 d\zeta_2}{(\zeta_1-\frac{1}{\bar z_1})(\zeta_2-z_2)}\\
&\quad-\frac{1}{2\pi i}\int_{\partial D_1}\Big[\frac{1}{\pi}\int_{D_2}\frac{\partial_{\bar\zeta_2}\Phi(\zeta)}{\zeta_2-z_2}d\sigma_{\zeta_2}\Big]\frac{d\zeta_1}{\zeta_1-\frac{1}{\bar z_1}}-\frac{1}{2\pi i}\int_{\partial D_2}\Big[\frac{1}{\pi}\int_{D_1}\frac{\partial_{\bar\zeta_1}\Phi(\zeta)}{\zeta_1-\frac{1}{\bar z_1}}d\sigma_{\zeta_1}\Big]\frac{d\zeta_2}{\zeta_2-z_2}
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\lambda-1}{4\lambda}\frac{1}{(2\pi i)^2}\int_{\partial_0 D} \frac{(\bar{\zeta}_1-\bar{z}_1)(\bar{\zeta}_2-\bar{z}_2)\varphi(\zeta)d\zeta_1 d\zeta_2}{(\zeta_1-\frac{1}{\bar{z}_1})(\zeta_2-z_2)} \\
&+ \frac{\lambda+1}{4\lambda}\frac{1}{(2\pi i)^2}\int_{\partial_0 D} \overline{\varphi(\zeta)}\Big(\frac{d\zeta_1}{\zeta_1-\frac{1}{\bar{z}_1}}+\frac{d\bar{\zeta}_1}{\bar{\zeta}_1-\frac{1}{z_1}}\Big)\Big(\frac{d\zeta_2}{\zeta_2-z_2}+\frac{d\bar{\zeta}_2}{\overline{\zeta_2-z_2}}\Big).
\end{aligned}
\tag{3.11}
$$

Similarly, as $|\frac{1}{\bar{z}_2}|>1$, we have that

$$
\begin{aligned}
0 =& \frac{1}{(2\pi i)^2}\int_{\partial_0 D} \frac{h(\zeta)d\zeta_1 d\zeta_2}{(\zeta_1-z_1)(\zeta_2-\frac{1}{\bar{z}_2})} \\
&- \frac{1}{2\pi i}\int_{\partial D_1}\Big[\frac{1}{\pi}\int_{D_2}\frac{\partial_{\bar{\zeta}_2}\Phi(\zeta)}{\zeta_2-\frac{1}{\bar{z}_2}}d\sigma_{\zeta_2}\Big]\frac{d\zeta_1}{\zeta_1-z_1} - \frac{1}{2\pi i}\int_{\partial D_2}\Big[\frac{1}{\pi}\int_{D_1}\frac{\partial_{\bar{\zeta}_1}\Phi(\zeta)}{\zeta_1-z_1}d\sigma_{\zeta_1}\Big]\frac{d\zeta_2}{\zeta_2-\frac{1}{\bar{z}_2}} \\
&+ \frac{\lambda-1}{4\lambda}\frac{1}{(2\pi i)^2}\int_{\partial_0 D} \frac{(\bar{\zeta}_1-\bar{z}_1)(\bar{\zeta}_2-\bar{z}_2)\varphi(\zeta)d\zeta_1 d\zeta_2}{(\zeta_1-z_1)(\zeta_2-\frac{1}{\bar{z}_2})} \\
&+ \frac{\lambda+1}{4\lambda}\frac{1}{(2\pi i)^2}\int_{\partial_0 D} \overline{\varphi(\zeta)}\Big(\frac{d\zeta_1}{\zeta_1-z_1}+\frac{d\bar{\zeta}_1}{\overline{\zeta_1-z_1}}\Big)\Big(\frac{d\zeta_2}{\zeta_2-\frac{1}{\bar{z}_2}}+\frac{d\bar{\zeta}_2}{\bar{\zeta}_2-\frac{1}{z_2}}\Big).
\end{aligned}
\tag{3.12}
$$

As $|\frac{1}{\bar{z}_1}|>1, |\frac{1}{\bar{z}_2}|>1$, we have that

$$
\begin{aligned}
0 =& \frac{1}{(2\pi i)^2}\int_{\partial_0 D} \frac{h(\zeta)d\zeta_1 d\zeta_2}{(\zeta_1-\frac{1}{\bar{z}_1})(\zeta_2-\frac{1}{\bar{z}_2})} \\
&- \frac{1}{2\pi i}\int_{\partial D_1}\Big[\frac{1}{\pi}\int_{D_2}\frac{\partial_{\bar{\zeta}_2}\Phi(\zeta)}{\zeta_2-\frac{1}{\bar{z}_2}}d\sigma_{\zeta_2}\Big]\frac{d\zeta_1}{\zeta_1-\frac{1}{\bar{z}_1}} - \frac{1}{2\pi i}\int_{\partial D_2}\Big[\frac{1}{\pi}\int_{D_1}\frac{\partial_{\bar{\zeta}_1}\Phi(\zeta)}{\zeta_1-\frac{1}{\bar{z}_1}}d\sigma_{\zeta_1}\Big]\frac{d\zeta_2}{\zeta_2-\frac{1}{\bar{z}_2}} \\
&+ \frac{\lambda-1}{4\lambda}\frac{1}{(2\pi i)^2}\int_{\partial_0 D} \frac{(\bar{\zeta}_1-\bar{z}_1)(\bar{\zeta}_2-\bar{z}_2)\varphi(\zeta)d\zeta_1 d\zeta_2}{(\zeta_1-\frac{1}{\bar{z}_1})(\zeta_2-\frac{1}{\bar{z}_2})} \\
&+ \frac{\lambda+1}{4\lambda}\frac{1}{(2\pi i)^2}\int_{\partial_0 D} \overline{\varphi(\zeta)}\Big(\frac{d\zeta_1}{\zeta_1-\frac{1}{\bar{z}_1}}+\frac{d\bar{\zeta}_1}{\bar{\zeta}_1-\frac{1}{z_1}}\Big)\Big(\frac{d\zeta_2}{\zeta_2-\frac{1}{\bar{z}_2}}+\frac{d\bar{\zeta}_2}{\bar{\zeta}_2-\frac{1}{z_2}}\Big).
\end{aligned}
\tag{3.13}
$$

Therefore, (3.10) minus (3.11) minus (3.12) plus (3.13) follows that

$$
\begin{aligned}
\Phi(z) =& \frac{1}{(2\pi i)^2}\int_{\partial_0 D}\Big[\frac{1}{(\zeta_1-z_1)(\zeta_2-z_2)} - \frac{1}{(\zeta_1-\frac{1}{\bar{z}_1})(\zeta_2-z_2)} \\
&- \frac{1}{(\zeta_1-z_1)(\zeta_2-\frac{1}{\bar{z}_2})} + \frac{1}{(\zeta_1-\frac{1}{\bar{z}_1})(\zeta_2-\frac{1}{\bar{z}_2})}\Big]h(\zeta)d\zeta \\
&+ \frac{1}{\pi}\int_{D_2}\Big[\frac{1}{2\pi i}\int_{\partial D_1}\Big(\frac{\partial_{\bar{\zeta}_2}\Phi(\zeta)}{\zeta_1-\frac{1}{\bar{z}_1}} - \frac{\partial_{\bar{\zeta}_2}\Phi(\zeta)}{\zeta_1-z_1}\Big)d\zeta_1\Big]\frac{d\sigma_{\zeta_2}}{\zeta_2-z_2} \\
&+ \frac{1}{\pi}\int_{D_2}\Big[\frac{1}{2\pi i}\int_{\partial D_1}\Big(\frac{\partial_{\bar{\zeta}_2}\Phi(\zeta)}{\zeta_1-z_1} - \frac{\partial_{\bar{\zeta}_2}\Phi(\zeta)}{\zeta_1-\frac{1}{\bar{z}_1}}\Big)d\zeta_1\Big]\frac{d\sigma_{\zeta_2}}{\zeta_2-\frac{1}{\bar{z}_2}} \\
&+ \frac{1}{\pi}\int_{D_1}\Big[\frac{1}{2\pi i}\int_{\partial D_2}\Big(\frac{\partial_{\bar{\zeta}_1}\Phi(\zeta)}{\zeta_2-z_2} - \frac{\partial_{\bar{\zeta}_1}\Phi(\zeta)}{\zeta_2-\frac{1}{\bar{z}_2}}\Big)d\zeta_2\Big]\frac{d\sigma_{\zeta_1}}{\zeta_1-\frac{1}{\bar{z}_1}} \\
&+ \frac{1}{\pi}\int_{D_1}\Big[\frac{1}{2\pi i}\int_{\partial D_2}\Big(\frac{\partial_{\bar{\zeta}_1}\Phi(\zeta)}{\zeta_2-\frac{1}{\bar{z}_2}} - \frac{\partial_{\bar{\zeta}_1}\Phi(\zeta)}{\zeta_2-z_2}\Big)d\zeta_2\Big]\frac{d\sigma_{\zeta_1}}{\zeta_1-z_1}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\lambda-1}{4\lambda}A+\frac{\lambda+1}{4\lambda}B\\
&=\frac{1}{(2\pi i)^2}\int_{\partial_0 D}\Big(\frac{\zeta_1+z_1}{\zeta_1-z_1}+\frac{\bar\zeta_1+\bar z_1}{\bar\zeta_1-\bar z_1}\Big)\Big(\frac{\zeta_2+z_2}{\zeta_2-z_2}+\frac{\bar\zeta_2+\bar z_2}{\bar\zeta_2-\bar z_2}\Big)\frac{h(\zeta)}{4\zeta_1\zeta_2}d\zeta\\
&+\frac{1}{2\pi i}\int_{\partial D_1}\Big(\frac{1}{\zeta_1-z_1}-\frac{1}{\zeta_1-\frac{1}{\bar z_1}}\Big)\Big[\frac{1}{\pi}\int_{D_2}\Big(\frac{1}{\zeta_2-\frac{1}{\bar z_2}}-\frac{1}{\zeta_2-z_2}\Big)\partial_{\bar\zeta_2}\Phi(\zeta)d\sigma_{\zeta_2}\Big]d\zeta_1\\
&+\frac{1}{2\pi i}\int_{\partial D_2}\Big(\frac{1}{\zeta_2-z_2}-\frac{1}{\zeta_2-\frac{1}{\bar z_2}}\Big)\Big[\frac{1}{\pi}\int_{D_1}\Big(\frac{1}{\zeta_1-\frac{1}{\bar z_1}}-\frac{1}{\zeta_1-z_1}\Big)\partial_{\bar\zeta_1}\Phi(\zeta)d\sigma_{\zeta_1}\Big]d\zeta_2\\
&+\frac{\lambda-1}{4\lambda}A+\frac{\lambda+1}{4\lambda}B,
\end{aligned}
\tag{3.14}
$$

where

$$
\begin{aligned}
A&=\frac{1}{(2\pi i)^2}\int_{\partial_0 D}\Big[\frac{(\bar\zeta_1-\bar z_1)(\bar\zeta_2-\bar z_2)}{(\zeta_1-z_1)(\zeta_2-z_2)}-\frac{(\bar\zeta_1-\bar z_1)(\bar\zeta_2-\bar z_2)}{(\zeta_1-\frac{1}{\bar z_1})(\zeta_2-z_2)}\\
&-\frac{(\bar\zeta_1-\bar z_1)(\bar\zeta_2-\bar z_2)}{(\zeta_1-z_1)(\zeta_2-\frac{1}{\bar z_2})}+\frac{(\bar\zeta_1-\bar z_1)(\bar\zeta_2-\bar z_2)}{(\zeta_1-\frac{1}{\bar z_1})(\zeta_2-\frac{1}{\bar z_2})}\Big]\varphi(\zeta)d\zeta\\
&=\frac{1}{(2\pi i)^2}\int_{\partial_0 D}\Big(\frac{\bar\zeta_1-\bar z_1}{\zeta_1-z_1}-\frac{\bar\zeta_1-\bar z_1}{\zeta_1-\frac{1}{\bar z_1}}\Big)\Big(\frac{\bar\zeta_2-\bar z_2}{\zeta_2-z_2}-\frac{\bar\zeta_2-\bar z_2}{\zeta_2-\frac{1}{\bar z_2}}\Big)\varphi(\zeta)d\zeta\\
&=\frac{1}{2\pi i}\int_{\partial D_1}\Big(\frac{\bar\zeta_1-\bar z_1}{\zeta_1-z_1}-\frac{\bar\zeta_1-\bar z_1}{\zeta_1-\frac{1}{\bar z_1}}\Big)\\
&\cdot\Big\{\frac{1}{2\pi i}\int_{\partial D_2}\Big[\frac{\bar\zeta_2}{\zeta_2-z_2}-\frac{\bar z_2}{\zeta_2-z_2}-\frac{\bar\zeta_2}{\zeta_2-\frac{1}{\bar z_2}}+\frac{\bar z_2}{\zeta_2-\frac{1}{\bar z_2}}\Big]\varphi(\zeta)d\zeta_2\Big\}d\zeta_1\\
&=\frac{1}{2\pi i}\int_{\partial D_1}\Big(\frac{\bar\zeta_1-\bar z_1}{\zeta_1-z_1}-\frac{\bar\zeta_1-\bar z_1}{\zeta_1-\frac{1}{\bar z_1}}\Big)\Big[\Big(\frac{\varphi(\zeta_1,z_2)}{z_2}-\frac{\varphi(\zeta_1,0)}{z_2}\Big)-\bar z_2\varphi(\zeta_1,z_2)+\bar z_2\varphi(\zeta_1,0)\Big]d\zeta_1\\
&=(\frac{1}{z_2}-\bar z_2)\Big[\frac{1}{2\pi i}\int_{\partial D_1}\Big(\frac{\bar\zeta_1-\bar z_1}{\zeta_1-z_1}-\frac{\bar\zeta_1-\bar z_1}{\zeta_1-\frac{1}{\bar z_1}}\Big)\varphi(\zeta_1,z_2)d\zeta_1\\
&-\frac{1}{2\pi i}\int_{\partial D_1}\Big(\frac{\bar\zeta_1-\bar z_1}{\zeta_1-z_1}-\frac{\bar\zeta_1-\bar z_1}{\zeta_1-\frac{1}{\bar z_1}}\Big)\varphi(\zeta_1,0)d\zeta_1\Big]\\
&=(\frac{1}{z_2}-\bar z_2)\Big\{(\frac{1}{z_1}-\bar z_1)[\varphi(z_1,z_2)-\varphi(0,z_2)]-(\frac{1}{z_1}-\bar z_1)[\varphi(z_1,0)-\varphi(0,0)]\Big\}\\
&=(\frac{1}{z_1}-\bar z_1)(\frac{1}{z_2}-\bar z_2)[\varphi(z_1,z_2)-\varphi(0,z_2)-\varphi(z_1,0)+\varphi(0,0)],
\end{aligned}
\tag{3.15}
$$

and

$$
\begin{aligned}
B=\frac{1}{(2\pi i)^2}\int_{\partial_0 D}\overline{\varphi(\zeta)}\Big[&\Big(\frac{d\zeta_1}{\zeta_1-z_1}+\frac{d\bar\zeta_1}{\overline{\zeta_1-z_1}}\Big)\Big(\frac{d\zeta_2}{\zeta_2-z_2}+\frac{d\bar\zeta_2}{\overline{\zeta_2-z_2}}\Big)-\Big(\frac{d\zeta_1}{\zeta_1-\frac{1}{\bar z_1}}+\frac{d\bar\zeta_1}{\bar\zeta_1-\frac{1}{z_1}}\Big)\\
&\cdot\Big(\frac{d\zeta_2}{\zeta_2-z_2}+\frac{d\bar\zeta_2}{\overline{\zeta_2-z_2}}\Big)-\Big(\frac{d\zeta_1}{\zeta_1-z_1}+\frac{d\bar\zeta_1}{\overline{\zeta_1-z_1}}\Big)\Big(\frac{d\zeta_2}{\zeta_2-\frac{1}{\bar z_2}}+\frac{d\bar\zeta_2}{\bar\zeta_2-\frac{1}{z_2}}\Big)\\
&+\Big(\frac{d\zeta_1}{\zeta_1-\frac{1}{\bar z_1}}+\frac{d\bar\zeta_1}{\bar\zeta_1-\frac{1}{z_1}}\Big)\Big(\frac{d\zeta_2}{\zeta_2-\frac{1}{\bar z_2}}+\frac{d\bar\zeta_2}{\bar\zeta_2-\frac{1}{z_2}}\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{(2\pi i)^2}\int_{\partial_0 D}\overline{\widetilde{\varphi}(\zeta)}\Big(\frac{d\zeta_1}{\zeta_1 - z_1} + \frac{d\bar{\zeta}_1}{\overline{\zeta_1 - z_1}}\Big)\Big[\Big(\frac{d\zeta_2}{\zeta_2 - z_2} + \frac{d\bar{\zeta}_2}{\overline{\zeta_2 - z_2}}\Big) - \Big(\frac{d\zeta_2}{\zeta_2 - \frac{1}{\bar{z}_2}} + \frac{d\bar{\zeta}_2}{\bar{\zeta}_2 - \frac{1}{z_2}}\Big)\Big] \\
&\quad - \frac{1}{(2\pi i)^2}\int_{\partial_0 D}\overline{\widetilde{\varphi}(\zeta)}\Big(\frac{d\zeta_1}{\zeta_1 - \frac{1}{\bar{z}_1}} + \frac{d\bar{\zeta}_1}{\bar{\zeta}_1 - \frac{1}{z_1}}\Big)\Big[\Big(\frac{d\zeta_2}{\zeta_2 - z_2} + \frac{d\bar{\zeta}_2}{\overline{\zeta_2 - z_2}}\Big) - \Big(\frac{d\zeta_2}{\zeta_2 - \frac{1}{\bar{z}_2}} + \frac{d\bar{\zeta}_2}{\bar{\zeta}_2 - \frac{1}{z_2}}\Big)\Big] \\
&= \frac{1}{2\pi i}\int_{\partial D_1}\Big(\frac{d\zeta_1}{\zeta_1 - z_1} + \frac{d\bar{\zeta}_1}{\overline{\zeta_1 - z_1}}\Big) \\
&\quad \cdot\Big\{\frac{1}{2\pi i}\int_{\partial D_2}\overline{\widetilde{\varphi}(\zeta)}\Big[\Big(\frac{d\zeta_2}{\zeta_2 - z_2} + \frac{d\bar{\zeta}_2}{\overline{\zeta_2 - z_2}}\Big) - \Big(\frac{d\zeta_2}{\zeta_2 - \frac{1}{\bar{z}_2}} + \frac{d\bar{\zeta}_2}{\bar{\zeta}_2 - \frac{1}{z_2}}\Big)\Big]\Big\} \\
&\quad - \frac{1}{2\pi i}\int_{\partial D_1}\Big(\frac{d\zeta_1}{\zeta_1 - \frac{1}{\bar{z}_1}} + \frac{d\bar{\zeta}_1}{\bar{\zeta}_1 - \frac{1}{z_1}}\Big) \\
&\quad \cdot\Big\{\frac{1}{2\pi i}\int_{\partial D_2}\overline{\widetilde{\varphi}(\zeta)}\Big[\Big(\frac{d\zeta_2}{\zeta_2 - z_2} + \frac{d\bar{\zeta}_2}{\overline{\zeta_2 - z_2}}\Big) - \Big(\frac{d\zeta_2}{\zeta_2 - \frac{1}{\bar{z}_2}} + \frac{d\bar{\zeta}_2}{\bar{\zeta}_2 - \frac{1}{z_2}}\Big)\Big]\Big\} \\
&= 0,
\end{aligned}
\tag{3.16}
$$

and the last equal sign in Eq (3.16) is due to

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{\partial D_2}\overline{\widetilde{\varphi}(\zeta)}\Big[\Big(\frac{d\zeta_2}{\zeta_2 - z_2} + \frac{d\bar{\zeta}_2}{\overline{\zeta_2 - z_2}}\Big) - \Big(\frac{d\zeta_2}{\zeta_2 - \frac{1}{\bar{z}_2}} + \frac{d\bar{\zeta}_2}{\bar{\zeta}_2 - \frac{1}{z_2}}\Big)\Big] \\
&= \overline{\frac{-1}{2\pi i}\int_{\partial D_2}\widetilde{\varphi}(\zeta)\frac{d\bar{\zeta}_2}{\overline{\zeta_2 - z_2}}} + \overline{\frac{-1}{2\pi i}\int_{\partial D_2}\widetilde{\varphi}(\zeta)\frac{d\zeta_2}{\zeta_2 - z_2}} \\
&\quad - \overline{\frac{-1}{2\pi i}\int_{\partial D_2}\widetilde{\varphi}(\zeta)\frac{d\bar{\zeta}_2}{\bar{\zeta}_2 - \frac{1}{z_2}}} - \overline{\frac{-1}{2\pi i}\int_{\partial D_2}\widetilde{\varphi}(\zeta)\frac{d\zeta_2}{\zeta_2 - \frac{1}{\bar{z}_2}}} \\
&= \overline{\frac{1}{2\pi i}\int_{\partial D_2}\frac{\widetilde{\varphi}(\zeta)d\zeta_2}{(1 - \bar{z}_2\zeta_2)\zeta_2}} - \overline{\widetilde{\varphi}(\zeta_1, z_2)} - \overline{\frac{1}{2\pi i}\int_{\partial D_2}\frac{z_2\widetilde{\varphi}(\zeta)d\zeta_2}{(z_2 - \zeta_2)\zeta_2}} - 0 \\
&= \overline{\widetilde{\varphi}(\zeta_1, 0)} - \overline{\widetilde{\varphi}(\zeta_1, z_2)} - [\overline{\widetilde{\varphi}(\zeta_1, 0)} - \overline{\widetilde{\varphi}(\zeta_1, z_2)}] = 0.
\end{aligned}
$$

Plugging (3.15) and (3.16) into (3.14), we obtain that

$$
\begin{aligned}
\Phi(z) &= \frac{1}{(2\pi i)^2}\int_{\partial_0 D}\Big(\frac{\zeta_1 + z_1}{\zeta_1 - z_1} + \frac{\bar{\zeta}_1 + \bar{z}_1}{\bar{\zeta}_1 - \bar{z}_1}\Big)\Big(\frac{\zeta_2 + z_2}{\zeta_2 - z_2} + \frac{\bar{\zeta}_2 + \bar{z}_2}{\bar{\zeta}_2 - \bar{z}_2}\Big)\frac{h(\zeta)}{4\zeta_1\zeta_2}d\zeta \\
&\quad + \frac{1}{2\pi i}\int_{\partial D_1}\Big(\frac{1}{\zeta_1 - z_1} - \frac{1}{\zeta_1 - \frac{1}{\bar{z}_1}}\Big)\Big[\frac{1}{\pi}\int_{D_2}\Big(\frac{1}{\zeta_2 - \frac{1}{\bar{z}_2}} - \frac{1}{\zeta_2 - z_2}\Big)\partial_{\bar{\zeta}_2}\Phi(\zeta)d\sigma_{\zeta_2}\Big]d\zeta_1 \\
&\quad + \frac{1}{2\pi i}\int_{\partial D_2}\Big(\frac{1}{\zeta_2 - z_2} - \frac{1}{\zeta_2 - \frac{1}{\bar{z}_2}}\Big)\Big[\frac{1}{\pi}\int_{D_1}\Big(\frac{1}{\zeta_1 - \frac{1}{\bar{z}_1}} - \frac{1}{\zeta_1 - z_1}\Big)\partial_{\bar{\zeta}_1}\Phi(\zeta)d\sigma_{\zeta_1}\Big]d\zeta_2 \\
&\quad + \frac{\lambda - 1}{4\lambda}\Big(\frac{1}{z_1} - \bar{z}_1\Big)\Big(\frac{1}{z_2} - \bar{z}_2\Big)[\varphi(z_1, z_2) - \varphi(0, z_2) - \varphi(z_1, 0) + \varphi(0, 0)].
\end{aligned}
\tag{3.17}
$$

On the other hand, as φ is analytic, so we have that

$$
\begin{aligned}
&\frac{1}{(2\pi i)^2}\int_{\partial_0 D}\frac{(\bar{\zeta}_1-\bar{z}_1)(\bar{\zeta}_2-\bar{z}_2)\varphi(\zeta)d\zeta_1 d\zeta_2}{(\zeta_1-\frac{1}{\bar{z}_1})(\zeta_2-z_2)}\\
={}&\frac{1}{2\pi i}\int_{\partial D_2}\Big[\frac{1}{2\pi i}\int_{\partial D_1}\frac{\bar{\zeta}_1-\bar{z}_1}{\zeta_1-\frac{1}{\bar{z}_1}}\varphi(\zeta)d\zeta_1\Big]\frac{\bar{\zeta}_2-\bar{z}_2}{\zeta_2-z_2}d\zeta_2\\
={}&\frac{1}{2\pi i}\int_{\partial D_2}\Big[\frac{1}{2\pi i}\int_{\partial D_1}\frac{\varphi(\zeta)d\zeta_1}{\zeta_1(\zeta_1-\frac{1}{\bar{z}_1})}\Big]\frac{\bar{\zeta}_2-\bar{z}_2}{\zeta_2-z_2}d\zeta_2\\
={}&\bar{z}_1\frac{-1}{2\pi i}\int_{\partial D_2}\varphi(0,\zeta_2)\frac{\bar{\zeta}_2-\bar{z}_2}{\zeta_2-z_2}d\zeta_2=\bar{z}_1\Big[\frac{1}{2\pi i}\int_{\partial D_2}\frac{\bar{z}_2\varphi(0,\zeta_2)}{\zeta_2-z_2}d\zeta_2-\frac{1}{2\pi i}\int_{\partial D_2}\frac{\varphi(0,\zeta_2)}{\zeta_2(\zeta_2-z_2)}d\zeta_2\Big]\\
={}&\begin{cases}\bar{z}_1[\bar{z}_2\varphi(0,z_2)+\frac{\varphi(0,0)}{z_2}-\frac{\varphi(0,z_2)}{z_2}],\ z_2\neq 0,\\ -\bar{z}_1\partial_{z_2}\varphi(0,0),\ z_2=0,\end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{(2\pi i)^2}\int_{\partial_0 D}\overline{\widetilde{\varphi}(\zeta)}\Big(\frac{d\zeta_1}{\zeta_1-\frac{1}{\bar{z}_1}}+\frac{d\bar{\zeta}_1}{\bar{\zeta}_1-\frac{1}{z_1}}\Big)\Big(\frac{d\zeta_2}{\zeta_2-z_2}+\frac{d\bar{\zeta}_2}{\overline{\zeta_2-z_2}}\Big)\\
={}&\frac{1}{(2\pi i)^2}\int_{\partial_0 D}\frac{\overline{\widetilde{\varphi}(\zeta)}d\zeta_1 d\zeta_2}{(\zeta_1-\frac{1}{\bar{z}_1})(\zeta_2-z_2)}+\frac{1}{(2\pi i)^2}\int_{\partial_0 D}\frac{\overline{\widetilde{\varphi}(\zeta)}d\zeta_1 d\bar{\zeta}_2}{(\zeta_1-\frac{1}{\bar{z}_1})(\overline{\zeta_2-z_2})}\\
&+\frac{1}{(2\pi i)^2}\int_{\partial_0 D}\frac{\overline{\widetilde{\varphi}(\zeta)}d\bar{\zeta}_1 d\zeta_2}{(\bar{\zeta}_1-\frac{1}{z_1})(\zeta_2-z_2)}+\frac{1}{(2\pi i)^2}\int_{\partial_0 D}\frac{\overline{\widetilde{\varphi}(\zeta)}d\bar{\zeta}_1 d\bar{\zeta}_2}{(\bar{\zeta}_1-\frac{1}{z_1})(\overline{\zeta_2-z_2})}\\
={}&\frac{1}{2\pi i}\int_{\partial D_2}\overline{\Big[\frac{-1}{2\pi i}\int_{\partial D_1}\frac{\widetilde{\varphi}(\zeta)d\bar{\zeta}_1}{\bar{\zeta}_1-\frac{1}{z_1}}\Big]}\frac{d\zeta_2}{\zeta_2-z_2}+\frac{1}{2\pi i}\int_{\partial D_2}\overline{\Big[\frac{-1}{2\pi i}\int_{\partial D_1}\frac{\widetilde{\varphi}(\zeta)d\bar{\zeta}_1}{\bar{\zeta}_1-\frac{1}{z_1}}\Big]}\frac{d\bar{\zeta}_2}{\overline{\overline{\zeta_2-z_2}}}\\
&+\frac{1}{2\pi i}\int_{\partial D_2}\overline{\Big[\frac{-1}{2\pi i}\int_{\partial D_1}\frac{\widetilde{\varphi}(\zeta)d\zeta_1}{\zeta_1-\frac{1}{\bar{z}_1}}\Big]}\frac{d\zeta_2}{\zeta_2-z_2}+\frac{1}{2\pi i}\int_{\partial D_2}\overline{\Big[\frac{-1}{2\pi i}\int_{\partial D_1}\frac{\widetilde{\varphi}(\zeta)d\zeta_1}{\zeta_1-\frac{1}{\bar{z}_1}}\Big]}\frac{d\bar{\zeta}_2}{\overline{\overline{\zeta_2-z_2}}}\\
={}&\frac{1}{2\pi i}\int_{\partial D_2}\Big[\overline{\widetilde{\varphi}(0,\zeta_2)}-\overline{\widetilde{\varphi}(z_1,\zeta_2)}\Big]\frac{d\zeta_2}{\zeta_2-z_2}+\frac{1}{2\pi i}\int_{\partial D_2}\Big[\overline{\widetilde{\varphi}(0,\zeta_2)}-\overline{\widetilde{\varphi}(z_1,\zeta_2)}\Big]\frac{d\bar{\zeta}_2}{\overline{\overline{\zeta_2-z_2}}}\\
={}&\overline{\frac{1}{2\pi i}\int_{\partial D_2}\Big[\widetilde{\varphi}(0,\zeta_2)-\widetilde{\varphi}(z_1,\zeta_2)\Big]\frac{d\zeta_2}{\zeta_2^2(\bar{\zeta}_2-\bar{z}_2)}}+\overline{\frac{-1}{2\pi i}\int_{\partial D_2}\Big[\widetilde{\varphi}(0,\zeta_2)-\widetilde{\varphi}(z_1,\zeta_2)\Big]\frac{d\zeta_2}{\zeta_2-z_2}}\\
={}&\overline{\widetilde{\varphi}(0,0)}-\overline{\widetilde{\varphi}(z_1,0)}+\overline{\widetilde{\varphi}(z)}-\overline{\widetilde{\varphi}(0,z_2)}.
\end{aligned}
$$

Therefore, we use Eq (3.11) to obtain that

$$
\begin{aligned}
0={}&\overline{\frac{1}{(2\pi i)^2}\int_{\partial_0 D}\frac{h(\zeta)d\zeta_1 d\zeta_2}{(\zeta_1-\frac{1}{\bar{z}_1})(\zeta_2-z_2)}}\\
&-\overline{\frac{1}{2\pi i}\int_{\partial D_1}\Big[\frac{1}{\pi}\int_{D_2}\frac{\partial_{\bar{\zeta}_2}\Phi(\zeta)}{\zeta_2-z_2}d\sigma_{\zeta_2}\Big]\frac{d\zeta_1}{\zeta_1-\frac{1}{\bar{z}_1}}}-\overline{\frac{1}{2\pi i}\int_{\partial D_2}\Big[\frac{1}{\pi}\int_{D_1}\frac{\partial_{\bar{\zeta}_1}\Phi(\zeta)}{\zeta_1-\frac{1}{\bar{z}_1}}d\sigma_{\zeta_1}\Big]\frac{d\zeta_2}{\zeta_2-z_2}}\\
&+\frac{\lambda-1}{4\lambda}\cdot\begin{cases}z_1[z_2\overline{\varphi(0,z_2)}+\frac{\overline{\varphi(0,0)}}{\bar{z}_2}-\frac{\overline{\varphi(0,z_2)}}{\bar{z}_2}],\ z_2\neq 0,\\ -z_1\overline{\partial_{z_2}\varphi(0,0)},\ z_2=0,\end{cases}\\
&+\frac{\lambda+1}{4\lambda}[\widetilde{\varphi}(0,0)-\widetilde{\varphi}(z_1,0)+\widetilde{\varphi}(z)-\widetilde{\varphi}(0,z_2)].
\end{aligned}
$$

Thus,

$$\begin{aligned}
&\widetilde{\varphi}(0,0)-\widetilde{\varphi}(z_1,0)+\widetilde{\varphi}(z)-\widetilde{\varphi}(0,z_2)\\
=&\frac{4\lambda}{\lambda+1}\Big\{-\overline{\frac{1}{(2\pi i)^2}\int_{\partial_0 D}\frac{h(\zeta)d\zeta_1 d\zeta_2}{(\zeta_1-\frac{1}{\bar z_1})(\zeta_2-z_2)}}\\
&+\overline{\frac{1}{2\pi i}\int_{\partial D_1}\Big[\frac{1}{\pi}\int_{D_2}\frac{\partial_{\bar\zeta_2}\Phi(\zeta)}{\zeta_2-z_2}d\sigma_{\zeta_2}\Big]\frac{d\zeta_1}{\zeta_1-\frac{1}{\bar z_1}}}+\overline{\frac{1}{2\pi i}\int_{\partial D_2}\Big[\frac{1}{\pi}\int_{D_1}\frac{\partial_{\bar\zeta_1}\Phi(\zeta)}{\zeta_1-\frac{1}{\bar z_1}}d\sigma_{\zeta_1}\Big]\frac{d\zeta_2}{\zeta_2-z_2}}\Big\}\\
&-\frac{\lambda-1}{\lambda+1}\cdot\begin{cases} z_1[z_2\overline{\varphi(0,z_2)}+\frac{\overline{\varphi(0,0)}}{\bar z_2}-\frac{\overline{\varphi(0,z_2)}}{\bar z_2}],\ z_2\neq 0,\\ -z_1\overline{\partial_{z_2}\varphi(0,0)},\ z_2=0.\end{cases}
\end{aligned}\tag{3.18}$$

As $\widetilde{\varphi}(z)=\int_0^{z_1}\int_0^{z_2}\varphi(\zeta)d\zeta_1 d\zeta_2$, and then $\partial_{z_1}\partial_{z_2}\widetilde{\varphi}(z)=\varphi(z)$. By taking the second-order partial derivative with respect to z_1 and z_2 at the left and right ends of (3.18), we get that

$$\begin{aligned}
\varphi(z)=&\frac{4\lambda}{\lambda+1}\partial_{z_1}\partial_{z_2}\Big\{-\overline{\frac{1}{(2\pi i)^2}\int_{\partial_0 D}\frac{h(\zeta)d\zeta_1 d\zeta_2}{(\zeta_1-\frac{1}{\bar z_1})(\zeta_2-z_2)}}\\
&+\overline{\frac{1}{2\pi i}\int_{\partial D_1}\Big[\frac{1}{\pi}\int_{D_2}\frac{\partial_{\bar\zeta_2}\Phi(\zeta)}{\zeta_2-z_2}d\sigma_{\zeta_2}\Big]\frac{d\zeta_1}{\zeta_1-\frac{1}{\bar z_1}}}+\overline{\frac{1}{2\pi i}\int_{\partial D_2}\Big[\frac{1}{\pi}\int_{D_1}\frac{\partial_{\bar\zeta_1}\Phi(\zeta)}{\zeta_1-\frac{1}{\bar z_1}}d\sigma_{\zeta_1}\Big]\frac{d\zeta_2}{\zeta_2-z_2}}\Big\}\\
&-\frac{\lambda-1}{\lambda+1}\cdot\begin{cases}\overline{\varphi(0,z_2)},\ z_2\neq 0,\\ 0,\ z_2=0\end{cases}\\
=&\frac{4\lambda}{\lambda+1}\partial_{z_1}\partial_{z_2}\Big\{\frac{-1}{(2\pi i)^2}\int_{\partial_0 D}\frac{\overline{h(\zeta)}d\bar\zeta}{(\bar\zeta_1-\frac{1}{z_1})(\bar\zeta_2-\bar z_2)}\\
&-\frac{1}{2\pi i}\int_{\partial D_1}\Big[\frac{1}{\pi}\int_{D_2}\frac{\overline{\partial_{\bar\zeta_2}\Phi(\zeta)}}{\overline{\zeta_2-z_2}}d\sigma_{\zeta_2}\Big]\frac{d\bar\zeta_1}{\bar\zeta_1-\frac{1}{z_1}}-\frac{1}{2\pi i}\int_{\partial D_2}\Big[\frac{1}{\pi}\int_{D_1}\frac{\overline{\partial_{\bar\zeta_1}\Phi(\zeta)}}{\bar\zeta_1-\frac{1}{z_1}}d\sigma_{\zeta_1}\Big]\frac{d\bar\zeta_2}{\overline{\zeta_2-z_2}}\Big\}\\
&-\frac{\lambda-1}{\lambda+1}\cdot\begin{cases}\overline{\varphi(0,z_2)},\ z_2\neq 0,\\ 0,\ z_2=0\end{cases}\\
=&\frac{4\lambda}{\lambda+1}\partial_{z_1}\Big\{\frac{-1}{2\pi i}\int_{\partial D_1}\partial_{z_2}\Big[\frac{1}{\pi}\int_{D_2}\frac{\overline{\partial_{\bar\zeta_2}\Phi(\zeta)}}{\overline{\zeta_2-z_2}}d\sigma_{\zeta_2}\Big]\frac{d\bar\zeta_1}{\bar\zeta_1-\frac{1}{z_1}}\Big\}-\frac{\lambda-1}{\lambda+1}\cdot\begin{cases}\overline{\varphi(0,z_2)},\ z_2\neq 0,\\ 0,\ z_2=0\end{cases}\\
=&\frac{4\lambda}{\lambda+1}\partial_{z_1}\Big[\frac{1}{2\pi i}\int_{\partial D_1}\overline{\partial_{\bar z_2}\Phi(\zeta_1,z_2)}\frac{z_1 d\zeta_1}{\zeta_1(\zeta_1-z_1)}\Big]-\frac{\lambda-1}{\lambda+1}\cdot\begin{cases}\overline{\varphi(0,z_2)},\ z_2\neq 0,\\ 0,\ z_2=0\end{cases}\\
=&\frac{4\lambda}{\lambda+1}\frac{1}{2\pi i}\int_{\partial D_1}\overline{\partial_{\bar z_2}\Phi(\zeta_1,z_2)}\frac{d\zeta_1}{(\zeta_1-z_1)^2}-\frac{\lambda-1}{\lambda+1}\cdot\begin{cases}\overline{\varphi(0,z_2)},\ z_2\neq 0,\\ 0,\ z_2=0,\end{cases}
\end{aligned}\tag{3.19}$$

which follows that

$$\begin{aligned}
\varphi(z_1,0)&=\frac{4\lambda}{\lambda+1}\frac{1}{2\pi i}\int_{\partial D_1}\overline{\partial_{\bar z_2}\Phi(\zeta_1,0)}\frac{d\zeta_1}{(\zeta_1-z_1)^2},\\
\varphi(0,z_2)&=\frac{4\lambda}{\lambda+1}\frac{1}{2\pi i}\int_{\partial D_1}\overline{\partial_{\bar z_2}\Phi(\zeta_1,z_2)}\frac{d\zeta_1}{\zeta_1^2}-\frac{\lambda-1}{\lambda+1}\overline{\varphi(0,z_2)},\\
\varphi(0,0)&=\frac{4\lambda}{\lambda+1}\frac{1}{2\pi i}\int_{\partial D_1}\overline{\partial_{\bar z_2}\Phi(\zeta_1,0)}\frac{d\zeta_1}{\zeta_1^2}.
\end{aligned}$$

Consequently, the above equations yield that

$$
\begin{aligned}
&\varphi(z_1,z_2)-\varphi(0,z_2)-\varphi(z_1,0)+\varphi(0,0)\\
=&\ \frac{4\lambda}{\lambda+1}\Big\{\frac{1}{2\pi i}\int_{\partial D_1}\overline{\partial_{\bar z_2}\Phi(\zeta_1,z_2)}\Big[\frac{d\zeta_1}{(\zeta_1-z_1)^2}-\frac{d\zeta_1}{\zeta_1^2}\Big]\\
&+\frac{1}{2\pi i}\int_{\partial D_1}\overline{\partial_{\bar z_2}\Phi(\zeta_1,0)}\Big[\frac{d\zeta_1}{\zeta_1^2}-\frac{d\zeta_1}{(\zeta_1-z_1)^2}\Big]\Big\}\\
=&\ \frac{4\lambda}{\lambda+1}\frac{1}{2\pi i}\int_{\partial D_1}\Big[\overline{\partial_{\bar z_2}\Phi(\zeta_1,z_2)}-\overline{\partial_{\bar z_2}\Phi(\zeta_1,0)}\Big]\Big[\frac{d\zeta_1}{(\zeta_1-z_1)^2}-\frac{d\zeta_1}{\zeta_1^2}\Big]\\
=&\ \frac{4\lambda}{\lambda+1}\frac{1}{2\pi i}\int_{\partial D_1}\Big[\overline{\varphi_2(\zeta_1,z_2)}-\overline{\varphi_1(\zeta_1,0)}\Big]\Big[\frac{1}{(\zeta_1-z_1)^2}-\frac{1}{\zeta_1^2}\Big]d\zeta_1.
\end{aligned}
\tag{3.20}
$$

As

$$
\partial_{\bar z_1}\Phi(z_1,\zeta_2)=\varphi_1(z_1,\zeta_2),\quad \partial_{\bar z_2}\Phi(\zeta_1,z_2)=\varphi_2(\zeta_1,z_2)\ \ (\zeta_1\in\partial D_1,\zeta_2\in\partial D_2),
$$

plugging (3.20) into (3.17), we obtain that $\Phi(z)$ is given by

$$
\begin{aligned}
\Phi(z)=&\ \frac{1}{(2\pi i)^2}\int_{\partial_0 D}\Big(\frac{\zeta_1+z_1}{\zeta_1-z_1}+\frac{\bar\zeta_1+\bar z_1}{\bar\zeta_1-\bar z_1}\Big)\Big(\frac{\zeta_2+z_2}{\zeta_2-z_2}+\frac{\bar\zeta_2+\bar z_2}{\bar\zeta_2-\bar z_2}\Big)\frac{h(\zeta)}{4\zeta_1\zeta_2}d\zeta\\
&+\frac{1}{2\pi i}\int_{\partial D_1}\Big(\frac{1}{\zeta_1-z_1}-\frac{1}{\zeta_1-\frac{1}{\bar z_1}}\Big)\Big[\frac{1}{\pi}\int_{D_2}\Big(\frac{1}{\zeta_2-\frac{1}{\bar z_2}}-\frac{1}{\zeta_2-z_2}\Big)\varphi_2(\zeta)d\sigma_{\zeta_2}\Big]d\zeta_1\\
&+\frac{1}{2\pi i}\int_{\partial D_2}\Big(\frac{1}{\zeta_2-z_2}-\frac{1}{\zeta_2-\frac{1}{\bar z_2}}\Big)\Big[\frac{1}{\pi}\int_{D_1}\Big(\frac{1}{\zeta_1-\frac{1}{\bar z_1}}-\frac{1}{\zeta_1-z_1}\Big)\varphi_1(\zeta)d\sigma_{\zeta_1}\Big]d\zeta_2\\
&+\frac{\lambda-1}{\lambda+1}\Big(\frac{1}{z_1}-\bar z_1\Big)\Big(\frac{1}{z_2}-\bar z_2\Big)\frac{1}{2\pi i}\int_{\partial D_1}\Big[\overline{\varphi_2(\zeta_1,z_2)}-\overline{\varphi_1(\zeta_1,0)}\Big]\Big[\frac{1}{(\zeta_1-z_1)^2}-\frac{1}{\zeta_1^2}\Big]d\zeta_1\\
=&\ \frac{1}{(2\pi i)^2}\int_{\partial_0 D}\Big(\frac{\zeta_1+z_1}{\zeta_1-z_1}+\frac{\bar\zeta_1+\bar z_1}{\bar\zeta_1-\bar z_1}\Big)\Big(\frac{\zeta_2+z_2}{\zeta_2-z_2}+\frac{\bar\zeta_2+\bar z_2}{\bar\zeta_2-\bar z_2}\Big)\frac{h(\zeta)}{4\zeta_1\zeta_2}d\zeta\\
&+\frac{1}{2\pi i}\int_{\partial D_1}\Big(\frac{\zeta_1+z_1}{\zeta_1-z_1}+\frac{\bar\zeta_1+\bar z_1}{\bar\zeta_1-\bar z_1}\Big)\Big[\frac{1}{\pi}\int_{D_2}\Big(\frac{1}{\zeta_2-\frac{1}{\bar z_2}}-\frac{1}{\zeta_2-z_2}\Big)\varphi_2(\zeta)d\sigma_{\zeta_2}\Big]\frac{d\zeta_1}{2\zeta_1}\\
&+\frac{1}{2\pi i}\int_{\partial D_2}\Big(\frac{\zeta_2+z_2}{\zeta_2-z_2}+\frac{\bar\zeta_2+\bar z_2}{\bar\zeta_2-\bar z_2}\Big)\Big[\frac{1}{\pi}\int_{D_1}\Big(\frac{1}{\zeta_1-\frac{1}{\bar z_1}}-\frac{1}{\zeta_1-z_1}\Big)\varphi_1(\zeta)d\sigma_{\zeta_1}\Big]\frac{d\zeta_2}{2\zeta_2}\\
&+\frac{\lambda-1}{\lambda+1}\Big(\frac{1}{z_1}-\bar z_1\Big)\Big(\frac{1}{z_2}-\bar z_2\Big)\frac{1}{2\pi i}\int_{\partial D_1}\Big[\overline{\varphi_2(\zeta_1,z_2)}-\overline{\varphi_1(\zeta_1,0)}\Big]\Big[\frac{1}{(\zeta_1-z_1)^2}-\frac{1}{\zeta_1^2}\Big]d\zeta_1.
\end{aligned}
\tag{3.21}
$$

On the other hand, if $z\in\partial_0 D$ (that is $|z_1|=|z_2|=1$), as

$$
\Big(\frac{\zeta_1+z_1}{\zeta_1-z_1}+\frac{\bar\zeta_1+\bar z_1}{\bar\zeta_1-\bar z_1}\Big)\Big(\frac{\zeta_2+z_2}{\zeta_2-z_2}+\frac{\bar\zeta_2+\bar z_2}{\bar\zeta_2-\bar z_2}\Big)\frac{1}{4\zeta_1\zeta_2}
$$

is the Poisson kernel on D, and from (3.21), we have that $\Phi(z)=h(z)$ for $z\in\partial_0 D$. If $z_2\in\partial D_2$, as

$$
\Big(\frac{\zeta_2+z_2}{\zeta_2-z_2}+\frac{\bar\zeta_2+\bar z_2}{\bar\zeta_2-\bar z_2}\Big)\frac{1}{2\zeta_2}
$$

is the Poisson kernel on D_2, and from (3.21), we have that

$$\partial_{\bar{z}_1}\Phi(z) = \frac{1}{2\pi i}\int_{\partial D_2}\Big(\frac{\zeta_2+z_2}{\zeta_2-z_2}+\frac{\bar{\zeta}_2+\bar{z}_2}{\bar{\zeta}_2-\bar{z}_2}\Big)\partial_{\bar{z}_1}\Big[\frac{-1}{\pi}\int_{D_1}\frac{1}{\zeta_1-z_1}\varphi_1(\zeta)d\sigma_{\zeta_1}\Big]\frac{d\zeta_2}{2\zeta_2} = \varphi_1(z).$$

Similarly, $\partial_{\bar{z}_2}\Phi(z) = \varphi_2(z)$ for $z_1 \in \partial D_1$.

The above results indicate that (3.2) is the solution to (3.1). □

If we denote that

$$D\Phi(z) = (1-\lambda)\partial^2_{\bar{z}_1}\partial^2_{\bar{z}_2}\Phi(z) + (1+\lambda)\partial_{z_1}\partial_{\bar{z}_1}\partial_{z_2}\partial_{\bar{z}_2}\overline{\Phi(z)},$$

then $D\Phi(z) = 0$ implies that

$$\partial_{\bar{z}_1}\partial_{\bar{z}_2}[(1-\lambda)\partial_{\bar{z}_1}\partial_{\bar{z}_2}\Phi(z) + (1+\lambda)\partial_{z_1}\partial_{z_2}\overline{\Phi(z)}] = 0.$$

Denote

$$(1-\lambda)\partial_{\bar{z}_1}\partial_{\bar{z}_2}\Phi(z) + (1+\lambda)\partial_{z_1}\partial_{z_2}\overline{\Phi(z)} \doteq \varphi(z),$$

then $\partial_{\bar{z}_1}\partial_{\bar{z}_2}\varphi(z) = 0$ and

$$\begin{aligned}\frac{\lambda-1}{4\lambda}\varphi(z) + \frac{\lambda+1}{4\lambda}\overline{\varphi(z)} &= \frac{\lambda-1}{4\lambda}[(1-\lambda)\partial_{\bar{z}_1}\partial_{\bar{z}_2}\Phi(z) + (1+\lambda)\partial_{z_1}\partial_{z_2}\overline{\Phi(z)}]\\ &\quad + \frac{\lambda+1}{4\lambda}[(1+\lambda)\partial_{\bar{z}_1}\partial_{\bar{z}_2}\Phi(z) + (1-\lambda)\partial_{z_1}\partial_{z_2}\overline{\Phi(z)}]\\ &= \partial_{\bar{z}_1}\partial_{\bar{z}_2}\Phi(z).\end{aligned}$$

On the other hand, provided that $\Phi(z)$ satisfies

$$\partial_{\bar{z}_1}\partial_{\bar{z}_2}\Phi(z) = \frac{\lambda-1}{4\lambda}\varphi(z) + \frac{\lambda+1}{4\lambda}\overline{\varphi(z)}, \quad \partial_{\bar{z}_1}\partial_{\bar{z}_2}\varphi(z) = 0,$$

then

$$\begin{aligned}D\Phi(z) &= (1-\lambda)\partial^2_{\bar{z}_1}\partial^2_{\bar{z}_2}\Phi(z) + (1+\lambda)\partial_{z_1}\partial_{\bar{z}_1}\partial_{z_2}\partial_{\bar{z}_2}\overline{\Phi(z)}\\ &= (1-\lambda)\partial_{\bar{z}_1}\partial_{\bar{z}_2}\Big[\frac{\lambda-1}{4\lambda}\varphi(z) + \frac{\lambda+1}{4\lambda}\overline{\varphi(z)}\Big]\\ &\quad + (1+\lambda)\partial_{z_1}\partial_{z_2}\overline{\Big[\frac{\lambda-1}{4\lambda}\varphi(z) + \frac{\lambda+1}{4\lambda}\overline{\varphi(z)}\Big]}\\ &= (1-\lambda)\frac{\lambda+1}{4\lambda}\overline{\partial_{z_1}\partial_{z_2}\varphi(z)} + (1+\lambda)\frac{\lambda-1}{4\lambda}\overline{\partial_{z_1}\partial_{z_2}\varphi(z)}\\ &= 0.\end{aligned}$$

The above results mean that

$$D\Phi(z) = (1-\lambda)\partial^2_{\bar{z}_1}\partial^2_{\bar{z}_2}\Phi(z) + (1+\lambda)\partial_{z_1}\partial_{\bar{z}_1}\partial_{z_2}\partial_{\bar{z}_2}\overline{\Phi(z)} = 0$$

can be decomposed into

$$\partial_{\bar{z}_1}\partial_{\bar{z}_2}\Phi(z) = \frac{\lambda-1}{4\lambda}\varphi(z) + \frac{\lambda+1}{4\lambda}\overline{\varphi(z)}, \quad \partial_{\bar{z}_1}\partial_{\bar{z}_2}\varphi(z) = 0,$$

where $\varphi(z)$ is called the associate function of $\Phi(z)$. Therefore, from Theorem 3.1, we can get the following conclusion:

Theorem 3.2. *Let $h \in C(\partial_0 D)$, $\varphi_1 \in C(D_1 \times \partial D_2)$ and $\varphi_2 \in C(\partial D_1 \times D_2)$ are given functions, and let $\lambda \in \mathbb{R} \setminus \{-1, 0, 1\}$. Then, (3.2) is the solution to the problem*

$$\begin{cases} D\Phi(z) = 0 (z \in D), \quad \Phi(\zeta) = h(\zeta) \ (\zeta \in \partial_0 D), \\ \partial_{\bar{z}_1}\Phi(z_1, \zeta_2) = \varphi_1(z_1, \zeta_2), \ \ \partial_{\bar{z}_2}\Phi(\zeta_1, z_2) = \varphi_2(\zeta_1, z_2) \ (\zeta_1 \in \partial D_1, \ \zeta_2 \in \partial D_2). \end{cases}$$

Corollary 3.3. *Let $\lambda \in \mathbb{R} \setminus \{-1, 0, 1\}$. Then, $\Phi(z) = 0$ is the unique solution to the problem*

$$\begin{cases} D\Phi(z) = 0 (z \in D), \quad \Phi(\zeta) = 0 \ (\zeta \in \partial_0 D), \\ \partial_{\bar{z}_1}\Phi(z_1, \zeta_2) = 0, \ \ \partial_{\bar{z}_2}\Phi(\zeta_1, z_2) = 0 \ (\zeta_1 \in \partial D_1, \ \zeta_2 \in \partial D_2). \end{cases}$$

Theorem 3.4. *Let $\theta \in L^p(D)$ $(p > 2)$, $h \in C(\partial_0 D)$, $\widetilde{\varphi_1} \in C(D_1 \times \partial D_2)$, and $\widetilde{\varphi_2} \in C(\partial D_1 \times D_2)$ be given functions, and let $\lambda \in \mathbb{R} \setminus \{-1, 0, 1\}$. Then, $w(z) = \Gamma\theta(z) - \Xi\theta(z) + \Phi(z)$ is the solution to*

$$\begin{cases} Dw(z) = \theta(z)(z \in D), \quad w(\zeta) = h(\zeta) \ (\zeta \in \partial_0 D), \\ \partial_{\bar{z}_1} w(z_1, \zeta_2) = \widetilde{\varphi_1}(z_1, \zeta_2), \ \ \partial_{\bar{z}_2} w(\zeta_1, z_2) = \widetilde{\varphi_2}(\zeta_1, z_2) \ (\zeta_1 \in \partial D_1, \zeta_2 \in \partial D_2), \end{cases} \tag{3.22}$$

where

$$Dw(z) = (1-\lambda)\partial^2_{\bar{z}_1}\partial^2_{\bar{z}_2} w(z) + (1+\lambda)\partial_{z_1}\partial_{\bar{z}_1}\partial_{z_2}\partial_{\bar{z}_2}\overline{w(z)}, \tag{3.23}$$

$$\begin{aligned} \Gamma\theta(z) = {} & \frac{\lambda-1}{4\lambda}\frac{1}{\pi^2}\int_D \theta(\zeta)\frac{\overline{\zeta_1 - z_1}}{\zeta_1 - z_1}\frac{\overline{\zeta_2 - z_2}}{\zeta_2 - z_2} d\sigma_{\zeta_1} d\sigma_{\zeta_2} \\ & + \frac{\lambda+1}{4\lambda}\frac{1}{\pi^2}\int_D \overline{\theta(\zeta)} \ln|\zeta_1 - z_1|^2 \ln|\zeta_2 - z_2|^2 d\sigma_{\zeta_1} d\sigma_{\zeta_2}, \end{aligned} \tag{3.24}$$

$$\begin{aligned} \Xi\theta(z) = {} & \frac{\lambda-1}{4\lambda}\frac{1}{\pi^2}\int_D \theta(\zeta)\frac{\overline{\zeta_1 - z_1}}{\zeta_1 - \frac{1}{\bar{z}_1}}\frac{\overline{\zeta_2 - z_2}}{\zeta_2 - \frac{1}{\bar{z}_2}} d\sigma_{\zeta_1} d\sigma_{\zeta_2} \\ & + \frac{\lambda+1}{4\lambda}\frac{1}{\pi^2}\int_D \overline{\theta(\zeta)} \ln|1 - z_1\bar{\zeta}_1|^2 \ln|1 - z_2\bar{\zeta}_2|^2 d\sigma_{\zeta_1} d\sigma_{\zeta_2} \\ & + \frac{\lambda-1}{\lambda+1}(1-|z_1|^2)(1-|z_2|^2)\Big\{\frac{\lambda-1}{4\lambda}\frac{1}{\pi^2}\int_D \overline{\theta(\zeta)}\Big[\frac{1-\zeta_1\bar{\zeta}_1}{1-z_1\bar{\zeta}_1} + \frac{1-\zeta_1\bar{\zeta}_1}{(1-z_1\bar{\zeta}_1)^2}\Big] \\ & \cdot\Big[\frac{1-\zeta_2\bar{\zeta}_2}{1-z_2\bar{\zeta}_2} + \frac{1-\zeta_2\bar{\zeta}_2}{(1-z_2\bar{\zeta}_2)^2}\Big] d\sigma_{\zeta_1} d\sigma_{\zeta_2} + \frac{\lambda+1}{4\lambda\pi^2}\int_D \theta(\zeta)\frac{\bar{\zeta}_1^{\,2}}{1-z_1\bar{\zeta}_1}\frac{\bar{\zeta}_2^{\,2}}{1-z_2\bar{\zeta}_2} d\sigma_{\zeta_1} d\sigma_{\zeta_2}\Big\}, \end{aligned} \tag{3.25}$$

and $\Phi(z)$ expressed by (3.2) (in which φ_1 and φ_2 are replaced by $\widetilde{\varphi_1} - \partial_{\bar{z}_1}\Gamma\theta + \partial_{\bar{z}_1}\Xi\theta$ and $\widetilde{\varphi_2} - \partial_{\bar{z}_2}\Gamma\theta + \partial_{\bar{z}_2}\Xi\theta$, respectively) is the solution of the problem

$$\begin{cases} D\Phi(z) = 0 (z \in D), \quad \Phi(\zeta) = h(\zeta) \ (\zeta \in \partial_0 D), \\ \partial_{\bar{z}_1}\Phi(z_1, \zeta_2) = [\widetilde{\varphi_1} - \partial_{\bar{z}_1}\Gamma\theta + \partial_{\bar{z}_1}\Xi\theta](z_1, \zeta_2) \ (\zeta_2 \in \partial D_2), \\ \partial_{\bar{z}_2}\Phi(\zeta_1, z_2) = [\widetilde{\varphi_2} - \partial_{\bar{z}_2}\Gamma\theta + \partial_{\bar{z}_2}\Xi\theta](\zeta_1, z_2) \ (\zeta_1 \in \partial D_1). \end{cases}$$

Proof. (1) It is obvious that $\Xi\theta(\zeta) = \Gamma\theta(\zeta)$ for $\zeta \in \partial_0 D$. Thus, $\Gamma\theta(\zeta) - \Xi\theta(\zeta) + \Phi(\zeta) = h(\zeta)$. In addition,

$$\partial_{\bar{z}_1}[\Gamma\theta - \Xi\theta + \Phi](z_1, \zeta_2) = \partial_{\bar{z}_1}[\Gamma\theta - \Xi\theta](z_1, \zeta_2) + [\widetilde{\varphi_1} - \partial_{\bar{z}_1}\Gamma\theta + \partial_{\bar{z}_1}\Xi\theta](z_1, \zeta_2) = \widetilde{\varphi_1}(z_1, \zeta_2).$$

Similarly, we have that

$$\partial_{\bar{z}_2}[\Gamma\theta - \Xi\theta + \Phi](\zeta_1, z_2) = \widetilde{\varphi_2}(\zeta_1, z_2).$$

The above results indicate that $\Gamma\theta - \Xi\theta + \Phi$ satisfies the boundary condition of (3.22).

(2) In the following, we prove $D[\Gamma\theta - \Xi\theta + \Phi] = \theta$.

First, (3.24) leads to

$$\partial_{\bar{z}_1}\partial_{\bar{z}_2}\Gamma\theta = \frac{\lambda-1}{4\lambda}\frac{1}{\pi^2}\int_D \frac{\theta(\zeta)d\sigma_{\zeta_1}d\sigma_{\zeta_2}}{(\zeta_1 - z_1)(\zeta_2 - z_2)} + \frac{\lambda+1}{4\lambda}\frac{1}{\pi^2}\int_D \frac{\overline{\theta(\zeta)}d\sigma_{\zeta_1}d\sigma_{\zeta_2}}{\overline{(\zeta_1 - z_1)}\overline{(\zeta_2 - z_2)}} \doteq \widetilde{\Gamma}\theta.$$

Combining this with (3.23) and applying Lemma 2.3, we conclude that

$$\begin{aligned} D\Gamma\theta &= (1-\lambda)\partial^2_{\bar{z}_1}\partial^2_{\bar{z}_2}\Gamma\theta(z) + (1+\lambda)\partial_{z_1}\partial_{\bar{z}_1}\partial_{z_2}\partial_{\bar{z}_2}\overline{\Gamma\theta(z)} \\ &= (1-\lambda)\partial_{\bar{z}_1}\partial_{\bar{z}_2}\widetilde{\Gamma}\theta(z) + (1+\lambda)\overline{\partial_{z_1}\partial_{z_2}\widetilde{\Gamma}\theta(z)} \\ &= (1-\lambda)\Big[\frac{\lambda-1}{4\lambda}\theta(z) + \frac{\lambda+1}{4\lambda}\overline{\frac{1}{\pi^2}\int_D \frac{\theta(\zeta)d\sigma_{\zeta_1}d\sigma_{\zeta_2}}{(\zeta_1 - z_1)^2(\zeta_2 - z_2)^2}}\Big] \\ &\quad + (1+\lambda)\Big[\frac{\lambda-1}{4\lambda}\overline{\frac{1}{\pi^2}\int_D \frac{\theta(\zeta)d\sigma_{\zeta_1}d\sigma_{\zeta_2}}{(\zeta_1 - z_1)^2(\zeta_2 - z_2)^2}} + \frac{\lambda+1}{4\lambda}\theta(z)\Big] \\ &= \theta(z). \end{aligned} \tag{3.26}$$

Second, (3.25) leads to

$$\begin{aligned} \partial^2_{\bar{z}_1}\partial^2_{\bar{z}_2}\Xi\theta &= \frac{\lambda-1}{4\lambda}\frac{1}{\pi^2}\int_D \theta(\zeta)\partial^2_{\bar{z}_1}\frac{\overline{\zeta_1 - z_1}}{\zeta_1 - \frac{1}{\bar{z}_1}}\partial^2_{\bar{z}_2}\frac{\overline{\zeta_2 - z_2}}{\zeta_2 - \frac{1}{\bar{z}_2}}d\sigma_{\zeta_1}d\sigma_{\zeta_2} \\ &\quad + \frac{\lambda+1}{4\lambda}\frac{1}{\pi^2}\int_D \overline{\theta(\zeta)}\partial^2_{\bar{z}_1}\ln|1 - z_1\bar{\zeta}_1|^2\partial^2_{\bar{z}_2}\ln|1 - z_2\bar{\zeta}_2|^2 d\sigma_{\zeta_1}d\sigma_{\zeta_2} \\ &= \frac{\lambda-1}{4\lambda}\frac{1}{\pi^2}\int_D \theta(\zeta)\frac{2(1-\zeta_1\bar{\zeta}_1)}{(1-\bar{z}_1\zeta_1)^3}\frac{2(1-\zeta_2\bar{\zeta}_2)}{(1-\bar{z}_2\zeta_2)^3}d\sigma_{\zeta_1}d\sigma_{\zeta_2} \\ &\quad + \frac{\lambda+1}{4\lambda}\frac{1}{\pi^2}\int_D \overline{\theta(\zeta)}\frac{-\bar{\zeta}_1^2}{(1-\bar{z}_1\zeta_1)^2}\frac{-\bar{\zeta}_2^2}{(1-\bar{z}_2\zeta_2)^2}d\sigma_{\zeta_1}d\sigma_{\zeta_2}. \end{aligned} \tag{3.27}$$

Denote

$$\begin{aligned} E &= \frac{\lambda-1}{4\lambda}\frac{1}{\pi^2}\int_D \overline{\theta(\zeta)}\Big[\frac{1-\zeta_1\bar{\zeta}_1}{1-z_1\bar{\zeta}_1} + \frac{1-\zeta_1\bar{\zeta}_1}{(1-z_1\bar{\zeta}_1)^2}\Big]\Big[\frac{1-\zeta_2\bar{\zeta}_2}{1-z_2\bar{\zeta}_2} \\ &\quad + \frac{1-\zeta_2\bar{\zeta}_2}{(1-z_2\bar{\zeta}_2)^2}\Big]d\sigma_{\zeta_1}d\sigma_{\zeta_2} + \frac{\lambda+1}{4\lambda}\frac{1}{\pi^2}\int_D \theta(\zeta)\frac{\bar{\zeta}_1{}^2}{1-z_1\bar{\zeta}_1}\frac{\bar{\zeta}_2{}^2}{1-z_2\bar{\zeta}_2}d\sigma_{\zeta_1}d\sigma_{\zeta_2}, \end{aligned} \tag{3.28}$$

which follows that

$$\begin{aligned} \partial_{z_1}E &= \frac{\lambda-1}{4\lambda}\frac{1}{\pi^2}\int_D \overline{\theta(\zeta)}\Big[\frac{\bar{\zeta}_1(1-\zeta_1\bar{\zeta}_1)}{(1-z_1\bar{\zeta}_1)^2}\frac{1-\zeta_2\bar{\zeta}_2}{1-z_2\bar{\zeta}_2} + \frac{2\bar{\zeta}_1(1-\zeta_1\bar{\zeta}_1)}{(1-z_1\bar{\zeta}_1)^3}\frac{1-\zeta_2\bar{\zeta}_2}{(1-z_2\bar{\zeta}_2)^2} \\ &\quad + \frac{\bar{\zeta}_1(1-\zeta_1\bar{\zeta}_1)}{(1-z_1\bar{\zeta}_1)^2}\frac{1-\zeta_2\bar{\zeta}_2}{(1-z_2\bar{\zeta}_2)^2} + \frac{1-\zeta_2\bar{\zeta}_2}{1-z_2\bar{\zeta}_2}\frac{2\bar{\zeta}_1(1-\zeta_1\bar{\zeta}_1)}{(1-z_1\bar{\zeta}_1)^3}\Big]d\sigma_{\zeta_1}d\sigma_{\zeta_2} \\ &\quad + \frac{\lambda+1}{4\lambda}\frac{1}{\pi^2}\int_D \theta(\zeta)\frac{\bar{\zeta}_1{}^3}{(1-z_1\bar{\zeta}_1)^2}\frac{\bar{\zeta}_2{}^2}{1-z_2\bar{\zeta}_2}d\sigma_{\zeta_1}d\sigma_{\zeta_2}, \end{aligned} \tag{3.29}$$

$$\partial_{z_2}E = \frac{\lambda-1}{4\lambda}\frac{1}{\pi^2}\int_D \overline{\theta(\zeta)}\Big[\frac{1-\zeta_1\bar{\zeta}_1}{1-z_1\bar{\zeta}_1}\frac{\bar{\zeta}_2(1-\zeta_2\bar{\zeta}_2)}{(1-z_2\bar{\zeta}_2)^2}+\frac{1-\zeta_1\bar{\zeta}_1}{(1-z_1\bar{\zeta}_1)^2}\frac{2\bar{\zeta}_2(1-\zeta_2\bar{\zeta}_2)}{(1-z_2\bar{\zeta}_2)^3}$$
$$+\frac{1-\zeta_1\bar{\zeta}_1}{(1-z_1\bar{\zeta}_1)^2}\frac{\bar{\zeta}_2(1-\zeta_2\bar{\zeta}_2)}{(1-z_2\bar{\zeta}_2)^2}+\frac{1-\zeta_1\bar{\zeta}_1}{1-z_1\bar{\zeta}_1}\frac{2\bar{\zeta}_2(1-\zeta_2\bar{\zeta}_2)}{(1-z_2\bar{\zeta}_2)^3}\Big]d\sigma_{\zeta_1}d\sigma_{\zeta_2} \tag{3.30}$$
$$+\frac{\lambda+1}{4\lambda}\frac{1}{\pi^2}\int_D \theta(\zeta)\frac{\bar{\zeta}_1{}^2}{1-z_1\bar{\zeta}_1}\frac{\bar{\zeta}_2{}^3}{(1-z_2\bar{\zeta}_2)^2}d\sigma_{\zeta_1}d\sigma_{\zeta_2},$$

and

$$\partial_{z_2}\partial_{z_1}E = \frac{\lambda-1}{4\lambda}\frac{1}{\pi^2}\int_D \overline{\theta(\zeta)}\Big[\frac{\bar{\zeta}_1(1-\zeta_1\bar{\zeta}_1)}{(1-z_1\bar{\zeta}_1)^2}\frac{\bar{\zeta}_2(1-\zeta_2\bar{\zeta}_2)}{(1-z_2\bar{\zeta}_2)^2}+\frac{2\bar{\zeta}_1(1-\zeta_1\bar{\zeta}_1)}{(1-z_1\bar{\zeta}_1)^3}\frac{2\bar{\zeta}_2(1-\zeta_2\bar{\zeta}_2)}{(1-z_2\bar{\zeta}_2)^3}$$
$$+\frac{2\bar{\zeta}_1(1-\zeta_1\bar{\zeta}_1)}{(1-z_1\bar{\zeta}_1)^3}\frac{\bar{\zeta}_2(1-\zeta_2\bar{\zeta}_2)}{(1-z_2\bar{\zeta}_2)^2}+\frac{\bar{\zeta}_1(1-\zeta_1\bar{\zeta}_1)}{(1-z_1\bar{\zeta}_1)^2}\frac{2\bar{\zeta}_2(1-\zeta_2\bar{\zeta}_2)}{(1-z_2\bar{\zeta}_2)^3}\Big]d\sigma_{\zeta_1}d\sigma_{\zeta_2} \tag{3.31}$$
$$+\frac{\lambda+1}{4\lambda}\frac{1}{\pi^2}\int_D \theta(\zeta)\frac{\bar{\zeta}_1{}^3}{(1-z_1\bar{\zeta}_1)^2}\frac{\bar{\zeta}_2{}^3}{(1-z_2\bar{\zeta}_2)^2}d\sigma_{\zeta_1}d\sigma_{\zeta_2}.$$

From (3.25) and (3.28), we get that

$$\begin{aligned}\partial_{\bar{z}_1}\partial_{z_1}\partial_{\bar{z}_2}\partial_{z_2}\Xi\theta &= \partial_{\bar{z}_2}\partial_{z_2}[\partial_{\bar{z}_1}\partial_{z_1}\Xi\theta]\\ &= \frac{\lambda-1}{\lambda+1}\partial_{\bar{z}_2}\partial_{z_2}\{\partial_{\bar{z}_1}\partial_{z_1}[(1-|z_1|^2)(1-|z_2|^2)E]\}\\ &= \frac{\lambda-1}{\lambda+1}\partial_{\bar{z}_2}\partial_{z_2}[(|z_2|^2-1)E-z_1(1-|z_2|^2)\partial_{z_1}E]\\ &= \frac{\lambda-1}{\lambda+1}[E+z_2\partial_{z_2}E+z_1\partial_{z_1}E+z_1z_2\partial_{z_2}\partial_{z_1}E].\end{aligned} \tag{3.32}$$

Therefore, from (3.27)–(3.32), we obtain that

$$\begin{aligned}D\Xi\theta &= (1-\lambda)\partial^2_{\bar{z}_1}\partial^2_{\bar{z}_2}\Xi\theta(z)+(1+\lambda)\partial_{z_1}\partial_{\bar{z}_1}\partial_{z_2}\partial_{\bar{z}_2}\overline{\Xi\theta(z)}\\ &= (1-\lambda)\partial^2_{\bar{z}_1}\partial^2_{\bar{z}_2}\Xi\theta(z)+(1+\lambda)\overline{\partial_{\bar{z}_1}\partial_{z_1}\partial_{\bar{z}_2}\partial_{z_2}\Xi\theta(z)}\\ &= (1-\lambda)\partial^2_{\bar{z}_1}\partial^2_{\bar{z}_2}\Xi\theta(z)+(\lambda-1)[\bar{E}+\overline{z_2\partial_{z_2}E}+\overline{z_1\partial_{z_1}E}+\overline{z_1z_2\partial_{z_2}\partial_{z_1}E}]\\ &= \frac{-(1-\lambda)^2}{4\lambda}\frac{1}{\pi^2}\int_D\theta(\zeta)\frac{2(1-\zeta_1\bar{\zeta}_1)}{(1-\bar{z}_1\zeta_1)^3}\frac{2(1-\zeta_2\bar{\zeta}_2)}{(1-\bar{z}_2\zeta_2)^3}d\sigma_{\zeta_1}d\sigma_{\zeta_2}\\ &\quad+\frac{1-\lambda^2}{4\lambda}\frac{1}{\pi^2}\int_D\overline{\theta(\zeta)}\frac{\bar{\zeta}_1^2}{(1-\bar{z}_1\zeta_1)^2}\frac{\bar{\zeta}_2^2}{(1-\bar{z}_2\zeta_2)^2}d\sigma_{\zeta_1}d\sigma_{\zeta_2}\\ &\quad+\frac{(\lambda-1)^2}{4\lambda}\frac{1}{\pi^2}\int_D\theta(\zeta)\Big[\frac{1-\zeta_1\bar{\zeta}_1}{1-\bar{z}_1\zeta_1}+\frac{1-\zeta_1\bar{\zeta}_1}{(1-\bar{z}_1\zeta_1)^2}\Big]\\ &\quad\cdot\Big[\frac{1-\zeta_2\bar{\zeta}_2}{1-\bar{z}_2\zeta_2}+\frac{1-\zeta_2\bar{\zeta}_2}{(1-\bar{z}_2\zeta_2)^2}\Big]d\sigma_{\zeta_1}d\sigma_{\zeta_2}+\frac{\lambda^2-1}{4\lambda}\frac{1}{\pi^2}\int_D\overline{\theta(\zeta)}\frac{\zeta_1^2}{1-\bar{z}_1\zeta_1}\frac{\zeta_2^2}{1-\bar{z}_2\zeta_2}d\sigma_{\zeta_1}d\sigma_{\zeta_2}\\ &\quad+\bar{z}_2\Big\{\frac{(\lambda-1)^2}{4\lambda}\frac{1}{\pi^2}\int_D\theta(\zeta)\Big[\frac{1-\zeta_1\bar{\zeta}_1}{1-\bar{z}_1\zeta_1}\frac{\zeta_2(1-\zeta_2\bar{\zeta}_2)}{(1-\bar{z}_2\zeta_2)^2}+\frac{1-\zeta_1\bar{\zeta}_1}{(1-\bar{z}_1\zeta_1)^2}\frac{2\zeta_2(1-\zeta_2\bar{\zeta}_2)}{(1-\bar{z}_2\zeta_2)^3}\\ &\quad+\frac{1-\zeta_1\bar{\zeta}_1}{(1-\bar{z}_1\zeta_1)^2}\frac{\zeta_2(1-\zeta_2\bar{\zeta}_2)}{(1-\bar{z}_2\zeta_2)^2}+\frac{1-\zeta_1\bar{\zeta}_1}{1-\bar{z}_1\zeta_1}\frac{2\zeta_2(1-\zeta_2\bar{\zeta}_2)}{(1-\bar{z}_2\zeta_2)^3}\Big]d\sigma_{\zeta_1}d\sigma_{\zeta_2}\end{aligned}$$

$$
\begin{aligned}
&+\frac{\lambda+1}{4\lambda}\frac{1}{\pi^2}\int_D\overline{\theta(\zeta)}\frac{\zeta_1^2}{1-\bar{z}_1\zeta_1}\frac{\zeta_2^3}{(1-\bar{z}_2\zeta_2)^2}d\sigma_{\zeta_1}d\sigma_{\zeta_2}\Big\}\\
&+\bar{z}_1\Big\{\frac{(\lambda-1)^2}{4\lambda}\frac{1}{\pi^2}\int_D\theta(\zeta)\Big[\frac{\zeta_1(1-\zeta_1\bar{\zeta}_1)}{(1-\bar{z}_1\zeta_1)^2}\frac{1-\zeta_2\bar{\zeta}_2}{1-\bar{z}_2\zeta_2}+\frac{2\zeta_1(1-\zeta_1\bar{\zeta}_1)}{(1-\bar{z}_1\zeta_1)^3}\frac{1-\zeta_2\bar{\zeta}_2}{(1-\bar{z}_2\zeta_2)^2}\\
&+\frac{\zeta_1(1-\zeta_1\bar{\zeta}_1)}{(1-\bar{z}_1\zeta_1)^2}\frac{1-\zeta_2\bar{\zeta}_2}{(1-\bar{z}_2\zeta_2)^2}+\frac{2\zeta_1(1-\zeta_1\bar{\zeta}_1)}{(1-\bar{z}_1\zeta_1)^3}\frac{1-\zeta_2\bar{\zeta}_2}{1-\bar{z}_2\zeta_2}\Big]d\sigma_{\zeta_1}d\sigma_{\zeta_2}\\
&+\frac{\lambda^2-1}{4\lambda}\frac{1}{\pi^2}\int_D\overline{\theta(\zeta)}\frac{\zeta_1^3}{(1-\bar{z}_1\zeta_1)^2}\frac{\zeta_2^2}{1-\bar{z}_2\zeta_2}d\sigma_{\zeta_1}d\sigma_{\zeta_2}\Big\}\\
&+\bar{z}_1\bar{z}_2\Big\{\frac{(\lambda-1)^2}{4\lambda}\frac{1}{\pi^2}\int_D\theta(\zeta)\Big[\frac{\zeta_1(1-\zeta_1\bar{\zeta}_1)}{(1-\bar{z}_1\zeta_1)^2}\frac{\zeta_2(1-\zeta_2\bar{\zeta}_2)}{(1-\bar{z}_2\zeta_2)^2}+\frac{2\zeta_1(1-\zeta_1\bar{\zeta}_1)}{(1-\bar{z}_1\zeta_1)^3}\frac{2\zeta_2(1-\zeta_2\bar{\zeta}_2)}{(1-\bar{z}_2\zeta_2)^3}\\
&+\frac{2\zeta_1(1-\zeta_1\bar{\zeta}_1)}{(1-\bar{z}_1\zeta_1)^3}\frac{\zeta_2(1-\zeta_2\bar{\zeta}_2)}{(1-\bar{z}_2\zeta_2)^2}+\frac{\zeta_1(1-\zeta_1\bar{\zeta}_1)}{(1-\bar{z}_1\zeta_1)^2}\frac{2\zeta_2(1-\zeta_2\bar{\zeta}_2)}{(1-\bar{z}_2\zeta_2)^3}\Big]d\sigma_{\zeta_1}d\sigma_{\zeta_2}\\
&+\frac{\lambda^2-1}{4\lambda}\frac{1}{\pi^2}\int_D\overline{\theta(\zeta)}\frac{\zeta_1^3}{(1-\bar{z}_1\zeta_1)^2}\frac{\zeta_2^3}{(1-\bar{z}_2\zeta_2)^2}d\sigma_{\zeta_1}d\sigma_{\zeta_2}\Big\}\\
=&\ 0.
\end{aligned}
$$

Hence, we obtain that

$$D[\Gamma\theta(z)-\Xi\theta(z)+\Phi(z)]=\theta(z).$$

(3) Corollary 3.3 shows that the solution of (3.22) is unique.

Combining the above results, we conclude that $\Gamma\theta(z)-\Xi\theta(z)+\Phi(z)$ is the solution of (3.22). □

Corollary 3.5. *Let $\theta\in L^p(D)$ $(p>2)$ be a given function, and let $\lambda\in\mathbb{R}\setminus\{-1,0,1\}$. Then, $w(z)=\Gamma\theta(z)-\Xi\theta(z)+\Phi(z)$ is the solution of*

$$
\begin{cases}
Dw(z)=\theta(z)(z\in D^2), w(\zeta)=0\ \ (\zeta\in\partial_0 D),\\
\partial_{\bar{z}_1}w(z_1,\zeta_2)=0,\ \ \partial_{\bar{z}_2}w(\zeta_1,z_2)=0\ \ (\zeta_1\in\partial D_1,\ \zeta_2\in\partial D_2),
\end{cases}
$$

where Dw, $\Gamma\theta$, and $\Xi\theta$ are defined by (3.23), (3.24), and (3.25), respectively, and $\Phi(z)$ expressed by (3.2) (in which φ_1, φ_2 and h are replaced by $-\partial_{\bar{z}_1}\Gamma\theta+\partial_{\bar{z}_1}\Xi\theta$, $-\partial_{\bar{z}_2}\Gamma\theta+\partial_{\bar{z}_2}\Xi\theta$ and 0, respectively) is the solution of the problem

$$
\begin{cases}
D\Phi(z)=0(z\in D),\quad \Phi(\zeta)=0\ \ (\zeta\in\partial_0 D),\\
\partial_{\bar{z}_1}\Phi(z_1,\zeta_2)=[-\partial_{\bar{z}_1}\Gamma\theta+\partial_{\bar{z}_1}\Xi\theta](z_1,\zeta_2)\ \ (\zeta_2\in\partial D_2),\\
\partial_{\bar{z}_2}\Phi(\zeta_1,z_2)=[-\partial_{\bar{z}_2}\Gamma\theta+\partial_{\bar{z}_2}\Xi\theta](\zeta_1,z_2)\ \ (\zeta_1\in\partial D_1).
\end{cases}
$$

4. The Riemann problems

In this section, we discuss several Riemann problems for $(\lambda,1)$ bi-analytic functions on the generalized bicylinder in $\mathbb{C}^2$.

Problem R1. Let G be the generalized bicylinder in $\mathbb{C}^2$. Find a $(\lambda,1)$ bi-analytic function $f(z)$ in

$G^{++}\cup G^{+-}\cup G^{-+}\cup G^{--}$ such that

$$\begin{cases} \partial_{\bar z_1}\partial_{\bar z_2}f(z)=\frac{\lambda_1-1}{4\lambda_1}\phi(z)+\frac{\lambda_1+1}{4\lambda_1}\overline{\phi(z)}, \;\; z\in G^{++},\\ \partial_{\bar z_1}\partial_{\bar z_2}f(z)=\frac{\lambda_2-1}{4\lambda_2}\phi(z)+\frac{\lambda_2+1}{4\lambda_2}\overline{\phi(z)}, \;\; z\in G^{+-},\\ \partial_{\bar z_1}\partial_{\bar z_2}f(z)=\frac{\lambda_3-1}{4\lambda_3}\phi(z)+\frac{\lambda_3+1}{4\lambda_3}\overline{\phi(z)}, \;\; z\in G^{-+},\\ \partial_{\bar z_1}\partial_{\bar z_2}f(z)=\frac{\lambda_4-1}{4\lambda_4}\phi(z)+\frac{\lambda_4+1}{4\lambda_4}\overline{\phi(z)}, \;\; z\in G^{--},\\ \partial_{\bar z_1}\partial_{\bar z_2}\phi(z)=0, \;\; z\in G^{++}\cup G^{+-}\cup G^{-+}\cup G^{--}\\ f^{++}(\zeta)-f^{+-}(\zeta)-f^{-+}(\zeta)+f^{--}(\zeta)=b(\zeta), \;\; \zeta\in L,\\ \phi^{++}(\zeta)-\alpha_1(\zeta)\phi^{+-}(\zeta)-\alpha_2(\zeta)\phi^{-+}(\zeta)-\alpha_3(\zeta)\phi^{--}(\zeta)=\beta(\zeta), \;\; \zeta\in L, \end{cases} \tag{4.1}$$

where $z=(z_1,z_2)$ and ϕ is the associate function of f, $\lambda_i\in\mathbb{R}\backslash\{-1,0,1\}$ $(i=1,2,3,4)$, $b(\zeta)$, $\alpha_1(\zeta)$, $\alpha_2(\zeta)$, $\alpha_3(\zeta)$, and $\beta(\zeta)$ are known functions.

Theorem 4.1. *Let G be the generalized bicylinder in $\mathbb{C}^2$. Let $\lambda_i\in\mathbb{R}\backslash\{-1,0,1\}$ $(i=1,2,3,4)$, and $b(\zeta),\alpha_1(\zeta),\alpha_2(\zeta),\alpha_3(\zeta),\beta(\zeta)\in H_\alpha(L)$ $(\alpha\in(0,1])$. Then*

$$f(z)=C+\begin{cases} F_{\lambda_1}(z), \;\; z\in G^{++},\\ F_{\lambda_2}(z), \;\; z\in G^{+-},\\ F_{\lambda_3}(z), \;\; z\in G^{-+},\\ F_{\lambda_4}(z), \;\; z\in G^{--} \end{cases} \tag{4.2}$$

is the solution to Problem R1, where C is an arbitrary complex constant and

$$\begin{cases} F_{\lambda_i}(z)=Tg(z)+\frac{\lambda_i-1}{4\lambda_i}X\theta(z)+\frac{\lambda_i+1}{4\lambda_i}Y\theta(z), \; i=1,2,3,4,\\ Tg(z)=\frac{1}{(2\pi i)^2}\int_L\frac{g(\tau_1,\tau_2)d\tau}{(\tau_1-z_1)(\tau_2-z_2)}, \; z\notin L,\\ X\theta(z)=\frac{1}{(2\pi i)^2}\int_L\theta(\tau_1,\tau_2)\frac{\overline{\tau_1-z_1}}{\tau_1-z_1}\frac{\overline{\tau_2-z_2}}{\tau_2-z_2}d\tau, \; z\notin L,\\ Y\theta(z)=\frac{1}{(2\pi i)^2}\int_L\overline{\theta(\tau_1,\tau_2)}\ln|\tau_1-z_1|^2\ln|\tau_2-z_2|^2d\overline{\tau_1}d\overline{\tau_2}, \; z\notin L, \end{cases} \tag{4.3}$$

and

$$\theta(\zeta)=\phi^{++}(\zeta)-\phi^{+-}(\zeta)-\phi^{-+}(\zeta)+\phi^{--}(\zeta), \;\; \zeta\in L, \tag{4.4}$$

$$g(\zeta)=b(\zeta)+\Big(\frac{1}{\lambda_1}-\frac{1}{\lambda_2}-\frac{1}{\lambda_3}+\frac{1}{\lambda_4}\Big)\frac{X\theta(\zeta)-Y\theta(\zeta)}{4}, \;\; \zeta\in L, \tag{4.5}$$

in which $\phi(z)$ is the solution to the problem (see [15, 16]):

$$\begin{cases} \partial_{\bar z_1}\partial_{\bar z_2}\phi(z)=0, \;\; z\in G^{++}\cup G^{+-}\cup G^{-+}\cup G^{--},\\ \phi^{++}(\zeta)=\alpha_1(\zeta)\phi^{+-}(\zeta)+\alpha_2(\zeta)\phi^{-+}(\zeta)+\alpha_3(\zeta)\phi^{--}(\zeta)+\beta(\zeta), \;\; \zeta\in L. \end{cases} \tag{4.6}$$

Proof. By (4.2) and (4.3), we get that

$$\begin{aligned} \partial_{\bar z_1}\partial_{\bar z_2}f(z)&=\partial_{\bar z_1}\partial_{\bar z_2}F_{\lambda_i}(z)\\ &=\frac{\lambda_i-1}{4\lambda_i}\frac{1}{(2\pi i)^2}\int_L\frac{\theta(\tau)d\tau}{(\tau_1-z_1)(\tau_2-z_2)}+\frac{\lambda_i+1}{4\lambda_i}\frac{1}{(2\pi i)^2}\int_L\overline{\theta(\tau)}\frac{1}{\overline{\tau_1-z_1}}\frac{d\overline{\tau_1}d\overline{\tau_2}}{\overline{\tau_2-z_2}}\\ &=\frac{\lambda_i-1}{4\lambda_i}\frac{1}{(2\pi i)^2}\int_L\frac{\theta(\tau)d\tau}{(\tau_1-z_1)(\tau_2-z_2)}+\frac{\lambda_i+1}{4\lambda_i}\overline{\frac{1}{(2\pi i)^2}\int_L\frac{\theta(\tau)d\tau}{(\tau_1-z_1)(\tau_2-z_2)}}. \end{aligned}$$

In consideration of (4.4), we have

$$\frac{1}{(2\pi i)^2}\int_L \frac{\theta(\tau)d\tau}{(\tau_1 - z_1)(\tau_2 - z_2)} = \phi(z),$$

which satisfies (4.6). Thus, $\phi(z)$ is analytic on D^2 and

$$\partial_{\bar{z}_1}\partial_{\bar{z}_2} f(z) = \frac{\lambda_i - 1}{4\lambda_i}\phi(z) + \frac{\lambda_i + 1}{4\lambda_i}\overline{\phi(z)},$$

therefore, $f(z)$ is a $(\lambda, 1)$ bi-analytic function associated with $\phi(z)$.

As $\alpha_1(\zeta), \alpha_2(\zeta), \alpha_3(\zeta), \beta(\zeta) \in H_\alpha(L)$, and $\phi(z)$ is the solution of (4.6), then $\phi^{\pm\pm} \in H_\alpha(L)$, which follows that $\theta(\zeta) \in H_\alpha(L)$. In addition, as $b(\zeta) \in H_\alpha(L)$, thus $g(\zeta) \in H_\alpha(L)$. Consequently, for $t \in L$, the singular integral

$$F_{\lambda_i}(t) = \frac{1}{(2\pi i)^2}\int_L \frac{g(\tau_1, \tau_2)d\tau}{(\tau_1 - t_1)(\tau_2 - t_2)} + \frac{\lambda_i - 1}{4\lambda_i}X\theta(t) + \frac{\lambda_i + 1}{4\lambda_i}Y\theta(t) \tag{4.7}$$

exists in the sense of the Cauchy principal value (see [17]), and its Cauchy principal value is

$$\begin{aligned} F_{\lambda_i}(t) = &\frac{1}{(2\pi i)^2}\int_L \frac{g(t_1, t_2)d\tau}{(\tau_1 - t_1)(\tau_2 - t_2)} + \frac{1}{(2\pi i)^2}\int_L \frac{[g(\tau_1, \tau_2) - g(\tau_1, t_2) - g(t_1, \tau_2) + g(t_1, t_2)]d\tau}{(\tau_1 - t_1)(\tau_2 - t_2)} \\ &+ \frac{1}{(2\pi i)^2}\int_L \frac{[g(\tau_1, t_2) - g(t_1, t_2)]d\tau}{(\tau_1 - t_1)(\tau_2 - t_2)} + \frac{1}{(2\pi i)^2}\int_L \frac{[g(t_1, \tau_2) - g(t_1, t_2)]d\tau}{(\tau_1 - t_1)(\tau_2 - t_2)} \\ &+ \frac{\lambda_i - 1}{4\lambda_i}X\theta(t) + \frac{\lambda_i + 1}{4\lambda_i}Y\theta(t). \end{aligned}$$

Moreover, applying Lemma 2.4, we get the following Plemelj formula:

$$\begin{cases} F^{++}_{\lambda_i}(t) = \frac{1}{4}[g + S_1 g + S_2 g + S_3 g] + \frac{\lambda_i - 1}{4\lambda_i}X\theta(t) + \frac{\lambda_i + 1}{4\lambda_i}Y\theta(t), \\ F^{+-}_{\lambda_i}(t) = \frac{1}{4}[-g - S_1 g + S_2 g + S_3 g] + \frac{\lambda_i - 1}{4\lambda_i}X\theta(t) + \frac{\lambda_i + 1}{4\lambda_i}Y\theta(t), \\ F^{-+}_{\lambda_i}(t) = \frac{1}{4}[-g + S_1 g - S_2 g + S_3 g] + \frac{\lambda_i - 1}{4\lambda_i}X\theta(t) + \frac{\lambda_i + 1}{4\lambda_i}Y\theta(t), \\ F^{--}_{\lambda_i}(t) = \frac{1}{4}[g - S_1 g - S_2 g + S_3 g] + \frac{\lambda_i - 1}{4\lambda_i}X\theta(t) + \frac{\lambda_i + 1}{4\lambda_i}Y\theta(t), \end{cases} \tag{4.8}$$

where

$$\begin{cases} S_1\varphi = \frac{1}{\pi i}\int_{L_1} \frac{\varphi(\tau_1, t_2)}{\tau_1 - t_1}d\tau_1, \\ S_2\varphi = \frac{1}{\pi i}\int_{L_2} \frac{\varphi(t_1, \tau_2)}{\tau_2 - t_2}d\tau_2, \\ S_3\varphi = \frac{1}{(\pi i)^2}\int_L \frac{\varphi(\tau_1, \tau_2)}{(\tau_1 - t_1)(\tau_2 - t_2)}d\tau_1 d\tau_2. \end{cases} \tag{4.9}$$

From (4.7) and (4.8), we have that

$$\begin{cases} F^{++}_{\lambda_i}(t) = F_{\lambda_i}(t) + \frac{1}{4}[g + S_1 g + S_2 g], \\ F^{+-}_{\lambda_i}(t) = F_{\lambda_i}(t) + \frac{1}{4}[-g - S_1 g + S_2 g], \\ F^{-+}_{\lambda_i}(t) = F_{\lambda_i}(t) + \frac{1}{4}[-g + S_1 g - S_2 g], \\ F^{--}_{\lambda_i}(t) = F_{\lambda_i}(t) + \frac{1}{4}[g - S_1 g - S_2 g]. \end{cases} \tag{4.10}$$

Combining (4.2), (4.5), (4.7), and (4.10), we conclude that

$$\begin{aligned}
&f^{++}(t) - f^{+-}(t) - f^{-+}(t) + f^{--}(t)\\
&= F^{++}_{\lambda_1}(t) - F^{+-}_{\lambda_2}(t) - F^{-+}_{\lambda_3}(t) + F^{--}_{\lambda_4}(t)\\
&= F_{\lambda_1}(t) - F_{\lambda_2}(t) - F_{\lambda_3}(t) + F_{\lambda_4}(t) + g(t)\\
&= \Big[\frac{\lambda_1 - 1}{4\lambda_1} X\theta(t) + \frac{\lambda_1 + 1}{4\lambda_1} Y\theta(t)\Big] - \Big[\frac{\lambda_2 - 1}{4\lambda_2} X\theta(t) + \frac{\lambda_2 + 1}{4\lambda_2} Y\theta(t)\Big]\\
&\quad - \Big[\frac{\lambda_3 - 1}{4\lambda_3} X\theta(t) + \frac{\lambda_3 + 1}{4\lambda_3} Y\theta(t)\Big] + \Big[\frac{\lambda_4 - 1}{4\lambda_4} X\theta(t) + \frac{\lambda_4 + 1}{4\lambda_4} Y\theta(t)\Big] + g(t)\\
&= g(t) - \Big(\frac{1}{\lambda_1} - \frac{1}{\lambda_2} - \frac{1}{\lambda_3} + \frac{1}{\lambda_4}\Big)\frac{X\theta(t) - Y\theta(t)}{4}\\
&= b(t)
\end{aligned}$$

for $t \in L$. Therefore, (4.2) is the solution to Problem R1. □

Now, we investigate the corresponding inverse Riemann problem.

Problem R2. Let G be the generalized bicylinder in $\mathbb{C}^2$. Find a $(\lambda, 1)$ bi-analytic function $f(z)$ in $G^{++} \cup G^{+-} \cup G^{-+} \cup G^{--}$ and the corresponding function $\phi(z), \psi(z)$, and the constants λ_i $(i = 1, 2, 3, 4)$ such that

$$\begin{cases}
\partial_{\bar z_1}\partial_{\bar z_2} f(z) = \frac{\lambda_1-1}{4\lambda_1}\phi(z) + \frac{\lambda_1+1}{4\lambda_1}\overline{\phi(z)}, \ \ z \in G^{++},\\
\partial_{\bar z_1}\partial_{\bar z_2} f(z) = \frac{\lambda_2-1}{4\lambda_2}\phi(z) + \frac{\lambda_2+1}{4\lambda_2}\overline{\phi(z)}, \ \ z \in G^{+-},\\
\partial_{\bar z_1}\partial_{\bar z_2} f(z) = \frac{\lambda_3-1}{4\lambda_3}\phi(z) + \frac{\lambda_3+1}{4\lambda_3}\overline{\phi(z)}, \ \ z \in G^{-+},\\
\partial_{\bar z_1}\partial_{\bar z_2} f(z) = \frac{\lambda_4-1}{4\lambda_4}\phi(z) + \frac{\lambda_4+1}{4\lambda_4}\overline{\phi(z)}, \ \ z \in G^{--},\\
\partial_{\bar z_1}\partial_{\bar z_2}\phi(z) = 0, \ \ z \in G^{++} \cup G^{+-} \cup G^{-+} \cup G^{--},\\
f^{++}(\zeta) - f^{+-}(\zeta) - f^{-+}(\zeta) + f^{--}(\zeta) = b(\zeta), \ \ \zeta \in L,\\
\phi^{++}(\zeta) = \alpha_1(\zeta)\phi^{+-}(\zeta) + \alpha_2(\zeta)\phi^{-+}(\zeta) + \alpha_3(\zeta)\phi^{--}(\zeta) + \beta_1(\zeta)\psi(\zeta), \ \ \zeta \in L,\\
\phi^{++}(\zeta) = \gamma_1(\zeta)\phi^{+-}(\zeta) + \gamma_2(\zeta)\phi^{-+}(\zeta) + \gamma_3(\zeta)\phi^{--}(\zeta) + \beta_2(\zeta)\psi(\zeta), \ \ \zeta \in L,\\
f^{++}(\zeta_0) = c_1, \ \ f^{+-}(\zeta_0) = c_2, \ \ f^{-+}(\zeta_0) = c_3,
\end{cases} \tag{4.11}$$

where $b(\zeta), \alpha_j(\zeta), \gamma_j(\zeta)$ $(j = 1, 2, 3)$, and $\beta_1(\zeta), \beta_2(\zeta)$ are known functions with $\beta_1(\zeta) \neq \beta_2(\zeta)$, and ζ_0 is a given point on L, and c_1, c_2 are given constants.

Theorem 4.2. *Let G be the generalized bicylinder in $\mathbb{C}^2$. Let $b(\zeta)$, $\alpha_j(\zeta)$, $\gamma_j(\zeta)$, $\beta_1(\zeta)$, $\beta_2(\zeta) \in H_\alpha(L)$ ($\alpha \in (0, 1]$, $j = 1, 2, 3$). Then, the solution to Problem R2 is*

$$\begin{cases}
f(z) = C + \begin{cases} F_{\lambda_1}(z), \ \ z \in G^{++},\\ F_{\lambda_2}(z), \ \ z \in G^{+-},\\ F_{\lambda_3}(z), \ \ z \in G^{-+},\\ F_{\lambda_4}(z), \ \ z \in G^{--},\end{cases}\\
\psi(z) = \frac{\phi^{++}(\zeta) - \alpha_1(\zeta)\phi^{+-}(\zeta) - \alpha_2(\zeta)\phi^{-+}(\zeta) - \alpha_3(\zeta)\phi^{--}(\zeta)}{\beta_1(\zeta)},
\end{cases} \tag{4.12}$$

where C is an arbitrary complex constant, $F_{\lambda_i}, Tg, X\theta, Y\theta, \theta(\zeta)$, and $g(\zeta)$ are defined by (4.3)–(4.5), and $\phi(z)$ is the solution to the problem (see [16, 17]):

$$\begin{cases}
\partial_{\bar z_1}\partial_{\bar z_2}\phi(z) = 0, \ \ z \in G^{++} \cup G^{+-} \cup G^{-+} \cup G^{--},\\
(\beta_2 - \beta_1)\phi^{++}(\zeta) = (\beta_2\alpha_1 - \beta_1\gamma_1)\phi^{+-}(\zeta) + (\beta_2\alpha_2 - \beta_1\gamma_2)\phi^{-+}(\zeta) + (\beta_2\alpha_3 - \beta_1\gamma_3)\phi^{--}(\zeta),
\end{cases} \tag{4.13}$$

and

$$\lambda_1 = \frac{|E|}{|E_1|}, \ \ \lambda_2 = \frac{|E|}{|E_2|}, \ \ \lambda_3 = \frac{|E|}{|E_3|}, \ \ \lambda_4 = \frac{|E|}{|E_4|}, \tag{4.14}$$

in which

$$|E| = \begin{vmatrix} A_1 & -B_1 & -B_1 & B_1 \\ A_2 & -B_2 & -A_2 & A_2 \\ A_3 & -A_3 & -B_3 & A_3 \\ B_4 & -B_4 & -B_4 & A_4 \end{vmatrix}, \quad |E_1| = \begin{vmatrix} D_1 & -B_1 & -B_1 & B_1 \\ D_2 & -B_2 & -A_2 & A_2 \\ D_3 & -A_3 & -B_3 & A_3 \\ D_4 & -B_4 & -B_4 & A_4 \end{vmatrix}, \quad |E_2| = \begin{vmatrix} A_1 & D_1 & -B_1 & B_1 \\ A_2 & D_2 & -A_2 & A_2 \\ A_3 & D_3 & -B_3 & A_3 \\ B_4 & D_4 & -B_4 & A_4 \end{vmatrix},$$

$$|E_3| = \begin{vmatrix} A_1 & -B_1 & D_1 & B_1 \\ A_2 & -B_2 & D_2 & A_2 \\ A_3 & -A_3 & D_3 & A_3 \\ B_4 & -B_4 & D_4 & A_4 \end{vmatrix}, \quad |E_4| = \begin{vmatrix} A_1 & -B_1 & -B_1 & D_1 \\ A_2 & -B_2 & -A_2 & D_2 \\ A_3 & -A_3 & -B_3 & D_3 \\ B_4 & -B_4 & -B_4 & D_4 \end{vmatrix}, \tag{4.15}$$

and

$$\begin{cases} A_m = T_m[Y\theta - X\theta](\zeta_0) + [X\theta + Y\theta](\zeta_0), \ \ m = 1, 4, \\ A_n = T_n[Y\theta - X\theta](\zeta_0), \ \ n = 2, 3, \\ B_i = A_i - [X\theta - Y\theta](\zeta_0), \ \ i = 1, 2, 3, 4, \\ D_i = 4[T_i b(\zeta_0) + \frac{(X\theta + Y\theta)(\zeta_0)}{4} - c_i], \ \ i = 1, 2, 3, 4. \end{cases} \tag{4.16}$$

Proof. From (4.11), we have that

$$(\beta_2 - \beta_1)\phi^{++}(\zeta) = (\beta_2\alpha_1 - \beta_1\gamma_1)\phi^{+-}(\zeta) + (\beta_2\alpha_2 - \beta_1\gamma_2)\phi^{-+}(\zeta) + (\beta_2\alpha_3 - \beta_1\gamma_3)\phi^{--}(\zeta), \tag{4.17}$$

therefore, $\phi(z)$ is the solution to (4.13), and thus $\psi(z)$ is represented in (4.12). Similar to the discussion in Theorem 4.1, we get $f(z)$ is represented in (4.12), in which C is an arbitrary complex constant, and $F_{\lambda_i}, Tg, X\theta, Y\theta, \theta(\zeta)$, and $g(\zeta)$ are defined by (4.3)–(4.5).

In the following, we determine λ_i $(i = 1, 2, 3, 4)$. Combining (4.12) and (4.10), we get that

$$\begin{aligned} f^{++}(\zeta_0) = F^{++}_{\lambda_1}(\zeta_0) &= F_{\lambda_1}(\zeta_0) + \frac{1}{4}[g + S_1 g + S_2 g](\zeta_0) \\ &= Tg(\zeta_0) + \frac{\lambda_1 - 1}{4\lambda_1} X\theta(\zeta_0) + \frac{\lambda_1 + 1}{4\lambda_1} Y\theta(\zeta_0) + \frac{1}{4}[g + S_1 g + S_2 g](\zeta_0) \\ &\doteq \frac{\lambda_1 - 1}{4\lambda_1} X\theta(\zeta_0) + \frac{\lambda_1 + 1}{4\lambda_1} Y\theta(\zeta_0) + T_1 g(\zeta_0), \end{aligned} \tag{4.18}$$

in which

$$T_1 g(\zeta_0) = Tg(\zeta_0) + \frac{1}{4}[g + S_1 g + S_2 g](\zeta_0). \tag{4.19}$$

Plugging

$$g(\zeta_0) = b(\zeta_0) + \Big(\frac{1}{\lambda_1} - \frac{1}{\lambda_2} - \frac{1}{\lambda_3} + \frac{1}{\lambda_4}\Big)\frac{X\theta(\zeta_0) - Y\theta(\zeta_0)}{4}$$

into (4.18) to deduce that

$$\begin{aligned} f^{++}(\zeta_0) = T_1 b(\zeta_0) &+ \Big(\frac{1}{\lambda_1} - \frac{1}{\lambda_2} - \frac{1}{\lambda_3} + \frac{1}{\lambda_4}\Big)\frac{T_1[X\theta - Y\theta](\zeta_0)}{4} \\ &+ \frac{\lambda_1 - 1}{4\lambda_1} X\theta(\zeta_0) + \frac{\lambda_1 + 1}{4\lambda_1} Y\theta(\zeta_0) \\ &= c_1, \end{aligned}$$

which follows that

$$
\begin{aligned}
&\frac{1}{\lambda_1}\{T_1[Y\theta - X\theta](\zeta_0) + [X\theta - Y\theta](\zeta_0)\} - \frac{1}{\lambda_2}T_1[Y\theta - X\theta](\zeta_0)\\
&- \frac{1}{\lambda_3}T_1[Y\theta - X\theta](\zeta_0) + \frac{1}{\lambda_4}T_1[Y\theta - X\theta](\zeta_0)\\
=&4[T_1 b(\zeta_0) + \frac{(X\theta + Y\theta)(\zeta_0)}{4} - c_1].
\end{aligned}
$$

Denote that

$$
\begin{cases}
A_1 = T_1[Y\theta - X\theta](\zeta_0) + [X\theta - Y\theta](\zeta_0),\\
B_1 = A_1 - [X\theta - Y\theta](\zeta_0),\\
D_1 = 4[T_1 b(\zeta_0) + \frac{(X\theta+Y\theta)(\zeta_0)}{4} - c_1],
\end{cases} \tag{4.20}
$$

and then we have that

$$
\frac{A_1}{\lambda_1} - \frac{B_1}{\lambda_2} - \frac{B_1}{\lambda_3} + \frac{B_1}{\lambda_4} = D_1. \tag{4.21}
$$

Similarly, combining (4.12) and (4.10), we get that

$$
\begin{aligned}
f^{+-}(\zeta_0) = F^{+-}_{\lambda_2}(\zeta_0) &= F_{\lambda_2}(\zeta_0) + \frac{1}{4}[-g - S_1 g + S_2 g](\zeta_0)\\
&= Tg(\zeta_0) + \frac{\lambda_2 - 1}{4\lambda_2}X\theta(\zeta_0) + \frac{\lambda_2 + 1}{4\lambda_2}Y\theta(\zeta_0) + \frac{1}{4}[-g - S_1 g + S_2 g](\zeta_0)\\
&\doteq \frac{\lambda_2 - 1}{4\lambda_2}X\theta(\zeta_0) + \frac{\lambda_2 + 1}{4\lambda_2}Y\theta(\zeta_0) + T_2 g(\zeta_0)\\
&= T_2 b(\zeta_0) + \Big(\frac{1}{\lambda_1} - \frac{1}{\lambda_2} - \frac{1}{\lambda_3} + \frac{1}{\lambda_4}\Big)\frac{T_2[X\theta - Y\theta](\zeta_0)}{4}\\
&\quad + \frac{\lambda_2 - 1}{4\lambda_2}X\theta(\zeta_0) + \frac{\lambda_2 + 1}{4\lambda_2}Y\theta(\zeta_0)\\
&= c_2,
\end{aligned}
$$

in which

$$
T_2 g(\zeta_0) = Tg(\zeta_0) + \frac{1}{4}[-g - S_1 g + S_2 g](\zeta_0). \tag{4.22}
$$

Therefore,

$$
\begin{aligned}
&\frac{1}{\lambda_1}T_2[Y\theta - X\theta](\zeta_0) - \frac{1}{\lambda_2}\{T_2[Y\theta - X\theta](\zeta_0) - [X\theta - Y\theta](\zeta_0)\}\\
&- \frac{1}{\lambda_3}T_2[Y\theta - X\theta](\zeta_0) + \frac{1}{\lambda_4}T_2[Y\theta - X\theta](\zeta_0)\\
=&4[T_2 b(\zeta_0) + \frac{(X\theta + Y\theta)(\zeta_0)}{4} - c_2],
\end{aligned}
$$

and thus we obtain that

$$
\frac{A_2}{\lambda_1} - \frac{B_2}{\lambda_2} - \frac{A_2}{\lambda_3} + \frac{A_2}{\lambda_4} = D_2, \tag{4.23}
$$

where

$$
\begin{cases}
A_2 = T_2[Y\theta - X\theta](\zeta_0),\\
B_2 = A_2 - [X\theta - Y\theta](\zeta_0),\\
D_2 = 4[T_2 b(\zeta_0) + \frac{(X\theta+Y\theta)(\zeta_0)}{4} - c_2].
\end{cases} \tag{4.24}
$$

Similarly, from $f^{-+}(\zeta_0) = c_3$ and $f^{--}(\zeta_0) = c_4$, we can get that

$$\begin{cases} \frac{A_3}{\lambda_1} - \frac{A_3}{\lambda_2} - \frac{B_3}{\lambda_3} + \frac{A_3}{\lambda_4} = D_3, \\ \frac{B_4}{\lambda_1} - \frac{B_4}{\lambda_2} - \frac{B_4}{\lambda_3} + \frac{A_4}{\lambda_4} = D_4, \end{cases} \tag{4.25}$$

where

$$\begin{cases} A_3 = T_3[Y\theta - X\theta](\zeta_0), \\ B_3 = A_3 - [X\theta - Y\theta](\zeta_0), \\ D_3 = 4[T_3 b(\zeta_0) + \frac{(X\theta+Y\theta)(\zeta_0)}{4} - c_3], \\ A_4 = T_4[Y\theta - X\theta](\zeta_0) + [X\theta + Y\theta](\zeta_0), \\ B_4 = A_4 - [X\theta - Y\theta](\zeta_0), \\ D_4 = 4[T_4 b(\zeta_0) + \frac{(X\theta+Y\theta)(\zeta_0)}{4} - c_4]. \end{cases} \tag{4.26}$$

Therefore, combining (4.21), (4.23), and (4.25) we obtain that λ_i is determined by (4.14), and A_i, B_i, and D_i are represented by (4.15). □

Problem R3. Let the bicylinder in $\mathbb{C}^2$ be $D = D_1 \times D_2 = \{(z_1, z_2) : |z_1| < 1, |z_2| < 1\}$. Find $f(z)$ in $D^{++} \cup D^{+-} \cup D^{-+} \cup D^{--}$ such that

$$\begin{cases} \partial_{\bar{z}_1}\partial_{\bar{z}_2} f(z) = \frac{\lambda_1 - 1}{4\lambda_1}\phi^{++}(z) + \frac{\lambda_1+1}{4\lambda_1}\overline{\phi^{++}(z)}, \quad z \in D^{++}, \\ \partial_{\bar{z}_1}\partial_{\bar{z}_2} f(z) = \frac{\lambda_2 - 1}{4\lambda_2}\phi^{+-}(z) + \frac{\lambda_2+1}{4\lambda_2}\overline{\phi^{+-}(z)}, \quad z \in D^{+-}, \\ \partial_{\bar{z}_1}\partial_{\bar{z}_2} f(z) = \frac{\lambda_3 - 1}{4\lambda_3}\phi^{-+}(z) + \frac{\lambda_3+1}{4\lambda_3}\overline{\phi^{-+}(z)}, \quad z \in D^{-+}, \\ \partial_{\bar{z}_1}\partial_{\bar{z}_2} f(z) = \frac{\lambda_4 - 1}{4\lambda_4}\phi^{--}(z) + \frac{\lambda_4+1}{4\lambda_4}\overline{\phi\phi^{--}(z)}, \quad z \in D^{--}, \\ \partial_{\bar{z}_1}\partial_{\bar{z}_2}\phi^{++}(z) = q(z), \quad z \in D^{++}, \\ \partial_{\bar{z}_1}\partial_{\bar{z}_2}\phi^{+-}(z) = 0, \quad z \in D^{+-}, \\ \partial_{\bar{z}_1}\partial_{\bar{z}_2}\phi^{-\pm}(z) = 0, \quad z \in D^{-\pm}, \\ f^{++}(\zeta) = f^{+-}(\zeta) + f^{-+}(\zeta) - f^{--}(\zeta) + b(\zeta), \quad \zeta \in L, \\ \phi^{++}(\zeta) = \alpha_1(\zeta)\phi^{+-}(\zeta) + \alpha_2(\zeta)\phi^{-+}(\zeta) + \alpha_3(\zeta)\phi^{--}(\zeta) + \beta(\zeta), \quad \zeta \in L, \end{cases} \tag{4.27}$$

where $z = (z_1, z_2)$, and ϕ is the associated function of f, $\lambda_i \in \mathbb{R}\backslash\{-1, 0, 1\}$ $(i = 1, 2, 3, 4)$, and $q(z)$, $b(\zeta), \alpha_j(\zeta)$ $(j = 1, 2, 3)$, and $\beta(\zeta)$ are known functions.

Theorem 4.3. *Let the bicylinder in $\mathbb{C}^2$ be $D = D_1 \times D_2 = \{(z_1, z_2) : |z_1| < 1, |z_2| < 1\}$. Let $\lambda_i \in \mathbb{R}\backslash\{-1, 0, 1\}$ $(i = 1, 2, 3, 4)$, and $b(\zeta), \alpha_j(\zeta), \beta(\zeta) \in H_\alpha(L)$ $(\alpha \in (0, 1], j = 1, 2, 3)$. Then,*

$$f(z) = \begin{cases} f_q(z) + f_c^{++}(z), \quad z \in D^{++}, \\ f_c^{+-}(z), \quad z \in D^{+-}, \\ f_c^{-+}(z), \quad z \in D^{-+}, \\ f_c^{--}(z), \quad z \in D^{--} \end{cases} \tag{4.28}$$

is the solution to Problem R3, where $f_q(z)$ is the solution (see Corollary 3.5) to the problem:

$$\begin{cases} (1-\lambda)\partial_{\bar{z}_1}^2\partial_{\bar{z}_2}^2 f_q(z) + (1+\lambda)\partial_{z_1}\partial_{\bar{z}_1}\partial_{z_2}\partial_{\bar{z}_2}\overline{f_q(z)} = q, \quad z \in D^{++}, \\ f_q(\zeta) = 0, \quad \partial_{\bar{z}_1} f_q(z_1, \zeta_2) = 0, \quad \partial_{\bar{z}_2} f_q(\zeta_1, z_2) = 0 \;\; (\zeta_1 \in \partial D_1, \; \zeta_2 \in \partial D_2), \end{cases} \tag{4.29}$$

and

$$f_c(z) = C + \begin{cases} F_{\lambda_1}^c(z), \quad z \in D^{++}, \\ F_{\lambda_2}^c(z), \quad z \in D^{+-}, \\ F_{\lambda_3}^c(z), \quad z \in D^{-+}, \\ F_{\lambda_4}^c(z), \quad z \in D^{--}, \end{cases} \tag{4.30}$$

in which C is an arbitrary complex constant and

$$\begin{cases} F^c_{\lambda_i}(z) = Tg(z) + \frac{\lambda_i-1}{4\lambda_i}X\theta(z) + \frac{\lambda_i+1}{4\lambda_i}Y\theta(z),\ i = 1,2,3,4, \\ Tg(z) = \frac{1}{(2\pi i)^2}\int_L \frac{g(\tau_1,\tau_2)d\tau}{(\tau_1-z_1)(\tau_2-z_2)},\ z \notin L, \\ X\theta(z) = \frac{1}{(2\pi i)^2}\int_L \theta(\tau_1,\tau_2)\frac{\overline{\tau_1-z_1}}{\tau_1-z_1}\frac{\overline{\tau_2-z_2}}{\tau_2-z_2}d\tau,\ z \notin L, \\ Y\theta(z) = \frac{1}{(2\pi i)^2}\int_L \overline{\theta(\tau_1,\tau_2)}\ln|\tau_1-z_1|^2\ln|\tau_2-z_2|^2 d\overline{\tau_1}d\overline{\tau_2},\ z \notin L, \end{cases} \tag{4.31}$$

and

$$\theta(\zeta) = \phi_c^{++}(\zeta) - \phi_c^{+-}(\zeta) - \phi_c^{-+}(\zeta) + \phi_c^{--}(\zeta),\quad \zeta \in L, \tag{4.32}$$

$$g(\zeta) = b(\zeta) + \Big(\frac{1}{\lambda_1} - \frac{1}{\lambda_2} - \frac{1}{\lambda_3} + \frac{1}{\lambda_4}\Big)\frac{X\theta(\zeta) - Y\theta(\zeta)}{4},\quad \zeta \in L, \tag{4.33}$$

in which $\phi_c(z)$ is the solution (see [16, 17]) to the problem:

$$\begin{cases} \partial_{\bar z_1}\partial_{\bar z_2}\phi_c^{\pm\pm}(z) = 0,\quad z \in D^{\pm\pm}, \\ \phi_c^{++}(\zeta) = \alpha_1(\zeta)\phi_c^{+-}(\zeta) + \alpha_2(\zeta)\phi_c^{-+}(\zeta) + \alpha_3(\zeta)\phi_c^{--}(\zeta) + \beta(\zeta) \\ \qquad -(1-\lambda_1)\partial_{\bar z_1}\partial_{\bar z_2}f_q(\zeta) - (1+\lambda_1)\partial_{z_1}\partial_{z_2}\overline{f_q(\zeta)},\quad \zeta \in L. \end{cases} \tag{4.34}$$

Proof. As $f_q(z)$ is the solution to the problem (4.29), let

$$\phi_q^{++} = (1-\lambda)\partial_{\bar z_1}\partial_{\bar z_2}f_q(z) + (1+\lambda)\partial_{z_1}\partial_{z_2}\overline{f_q(z)}, \tag{4.35}$$

then we get that

$$\partial_{\bar z_1}\partial_{\bar z_2}\phi_q^{++}(z) = q(z) \tag{4.36}$$

and

$$\begin{aligned} \frac{\lambda_1-1}{4\lambda_1}\phi_q^{++} + \frac{\lambda_1+1}{4\lambda_1}\overline{\phi_q^{++}} &= \frac{\lambda_1-1}{4\lambda_1}[(1-\lambda)\partial_{\bar z_1}\partial_{\bar z_2}f_q(z) + (1+\lambda)\partial_{z_1}\partial_{z_2}\overline{f_q(z)}] \\ &\quad + \frac{\lambda_1+1}{4\lambda_1}[(1-\lambda)\partial_{z_1}\partial_{z_2}\overline{f_q(z)} + (1+\lambda)\partial_{\bar z_1}\partial_{\bar z_2}f_q(z)] \\ &= \partial_{\bar z_1}\partial_{\bar z_2}f_q(z). \end{aligned}$$

That is, f_q satisfies that

$$\partial_{\bar z_1}\partial_{\bar z_2}f_q(z) = \frac{\lambda_1-1}{4\lambda_1}\phi_q^{++}(z) + \frac{\lambda_1+1}{4\lambda_1}\overline{\phi_q^{++}(z)}. \tag{4.37}$$

Applying Theorem 4.1, we obtain that (4.30) is the solution to the problem:

$$\begin{cases} \partial_{\bar z_1}\partial_{\bar z_2}f_c^{++}(z) = \frac{\lambda_1-1}{4\lambda_1}\phi_c^{++}(z) + \frac{\lambda_1+1}{4\lambda_1}\overline{\phi_c^{++}(z)},\quad z \in D^{++}, \\ \partial_{\bar z_1}\partial_{\bar z_2}f_c^{+-}(z) = \frac{\lambda_2-1}{4\lambda_2}\phi_c^{+-}(z) + \frac{\lambda_2+1}{4\lambda_2}\overline{\phi_c^{+-}(z)},\quad z \in D^{+-}, \\ \partial_{\bar z_1}\partial_{\bar z_2}f_c^{-+}(z) = \frac{\lambda_3-1}{4\lambda_3}\phi_c^{-+}(z) + \frac{\lambda_3+1}{4\lambda_3}\overline{\phi_c^{-+}(z)},\quad z \in D^{-+}, \\ \partial_{\bar z_1}\partial_{\bar z_2}f_c^{--}(z) = \frac{\lambda_4-1}{4\lambda_4}\phi_c^{--}(z) + \frac{\lambda_4+1}{4\lambda_4}\overline{\phi_c^{--}(z)},\quad z \in D^{--}, \\ \partial_{\bar z_1}\partial_{\bar z_2}\phi_c^{\pm\pm}(z) = 0,\quad z \in D^{\pm\pm}, \\ f_c^{++}(\zeta) = f_c^{+-}(\zeta) + f_c^{-+}(\zeta) - f_c^{--}(\zeta) + b(\zeta),\quad \zeta \in L, \\ \phi_c^{++}(\zeta) = \alpha_1(\zeta)\phi_c^{+-}(\zeta) + \alpha_2(\zeta)\phi_c^{-+}(\zeta) + \alpha_3(\zeta)\phi_c^{--}(\zeta) + \beta(\zeta) \\ \qquad -(1-\lambda_1)\partial_{\bar z_1}\partial_{\bar z_2}f_q(\zeta) - (1+\lambda_1)\partial_{z_1}\partial_{z_2}\overline{f_q(\zeta)},\quad \zeta \in L, \end{cases} \tag{4.38}$$

(1) In the case of $z \in D^{++}$, let $\phi^{++}(z) = \phi_q^{++}(z) + \phi_c^{++}(z)$. By (4.36) and (4.38), we get that

$$\partial_{\bar{z}_1}\partial_{\bar{z}_2}\phi^{++}(z) = q(z). \tag{4.39}$$

By (4.28), (4.37) and (4.38), we get that

$$\begin{aligned}\partial_{\bar{z}_1}\partial_{\bar{z}_2}f(z) &= \partial_{\bar{z}_1}\partial_{\bar{z}_2}f_q(z) + \partial_{\bar{z}_1}\partial_{\bar{z}_2}f_c^{++}(z)\\ &= \frac{\lambda_1 - 1}{4\lambda_1}(\phi_q^{++}(z) + \phi_c^{++}(z)) + \frac{\lambda_1 + 1}{4\lambda_1}(\overline{\phi_q^{++}(z)} + \overline{\phi_c^{++}(z)})\\ &= \frac{\lambda_1 - 1}{4\lambda_1}\phi^{++}(z) + \frac{\lambda_1 + 1}{4\lambda_1}\overline{\phi^{++}(z)}.\end{aligned} \tag{4.40}$$

(2) In the case of $z \in D^{+-}$, let $\phi^{+-}(z) = \phi_c^{+-}(z)$. Then, (4.4) leads to that

$$\partial_{\bar{z}_1}\partial_{\bar{z}_2}\phi^{+-}(z) = 0. \tag{4.41}$$

Additionally, by (4.28) and (4.38), we get that

$$\partial_{\bar{z}_1}\partial_{\bar{z}_2}f(z) = \partial_{\bar{z}_1}\partial_{\bar{z}_2}f_c^{+-}(z) = \frac{\lambda_2 - 1}{4\lambda_2}\phi^{+-}(z) + \frac{\lambda_2 + 1}{4\lambda_2}\overline{\phi^{+-}(z)}. \tag{4.42}$$

(3) In the case of $z \in D^{-\pm}$, let $\phi^{-\pm}(z) = \phi_c^{-\pm}(z)$. Similar to the above discussion, we have that

$$\begin{cases}\partial_{\bar{z}_1}\partial_{\bar{z}_2}f(z) = \frac{\lambda_3-1}{4\lambda_3}\phi^{-+}(z) + \frac{\lambda_3+1}{4\lambda_3}\overline{\phi^{-+}(z)}, \quad z \in D^{-+},\\ \partial_{\bar{z}_1}\partial_{\bar{z}_2}f(z) = \frac{\lambda_4-1}{4\lambda_4}\phi^{--}(z) + \frac{\lambda_4+1}{4\lambda_4}\overline{\phi^{--}(z)}, \quad z \in D^{--},\\ \partial_{\bar{z}_1}\partial_{\bar{z}_2}\phi^{-\pm}(z) = 0, \quad z \in D^{-\pm}.\end{cases} \tag{4.43}$$

In addition, (4.28) and (4.38) lead to that

$$\begin{aligned}f^{++}(\zeta) &= f_c^{++}(\zeta) = f_c^{+-}(\zeta) + f_c^{-+}(\zeta) - f_c^{--}(\zeta) + b(\zeta)\\ &= f^{+-}(\zeta) + f^{-+}(\zeta) - f^{--}(\zeta) + b(\zeta), \quad \zeta \in L.\end{aligned} \tag{4.44}$$

(4.35) and (4.38) follow that

$$\begin{aligned}\phi^{++}(\zeta) &= \phi_q^{++}(\zeta) + \phi_c^{++}(\zeta)\\ &= (1-\lambda)\partial_{\bar{\zeta}_1}\partial_{\bar{\zeta}_2}f_q(z) + (1+\lambda)\partial_{\zeta_1}\partial_{\zeta_2}\overline{f_q(\zeta)} + [\alpha_1(\zeta)\phi_c^{+-}(\zeta) + \alpha_2(\zeta)\phi_c^{-+}(\zeta)\\ &\quad + \alpha_3(\zeta)\phi_c^{--}(\zeta) + \beta(\zeta) - (1-\lambda_1)\partial_{\bar{\zeta}_1}\partial_{\bar{\zeta}_2}f_q(\zeta) - (1+\lambda_1)\partial_{\zeta_1}\partial_{\zeta_2}\overline{f_q(\zeta)}]\\ &= \alpha_1(\zeta)\phi^{+-}(\zeta) + \alpha_2(\zeta)\phi^{-+}(\zeta) + \alpha_3(\zeta)\phi^{--}(\zeta) + \beta(\zeta).\end{aligned} \tag{4.45}$$

Combining (4.39)–(4.45) concludes that (4.28) is the solution to Problem R3. □

5. Conclusions

We first investigate a type of homogeneous Dirichlet problems for second order complex partial differential systems of $(\lambda, 1)$ bi-analytic functions on the bicylinder. With the help of the Cauchy-Pompeiu formula and the properties of the Poisson kernel, we get the specific representation of the solution. On this basis, with the help of the properties of the T-operator, we discuss the nonhomogeneous Dirichlet problems for $(\lambda, 1)$ bi-analytic functions on the bicylinder. In addition, applying the Plemelj formula, we obtain the expressions of the solutions to a kind of homogeneous Riemann problem and the inverse problems for systems of $(\lambda, 1)$ bi-analytic functions on the generalized bicylinder in $\mathbb{C}^2$. Thereafter, we discuss the corresponding nonhomogeneous Riemann problems of $(\lambda, 1)$ bi-analytic functions on the bicylinder. Expect for the above results, we also discuss the higher order complex partial differential systems of $(\lambda, 1)$ bi-analytic functions with the same method used in this paper. The conclusions obtained here make a solid foundation for studying other boundary value problems of bi-analytic functions or bi-ployanalytic functions.

Author contributions

Yanyan Cui: Conceptualization, Project administration, Writing original draft, Writing–review and editing; Chaojun Wang: Investigation, Writing original draft, Writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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