Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(7): 17012–17048.
DOI:10.3934/math.2025764
Received: 02 May 2025
Revised: 09 July 2025
Accepted: 22 July 2025
Published: 30 July 2025

Research article

Uniform estimate for strongly elliptic equations in high-contrast composites

Chiuyao He, Shiahsen Wang and Liming Yeh*

Department of Applied Mathematics, National Yang Ming Chiao Tung University, Hsinchu, 30050, Taiwan

* **Correspondence:** Email: liming@math.nctu.edu.tw.

Abstract: This paper considers uniform estimates for strongly elliptic equations in high-contrast composites. The composites consist of an ϵ-periodic lattice of fibers with high conductivity, included in a connected material with normal conductivity. The diffusion coefficients of the elliptic equations, depending on the conductivities, are not bounded above. The equations have fast diffusion inside the fibers and slow diffusion elsewhere. Let $\omega^2 \in (1, \infty)$ denote the conductivity ratio of the fibers to the connected material and let $\epsilon\frac{\mu}{2} \in (0, \frac{1}{2})$ be the diameter of the horizontal cross-section of each fiber. This work presents $W^{1,p}$ estimates uniformly in ϵ, ω, μ for the solutions of the elliptic equations.

Keywords: strongly elliptic equations, conductivity, capacity, reverse Hölder inequality
Mathematics Subject Classification: 35J05, 35J15, 35J25, 35J75

1. Introduction

Uniform estimates for strongly elliptic equations in high-contrast composites are considered here. The composites consist of an ϵ-periodic lattice of fibers with high conductivity, included in a connected material with normal conductivity. The diffusion coefficients of the elliptic equations, depending on the conductivities, are not bounded above. Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain with boundary $\partial\Omega$, $\epsilon, \mu \in (0, 1)$, $\Omega(\epsilon) \equiv \{x \in \Omega | \; dist(x, \partial\Omega) > \epsilon\}$, $\mathcal{I}_\epsilon \equiv \{\mathbf{j} \in \mathbb{Z}^2 | \; \epsilon(\mathcal{Y} + \mathbf{j}) \subset \Omega(\epsilon)\}$, where $\mathcal{Y} \equiv [\frac{-1}{2}, \frac{1}{2})^2$, $\mathcal{Y}_{\mu,m}$ is a disc centered at 0 with a radius $\frac{\mu}{4}$, $\mathcal{Y}_{\mu,f} \equiv \mathcal{Y} \setminus \mathcal{Y}_{\mu,m}$, $\Omega^\epsilon_{\mu,m} \equiv \bigcup_{\mathbf{j}\in\mathcal{I}_\epsilon} \epsilon(\mathcal{Y}_{\mu,m} + \mathbf{j})$ is a disconnected subset of Ω, and $\Omega^\epsilon_{\mu,f}$ $(\equiv \Omega \setminus \Omega^\epsilon_{\mu,m})$ is a connected sub-region of Ω. Moreover, $\mathfrak{D} \equiv \Omega \times (0, \mathbf{L}) \subset \mathbb{R}^3$ is the whole composite, $\mathfrak{D}^\epsilon_{\mu,m} \equiv \Omega^\epsilon_{\mu,m} \times (0, \mathbf{L})$ is the union of the ϵ-periodic lattice of fibers, and $\mathfrak{D}^\epsilon_{\mu,f} \equiv \Omega^\epsilon_{\mu,f} \times (0, \mathbf{L})$ is a connected region. Let $\mathbf{E}^\epsilon_{\theta,\mu} \equiv \mathcal{X}_{\mathfrak{D}^\epsilon_{\mu,f}} + \theta \mathcal{X}_{\mathfrak{D}^\epsilon_{\mu,m}}$ denote the diffusion coefficient for $\epsilon, \mu \in (0, 1)$ and $\theta > 0$, where $\mathcal{X}_S$ is a characteristic function on S. The strongly elliptic equations are

$$\begin{cases} -\nabla \cdot (\mathbf{E}^\epsilon_{\omega^2,\mu} \nabla U) = F & \text{in } \mathfrak{D}, \\ U = 0 & \text{on } \partial\mathfrak{D}, \end{cases} \tag{1.1}$$

where $\epsilon, \mu \in (0,1)$ and $\omega \in (1,\infty)$. Note $1 \leq \mathbf{E}^{\epsilon}_{\omega^2,\mu} \leq \omega^2 < \infty$. The unique existence of an $H^1_0(\mathfrak{D})$ solution for problem (1.1) is by the Lax-Milgram Theorem [17].

Problem (1.1) has applications in flows in fractured media, photonic crystal fibers, and the electrical conductivity of fiber-reinforced composites (see [5, 18, 20] and references therein). Problem (1.1) contains elliptic equations with fiber-like stiff inclusions, Eq (1.1) has fast diffusion inside the fibers and slow diffusion elsewhere, and the homogenized macroscopic equations of (1.1) can have nonlocal terms. For reference, some homogenized macroscopic equations of (1.1) are listed in the last section of this work (or see [4, Theorem A], [7, Eq (1.2)], and [8, Theorem 3]). Similar nonlocal behaviors are also found in the homogenized equations for other elliptic equations in high-contrast composites [6,9,25]. However, Briane (see [7, Theorem 1]) shows that in the case of two-dimensional (2-D) space, nonlocal macroscopic behavior does not appear in the homogenized solutions for the elliptic equations in high-contrast composites.

We now recall the regularity results related to (1.1). The Lipschitz estimate uniformly in ϵ for the Laplace equation in periodic perforated domains is shown in [21]. By the periodicity of the domains, the L^∞ norm is established uniformly in ϵ for discrete difference quotients of the Laplace equation. Then we solve local elliptic problems with boundary conditions from the solutions of the abovementioned discrete difference quotients. Lipschitz estimates are derived uniformly in ϵ for the Laplace equation by a scaling argument to the solutions of the abovementioned local elliptic problems. The Hölder, $W^{1,p}$, and Lipschitz estimates uniformly in ϵ for uniform elliptic equations with Hölder periodic coefficients are shown in [2, 22, 26] by a three-step compactness argument. These estimates are proved by studying the homogenization problems of uniform elliptic equations first. Then we solve local elliptic problems from some scaled given functions. The expected estimates are derived by applying a scaling argument to the solutions of the local elliptic problems above. Similar ideas to the three-step compactness argument are also employed in the Hölder, $W^{1,p}$, and Lipschitz estimates uniformly in ϵ, ω for degenerate elliptic equations with grain-like soft inclusions [28]. Recently, two papers concerned uniform estimates for degenerate elliptic equations with grain-like soft inclusions (see [15, 24]); uniform non-tangential maximal function estimates for the Dirichlet, regularity, and Neumann problems with L^p boundary data are obtained.

A few results on the regularity for strongly elliptic equations with fiber-like stiff inclusions are reported. Equation $(1.1)_1$ (that is, the first equation in (1.1) or $-\nabla \cdot (\mathbf{E}^{\epsilon}_{\omega^2,\mu} \nabla U) = F$ in $\mathfrak{D}$) with nonhomogeneous Dirichlet boundary conditions in the case of $\omega \geq 1$, $\omega^2 |\mathfrak{D}^{\epsilon}_{\mu,m}| \approx 1$, and $\epsilon^2 |\ln \mu| \approx 1$ are studied in [8] (here, $a \approx b$ means $\mathbf{m}_0\, a \leq b \leq \mathbf{m}_1\, a$ for some positive constants $\mathbf{m}_0, \mathbf{m}_1$ independent of ϵ, ω, μ). A $W^{1,p}$ (for $p > 2$) estimate can not be expected uniformly in ϵ, ω, μ for Eq $(1.1)_1$ with nonhomogeneous Dirichlet boundary condition in the whole domain $\mathfrak{D}$ in general (see [8, Corollary 2]). Moreover, nonlocal behaviors appear in the homogenized macroscopic solutions of (1.1) (see [8, Theorem 3]). Local $W^{1,6}$ estimates and local $C^{1,\alpha}$ convergence are obtained uniformly in ϵ, ω, μ in the interior of the sub-region $\mathfrak{D}^{\epsilon}_{\mu,f}$ of $\mathfrak{D}$ for the solutions of $(1.1)_1$ with nonhomogeneous Dirichlet boundary conditions. Shen [23] considers elliptic systems for elasticity with periodic grain-like stiff inclusions in $\mathbb{R}^n$ for $n \geq 3$. By the periodicity of the diffusion coefficients, local Lipschitz estimates are proved uniformly in ϵ, ω for the solutions of the elliptic system. For strongly elliptic equations in two-dimensional space, Lipschitz estimates for the solutions of the elliptic equations in the whole domain can be found in [29].

Under $\omega > 1$ and $\omega^2 |\mathfrak{D}^{\epsilon}_{\mu,m}| \approx 1$, this work derives a $W^{1,p}$ (for $p > 2$) estimate uniformly in

ϵ, ω, μ for (1.1) in the whole domain. Here, the parameters ϵ, ω, μ satisfy (i) $\omega\mu \approx 1$ (equivalent to $\omega^2|\mathfrak{D}^{\epsilon}_{\mu,m}| \approx 1$) and (ii) ϵ, μ are free. By [4, Theorem A], [7, Eq (1.2)], and [8, Theorem 3], we know that under $\omega > 1$, $\omega^2|\mathfrak{D}^{\epsilon}_{\mu,m}| \approx 1$, and $\epsilon^2|\ln\mu| \approx 1$, the nonlocal term appears in the homogenized macroscopic equations of (1.1). In other words, our results can be applied to strongly elliptic equations that their homogenized macroscopic equations may have nonlocal behaviors. Next, we describe the idea of derivation for the $W^{1,p}$ estimate. As mentioned above, (i) the solutions of the homogenized macroscopic equations of (1.1) in three-dimensional (3-D) space have nonlocal behaviors (which explains the difficulty of obtaining the regularity of solutions of (1.1) in 3-D space case) and (ii) the solutions of the homogenized macroscopic equations for the 2-D elliptic equations in high-contrast composites do not have nonlocal behaviors [7]. Therefore, to avoid the difficulty from the nonlocal behaviors, we transform the 3-D problem (1.1) into 2-D problems but with strange external sources. More precisely, we apply separation of variables to replace the 3-D problem (1.1) by 2-D Helmholtz-type strongly elliptic equations in ϵ-periodic composites Ω. Then we study the uniform convergence, the homogenization problems, and the Lipschitz estimates for the strongly elliptic 2-D equations. This is achieved by employing the layer potentials [11], the reverse Hölder inequality [16], and a three-step compactness argument [2, 3]. In order to employ the three-step compactness argument, we need uniform convergence of the solutions of the 2-D elliptic equations with "$uniform\ L^1$" (or "$uniform\ L^1 \setminus L^s\ for\ s > 1$") external sources (see Remark 4.1 for the definition of "$uniform\ L^1$" (or "$uniform\ L^1 \setminus L^s\ for\ s > 1$")). Indeed, this is an essential step used to obtain the uniform estimate for the problem (1.1). The uniform convergence of the 2-D elliptic equations with "$uniform\ L^1$" (or "$uniform\ L^1 \setminus L^s\ for\ s > 1$") external sources for the case of *periodic size* $\le \mu$ and for $\mu \le$ *periodic size* is considered separately. To study the boundary Lipschitz estimate for the 2-D strongly elliptic equations, we need the corrector functions of the 2-D elliptic operators (see [29, Lemma 6.9]). Finally, the $W^{1,p}$ (for $p > 2$) estimate uniformly in $\epsilon, \mu (= \frac{1}{\omega})$ for the problem (1.1) in $\mathfrak{D} \subset \mathbb{R}^3$ is proved by combining the estimates obtained from the abovementioned 2-D strongly elliptic equations. Note that the Lipschitz estimate for the solutions of 2-D strongly elliptic equations with L^s for $s > 1$ external sources can be found in [29].

The rest of this work is organized as follows: The notation and main results are stated in Section 2. The main results are proved in Section 3 by the separation of variables and the Lipschitz estimate for 2-D Helmholtz-type strongly elliptic equations (i.e., Lemma 3.2). Next, we study two convergence results for the 2-D elliptic equations with "$uniform\ L^1$" (or "$uniform\ L^1 \setminus L^s\ for\ s > 1$") external sources. The first one is the uniform convergence of the 2-D elliptic equations shown in Section 4; the second one is the L^2-gradient convergence of the 2-D elliptic equations shown in Section 5. Furthermore, a Lipschitz estimate for 2-D diffraction problems (i.e., Lemma 4.1), L^∞-estimates for 2-D elliptic equations, and a local $W^{1,2+\delta}$ estimate for the 2-D elliptic equations are derived in Section 4. Lipschitz estimates for the 2-D elliptic equations are given in Section 6 and shown by a three-step compactness argument [2, 3] and the results from Sections 4 and 5. Lemma 3.2 is proved in Section 7; Lemma 4.1 is in Section 8; some of the homogenized macroscopic equations of (1.1) are listed in Section 9 for reference.

2. Notation and main results

$C^{k,\alpha}, L^p, W^{k,p}, H^k$ denote the Hölder space, Lebesgue space, Sobolev space, and Hilbert space (see [17]). $H^1_0(S)$ (resp. $C^1_0(S)$) is the closure of $C^\infty_0(S)$ under the H^1 norm (resp. C^1 norm). $[\varphi]_{C^\alpha}$

is the α-th Hölder seminorm of φ; $\|\varphi\|_{C^\alpha} = \|\varphi\|_{L^\infty} + [\varphi]_{C^\alpha}$; $supp(\varphi)$ is the support of function φ. $B_r(z) \equiv \{x \in \mathbb{R}^2 | \ \|x - z\| < r\}$ is a disk centered at z with a radius r. For a set S, $\overline{S}$ is the closure of S, ∂S is the boundary of S, $|S|$ is the volume of S, and $S/r \equiv \frac{1}{r}S \equiv \{x | rx \in S\}$ for $r > 0$. $S_1 \Subset S_2$ means that $\overline{S_1}$ is a compact subset of the interior of S_2. If $\varphi \in L^1(S)$,

$$(\varphi)_S \equiv \fint_S \varphi(x)dx \equiv \frac{1}{|S|}\int_S \varphi(x)dx. \tag{2.1}$$

If $x \in \mathbb{R}^3$, $x = (x_1, x_2, x_3) = (x', x_3)$. Since $\mathbf{E}^\epsilon_{\omega,\mu}$ is independent of x_3 (see Section 1), $\mathbf{E}^\epsilon_{\omega,\mu}$ can be regarded as a function defined in $\mathbb{R}^2$. Define $\mathbf{E}^{\epsilon,r}_{\theta,\mu}(x) \equiv \mathbf{E}^\epsilon_{\theta,\mu}(rx)$ on Ω/r for any $\epsilon, r, \theta, \mu > 0$. Also define $\mathbf{S}^{;r}_R(x) \equiv B_R(x) \cap \Omega/r$, $\mathbf{S}^{\epsilon,r}_{\mu,f,R}(x) \equiv B_R(x) \cap \Omega^\epsilon_{\mu,f}/r$, and $\mathbf{S}^{\epsilon,r}_{\mu,m,R}(x) \equiv B_R(x) \cap \Omega^\epsilon_{\mu,m}/r$.

Now we make the following statements:

(A1). Ω is a bounded $C^{1,\gamma}$ domain in $\mathbb{R}^2$;
(A2). $\omega \in (1, \infty)$, $\omega\mu = 1$;
(A3). $\epsilon \in (0, 1)$, $p \in (2, \infty)$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{2}{p'} = \tau \in (1, 2)$, $\gamma = \frac{p-2}{p}$.

(A2) and (A3) imply (i) $\omega^2|\mathfrak{D}^\epsilon_{\mu,m}| \approx 1$ and (ii) $\epsilon, \mu (= \frac{1}{\omega})$ are free. We know that under $\omega > 1$, $\omega^2|\mathfrak{D}^\epsilon_{\mu,m}| \approx 1$, and $\epsilon^2|\ln \mu| \approx 1$, the nonlocal term appears in the homogenized macroscopic equations of (1.1) (see [4, Theorem A], [7, Eq (1.2)], and [8, Theorem 3]). Our results can be applied to the equations considered in [4, 7, 8]; the results are below.

Theorem 2.1. *Under (A1)–(A3),* $\mathbf{L} > 0$, $q \in (2, \infty)$, *and* $F \in L^p(\mathfrak{D})$, *there is a constant c independent of* ϵ, ω, μ *such that the solution of (1.1) satisfies*

$$\|\mathbf{E}^\epsilon_{\omega^{2/q},\mu}\partial_{x'} U\|_{L^q(\mathfrak{D})} + \|\mathbf{E}^\epsilon_{\omega^{2/p},\mu}\partial_{x_3} U\|_{L^p(\mathfrak{D})} \le c\Big(\sum_{k=1}^\infty \|\mathbf{E}^\epsilon_{1/\omega,\mu}\mathbb{F}_k\|^s_{L^2(\Omega)}\Big)^{1/s} + c\Big(\sum_{k=1}^\infty \|\mathbf{E}^\epsilon_{1/\omega^\tau,\mu}\mathbb{F}_k\|^t_{L^p(\Omega)}\Big)^{1/t},$$

where $s, t \in (1, \infty)$ *and* $\vartheta \in (0, \frac{1}{\frac{q}{2}-1})$ *satisfy* $\frac{1}{s} + \frac{1}{s'} = 1$, $(1 + \vartheta - \frac{q}{2}\vartheta)s' > 1$, $\frac{2s'}{q} > 1$, $\frac{1}{t} + \frac{1}{t'} = 1$, *and* $\vartheta t' > 1$. *Here,* ∂_{x_3} *(resp.* $\partial_{x'}$*) is the partial derivative with respect to the* x_3 *(resp.* x'*) variable. If* π *is the ratio of the circumference of a circle to its diameter,* $\mathbb{F}_k : \Omega \to \mathbb{R}$ *for* $k \in \mathbb{N}$ *are the Fourier sine coefficients of F in (1.1) satisfying* $F(x', x_3) \equiv \sum_{k=1}^\infty \mathbb{F}_k(x') \sin(\frac{k\pi}{\mathbf{L}}x_3)$.

Theorem 2.1 shows the $W^{1,q}$ estimate uniformly in $\epsilon, \mu (= \frac{1}{\omega})$ for problem (1.1) in the domain $\mathfrak{D}$. High integrability for the derivatives of the solutions of (1.1) is derived. The solutions are less oscillatory in the fibers $\mathfrak{D}^\epsilon_{\mu,m}$ than in the connected region $\mathfrak{D}^\epsilon_{\mu,f}$. Moreover, to obtain high integrability for the derivatives of the solutions, more constraints are required in the horizontal directions than in the vertical direction. Theorem 2.1 implies the following result.

Corollary 2.1. *If (A1)–(A3),* $\mathbf{L} > 0$, $q \in (2, \infty)$, $F(x', 0) = F(x', \mathbf{L}) = 0$, $F \in L^p(\mathfrak{D})$, *and* $\partial_{x_3}F \in L^p(\mathfrak{D})$, *there is a constant c independent of* ϵ, ω, μ *such that the solution of (1.1) satisfies*

$$\begin{aligned}\|\mathbf{E}^\epsilon_{\omega^{2/q},\mu}\partial_{x'} U\|_{L^q(\mathfrak{D})} + \|\mathbf{E}^\epsilon_{\omega^{2/p},\mu}\partial_{x_3} U\|_{L^p(\mathfrak{D})} &\le c\Big(\int_\Omega\Big|\int_0^{\mathbf{L}} |\mathbf{E}^\epsilon_{1/\omega,\mu}(x')\partial_{x_3}F(x', x_3)|\, dx_3\Big|^2 dx'\Big)^{1/2}\\ &+c\Big(\int_\Omega\Big|\int_0^{\mathbf{L}} |\mathbf{E}^\epsilon_{1/\omega^\tau,\mu}(x')\partial_{x_3}F(x', x_3)|\, dx_3\Big|^p dx'\Big)^{1/p}.\end{aligned}$$

Remark 2.1. *The constant c is independent of ϵ, ω, μ but depends on p, q.*

Proof. The Fourier sine coefficients $\mathbb{F}_k$ of F in (1.1) can be written, by integration by parts and $F(x', 0) = F(x', \mathbf{L}) = 0$, as

$$\mathbb{F}_k(x') \equiv \frac{2}{\mathbf{L}} \int_0^{\mathbf{L}} F(x', x_3) \sin(\frac{k\pi}{\mathbf{L}} x_3)\, dx_3 = \frac{-2}{k\pi} \int_0^{\mathbf{L}} \partial_{x_3} F(x', x_3) \cos(\frac{k\pi}{\mathbf{L}} x_3)\, dx_3.$$

Therefore, for $s \in (1, \infty)$,

$$\begin{aligned}\sum_{k=1}^{\infty} \|\mathbf{E}^{\epsilon}_{1/\omega,\mu} \mathbb{F}_k\|^s_{L^2(\Omega)} &= \sum_{k=1}^{\infty} \Big(\int_{\Omega} |\mathbf{E}^{\epsilon}_{1/\omega,\mu} \mathbb{F}_k|^2 dx' \Big)^{s/2} \\ &\le c \sum_{k=1}^{\infty} \frac{1}{k^s} \Big(\int_{\Omega} \Big| \int_0^{\mathbf{L}} \big| \mathbf{E}^{\epsilon}_{1/\omega,\mu}(x') \partial_{x_3} F(x', x_3) \big|\, dx_3 \Big|^2 dx' \Big)^{s/2} \\ &\le c \Big(\int_{\Omega} \Big| \int_0^{\mathbf{L}} \big| \mathbf{E}^{\epsilon}_{1/\omega,\mu}(x') \partial_{x_3} F(x', x_3) \big|\, dx_3 \Big|^2 dx' \Big)^{s/2}. \end{aligned} \tag{2.2}$$

Similarly, for $t \in (1, \infty)$,

$$\sum_{k=1}^{\infty} \|\mathbf{E}^{\epsilon}_{1/\omega^\tau,\mu} \mathbb{F}_k\|^t_{L^p(\Omega)} \le c \Big(\int_{\Omega} \Big| \int_0^{\mathbf{L}} \big| \mathbf{E}^{\epsilon}_{1/\omega^\tau,\mu}(x') \partial_{x_3} F(x', x_3) \big|\, dx_3 \Big|^p dx' \Big)^{t/p}. \tag{2.3}$$

Corollary 2.1 follows from Theorem 2.1, (2.2), and (2.3). □

3. Proof of the main result

This section aims to prove Theorem 2.1.

Lemma 3.1. *Suppose (A1), $\omega \in (1, \infty)$, $\epsilon, \mu \in (0, 1)$, $\beta > 0$, and $G \in L^\lambda(\Omega)$ for $\lambda \in [2, \infty)$. A solution of*

$$\begin{cases} -\nabla \cdot (\mathbf{E}^{\epsilon}_{\omega^2,\mu} \nabla V) + \beta^2 \mathbf{E}^{\epsilon}_{\omega^2,\mu} V = G & \text{in } \Omega, \\ V = 0 & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

exists uniquely in $H^1(\Omega)$ and satisfies

$$\begin{cases} \big\|\beta^2 \mathbf{E}^{\epsilon}_{\omega^{2/\lambda},\mu} V\big\|_{L^\lambda(\Omega)} \le c \big\|\mathbf{E}^{\epsilon}_{\omega^{-2/\lambda'},\mu} G\big\|_{L^\lambda(\Omega)}, \\ \big\|\beta \mathbf{E}^{\epsilon}_{\omega,\mu} \nabla V\big\|_{L^2(\Omega)} \le c \big\|\mathbf{E}^{\epsilon}_{1/\omega,\mu} G\big\|_{L^2(\Omega)}, \end{cases} \tag{3.2}$$

where $\frac{1}{\lambda} + \frac{1}{\lambda'} = 1$ and c is a constant independent of $\epsilon, \omega, \mu, \beta$.

Proof. The unique existence of (3.1) is by the Lax-Milgram Theorem [17], and (3.2) is obtained by testing (3.1) against $|\beta^2 V|^{\lambda-2} \beta^2 V$ and applying the Hölder inequality. □

Lemma 3.2. *Under (A1)–(A3), $\beta > 1$, and $G \in L^p(\Omega)$, any solution of (3.1) satisfies*

$$\|\nabla V\|_{L^\infty(\Omega)} \le c \Big(\|\mathbf{E}^{\epsilon}_{1/\omega,\mu} G\|_{L^2(\Omega)} + \|\mathbf{E}^{\epsilon}_{1/\omega^\tau,\mu} G\|_{L^p(\Omega)} \Big),$$

where c is a constant independent of $\epsilon, \mu (= \frac{1}{\omega}), \beta$. See (A3) for τ, p.

Proof of Lemma 3.2 is given in Section 7. Lemmas 3.1 and 3.2 imply the following.

Lemma 3.3. *Under (A1)–(A3), $\beta > 1$, $G \in L^p(\Omega)$, and $q \in (2,\infty)$, any solution of (3.1) satisfies*

$$\left\|\beta^{\frac{2}{q}}\,\mathbf{E}^{\epsilon}_{\omega^{2/q},\mu}\nabla V\right\|_{L^q(\Omega)} \le c\left(\|\mathbf{E}^{\epsilon}_{1/\omega,\mu}G\|_{L^2(\Omega)} + \|\mathbf{E}^{\epsilon}_{1/\omega,\mu}G\|^{\frac{2}{q}}_{L^2(\Omega)}\|\mathbf{E}^{\epsilon}_{1/\omega^{\tau},\mu}G\|^{1-\frac{2}{q}}_{L^p(\Omega)}\right),$$

where c is a constant independent of $\epsilon, \mu\,(=\frac{1}{\omega}), \beta, q$. See (A3) for τ, p.

Proof. By $(3.2)_2$ and Lemma 3.2,

$$\|\beta^{2/q}\,\mathbf{E}^{\epsilon}_{\omega^{2/q},\mu}\nabla V\|^q_{L^q(\Omega)} \le c\,\|\beta\,\mathbf{E}^{\epsilon}_{\omega,\mu}\nabla V\|^2_{L^2(\Omega)}\,\|\nabla V\|^{q-2}_{L^\infty(\Omega)} \le c\left(\|\mathbf{E}^{\epsilon}_{1/\omega,\mu}G\|^q_{L^2(\Omega)} + \|\mathbf{E}^{\epsilon}_{1/\omega,\mu}G\|^2_{L^2(\Omega)}\|\mathbf{E}^{\epsilon}_{1/\omega^{\tau},\mu}G\|^{q-2}_{L^p(\Omega)}\right).$$

□

The Fourier sine representations of U, F in (1.1) are

$$\begin{cases} U(x', x_3) = \displaystyle\sum_{k=1}^{\infty} \mathbb{U}_k(x')\sin\left(\frac{k\pi}{\mathbf{L}}x_3\right), \\ F(x', x_3) = \displaystyle\sum_{k=1}^{\infty} \mathbb{F}_k(x')\sin\left(\frac{k\pi}{\mathbf{L}}x_3\right). \end{cases}$$

Comparing the Fourier coefficients, we obtain, for any $k \in \mathbb{N}$,

$$\begin{cases} -\nabla\cdot(\mathbf{E}^{\epsilon}_{\omega^2,\mu}\nabla\mathbb{U}_k) + \frac{k^2\pi^2}{\mathbf{L}^2}\mathbf{E}^{\epsilon}_{\omega^2,\mu}\mathbb{U}_k = \mathbb{F}_k & \text{in } \Omega, \\ \mathbb{U}_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Proof of Theorem 2.1. By Lemmas 3.1 and 3.3, the solution of (3.3) satisfies

$$\begin{cases} k^2\|\mathbf{E}^{\epsilon}_{\omega^{2/p},\mu}\mathbb{U}_k\|_{L^p(\Omega)} \le c\|\mathbf{E}^{\epsilon}_{\omega^{-2/p'},\mu}\mathbb{F}_k\|_{L^p(\Omega)} = c\|\mathbf{E}^{\epsilon}_{1/\omega^{\tau},\mu}\mathbb{F}_k\|_{L^p(\Omega)}, \\ k^{2/q}\|\mathbf{E}^{\epsilon}_{\omega^{2/q},\mu}\nabla\mathbb{U}_k\|_{L^q(\Omega)} \le c\left(\|\mathbf{E}^{\epsilon}_{1/\omega,\mu}\mathbb{F}_k\|_{L^2(\Omega)} + \|\mathbf{E}^{\epsilon}_{1/\omega,\mu}\mathbb{F}_k\|^{2/q}_{L^2(\Omega)}\|\mathbf{E}^{\epsilon}_{1/\omega^{\tau},\mu}\mathbb{F}_k\|^{1-2/q}_{L^p(\Omega)}\right), \end{cases} \tag{3.4}$$

where $k \in \mathbb{N}$, $q \in (2,\infty)$, and c is independent of $\epsilon, \mu(=\frac{1}{\omega}), k, q$. See (A3) for τ, p, p'. By (3.4) and Young's inequality,

$$\begin{aligned} &\|\mathbf{E}^{\epsilon}_{\omega^{2/q},\mu}\partial_{x'}U\|_{L^q(\mathfrak{D})} + \|\mathbf{E}^{\epsilon}_{\omega^{2/p},\mu}\partial_{x_3}U\|_{L^p(\mathfrak{D})} \le c\sum_{k=1}^{\infty}\left(\|\mathbf{E}^{\epsilon}_{\omega^{2/q},\mu}\nabla\mathbb{U}_k\|_{L^q(\Omega)} + k\|\mathbf{E}^{\epsilon}_{\omega^{2/p},\mu}\mathbb{U}_k\|_{L^p(\Omega)}\right) \\ &\quad\le c\sum_{k=1}^{\infty}\left(k^{\frac{-2}{q}}\|\mathbf{E}^{\epsilon}_{1/\omega,\mu}\mathbb{F}_k\|^{2/q}_{L^2(\Omega)}\|\mathbf{E}^{\epsilon}_{1/\omega^{\tau},\mu}\mathbb{F}_k\|^{1-2/q}_{L^p(\Omega)} + k^{\frac{-2}{q}}\|\mathbf{E}^{\epsilon}_{1/\omega,\mu}\mathbb{F}_k\|_{L^2(\Omega)} + k^{-1}\|\mathbf{E}^{\epsilon}_{1/\omega^{\tau},\mu}\mathbb{F}_k\|_{L^p(\Omega)}\right) \\ &\quad\le c\sum_{k=1}^{\infty}\left((k^{\frac{q}{2}\vartheta-1-\vartheta} + k^{\frac{-2}{q}})\|\mathbf{E}^{\epsilon}_{1/\omega,\mu}\mathbb{F}_k\|_{L^2(\Omega)} + (k^{-\vartheta} + k^{-1})\|\mathbf{E}^{\epsilon}_{1/\omega^{\tau},\mu}\mathbb{F}_k\|_{L^p(\Omega)}\right), \end{aligned}$$

for any $\vartheta > 0$. Suppose $\vartheta \in (0, \frac{1}{\frac{q}{2}-1})$, there exist $s, t \in (1,\infty)$ such that $\frac{1}{s} + \frac{1}{s'} = 1$, $(1 + \vartheta - \frac{q}{2}\vartheta)s' > 1$, $\frac{2s'}{q} > 1$, $\frac{1}{t} + \frac{1}{t'} = 1$, and $\vartheta t' > 1$. By the Cauchy inequality,

$$\sum_{k=1}^{\infty}(k^{\frac{q}{2}\vartheta-1-\vartheta} + k^{\frac{-2}{q}})\|\mathbf{E}^{\epsilon}_{1/\omega,\mu}\mathbb{F}_k\|_{L^2(\Omega)} \le c\left(\sum_{k=1}^{\infty}\|\mathbf{E}^{\epsilon}_{1/\omega,\mu}\mathbb{F}_k\|^s_{L^2(\Omega)}\right)^{1/s},$$

$$\sum_{k=1}^{\infty}(k^{-\vartheta}+k^{-1})\|\mathbf{E}^{\epsilon}_{1/\omega^{\tau},\mu}\mathbb{F}_k\|_{L^p(\Omega)}\leq c\Big(\sum_{k=1}^{\infty}\|\mathbf{E}^{\epsilon}_{1/\omega^{\tau},\mu}\mathbb{F}_k\|^t_{L^p(\Omega)}\Big)^{1/t}.$$

Theorem 2.1 follows from the above estimates. □

We are ready to show Lemma 3.2. From now on, the functions are defined in 2-D space. In Sections 4 and 5 below, we study the convergence of strongly elliptic 2-D equations with "$uniform\ L^1$" (or "$uniform\ L^1 \setminus L^s\ for\ s>1$") external sources. See Remark 4.1 for the definition of "$uniform\ L^1$" (or "$uniform\ L^1 \setminus L^s\ for\ s>1$"). Consider the following problem:

$$\begin{cases}-\nabla\cdot(\mathbf{E}^{\epsilon,r}_{\omega^2,\mu}\nabla\Psi)=G & \text{in } B_1(0)\cap\Omega/r,\\ \Psi=\Psi_b & \text{on } B_1(0)\cap\partial\Omega/r,\end{cases}\tag{3.5}$$

where $\epsilon,\mu,r\in(0,1)$, $\omega\in(1,\infty)$, Ψ_b is smooth, and $G\in$ "$uniform\ L^1$" (or "$uniform\ L^1\setminus L^s\ for\ s>1$"). Here, $\mathbf{E}^{\epsilon,r}_{\omega^2,\mu}$ is a highly oscillatory function and G is a non-smooth function. We plan to show uniform convergence for Eq (3.5) in Section 4 and L^2-gradient convergence for Eq (3.5) in Section 5.

4. Uniform convergence

This section studies uniform convergence for the elliptic Eq (3.5). Convergence analysis for the case of *periodic size* $\leq\mu$ (resp. $\mu\leq$ *periodic size*) is in Subsection 4.1 (resp. Subsection 4.2).

4.1. Convergence for periodic size $\leq\mu$

We first show a uniform Lipschitz estimate for diffraction problems. Then we show a maximal principle and uniform convergence for Eq (3.5). Lemmas 4.3 and 4.4, and Remark 4.2 are true under *periodic size* $\leq\mu$; the external sources G in Lemma 4.4 and Remark 4.2 belong to "$uniform\ L^1$" (or "$uniform\ L^1\setminus L^s\ for\ s>1$").

For any $\theta,\nu\in(0,\infty)$ and $\mu\in(0,1)$, define a periodic function $\mathbf{K}_{\theta,\mu}$ (resp. $\mathbf{K}^{\nu}_{\theta,\mu}$) with a period $\mathcal{Y}$ (resp. $\nu\mathcal{Y}$) in $\mathbb{R}^2$ as

$$\begin{cases}\mathbf{K}_{\theta,\mu}(z)\equiv\theta\mathcal{X}_{\mathcal{Y}_{\mu,m}}(z)+\mathcal{X}_{\mathcal{Y}_{\mu,f}}(z) & \text{if } z\in\mathcal{Y},\\ \mathbf{K}^{\nu}_{\theta,\mu}(x)\equiv\mathbf{K}_{\theta,\mu}(\frac{x}{\nu}) & \text{if } x\in\mathbb{R}^2.\end{cases}\tag{4.1}$$

Lemma 4.1. *Suppose that $\omega\in(1,\infty)$, $\mu\in(0,1)$, $\frac{|\ln\mu|}{\omega^2\mu}\leq 1$, $p>2$, $G\in L^p(\mathcal{Y})$, and $Q\in C^{\gamma}(\mathcal{Y})$ for $\gamma\equiv\frac{p-2}{p}$. Any solution of*

$$-\nabla\cdot(\mathbf{K}_{\omega^2,\mu}(\nabla\Psi+Q))=G\qquad in\ \mathcal{Y}\tag{4.2}$$

satisfies $\|\nabla\Psi\|_{L^{\infty}(B_{2/5}(0))}\leq c(\|\Psi\|_{L^2(\mathcal{Y}\setminus B_{1/4}(0))}+\|Q\|_{C^{\gamma}(\mathcal{Y})}+\|\mathbf{K}_{1/\omega^2,\mu}G\|_{L^p(\mathcal{Y})})$, where c is a constant independent of ω,μ.

Lemma 4.1 is proved in Section 8. We follow the proof of Lemma 12 in [8] to see the following.

Lemma 4.2. *Suppose that (A2), $\epsilon,\frac{\epsilon}{r}\in(0,1)$, $r>0$, and $\varkappa\geq 2$ and suppose that $\mathbf{D}$ is a Lipschitz subdomain of Ω/r. In this case, there is a constant c (independent of $\epsilon,\omega,\mu,r,|\mathbf{D}|$) such that, for any $\zeta\in H^1_0(\mathbf{D})$,*

$$\|\mathbf{E}^{\epsilon,r}_{\omega^{2/\varkappa},\mu}\zeta\|_{L^{\varkappa}(\mathbf{D})}\leq c\max\left\{1,|\frac{\epsilon}{r}|^{\frac{2}{\varkappa}}|\ln\mu|^{\frac{1}{2}}\right\}\|\mathbf{E}^{\epsilon,r}_{\omega,\mu}\nabla\zeta\|_{L^2(\mathbf{D})}.$$

Consider the equation

$$-\nabla\cdot(\mathbf{E}^{\epsilon,r}_{\omega^2,\mu}\nabla\Psi)=G \qquad \text{in } \mathbf{D}\ (\subset\Omega/r). \tag{4.3}$$

Modifying the argument of Lemma 11 [8], we have the following result.

Lemma 4.3. *If (A2), (A3), $\frac{\epsilon}{r}\in(0,1)$, $r>0$, $G\in L^p(\mathbf{D})$, and $\mathbf{D}$ is a bounded Lipschitz subdomain of Ω/r and if $\varkappa>2$, $0<\frac{6}{\varkappa}<\tau$, and $|\frac{\epsilon}{r}|^{\frac{4}{\varkappa}}|\ln\mu|$ are bounded independent of ϵ,ω,μ,r, then any weak solution $\Psi\in H^1(\mathbf{D})$ of (4.3) satisfies*

$$\begin{cases}\sup\limits_{\mathbf{D}}\Psi\le\sup\limits_{\partial\mathbf{D}}\Psi^+ + c(p,\varkappa)|\mathbf{D}|^{1-\frac{1}{p}-\frac{2}{\varkappa}}\left\|\mathbf{E}^{\epsilon,r}_{1/\omega^\tau,\mu}G\right\|_{L^p(\mathbf{D})},\\ \sup\limits_{\mathbf{D}}(-\Psi)\le\sup\limits_{\partial\mathbf{D}}(\Psi^-) + c(p,\varkappa)|\mathbf{D}|^{1-\frac{1}{p}-\frac{2}{\varkappa}}\left\|\mathbf{E}^{\epsilon,r}_{1/\omega^\tau,\mu}G\right\|_{L^p(\mathbf{D})},\end{cases} \tag{4.4}$$

where $\Psi^+\equiv\max\{0,\Psi\},\Psi^-\equiv\max\{0,-\Psi\}$, and c is a constant independent of $\epsilon,\omega,\mu,r,|\mathbf{D}|$. Note $\frac{1}{\varkappa}<1-\frac{1}{p}-\frac{2}{\varkappa}<1$ and see (A3) for p,p',τ.

Proof. Consider $(4.4)_1$ first. By Lemma 4.1, any solution of (4.3) is a Lipschitz function. Set $\mathfrak{X}\equiv\|\mathbf{E}^{\epsilon,r}_{1/\omega^\tau,\mu}G\|_{L^p(\mathbf{D})}$, $\mathscr{C}\equiv\sup\limits_{\partial\mathbf{D}}\Psi^+$, and $\mathfrak{A}(k)\equiv\{x\in\mathbf{D}|\ \Psi(x)\ge k\}$ for any $k>\mathscr{C}$. Note that $\mathfrak{A}(k)$ is a Lipschitz subdomain of $\mathbf{D}$. Since $\varkappa>2$ and $\frac{6}{\varkappa}<\tau=\frac{2}{p'}$, there is a $\theta\in(1,\frac{\varkappa}{2})$ satisfying $\frac{1}{p}+\frac{1}{\varkappa}+\frac{1}{\theta}=1$. Test (4.3) against $\zeta=(\Psi-k)^+\in H^1_0(\mathbf{D})$ and apply Lemma 4.2 to obtain

$$\begin{aligned}\mathbf{m}_2\left\|\mathbf{E}^{\epsilon,r}_{\omega^\sigma,\mu}(\Psi-k)^+\right\|^2_{L^\varkappa(\mathfrak{A}(k))}&\le\int_{\mathfrak{A}(k)}\mathbf{E}^{\epsilon,r}_{\omega^2,\mu}\nabla\Psi\nabla\zeta\,dz=\int_{\mathfrak{A}(k)}G\,(\Psi-k)^+dz\\ &\le\mathfrak{X}\left\|\mathbf{E}^{\epsilon,r}_{\omega^\sigma,\mu}(\Psi-k)^+\right\|_{L^\varkappa(\mathfrak{A}(k))}\left\|\mathbf{E}^{\epsilon,r}_{\omega^{\tau-\sigma},\mu}\right\|_{L^\theta(\mathfrak{A}(k))}\le\mathfrak{X}\left\|\mathbf{E}^{\epsilon,r}_{\omega^\sigma,\mu}(\Psi-k)^+\right\|_{L^\varkappa(\mathfrak{A}(k))}|2\,\mathfrak{A}(k)|^{\frac{\lambda}{\varkappa}},\end{aligned} \tag{4.5}$$

where $\sigma\equiv\frac{2}{\varkappa}$, $\lambda\equiv\frac{\varkappa}{\theta}>1$, and $\mathbf{m}_2$ is a constant independent of $\epsilon,\omega,\mu,r,|\mathfrak{A}(k)|$. Note that $\frac{1}{p}+\frac{1}{\varkappa}+\frac{1}{\theta}=1$ implies $(\tau-\sigma)\theta=2$. For any $\mathscr{C}<h<k$,

$$|\mathfrak{A}(k)|\le\int_{\mathfrak{A}(k)}\frac{|(\Psi-h)^+|^\varkappa}{(k-h)^\varkappa}dz=\frac{\|(\Psi-h)^+\|^\varkappa_{L^\varkappa(\mathfrak{A}(k))}}{(k-h)^\varkappa}.$$

For any $\mathscr{C}<h<k$, by (4.5),

$$\|(\Psi-k)^+\|_{L^\varkappa(\mathfrak{A}(k))}\le\frac{\mathfrak{X}}{\mathbf{m}_2}\,2^{\frac{\lambda}{\varkappa}}\,\frac{\|(\Psi-h)^+\|^\lambda_{L^\varkappa(\mathfrak{A}(k))}}{(k-h)^\lambda}. \tag{4.6}$$

Let us define

$$\delta\equiv\left|\frac{\mathfrak{X}}{\mathbf{m}_2}\right|^{1/\lambda}2^{\frac{\lambda}{\lambda-1}}2^{\frac{1}{\varkappa}}\left\|(\Psi-\mathscr{C})^+\right\|^{\frac{\lambda-1}{\lambda}}_{L^\varkappa(\mathfrak{A}(\mathscr{C}))}. \tag{4.7}$$

If $(\Psi-\mathscr{C})^+=0$ or $G=0$, then (4.5) implies $(4.4)_1$ true. By (4.7), $(\Psi-\mathscr{C})^+=0$ if and only if $\delta=0$. We consider $\delta>0$ below. Define

$$\begin{cases}t_j\equiv2\delta\,(1-\frac{1}{2^{j+1}})+\mathscr{C},\\ \aleph\equiv\frac{\mathfrak{X}}{\mathbf{m}_2}\left|\frac{2}{\delta}\right|^\lambda,\\ \ell_j\equiv\aleph^{\frac{1}{\lambda-1}}\|(\Psi-t_j)^+\|_{L^\varkappa(\mathfrak{A}(t_j))},\end{cases}\qquad\text{for } j\ge0. \tag{4.8}$$

By (4.7) and $(4.8)_2$,

$$\aleph = 2^{\frac{-\lambda}{\varkappa}} 2^{\frac{-\lambda}{\lambda-1}} \left\|(\Psi - \mathscr{C})^+\right\|^{1-\lambda}_{L^\varkappa(\mathfrak{A}(\mathscr{C}))}. \tag{4.9}$$

By $(4.8)_3$ with $j = 0$ and (4.9), $\ell_0 \le 2^{\frac{-\lambda}{\varkappa(\lambda-1)}} 2^{\frac{-\lambda}{(\lambda-1)^2}}$. By $(4.8)_{2,3}$, set $k = t_{j+1}$ and $h = t_j$ in (4.6) to see

$$\ell_{j+1} \le 2^{\frac{\lambda}{\varkappa}} 2^{j\lambda} \aleph^{\frac{\lambda}{\lambda-1}} \|(\Psi - t_j)^+\|^{\lambda}_{L^\varkappa(\mathfrak{A}(t_{j+1}))} \le 2^{\frac{\lambda}{\varkappa}} 2^{j\lambda}\, \ell_j^{\lambda}.$$

Lemma 4.7 in [19] implies $\ell_{j+1} \le 2^{\frac{-\lambda}{\varkappa(\lambda-1)}} 2^{\frac{-\lambda}{(\lambda-1)^2}} 2^{\frac{-j\lambda}{\lambda-1}}$. Since $\lambda \equiv \frac{\varkappa}{\theta} > 1$, we obtain $\lim_{j\to\infty} \ell_j = 0$. Since $\lim_{j\to\infty} t_j = 2\delta + \mathscr{C}$, by $(4.8)_3$,

$$\|(\Psi - \mathscr{C})^+\|_{L^\infty(\mathbf{D})} \le 2\delta. \tag{4.10}$$

Equations (4.7) and (4.10) imply $\delta \le |\mathbf{m}_2|^{\frac{-1}{\lambda}}\ 2^{\frac{\lambda}{\lambda-1}} 2^{\frac{1}{\varkappa}} 2^{\frac{\lambda-1}{\lambda}} \left|\mathfrak{X}\right|^{\frac{1}{\lambda}} \delta^{\frac{\lambda-1}{\lambda}} |\mathfrak{A}(\mathscr{C})|^{\frac{\lambda-1}{\varkappa\lambda}}$. So

$$\delta \le c(p, \varkappa, \mathbf{m}_2)|\mathfrak{A}(\mathscr{C})|^{\frac{\lambda-1}{\varkappa}}\,\mathfrak{X}. \tag{4.11}$$

Since $\lambda \equiv \frac{\varkappa}{\theta}$, $\frac{6}{\varkappa} < \frac{2}{p'}$, and $\frac{1}{p} + \frac{1}{\varkappa} + \frac{1}{\theta} = 1$, we have $\frac{\lambda-1}{\varkappa} = 1 - \frac{1}{p} - \frac{2}{\varkappa} > \frac{1}{\varkappa} > 0$. By (4.10) and (4.11),

$$\|(\Psi - \mathscr{C})^+\|_{L^\infty(\mathbf{D})} \le c(p, \varkappa, \mathbf{m}_2)|\mathbf{D}|^{1-\frac{1}{p}-\frac{2}{\varkappa}}\,\mathfrak{X}. \tag{4.12}$$

Equation $(4.4)_1$ follows from (4.12). To obtain $(4.4)_2$, we simply replace Ψ by $-\Psi$ and repeat the argument for Case $(4.4)_1$. □

By translation, we can move any point $z \in \partial\Omega$ to 0. By (A1), there is a $\gamma_* > 0$ (independent of z) and a $C^{1,\gamma}$ function $\Upsilon : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} \Upsilon(0) = 0, \\ \|\nabla\Upsilon\|_{L^\infty(\mathbb{R})} \le \mathbf{m}_3, \\ B_{\gamma_*}(0) \cap \Omega = B_{\gamma_*}(0) \cap \{(x_1, x_2)|\ x_1 \in \mathbb{R},\ x_2 > \Upsilon(x_1)\}, \\ \mathbf{m}_4\, x_2 \le dist(x, \partial\Omega) \le \mathbf{m}_5\, x_2 \quad \text{for any } x = (0, x_2) \in B_{\gamma_*}(0) \cap \Omega. \end{cases} \tag{4.13}$$

Here, $dist(x, \partial\Omega)$ is the distance from x to $\partial\Omega$; $\gamma_*, \mathbf{m}_3, \mathbf{m}_4, \mathbf{m}_5$ are positive numbers independent of $z\,(\in \partial\Omega)$, $x\,(\in B_{\gamma_*}(0) \cap \Omega)$. Recall Section 2 for $\mathbf{S}^{;r}_R(x) \equiv B_R(x) \cap \Omega/r$.

Lemma 4.4. *Consider the following problem:*

$$\begin{cases} -\nabla \cdot (\mathbf{E}^{\epsilon,r}_{\omega^2,\mu}\nabla\Psi) = G & \text{in } B_1(0) \cap \Omega/r, \\ \Psi = \Psi_b & \text{on } B_1(0) \cap \partial\Omega/r. \end{cases} \tag{4.14}$$

Suppose that a sequence $\{\epsilon_k, \omega_k, \mu_k, r_k, \Psi_k, G_k, \Psi_{b,k}, \mathbf{S}^{;r_k}_1(0)\}$ *and a set* $\mathbf{S}$ *satisfy (A1)–(A3),* $\frac{\epsilon_k}{r_k}, r_k, \alpha \in (0, 1)$, $\mathbf{d} \in (1, 2)$*, (4.13), and (4.14) and suppose the following:*

(i) $\varkappa > 2$, $0 < \frac{6}{\varkappa} < \tau$, *and* $|\frac{\epsilon_k}{r_k}|^{\frac{4}{\varkappa}}|\ln\mu_k|$ *are bounded independent of* k*;*
(ii) $\|\Psi_k\|_{W^{1,\mathbf{d}}(\mathbf{S}^{;r_k}_1(0))}$, $\|\mathbf{E}^{\epsilon_k,r_k}_{1/\omega^\tau_k,\mu_k} G_k\|_{L^p(\mathbf{S}^{;r_k}_1(0))}$, *and* $\|\Psi_{b,k}\|_{C^{1,\alpha}(\mathbf{S}^{;r_k}_1(0))}$ *are bounded independent of* k*;*
(iii) $|\mathbf{S}^{;r_k}_1(0) \setminus \mathbf{S}| + |\mathbf{S} \setminus \mathbf{S}^{;r_k}_1(0)| \to 0$ *as* $k \to \infty$*;*
(iv) $\Psi_k \in W^{1,\mathbf{d}}(\mathbf{S}^{;r_k}_1(0)) \cap C(\mathbf{S}^{;r_k}_1(0))$ *converges to* $\Psi \in W^{1,\mathbf{d}}(\mathbf{S}) \cap C(\mathbf{S})$ *weakly in* $W^{1,\mathbf{d}}(\mathbf{S})$ *as* $k \to \infty$*,*

then $\|\Psi_k - \Psi\|_{L^\infty(\mathbf{S}_1^{;r_k}(0)\cap\mathbf{S})} \to 0$ *as* $k \to \infty$*. See (A3) for* p, τ*.*

Remark 4.1. *Although each source function* G_k *in Lemma 4.4 is in* L^p *space, the whole sequence* $\{G_k\}_{k=1}^\infty$ *is bounded (uniformly in* $k, \epsilon_k, \omega_k, \mu_k, r_k$*) only in* L^1 *space. This is the reason we say that* $G_k \in$ *"uniform* L^1*" (or "uniform* $L^1 \setminus L^s$ *for* $s > 1$*"). We now show that the sequence* $\{G_k\}_{k=1}^\infty$ *in Lemma 4.4 is bounded (uniformly in* $k, \epsilon_k, \omega_k, \mu_k, r_k$*) in* L^1 *space. By the Hölder inequality, (ii) of Lemma 4.4, and (A3),*

$$\|G_k\|_{L^1(\mathbf{S}_1^{;r_k}(0))} \le \|\mathbf{E}^{\epsilon_k, r_k}_{1/\omega_k^\tau, \mu_k} G_k\|_{L^p(\mathbf{S}_1^{;r_k}(0))} \|\mathbf{E}^{\epsilon_k, r_k}_{\omega_k^\tau, \mu_k}\|_{L^{p'}(\mathbf{S}_1^{;r_k}(0))} \le c,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ *and* c *is a constant independent of* $k, \epsilon_k, \omega_k, \mu_k, r_k$*.*

Proof. **Step I.** By Lemma 6.37 [17], for any $r, \alpha \in (0,1)$, there exists an extension operator Π_r : $C^{1,\alpha}(\mathbf{S}_1^{;r}(0)) \to C^{1,\alpha}(B_1(0))$ such that, for any $\zeta \in C^{1,\alpha}(\overline{\mathbf{S}_1^{;r}(0)})$,

$$\begin{cases} \Pi_r\, \zeta = \zeta & \text{in } \mathbf{S}_1^{;r}(0), \\ \|\Pi_r\, \zeta\|_{C^{1,\alpha}(B_1(0))} \le c\|\zeta\|_{C^{1,\alpha}(\mathbf{S}_1^{;r}(0))}, \end{cases}$$

where c is a constant independent of r. Define $\phi_k \equiv \Psi_k \mathcal{X}_{\mathbf{S}_1^{;r_k}(0)} + \Pi_{r_k}\Psi_{b,k}\mathcal{X}_{B_1(0)\setminus\mathbf{S}_1^{;r_k}(0)}$. By (ii) of the assumptions, $\|\phi_k\|_{W^{1,\mathbf{d}}(B_1(0))} + \|\Pi_{r_k}\Psi_{b,k}\|_{C^{1,\alpha}(B_1(0))}$ are bounded independent of k. There are $\widehat{\Psi}, \Psi_b \in C(B_1(0))$ and a subsequence of $\{\phi_k, \Pi_{r_k}\Psi_{b,k}\}$ (with the same notation for the subsequence) such that, by (iii) and (iv) of the assumptions, and Theorem 7 in [14], we have

$$\begin{cases} \phi_k \text{ converges quasi-uniformly to } \widehat{\Psi} \in C(B_1(0)) \text{ and } \widehat{\Psi}|_{\mathbf{S}} = \Psi, \\ \lim\limits_{k\to 0} \|\Pi_{r_k}\Psi_{b,k} - \Psi_b\|_{C^1(B_1(0))} + \lim\limits_{k\to 0} \|\phi_k - \Psi_b\|_{C^1(B_1(0)\setminus\mathbf{S}_1^{;r_k}(0))} = 0, \\ |\Psi_b(x) - \widehat{\Psi}(x)| = 0 \quad \text{if } x \in B_1(0) \setminus \bigcup_{k=1}^\infty \mathbf{S}_1^{;r_k}(0). \end{cases} \tag{4.15}$$

Step II. Claim a subsequence of ϕ_k converges to $\widehat{\Psi}$ in $C(B_{\frac{1}{2}}(0))$. If this is true, the whole sequence ϕ_k converges to $\widehat{\Psi}$ in $C(B_{\frac{1}{2}}(0))$ by a contradiction argument.

Proof of the claim. If $m\, (\in \mathbb{N}) > 6$, by $\widehat{\Psi}, \Psi_b \in C(B_1(0))$, there is a $\delta_m \in (0, \frac{1}{m})$ satisfying

$$\begin{cases} |\widehat{\Psi}(x) - \widehat{\Psi}(y)| < \frac{1}{2m}, \\ |\Psi_b(x) - \widehat{\Psi}(x) - \Psi_b(y) + \widehat{\Psi}(y)| < \frac{1}{4m}, \end{cases} \quad \text{if } x, y \in \overline{B_{2/3}(0)},\ |x-y| \le \delta_m. \tag{4.16}$$

Let $\mathfrak{C}_s(\mathfrak{L})$ for $s \in (1, \mathbf{d})$ be the s-capacity of a unit segment $\mathfrak{L}$ in $\mathbb{R}^2$ (see [10, page 458]). For any $s \in (1, \mathbf{d})$, by [14, Theorem 7] and $(4.15)_1$, there is an open subset $O_{s,m} \subset B_{2/3}(0)$ and a number $\mathcal{N}_{s,m} \in \mathbb{N}$ so that

$$\begin{cases} \mathfrak{C}_s(O_{s,m}) < \mathfrak{C}_s(\mathfrak{L})|\delta_m|^{2-s}, \\ |\phi_k(x) - \widehat{\Psi}(x)| < |2m|^{-1} \quad \text{if } x \in B_{2/3}(0) \setminus O_{s,m},\ k \ge \mathcal{N}_{s,m}, \end{cases} \tag{4.17}$$

where $\mathfrak{C}_s(O_{s,m})$ is the s-capacity of $O_{s,m}$. Moreover, by $(4.15)_3$, $(4.16)_2$, and (A1),

$$|\Psi_b(x) - \widehat{\Psi}(x)| < |4m|^{-1} \quad \text{if } x \in B_{2/3}(0) \setminus \mathbf{S}_1^{;r_k}(0),\ k \ge \mathcal{N}_{s,m}. \tag{4.18}$$

Then $(4.15)_2$ and (4.18) imply

$$|\phi_k(x) - \widehat{\Psi}(x)| < |2m|^{-1} \qquad \text{if } x \in B_{2/3}(0) \setminus \mathbf{S}_1^{;r_k}(0),\ k \geq \mathcal{N}_{s,m}. \tag{4.19}$$

Find a connected component O of $O_{s,m}$ so that $\overline{O} \cap \overline{B_{1/2}(0)} \neq \emptyset$ (see Section 2 for the definition of the closure of a set). Since O is connected, for any $y, z \in O$, there is a curve $\mathbb{L}$ ($\subset O \subset O_{s,m}$) connecting y and z. By [10, Lemma 2.8] and $(4.17)_1$,

$$\mathfrak{C}_s(\mathfrak{L})|y-z|^{2-s} \leq \mathfrak{C}_s(\mathbb{L}) \leq \mathfrak{C}_s(O_{s,m}) \leq \mathfrak{C}_s(\mathfrak{L})|\delta_m|^{2-s}. \tag{4.20}$$

In this case, (4.20) implies $diam(O) \leq \delta_m$ (here, $diam(O)$ is the diameter of O). Therefore,

$$|O| \leq \pi|\delta_m|^2, \qquad \overline{O} \subset \overline{B_{1/2+\delta_m}(0)}, \qquad \partial O \subset \overline{B_{2/3}(0)} \setminus O_{s,m}. \tag{4.21}$$

If a sequence $\{\epsilon_k, \omega_k, \mu_k, r_k, \phi_k, G_k, \Psi_{b,k}, \mathbf{S}_1^{;r_k}(0)\}$ satisfies (4.14), then Lemma 4.3 and (4.21) imply, for $x \in O \cap \Omega/r_k$,

$$\begin{aligned} \min_{\partial(O\cap\Omega/r_k)} \phi_k - c|\delta_m|^{2-\frac{2}{p}-\frac{4}{\varkappa}} \|\mathbf{E}^{\epsilon_k, r_k}_{1/\omega_k^\tau, \mu_k} G_k\|_{L^p(\mathbf{S}_1^{;r_k}(0))} &\leq \phi_k(x) \\ &\leq \max_{\partial(O\cap\Omega/r_k)} \phi_k + c|\delta_m|^{2-\frac{2}{p}-\frac{4}{\varkappa}} \|\mathbf{E}^{\epsilon_k, r_k}_{1/\omega_k^\tau, \mu_k} G_k\|_{L^p(\mathbf{S}_1^{;r_k}(0))}, \end{aligned} \tag{4.22}$$

where c is independent of $\epsilon_k, \omega_k, \mu_k, r_k, m, k, \delta_m$. By (4.16), $(4.17)_2$, (4.19), and (4.21),

$$\begin{cases} \min_{\partial(O\cap\Omega/r_k)} \phi_k \geq \min_{\partial(O\cap\Omega/r_k)} \widehat{\Psi} - \frac{1}{2m} \geq \widehat{\Psi}(x) - \frac{1}{m}, \\ \max_{\partial(O\cap\Omega/r_k)} \phi_k \leq \max_{\partial(O\cap\Omega/r_k)} \widehat{\Psi} + \frac{1}{2m} \leq \widehat{\Psi}(x) + \frac{1}{m}, \end{cases} \qquad \text{for } k \geq \mathcal{N}_{s,m},\ x \in \overline{O}. \tag{4.23}$$

By (4.22) and (4.23), for any $m \in \mathbb{N}$ and $k \geq \mathcal{N}_{s,m}$,

$$\|\phi_k - \widehat{\Psi}\|_{L^\infty(O\cap\Omega/r_k)} \leq \frac{1}{m} + c|\delta_m|^{2-\frac{2}{p}-\frac{4}{\varkappa}} \|\mathbf{E}^{\epsilon_k, r_k}_{1/\omega_k^\tau, \mu_k} G_k\|_{L^p(\mathbf{S}_1^{;r_k}(0))}. \tag{4.24}$$

Since $O_{s,m}$ is the union of its connected components, $(4.17)_2$ and (4.24) imply

$$\|\phi_k - \widehat{\Psi}\|_{L^\infty(B_{1/2}(0)\cap\Omega/r_k)} \leq \frac{1}{m} + c|\delta_m|^{2-\frac{2}{p}-\frac{4}{\varkappa}} \|\mathbf{E}^{\epsilon_k, r_k}_{1/\omega_k^\tau, \mu_k} G_k\|_{L^p(\mathbf{S}_1^{;r_k}(0))}, \tag{4.25}$$

for $m \in \mathbb{N},\ k \geq \mathcal{N}_{s,m}$. Note $\frac{1}{\varkappa} < 1 - \frac{1}{p} - \frac{2}{\varkappa} < 1$ by Lemma 4.3. By $\delta_m < \frac{1}{m}$, (4.19), (4.25), and (ii) of the assumptions, we prove the uniform convergence of ϕ_k. So we prove the claim, and Lemma 4.4 is proved. □

Remark 4.2. *Lemma 4.4 proves a convergence result for Eq (4.14) around the boundary of the domain. A straightforward modification of the argument of Lemma 4.4, we see that the convergence result is also true in the interior of the domain.*

Consider the following problem:

$$-\nabla \cdot (\mathbf{E}^{\epsilon, r}_{\omega^2, \mu} \nabla \Psi) = G \qquad \text{in } B_1(0)\ (\Subset \Omega/r). \tag{4.26}$$

Suppose that a sequence $\{\epsilon_k, \omega_k, \mu_k, r_k, \Psi_k, G_k\}$ *satisfies (A2) and (A3),* $\frac{\epsilon_k}{r_k}, r_k \in (0,1)$, $\mathbf{d} \in (1,2)$, *and (4.26) and suppose the following:*

(i) $\varkappa > 2$, $0 < \frac{6}{\varkappa} < \tau$, *and* $|\frac{\epsilon_k}{r_k}|^{\frac{4}{\varkappa}} |\ln \mu_k|$ *are bounded independent of* k;

(ii) $\|\Psi_k\|_{W^{1,\mathbf{d}}(B_1(0))}, \|\mathbf{E}^{\epsilon_k, r_k}_{1/\omega_k^\tau, \mu_k} G_k\|_{L^p(B_1(0))}$ *are bounded independent of* k;

(iii) $\Psi_k \in W^{1,\mathbf{d}}(B_1(0)) \cap C(B_1(0))$ *converges to* $\Psi \in W^{1,\mathbf{d}}(B_1(0)) \cap C(B_1(0))$ *weakly in* $W^{1,\mathbf{d}}(B_1(0))$.

In this case, $\|\Psi_k - \Psi\|_{L^\infty(B_1(0))} \to 0$ *as* $k \to \infty$. *See (A3) for* p, τ.

4.2. Convergence for $\mu \le$ periodic size

We show a regularity result for interface problems and show local $L^{2+\delta}$-gradient estimates and a uniform convergence for (3.5). Lemmas 4.10–4.12 are true under $\mu \le$ *periodic size*; the external sources G in Lemmas 4.10–4.12 are "$uniform\ L^1$" (or "$uniform\ L^1 \setminus L^s\ for\ s > 1$"). Convergence (i.e., Corollary 4.1) is from Lemma 4.12 and Sobolev's embedding theorem [17].

Assume that A_1, A_2 are two positive functions and $\theta > 0$, and define

$$\begin{cases} \mathbf{T}_\theta(x) \equiv \mathcal{X}_{\{(z_1,z_2)|z_2\ge 0\}}(x) + \theta\mathcal{X}_{\{(z_1,z_2)|z_2<0\}}(x), \\ \mathcal{P}_\theta(x) \equiv A_1\mathcal{X}_{\{(z_1,z_2)|z_2\ge 0\}}(x) + \theta A_2\mathcal{X}_{\{(z_1,z_2)|z_2<0\}}(x). \end{cases} \tag{4.27}$$

Following the proof of Lemma 3.2 [29], we have the following result.

Lemma 4.5. *If $\omega, t, q \in (1,\infty)$, $\sigma \in [0,2]$, $0 < \mathbf{m}_6 < A_1, A_2 \in C^2(B_2(0))$ in (4.27), $Q \in L^2(B_1(0))$, and $G \in L^t(B_1(0))$, then any solution of $-\nabla\cdot(\mathcal{P}_{\omega^2}\nabla\Psi + Q) = G$ in $B_2(0)$ satisfies*

$$\|\mathbf{T}_{\omega^\sigma}\nabla\Psi\|_{L^2(B_{1/2}(0))} \le c\left(\|\mathbf{T}_{\omega^{\sigma-2}}Q\|_{L^2(B_1(0))} + \|\mathbf{T}_{\omega^{\sigma-2}}G\|_{L^t(B_1(0))}\right) + c\min\left\{\|\mathbf{T}_{\omega^\sigma}\Psi\|_{L^2(B_1(0))},\ |(|\mathbf{T}_{\omega^\sigma}\nabla\Psi|^q)_{B_1(0)}|^{1/q}\right\},$$

where c is a constant independent of ω, t, q, σ. See (2.1) for $(|\mathbf{T}_{\omega^\sigma}\nabla\Psi|^q)_{B_1(0)}$ and (4.27) for $\mathbf{T}_\theta, \mathcal{P}_\theta$.

$H^1(\mathbf{D})/\mathbb{R} \equiv \{\zeta \in H^1(\mathbf{D})|\ (\zeta)_{\mathbf{D}} = 0\}$ for any set $\mathbf{D}$. See (2.1) for $(\zeta)_{\mathbf{D}}$. Below is an extension result.

Lemma 4.6. *If $\mu \in (0,1)$, there is an operator $\widetilde{\Pi}_\mu : H^1(\mathcal{Y}_{\mu,m})/\mathbb{R} \to H^1_0(2\mathcal{Y}_{\mu,m})$ such that*

$$\begin{cases} \widetilde{\Pi}_\mu\zeta = \zeta & \text{in } \mathcal{Y}_{\mu,m}, \\ \|\widetilde{\Pi}_\mu\zeta\|_{H^1(2\mathcal{Y}_{\mu,m})} \le c\|\nabla\zeta\|_{L^2(\mathcal{Y}_{\mu,m})}, \end{cases} \quad \text{for } \zeta \in H^1(\mathcal{Y}_{\mu,m})/\mathbb{R},$$

where c is a constant independent of μ.

Proof. Let c be a constant independent of μ. By [17, Theorem 7.25 and the Poincaré inequality], there is an operator $\widetilde{\Pi} : H^1(B_1(0))/\mathbb{R} \to H^1_0(B_2(0))$ such that

$$\begin{cases} \widetilde{\Pi}\phi = \phi & \text{in } B_1(0), \\ \|\widetilde{\Pi}\phi\|_{H^1(B_2(0))} \le c\|\nabla\phi\|_{L^2(B_1(0))}, \end{cases} \quad \text{for } \phi \in H^1(B_1(0))/\mathbb{R}.$$

Extend $\widetilde{\Pi}\phi \in H^1_0(B_2(0))$ from $B_2(0)$ to $\mathbb{R}^2$ by 0 and consider $H^1_0(B_2(0)) \subset H^1(\mathbb{R}^2)$. Recall that $\mathcal{Y}_{\mu,m} = B_{\mu/4}(0)$. Define an operator $\widetilde{\Pi}_\mu : H^1(\mathcal{Y}_{\mu,m})/\mathbb{R} \to H^1_0(\mathcal{Y})$ as follows: Set $\phi(x) \equiv \zeta(\frac{\mu}{4}x)$ for any $\zeta \in H^1(\mathcal{Y}_{\mu,m})/\mathbb{R}$ and $x \in B_1(0)$, and set $\widetilde{\Pi}_\mu\zeta(y) \equiv \widetilde{\Pi}\phi(\frac{4}{\mu}y)$ for $\widetilde{\Pi}\phi \in H^1(B_2(0))$ and $y \in 2\mathcal{Y}_{\mu,m}$. Then

$$\begin{cases} \widetilde{\Pi}_\mu\zeta = \zeta & \text{in } \mathcal{Y}_{\mu,m}, \\ \|\widetilde{\Pi}_\mu\zeta\|_{H^1(2\mathcal{Y}_{\mu,m})} \le c\|\widetilde{\Pi}\phi\|_{H^1(B_2(0))} \le c\|\nabla\phi\|_{L^2(B_1(0))} \le c\|\nabla\zeta\|_{L^2(\mathcal{Y}_{\mu,m})}. \end{cases}$$

Thus we prove the lemma. □

Next is a local L^2-gradient estimate for the solutions of elliptic equations. The idea is from [23].

Lemma 4.7. *Suppose $\omega, t \in (1,\infty)$, $\frac{\epsilon}{r}, r, \mu \in (0,1)$, $Q \in L^2(\frac{2\epsilon}{r}\mathcal{Y}_{\mu,m})$, and $G \in L^t(\frac{2\epsilon}{r}\mathcal{Y}_{\mu,m})$. Then any weak solution of*

$$-\nabla\cdot(\mathbf{K}^{\epsilon/r}_{\omega^2,\mu}\nabla\Psi + Q) = G \qquad \text{in } \tfrac{\epsilon}{r}\mathcal{Y} \tag{4.28}$$

satisfies

$$\|\nabla\Psi\|_{L^2(\frac{\epsilon}{r}\mathcal{Y}_{\mu,m})} \le \frac{c}{\omega^2}\Big(\|\nabla\Psi\|_{L^2(\frac{2\epsilon}{r}\mathcal{Y}_{\mu,m}\setminus\frac{\epsilon}{r}\mathcal{Y}_{\mu,m})} + \|Q\|_{L^2(\frac{2\epsilon}{r}\mathcal{Y}_{\mu,m})} + \Big|\frac{\epsilon\mu}{r}\Big|^{\frac{2}{t'}}\|G\|_{L^t(\frac{2\epsilon}{r}\mathcal{Y}_{\mu,m})}\Big),$$

where $\frac{1}{t}+\frac{1}{t'}=1$ and c is a constant independent of ϵ,ω,μ,r,t. See $(4.1)_2$ for $\mathbf{K}^{\epsilon/r}_{\omega^2,\mu}$.

Proof. Take a constant h such that $(\Psi-h)_{\frac{\epsilon}{r}\mathcal{Y}_{\mu,m}} = 0$ (see (2.1)). By Lemma 4.6, there is a $V \in H^1_0(\frac{2\epsilon}{r}\mathcal{Y}_{\mu,m})$ satisfying

$$\begin{cases} V = \Psi - h & \text{in } \frac{\epsilon}{r}\mathcal{Y}_{\mu,m},\\ \|V\|_{H^1(\frac{2\epsilon}{r}\mathcal{Y}_{\mu,m})} \le c\|\nabla\Psi\|_{L^2(\frac{\epsilon}{r}\mathcal{Y}_{\mu,m})}, \end{cases} \tag{4.29}$$

where c is a constant independent of ϵ,ω,μ,r. Extend V from $\frac{2\epsilon}{r}\mathcal{Y}_{\mu,m}$ to $\frac{\epsilon}{r}\mathcal{Y}$ by 0 and test (4.28) against V to get

$$\int_{\frac{2\epsilon}{r}\mathcal{Y}_{\mu,m}} \mathbf{K}^{\epsilon/r}_{\omega^2,\mu}\nabla\Psi\nabla V\,dx + \int_{\frac{2\epsilon}{r}\mathcal{Y}_{\mu,m}} Q\nabla V\,dx = \int_{\frac{2\epsilon}{r}\mathcal{Y}_{\mu,m}} GV\,dx.$$

We then have

$$\omega^2\|\nabla\Psi\|^2_{L^2(\frac{\epsilon}{r}\mathcal{Y}_{\mu,m})} \le \Big(\|\nabla\Psi\|_{L^2(\frac{2\epsilon}{r}\mathcal{Y}_{\mu,m}\setminus\frac{\epsilon}{r}\mathcal{Y}_{\mu,m})} + \|Q\|_{L^2(\frac{2\epsilon}{r}\mathcal{Y}_{\mu,m})}\Big)\|\nabla V\|_{L^2(\frac{2\epsilon}{r}\mathcal{Y}_{\mu,m})} + \|G\|_{L^t(\frac{2\epsilon}{r}\mathcal{Y}_{\mu,m})}\|V\|_{L^{t'}(\frac{2\epsilon}{r}\mathcal{Y}_{\mu,m})}.$$

The lemma follows from the inequality above and (4.29). □

Lemma 4.8. *Suppose (A1), $\omega, t, q \in (1,\infty)$, $\frac{1}{t}+\frac{1}{t'}=1$, $\frac{\epsilon}{r}, r, \mu \in (0,1)$, $\sigma\in[0,2]$, $\mathscr{P}\in\Omega/r$, $Q \in L^2(B_R(\mathscr{P})\cap\Omega/r)$, and $G\in L^t(B_R(\mathscr{P})\cap\Omega/r)$. Consider the following problem:*

$$\begin{cases} -\nabla\cdot(\mathbf{E}^{\epsilon,r}_{\omega^2,\mu}\nabla\Psi + Q) = G & \text{in } B_R(\mathscr{P})\cap\Omega/r,\\ \Psi = 0 & \text{on } B_R(\mathscr{P})\cap\partial\Omega/r. \end{cases} \tag{4.30}$$

(I) *If $R\in(0,\frac{\epsilon\mu}{r}]$, any solution of (4.30) satisfies*

$$\begin{aligned}\|\mathbf{E}^{\epsilon,r}_{\omega^\sigma,\mu}\nabla\Psi\|_{L^2(\mathbf{S}^r_{R/2}(\mathscr{P}))} &\le c\Big(\|\mathbf{E}^{\epsilon,r}_{\omega^{\sigma-2},\mu}Q\|_{L^2(\mathbf{S}^r_R(\mathscr{P}))} + R^{2/t'}\|\mathbf{E}^{\epsilon,r}_{\omega^{\sigma-2},\mu}G\|_{L^t(\mathbf{S}^r_R(\mathscr{P}))}\Big)\\ &\quad + c\,\min\Big\{R^{-1}\|\mathbf{E}^{\epsilon,r}_{\omega^\sigma,\mu}\Psi\|_{L^2(\mathbf{S}^r_R(\mathscr{P}))},\ R\Big|(|\mathbf{E}^{\epsilon,r}_{\omega^\sigma,\mu}\nabla\Psi|^q)_{\mathbf{S}^r_R(\mathscr{P})}\Big|^{1/q}\Big\},\end{aligned} \tag{4.31}$$

where c is independent of $\epsilon,\omega,\mu,r,\sigma,t,q,R,\mathscr{P}$. See Section 2 for $\mathbf{S}^r_R(\mathscr{P})$ and see (2.1) for $(|\mathbf{E}^{\epsilon,r}_{\omega^\sigma,\mu}\nabla\Psi|^q)_{\mathbf{S}^r_R(\mathscr{P})}$.

(II) *If $R\in(\frac{\epsilon\mu}{r}, 32\frac{\epsilon\mu}{r}]$, any solution of (4.30) satisfies*

$$\|\mathbf{E}^{\epsilon,r}_{\omega^\sigma,\mu}\nabla\Psi\|_{L^2(\mathbf{S}^r_{R/2}(\mathscr{P}))} \le c\Big(R\Big|(|\nabla\Psi|^q)_{\mathbf{S}^r_R(\mathscr{P})}\Big|^{1/q} + \|\mathbf{E}^{\epsilon,r}_{\omega^{\sigma-2},\mu}Q\|_{L^2(\mathbf{S}^r_R(\mathscr{P}))} + R^{2/t'}\|\mathbf{E}^{\epsilon,r}_{\omega^{\sigma-2},\mu}G\|_{L^t(\mathbf{S}^r_R(\mathscr{P}))}\Big), \tag{4.32}$$

where c is independent of $\epsilon,\omega,\mu,r,\sigma,t,q,R,\mathscr{P}$. See (2.1) for $(|\nabla\Psi|^q)_{\mathbf{S}^r_R(\mathscr{P})}$.

Proof. **For Case (I).** If $B_R(\mathscr{P}) \subset \Omega^{\epsilon}_{\mu,f}/r$ or $B_R(\mathscr{P}) \subset \Omega^{\epsilon}_{\mu,m}/r$ or $\mathscr{P} \in \partial\Omega/r$, then (4.31) is proved by the energy method. Consider $\mathscr{P} \in \frac{\epsilon}{r}\partial(\mathcal{Y}_{\mu,m} + \mathbf{j})$ for $\mathbf{j} \in \mathcal{I}_\epsilon$. By translation, let $\mathscr{P} = 0$. Define $\phi(x) \equiv \Psi(Rx), \mathbb{Q}(x) = RQ(Rx)$, and $\mathbb{G}(x) = R^2G(Rx)$. Then $\phi, \mathbb{Q}, \mathbb{G}$ satisfy

$$-\nabla\cdot(\mathbf{E}^{\epsilon,rR}_{\omega^2,\mu}\nabla\phi + \mathbb{Q}) = \mathbb{G} \qquad \text{in } B_1(0). \tag{4.33}$$

Any solution of (4.33) satisfies, by [1, pages 3964 and 3965] and Lemma 4.5,

$$\begin{aligned}\|\mathbf{E}^{\epsilon,rR}_{\omega^\sigma,\mu}\nabla\phi\|_{L^2(B_{1/2}(0))} &\le c\left(\|\mathbf{E}^{\epsilon,rR}_{\omega^{\sigma-2},\mu}\mathbb{Q}\|_{L^2(B_1(0))} + \|\mathbf{E}^{\epsilon,rR}_{\omega^{\sigma-2},\mu}\mathbb{G}\|_{L^t(B_1(0))}\right)\\ &\quad + c\ \min\left\{\|\mathbf{E}^{\epsilon,rR}_{\omega^\sigma,\mu}\phi\|_{L^2(B_1(0))},\ \left|(|\mathbf{E}^{\epsilon,rR}_{\omega^\sigma,\mu}\nabla\phi|^q)_{B_1(0)}\right|^{1/q}\right\},\end{aligned} \tag{4.34}$$

where $t, q > 1$ and c is a constant independent of $\epsilon, \omega, \mu, r, \sigma, t, q, R$. Here, (4.31) is from (4.34) and a change in the variables.

For Case (II). By translation, set $\mathscr{P} = 0$. Define $h \equiv \frac{\epsilon\mu}{r}$, $\phi(x) \equiv \Psi(hx)$, $\mathbb{Q}(x) \equiv hQ(hx)$, and $\mathbb{G}(x) \equiv h^2G(hx)$. Then $\phi, \mathbb{Q}, \mathbb{G}$ satisfy

$$\begin{cases}-\nabla\cdot(\mathbf{E}^{\epsilon,rh}_{\omega^2,\mu}\nabla\phi + \mathbb{Q}) = \mathbb{G} & \text{in } B_{R/h}(0)\cap\Omega/rh,\\ \phi = 0 & \text{on } B_{R/h}(0)\cap\partial\Omega/rh.\end{cases}$$

Next, we take some finite points $\{\mathscr{P}_i\}_{i=1}^k$ such that (i) $B_{2/3}(\mathscr{P}_i)\cap\partial\frac{1}{\mu}(\mathcal{Y}_{\mu,m}+\mathbf{j}) = \emptyset$ or $\mathscr{P}_i \in \partial\frac{1}{\mu}(\mathcal{Y}_{\mu,m}+\mathbf{j})$ for $\mathbf{j} \in \mathcal{I}_\epsilon$ and (ii) $B_{R/2h}(0) \subset \bigcup_{i=1}^k B_{1/2}(\mathscr{P}_i) \subset \bigcup_{i=1}^k B_{2/3}(\mathscr{P}_i) \subset B_{R/h}(0)$. For each point $\mathscr{P}_i \in \partial\frac{1}{\mu}(\mathcal{Y}_{\mu,m} + \mathbf{j})$, we consider

$$-\nabla\cdot(\mathbf{E}^{\epsilon,rh}_{\omega^2,\mu}\nabla\phi + \mathbb{Q}) = \mathbb{G} \qquad \text{in } B_{2/3}(\mathscr{P}_i). \tag{4.35}$$

Following the argument in [1, pages 3964 and 3965] and applying Lemma 4.5 with $\sigma = 0$, any solution of (4.35) satisfies the following, for $t > 1$,

$$\|\nabla\phi\|_{L^2(B_{1/2}(\mathscr{P}_i))} \le c\left(\|\phi\|_{L^2(B_{2/3}(\mathscr{P}_i))} + \|\mathbf{E}^{\epsilon,rh}_{\omega^{-2},\mu}\mathbb{Q}\|_{L^2(B_{2/3}(\mathscr{P}_i))} + \|\mathbf{E}^{\epsilon,rh}_{\omega^{-2},\mu}\mathbb{G}\|_{L^t(B_{2/3}(\mathscr{P}_i))}\right). \tag{4.36}$$

For each point $\mathscr{P}_i$ with $B_{2/3}(\mathscr{P}_i) \cap \partial\frac{1}{\mu}(\mathcal{Y}_{\mu,m} + \mathbf{j}) = \emptyset$, we consider

$$\begin{cases}-\nabla\cdot(\mathbf{E}^{\epsilon,rh}_{\omega^2,\mu}\nabla\phi + \mathbb{Q}) = \mathbb{G} & \text{in } \mathbf{S}^{rh}_{2/3}(\mathscr{P}_i) \equiv B_{2/3}(\mathscr{P}_i)\cap\Omega/rh,\\ \phi = 0 & \text{on } B_{2/3}(\mathscr{P}_i)\cap\partial\Omega/rh.\end{cases} \tag{4.37}$$

By the energy method, any solution of (4.37) satisfies, for $t > 1$,

$$\|\nabla\phi\|_{L^2(\mathbf{S}^{rh}_{1/2}(\mathscr{P}_i))} \le c\left(\|\phi\|_{L^2(\mathbf{S}^{rh}_{2/3}(\mathscr{P}_i))} + \|\mathbf{E}^{\epsilon,rh}_{\omega^{-2},\mu}\mathbb{Q}\|_{L^2(\mathbf{S}^{rh}_{2/3}(\mathscr{P}_i))} + \|\mathbf{E}^{\epsilon,rh}_{\omega^{-2},\mu}\mathbb{G}\|_{L^t(\mathbf{S}^{rh}_{2/3}(\mathscr{P}_i))}\right). \tag{4.38}$$

Square both sides for (4.36) and (4.38); sum these equations for $i = 1, \cdots, k$ to get

$$\|\nabla\phi\|^2_{L^2(\mathbf{S}^{rh}_{R/2h}(0))} \le c\left(\|\phi\|^2_{L^2(\mathbf{S}^{rh}_{R/h}(0))} + \|\mathbf{E}^{\epsilon,rh}_{\omega^{-2},\mu}\mathbb{Q}\|^2_{L^2(\mathbf{S}^{rh}_{R/h}(0))} + \|\mathbf{E}^{\epsilon,rh}_{\omega^{-2},\mu}\mathbb{G}\|^2_{L^t(\mathbf{S}^{rh}_{R/h}(0))}\right). \tag{4.39}$$

In (4.39), the sum is over a finite numbers. After a change in the variables,

$$\|\nabla\Psi\|^2_{L^2(\mathbf{S}^r_{R/2}(0))} \le c\left(h^{-2}\|\Psi\|^2_{L^2(\mathbf{S}^r_R(0))} + \|\mathbf{E}^{\epsilon,r}_{\omega^{-2},\mu}Q\|^2_{L^2(\mathbf{S}^r_R(0))} + h^{\frac{4}{t'}}\|\mathbf{E}^{\epsilon,r}_{\omega^{-2},\mu}G\|^2_{L^t(\mathbf{S}^r_R(0))}\right). \tag{4.40}$$

By Lemma 4.7, (4.40), and $R \in (\frac{\epsilon\mu}{r}, 32\frac{\epsilon\mu}{r}]$,

$$\begin{aligned}||\mathbf{E}^{\epsilon,r}_{\omega^{\sigma},\mu}\nabla\Psi||^2_{L^2(\mathbf{S}^{r}_{R/4}(0))} &\le c\left(||\nabla\Psi||^2_{L^2(\mathbf{S}^{r}_{R/2}(0))} + ||\omega^{\sigma-2}Q||^2_{L^2(\mathbf{S}^{r}_{R}(0))} + h^{\frac{4}{t'}}||\omega^{\sigma-2}G||^2_{L^t(\mathbf{S}^{r}_{R}(0))}\right)\\ &\le c\left(h^2\left|(|\nabla\Psi|^q)_{\mathbf{S}^{r}_{R}(0)}\right|^{\frac{2}{q}} + ||\mathbf{E}^{\epsilon,r}_{\omega^{\sigma-2},\mu}Q||^2_{L^2(\mathbf{S}^{r}_{R}(0))} + h^{\frac{4}{t'}}||\mathbf{E}^{\epsilon,r}_{\omega^{\sigma-2},\mu}G||^2_{L^t(\mathbf{S}^{r}_{R}(0))}\right),\end{aligned}$$

for $t, q > 1$. Inequality (4.32) follows from the inequality above. □

Lemma 4.9. *Suppose (A1), (A3), $\frac{\epsilon}{r}, r \in (0,1)$, $q \in (1,\infty)$, $\mathscr{P} \in \Omega/r$, $Q \in L^2(\mathbf{S}^{r}_{R}(\mathscr{P}))$, $\mathbf{E}^{\epsilon,r}_{\omega^{-2},\mu}Q \in C^{\gamma}(\mathbf{S}^{r}_{R}(\mathscr{P}))$, and $G \in L^p(\mathbf{S}^{r}_{R}(\mathscr{P}))$. If $32\frac{\epsilon\mu}{r} < \frac{\epsilon}{r}$ and $R \in (32\frac{\epsilon\mu}{r}, \frac{\epsilon}{r}]$, any weak solution Ψ of (4.30) satisfies*

$$\begin{aligned}||\mathbf{E}^{\epsilon,r}_{\omega^{2-\tau},\mu}\nabla\Psi||_{L^2(\mathbf{S}^{r}_{R/2}(\mathscr{P}))} \le c\bigg(R\left|(|\nabla\Psi|^q)_{\mathbf{S}^{r}_{R}(\mathscr{P})}\right|^{1/q} &+ ||\omega^{-\tau}Q||_{L^2(\frac{2\epsilon}{r}(\mathcal{Y}_{\mu,m}+\mathbf{j})\cap\mathbf{S}^{r}_{R}(\mathscr{P}))}\\ &+R||\mathbf{E}^{\epsilon,r}_{\omega^{-2},\mu}Q||_{C^{\gamma}(\mathbf{S}^{r}_{R}(\mathscr{P}))} + R^{\tau}||\mathbf{E}^{\epsilon,r}_{1/\omega^{\tau},\mu}G||_{L^p(\mathbf{S}^{r}_{R}(\mathscr{P}))}\bigg),\end{aligned}$$

where c is independent of $\epsilon, \mu (= \frac{1}{\omega}), r, p, q, R, \mathscr{P}$. See (A3) for p, p', τ, γ, and Section 2 for $\mathbf{S}^{r}_{R}(\mathscr{P})$ and $(|\nabla\Psi|^q)_{\mathbf{S}^{r}_{R}(\mathscr{P})}$.

Proof. **For $\frac{\epsilon}{r}(\mathcal{Y}_{\mu,m} + \mathbf{j}) \subset B_{R/2}(\mathscr{P}) \subset \Omega/r$.** By translation and dilation, assume $\frac{\epsilon}{rR}\mathcal{Y}_{\mu,m} \subset B_{1/2}(0) \subset \Omega/rR$. Define $\phi(x) \equiv \Psi(Rx)$, $\mathbb{Q}(x) = RQ(Rx)$, and $\mathbb{G}(x) = R^2G(Rx)$. Then $\phi, \mathbb{Q}, \mathbb{G}$ satisfy

$$-\nabla\cdot(\mathbf{E}^{\epsilon,rR}_{\omega^2,\mu}\nabla\phi + \mathbb{Q}) = \mathbb{G} \qquad \text{in } B_{1/2}(0).$$

By (A2) and Lemma 4.1,

$$||\nabla\phi||_{L^2(B_{1/4}(0))} \le ||\nabla\phi||_{L^\infty(B_{1/4}(0))} \le c\left(||\phi||_{L^2(B_1(0)\cap\frac{\epsilon}{rR}\mathcal{Y}_{\mu,f})} + ||\mathbf{E}^{\epsilon,rR}_{1/\omega^2,\mu}\mathbb{Q}||_{C^{\gamma}(B_1(0))} + ||\mathbf{E}^{\epsilon,rR}_{1/\omega^2,\mu}\mathbb{G}||_{L^p(B_1(0))}\right),$$

where $\gamma = \frac{p-2}{p}$ and c is independent of $\epsilon, \mu (= \frac{1}{\omega})$. After a change in the variables,

$$\begin{aligned}||\nabla\Psi||_{L^2(B_{2R/3}(0))} &\le c\left(R^{-1}||\Psi||_{L^2(B_R(0)\cap\frac{\epsilon}{r}\mathcal{Y}_{\mu,f})} + R||\mathbf{E}^{\epsilon,r}_{\omega^{-2},\mu}Q||_{C^{\gamma}(B_R(0))} + R^{\tau}||\mathbf{E}^{\epsilon,r}_{\omega^{-2},\mu}G||_{L^p(B_R(0))}\right)\\ &\le c\left(R\left|(|\nabla\Psi|^q)_{B_R(0)}\right|^{1/q} + R||\mathbf{E}^{\epsilon,r}_{\omega^{-2},\mu}Q||_{C^{\gamma}(B_R(0))} + R^{\tau}||\mathbf{E}^{\epsilon,r}_{\omega^{-2},\mu}G||_{L^p(B_R(0))}\right),\end{aligned}$$

for $q \in (1,\infty)$. By Lemma 4.7 with $t = p$,

$$\begin{aligned}||\mathbf{E}^{\epsilon,r}_{\omega^{2-\tau},\mu}\nabla\Psi||^2_{L^2(B_{R/2}(0))} &\le c\left(||\nabla\Psi||^2_{L^2(\mathbf{S}^{\epsilon,r}_{\mu,f,2R/3}(0))} + ||\omega^{-\tau}Q||^2_{L^2(\frac{2\epsilon}{r}\mathcal{Y}_{\mu,m})} + \left|\frac{\epsilon\mu}{r}\right|^{2\tau}||\omega^{-\tau}G||^2_{L^p(\frac{2\epsilon}{r}\mathcal{Y}_{\mu,m})}\right)\\ &\le c\left(R^2\left|(|\nabla\Psi|^q)_{B_R(0)}\right|^{2/q} + ||\omega^{-\tau}Q||^2_{L^2(\frac{2\epsilon}{r}\mathcal{Y}_{\mu,m})} + R^2||\mathbf{E}^{\epsilon,r}_{\omega^{-2},\mu}Q||^2_{C^{\gamma}(B_R(0))} + R^{2\tau}||\mathbf{E}^{\epsilon,r}_{1/\omega^{\tau},\mu}G||^2_{L^p(B_R(0))}\right).\end{aligned}$$

So Lemma 4.9 is true for the case of $\frac{\epsilon}{r}(\mathcal{Y}_{\mu,m} + \mathbf{j}) \subset B_{R/2}(\mathscr{P}) \subset \Omega/r$.

For $\mathscr{P} \in \partial\Omega/r$ or $\frac{\epsilon}{r}(\mathcal{Y}_{\mu,m} + \mathbf{j}) \cap B_{R/2}(\mathscr{P}) = \emptyset$. In this case, Lemma 4.9 can be proved by the energy method. □

Lemma 4.10. *Suppose (A1)–(A3), $\frac{\epsilon}{r}, r \in (0, \frac{1}{2})$, $R \in (\max\{\frac{\epsilon}{2r}, 16\frac{\epsilon\mu}{r}\}, 2)$, $\mu \le \left|\frac{\epsilon}{r}\right|^{\tau-1}$, $\mathscr{P} \in \Omega/r$, $Q \in L^2(\mathbf{S}^{r}_{R}(\mathscr{P}))$, $G \in L^p(\mathbf{S}^{r}_{R}(\mathscr{P}))$, and Ψ is a weak solution of (4.30). If $\ell \in [\frac{R}{2}, R - 8\frac{\epsilon\mu}{r}]$ and $\theta \in (0,1)$, then*

$$\begin{aligned}||\mathbf{E}^{\epsilon,r}_{\omega^{2-\tau},\mu}\nabla\Psi||^2_{L^2(\mathbf{S}^{r}_{\ell}(\mathscr{P}))} \le \frac{c}{\theta(R-\ell)^2}||\Psi||^2_{L^2(\mathbf{S}^{r}_{R}(\mathscr{P}))} &+ \left(\frac{c}{\omega^2} + \theta\right)||\nabla\Psi||^2_{L^2(\mathbf{S}^{r}_{R}(\mathscr{P}))}\\ &+c\,\theta^{-1}\left(||\mathbf{E}^{\epsilon,r}_{1/\omega^{\tau},\mu}Q||^2_{L^2(\mathbf{S}^{r}_{R}(\mathscr{P}))} + R^{2\tau}||\mathbf{E}^{\epsilon,r}_{1/\omega^{\tau},\mu}G||^2_{L^p(\mathbf{S}^{r}_{R}(\mathscr{P}))}\right),\end{aligned}$$

where c is independent of $\epsilon, \mu (= \frac{1}{\omega}), r, p, R, \theta, \ell, \mathscr{P}$. See (A2) and (A3) for ω, μ, p, p', τ.

Proof. Recall $\mathbf{S}^{;r}_{R}(\mathscr{P}), \mathbf{S}^{\epsilon,r}_{\mu,f,R}(\mathscr{P}), \mathbf{S}^{\epsilon,r}_{\mu,m,R}(\mathscr{P})$, and $(\varphi)_D$ from Section 2. By Lemma 4.7 with $t = 2$ and (A3) and summing up all $\frac{\epsilon}{r}(\mathscr{Y} + \mathbf{j})$ with $\frac{\epsilon}{r}(\mathscr{Y} + \mathbf{j}) \cap \mathbf{S}^{;r}_{\ell}(\mathscr{P}) \neq \emptyset$, we have

$$\|\mathbf{E}^{\epsilon,r}_{\omega^{2-\tau},\mu}\nabla\Psi\|^2_{L^2(\mathbf{S}^{;r}_{\ell}(\mathscr{P}))} \leq c\left(\|\nabla\Psi\|^2_{L^2(\mathbf{S}^{\epsilon,r}_{\mu,f,\ell}(\mathscr{P}))} + \|\omega^{-\tau}Q\|^2_{L^2(\mathbf{S}^{;r}_{\ell}(\mathscr{P}))} + \left|\frac{\epsilon\mu}{r}\right|^2\|\omega^{-\tau}G\|^2_{L^2(\mathbf{S}^{;r}_{\ell}(\mathscr{P}))}\right), \tag{4.41}$$

where c is a constant independent of $\epsilon, \omega, \mu, r, \ell, \mathscr{P}$. Since $R - \ell \geq 8\frac{\epsilon\mu}{r}$, there is a bell-shaped function $\eta \in C^1_0(B_{R-3\frac{\epsilon\mu}{r}}(\mathscr{P}))$ so that $\eta = 1$ in $B_{\ell+3\frac{\epsilon\mu}{r}}(\mathscr{P})$ and

$$|\nabla\eta| \leq c\left|R - \ell - 6\frac{\epsilon\mu}{r}\right|^{-1} \leq c|R - \ell|^{-1}.$$

For each $\frac{\epsilon}{r}(\mathscr{Y}_{\mu,m} + \mathbf{j})$ with $\mathbf{j} \in \mathcal{I}_{\epsilon}$, there exist $V_{\mathbf{j}} \in H^1_0(\frac{2\epsilon}{r}\mathscr{Y}_{\mu,m} + \frac{\epsilon}{r}\mathbf{j})$ and $h_{\mathbf{j}} \in \mathbb{R}$ such that, by Lemma 4.6,

$$\begin{cases} V_{\mathbf{j}} = \Psi\eta^2 - h_{\mathbf{j}} & \text{in } \frac{\epsilon}{r}(\mathscr{Y}_{\mu,m} + \mathbf{j}), \\ \|V_{\mathbf{j}}\|_{H^1(\frac{2\epsilon}{r}\mathscr{Y}_{\mu,m}+\frac{\epsilon}{r}\mathbf{j})} \leq c\|\nabla(\Psi\eta^2)\|_{L^2(\frac{\epsilon}{r}(\mathscr{Y}_{\mu,m}+\mathbf{j}))}. \end{cases} \tag{4.42}$$

Note that $h_{\mathbf{j}} = \begin{cases} 0 & \text{if } \overline{\frac{\epsilon}{r}(\mathscr{Y}_{\mu,m} + \mathbf{j})} \setminus B_R(\mathscr{P}) \neq \emptyset \\ (\Psi\eta^2)_{\frac{\epsilon}{r}(\mathscr{Y}_{\mu,m}+\mathbf{j})} & \text{otherwise} \end{cases}$. Extend $V_{\mathbf{j}}$ from $\frac{2\epsilon}{r}\mathscr{Y}_{\mu,m} + \frac{\epsilon}{r}\mathbf{j}$ to $\mathbb{R}^2$ by 0 and set $\zeta = \Psi\eta^2 - \sum_{\mathbf{j}\in\mathcal{I}_{\epsilon}} V_{\mathbf{j}}$ in Ω/r. Then $\zeta \in H^1_0(B_R(\mathscr{P}))$ and $\nabla\zeta = 0$ in $\mathbf{S}^{\epsilon,r}_{\mu,m,R}(\mathscr{P})$. Test (4.30) against ζ to get

$$\left|\int_{\mathbf{S}^{\epsilon,r}_{\mu,f,R}(\mathscr{P})} \nabla\Psi\nabla\zeta\, dx\right| \leq \|Q\nabla\zeta\|_{L^1(\mathbf{S}^{\epsilon,r}_{\mu,f,R}(\mathscr{P}))} + \|G\,\zeta\|_{L^1(\mathbf{S}^{;r}_{R}(\mathscr{P}))}. \tag{4.43}$$

By Hölder's inequality and (4.42),

$$\|Q\nabla\zeta\|_{L^1(\mathbf{S}^{\epsilon,r}_{\mu,f,R}(\mathscr{P}))} \leq \|Q\|_{L^2(\mathbf{S}^{\epsilon,r}_{\mu,f,R}(\mathscr{P}))}\|\nabla(\Psi\eta^2)\|_{L^2(\mathbf{S}^{;r}_{R}(\mathscr{P}))}. \tag{4.44}$$

Apply Hölder's inequality and Sobolev's embedding theorem to get, by $\tau p' = 2$,

$$\begin{aligned} \|G\,\zeta\|_{L^1(\mathbf{S}^{;r}_{R}(\mathscr{P}))} &= \left\|G\,\zeta\mathcal{X}_{\mathbf{S}^{\epsilon,r}_{\mu,f,R}(\mathscr{P})} + G\,\zeta\mathcal{X}_{\mathbf{S}^{\epsilon,r}_{\mu,m,R}(\mathscr{P})}\right\|_{L^1(\mathbf{S}^{;r}_{R}(\mathscr{P}))} \\ &\leq \|G\|_{L^p(\mathbf{S}^{\epsilon,r}_{\mu,f,R}(\mathscr{P}))}\|\zeta\|_{L^{p'}(\mathbf{S}^{\epsilon,r}_{\mu,f,R}(\mathscr{P}))} + \|G\|_{L^p(\mathbf{S}^{\epsilon,r}_{\mu,m,R}(\mathscr{P}))}\|\zeta\|_{L^{p'}(\mathbf{S}^{\epsilon,r}_{\mu,m,R}(\mathscr{P}))} \\ &\leq c\|G\|_{L^p(\mathbf{S}^{\epsilon,r}_{\mu,f,R}(\mathscr{P}))}R^{\tau}\|\nabla\zeta\|_{L^2(\mathbf{S}^{\epsilon,r}_{\mu,f,R}(\mathscr{P}))} + c\|\mathbf{E}^{\epsilon,r}_{1/\omega^{\tau},\mu}G\|_{L^p(\mathbf{S}^{;r}_{R}(\mathscr{P}))}\|\omega^{\tau}\zeta\|_{L^{p'}(\mathbf{S}^{\epsilon,r}_{\mu,m,R}(\mathscr{P}))}. \end{aligned} \tag{4.45}$$

Since $\mu \leq \left|\frac{\epsilon}{r}\right|^{\tau-1} < \left|\frac{1}{2}\right|^{\tau-1}$, there is a number $s > 1$ satisfying $\mu^{\frac{-2}{s}} = 2^{\tau-1}$. There is a constant c (independent of $\epsilon, \mu\,(= \frac{1}{\omega}), r, R, s, p, \mathscr{P}$) so that, by Hölder's inequality, (4.42), $\frac{1}{s} + \frac{1}{s'} = 1$, and (A2) and (A3),

$$\begin{aligned} \|\omega^{\tau}\zeta\|^{p'}_{L^{p'}(\mathbf{S}^{\epsilon,r}_{\mu,m,R}(\mathscr{P}))} &\leq c\sum_{\frac{\epsilon}{r}(\mathscr{Y}_{\mu,m}+\mathbf{j})\cap\mathbf{S}^{\epsilon,r}_{\mu,m,R}(\mathscr{P})\neq\emptyset} \omega^2 h^{p'}_{\mathbf{j}}\left|\frac{\epsilon\mu}{r}\right|^2 \leq c\,\omega^2\int_{\mathbf{S}^{\epsilon,r}_{\mu,m,R}(\mathscr{P})}|\Psi\eta^2|^{p'} \\ &\leq c\,\omega^2|\mu R|^{\frac{2}{s'}}\|\Psi\eta^2\|^{p'}_{L^{p's}(\mathbf{S}^{;r}_{R}(\mathscr{P}))} \leq c\,R^2\|\nabla(\Psi\eta^2)\|^{p'}_{L^2(\mathbf{S}^{;r}_{R}(\mathscr{P}))}. \end{aligned} \tag{4.46}$$

In (4.46), we use $\mu^{\frac{-2}{s}} = 2^{\tau-1}$. Define $\widetilde{\mathcal{I}}_{\epsilon,r} \equiv \{\mathbf{j} \in \mathcal{I}_{\epsilon} |\ \frac{\epsilon}{r}(\mathscr{Y} + \mathbf{j}) \cap \mathbf{S}^{;r}_{R}(\mathscr{P}) \neq \emptyset\}$. By (4.42)–(4.46) and Lemma 4.7 with $t = 2$,

$$\left|\int_{\mathbf{S}^{\epsilon,r}_{\mu,f,R}(\mathscr{P})} \nabla\Psi\nabla(\Psi\eta^2)\, dx\right| \leq \sum_{\mathbf{j}\in\widetilde{\mathcal{I}}_{\epsilon,r}}\left|\int_{\frac{\epsilon}{r}(\mathscr{Y}_{\mu,f}+\mathbf{j})} \nabla\Psi\nabla V_{\mathbf{j}}\, dx\right| + \|Q\nabla\zeta\|_{L^1(\mathbf{S}^{\epsilon,r}_{\mu,f,R}(\mathscr{P}))} + \|G\,\zeta\|_{L^1(\mathbf{S}^{;r}_{R}(\mathscr{P}))}$$

$$\leq c \sum_{\mathbf{j}\in\widetilde{\mathcal{I}}_{\epsilon,r}} \|\nabla\Psi\|_{L^2(\frac{\epsilon}{r}(\mathcal{Y}_{\mu,f}+\mathbf{j}))}\|\nabla(\Psi\eta^2)\|_{L^2(\frac{\epsilon}{r}(\mathcal{Y}_{\mu,m}+\mathbf{j}))} + \|Q\nabla\zeta\|_{L^1(\mathbf{S}^{\epsilon,r}_{\mu,f,R}(\mathcal{P}))} + \|G\,\zeta\|_{L^1(\mathbf{S}^r_R(\mathcal{P}))}$$

$$\leq c \sum_{\mathbf{j}\in\widetilde{\mathcal{I}}_{\epsilon,r}} \left\|\nabla\Psi\right\|_{L^2(\frac{\epsilon}{r}(\mathcal{Y}_{\mu,f}+\mathbf{j}))}\Big(\|\Psi\nabla\eta\|_{L^2(\frac{\epsilon}{r}(\mathcal{Y}_{\mu,m}+\mathbf{j}))} + \left\|\omega^{-2}\nabla\Psi\right\|_{L^2(\frac{\epsilon}{r}(\mathcal{Y}_{\mu,f}+\mathbf{j}))}$$

$$+\|\omega^{-2}Q\|^2_{L^2(\frac{\epsilon}{r}(\mathcal{Y}+\mathbf{j}))} + \frac{\epsilon\mu}{r}\left\|\omega^{-2}G\right\|_{L^2(\frac{\epsilon}{r}(\mathcal{Y}+\mathbf{j}))}\Big)$$

$$+\|Q\|_{L^2(\mathbf{S}^{\epsilon,r}_{\mu,f,R}(\mathcal{P}))}\|\nabla(\Psi\eta^2)\|_{L^2(\mathbf{S}^r_R(\mathcal{P}))} + c\,R^\tau\|\mathbf{E}^{\epsilon,r}_{1/\omega^\tau,\mu}G\|_{L^p(\mathbf{S}^r_R(\mathcal{P}))}\|\nabla(\Psi\eta^2)\|_{L^2(\mathbf{S}^r_R(\mathcal{P}))}.$$

Note that $\mu \leq \left|\frac{\epsilon}{r}\right|^{\tau-1}$ implies $\frac{\epsilon\mu}{r} \leq \left|\frac{\epsilon}{r}\right|^{\tau} \leq cR^\tau$. For any $\theta \in (0,1)$,

$$\Big|\int_{\mathbf{S}^{\epsilon,r}_{\mu,f,R}(\mathcal{P})} \nabla\Psi\nabla(\Psi\eta^2)\,dx\Big| \leq \frac{c}{\theta}\int_{\mathbf{S}^r_R(\mathcal{P})}|\Psi\nabla\eta|^2dx + \left(\frac{c}{\omega^2}+\theta\right)\int_{\mathbf{S}^r_R(\mathcal{P})}|\nabla\Psi|^2dx$$

$$+c\,\theta^{-1}\left(\|Q\|^2_{L^2(\mathbf{S}^{\epsilon,r}_{\mu,f,R}(\mathcal{P}))} + R^{2\tau}\|\mathbf{E}^{\epsilon,r}_{1/\omega^\tau,\mu}G\|^2_{L^p(\mathbf{S}^r_R(\mathcal{P}))}\right). \tag{4.47}$$

Moreover, (4.41), (4.47), and $\frac{\epsilon\mu}{r} \leq cR^\tau$ imply Lemma 4.10. □

Lemma 4.11. *Suppose that (A1)–(A3),* $\frac{\epsilon}{r}, r \in (0,\frac{1}{2})$, $R \in (\max\{\frac{\epsilon}{r}, 32\frac{\epsilon\mu}{r}\}, 2)$, $\mu \leq \left|\frac{\epsilon}{r}\right|^{\tau-1}$, $\mathcal{P} \in \Omega/r$, $Q \in L^2(\mathbf{S}^r_R(\mathcal{P}))$, $G \in L^p(\mathbf{S}^r_R(\mathcal{P}))$, *and* $q \in (1,\infty)$ *and suppose that* Ψ *is a weak solution of (4.30). In this case,*

$$\|\mathbf{E}^{\epsilon,r}_{\omega^{2-\tau},\mu}\nabla\Psi\|_{L^2(\mathbf{S}^r_{R/2}(\mathcal{P}))} \leq cR\left|(|\nabla\Psi|^q)_{\mathbf{S}^r_R(\mathcal{P})}\right|^{1/q} + \frac{1}{2}\|\nabla\Psi\|_{L^2(\mathbf{S}^r_R(\mathcal{P}))} + c\left(\|\mathbf{E}^{\epsilon,r}_{1/\omega^\tau}Q\|_{L^2(\mathbf{S}^r_R(\mathcal{P}))} + R^\tau\|\mathbf{E}^{\epsilon,r}_{1/\omega^\tau}G\|_{L^p(\mathbf{S}^r_R(\mathcal{P}))}\right),$$

where c is a constant independent of $\epsilon,\mu\,(=\frac{1}{\omega}), r, p, q, R, \mathcal{P}$. *See (A2) and (A3) for* ω,μ,p,p',τ *and Section 2 for* $\mathbf{S}^r_R(\mathcal{P})$ *and* $(|\nabla\Psi|^q)_{\mathbf{S}^r_R(\mathcal{P})}$.

Proof. Let k be the integer satisfying $\frac{R}{2^{k+2}} < 8\frac{\epsilon\mu}{r} \leq \frac{R}{2^{k+1}}$. In this case, $2^k\frac{\epsilon\mu}{r} \approx R$ (see Section 1). Set $\rho_i \equiv R(1-\frac{1}{2^i})$ for $i \in \mathbb{N}$. Then $\rho_i \geq \frac{R}{2}$ and $\frac{\rho_{i+1}}{2} \leq \rho_i \leq \rho_{i+1} - 8\frac{\epsilon\mu}{r}$ for $i = 1,\cdots,k$. By Lemma 4.10 with $\ell = \rho_i$ and $R = \rho_{i+1}$, for any $\theta \in (0,1)$,

$$\|\mathbf{E}^{\epsilon,r}_{\omega^{2-\tau},\mu}\nabla\Psi\|^2_{L^2(\mathbf{S}^r_{\rho_i}(\mathcal{P}))} \leq \frac{c_0}{\theta(\rho_{i+1}-\rho_i)^2}\|\Psi\|^2_{L^2(\mathbf{S}^r_{\rho_{i+1}}(\mathcal{P}))} + \left(\frac{c_0}{\omega^2}+\theta\right)\|\nabla\Psi\|^2_{L^2(\mathbf{S}^r_{\rho_{i+1}}(\mathcal{P}))}$$

$$+\frac{c_0}{\theta}\left(\|\mathbf{E}^{\epsilon,r}_{1/\omega^\tau,\mu}Q\|^2_{L^2(\mathbf{S}^r_{\rho_{i+1}}(\mathcal{P}))} + \rho^{2\tau}_{i+1}\|\mathbf{E}^{\epsilon,r}_{1/\omega^\tau,\mu}G\|^2_{L^p(\mathbf{S}^r_{\rho_{i+1}}(\mathcal{P}))}\right),$$

where c_0 is independent of $\epsilon,\mu\,(=\frac{1}{\omega}), r, p, \theta, \rho_i, \rho_{i+1}, \mathcal{P}$. By an iteration argument,

$$\|\mathbf{E}^{\epsilon,r}_{\omega^{2-\tau},\mu}\nabla\Psi\|^2_{L^2(\mathbf{S}^r_{\rho_1}(\mathcal{P}))} \leq \frac{c_0}{\theta}\sum_{i=1}^k\frac{(\frac{c_0}{\omega^2}+\theta)^{i-1}}{(\rho_{i+1}-\rho_i)^2}\|\Psi\|^2_{L^2(\mathbf{S}^r_R(\mathcal{P}))} + \left(\frac{c_0}{\omega^2}+\theta\right)^k\|\nabla\Psi\|^2_{L^2(\mathbf{S}^r_R(\mathcal{P}))}$$

$$+\frac{c_0}{\theta}\sum_{i=1}^k\left(\frac{c_0}{\omega^2}+\theta\right)^{i-1}\left(\|\mathbf{E}^{\epsilon,r}_{1/\omega^\tau,\mu}Q\|^2_{L^2(\mathbf{S}^r_R(\mathcal{P}))} + R^{2\tau}\|\mathbf{E}^{\epsilon,r}_{1/\omega^\tau,\mu}G\|^2_{L^p(\mathbf{S}^r_R(\mathcal{P}))}\right).$$

Since $\rho_{i+1}-\rho_i = \frac{R}{2^{i+1}}$, we obtain

$$\|\mathbf{E}^{\epsilon,r}_{\omega^{2-\tau},\mu}\nabla\Psi\|^2_{L^2(\mathbf{S}^r_{R/2}(\mathcal{P}))} \leq 4c_0\frac{\sum_{i=1}^k\left(4\frac{c_0}{\omega^2}+4\theta\right)^i}{\theta(\frac{c_0}{\omega^2}+\theta)R^2}\|\Psi\|^2_{L^2(\mathbf{S}^r_R(\mathcal{P}))} + \left(\frac{c_0}{\omega^2}+\theta\right)^k\|\nabla\Psi\|^2_{L^2(\mathbf{S}^r_R(\mathcal{P}))}$$

$$+\frac{c_0}{\theta}\sum_{i=1}^{k}\left(\frac{c_0}{\omega^2}+\theta\right)^{i-1}\left(\|\mathbf{E}^{\epsilon,r}_{1/\omega^\tau,\mu}Q\|^2_{L^2(\mathbf{S}^r_R(\mathscr{P}))}+R^{2\tau}\|\mathbf{E}^{\epsilon,r}_{1/\omega^\tau,\mu}G\|^2_{L^p(\mathbf{S}^r_R(\mathscr{P}))}\right).$$

Suppose $\frac{4c_0}{\omega^2}\le\frac{1}{2^2}$. Take $\theta=\frac{1}{2^4}$ and apply Sobolev's embedding theorem [17] to get

$$\begin{aligned}&\|\mathbf{E}^{\epsilon,r}_{\omega^{2-\tau},\mu}\nabla\Psi\|^2_{L^2(\mathbf{S}^r_{R/2}(\mathscr{P}))}\\ &\le cR^{-2}\|\Psi\|^2_{L^2(\mathbf{S}^r_R(\mathscr{P}))}+2^{-3k}\|\nabla\Psi\|^2_{L^2(\mathbf{S}^r_R(\mathscr{P}))}+c\left(\|\mathbf{E}^{\epsilon,r}_{1/\omega^\tau,\mu}Q\|^2_{L^2(\mathbf{S}^r_R(\mathscr{P}))}+R^{2\tau}\|\mathbf{E}^{\epsilon,r}_{1/\omega^\tau,\mu}G\|^2_{L^p(\mathbf{S}^r_R(\mathscr{P}))}\right)\\ &\le cR^2\left|(|\nabla\Psi|^q)_{\mathbf{S}^r_R(\mathscr{P})}\right|^{2/q}+2^{-2}\|\nabla\Psi\|^2_{L^2(\mathbf{S}^r_R(\mathscr{P}))}+c\left(\|\mathbf{E}^{\epsilon,r}_{1/\omega^\tau,\mu}Q\|^2_{L^2(\mathbf{S}^r_R(\mathscr{P}))}+R^{2\tau}\|\mathbf{E}^{\epsilon,r}_{1/\omega^\tau,\mu}G\|^2_{L^p(\mathbf{S}^r_R(\mathscr{P}))}\right).\end{aligned}$$

This proves Lemma 4.11 for $\frac{4c_0}{\omega^2}\le\frac{1}{2^2}$ case.

Suppose $\frac{4c_0}{\omega^2}>\frac{1}{2^2}$. Note $1<\omega^2<2^4c_0$ and $(4.30)_1$ is a uniform elliptic equation. Let $\eta\in C_0^\infty(B_R(\mathscr{P}))$ be a non-negative bell-shaped function with $\eta=1$ in $B_{R/2}(\mathscr{P})$. Lemma 4.11 for the case of $\frac{4c_0}{\omega^2}>\frac{1}{2^2}$ is obtained by testing (4.30) against $\Psi\eta^2$. □

Next, we state a local gradient estimate for strongly elliptic equations.

Lemma 4.12. *If (A1)–(A3), $\frac{\epsilon}{r}, r, R\in(0,\frac{1}{2})$, $\mu\le\left|\frac{\epsilon}{r}\right|^{\tau-1}$, $\mathscr{P}\in\Omega/r$, $Q\in L^2(\mathbf{S}^r_R(\mathscr{P}))$, $\mathbf{E}^{\epsilon,r}_{\omega^{-2},\mu}Q\in C^\gamma(\mathbf{S}^r_R(\mathscr{P}))$, and $G\in L^p(\mathbf{S}^r_R(\mathscr{P}))$ and if Ψ is a weak solution of (4.30), then there is a number $\lambda_*>2$ such that, for any $\lambda\in(2,\lambda_*)$,*

$$\begin{aligned}\|\mathbf{E}^{\epsilon,r}_{\omega^{2-\tau},\mu}\nabla\Psi\|_{L^\lambda(\mathbf{S}^r_{R/2}(\mathscr{P}))}\le c\Big(&R^{\frac{2}{\lambda}-1}\|\mathbf{E}^{\epsilon,r}_{\omega^{2-\tau},\mu}\nabla\Psi\|_{L^2(\mathbf{S}^r_R(\mathscr{P}))}+\|\mathbf{E}^{\epsilon,r}_{\omega^{-\tau},\mu}Q\|_{L^\lambda(\mathbf{S}^r_R(\mathscr{P}))}\\ &+R^{\frac{2}{\lambda}}\|\mathbf{E}^{\epsilon,r}_{\omega^{-2},\mu}Q\|_{C^\gamma(\mathbf{S}^r_R(\mathscr{P}))}+R^{\frac{2}{\lambda}}\|\mathbf{E}^{\epsilon,r}_{\omega^{-\tau},\mu}G\|_{L^p(\mathbf{S}^r_R(\mathscr{P}))}\Big),\end{aligned}\tag{4.48}$$

where c is independent of $\epsilon,\mu,\omega,r,p,R,\mathscr{P}$. See (A3) for p,p',τ,γ and Section 2 for $\mathbf{S}^r_R(\mathscr{P})$.

Proof. Since $\tau>1$, $R^\tau\le cR$. Lemma 4.8 (with $\sigma=2-\tau, t=p$) and Lemmas 4.9 and 4.11 imply that a weak solution of (4.30) satisfies

$$\begin{aligned}\fint_{\mathbf{S}^r_{R/2}(\mathscr{P})}|\mathbf{E}^{\epsilon,r}_{\omega^{2-\tau},\mu}\nabla\Psi|^2dy\le c\left|(|\mathbf{E}^{\epsilon,r}_{\omega^{2-\tau},\mu}\nabla\Psi|^q)_{\mathbf{S}^r_R(\mathscr{P})}\right|^{2/q}+\frac{1}{4}\fint_{\mathbf{S}^r_R(\mathscr{P})}|\nabla\Psi|^2dy\\ +\frac{c}{R^2}\Big(\|\mathbf{E}^{\epsilon,r}_{\omega^{-\tau},\mu}Q\|^2_{L^2(\mathbf{S}^r_R(\mathscr{P}))}+R^2\|\mathbf{E}^{\epsilon,r}_{\omega^{-2},\mu}Q\|^2_{C^\gamma(\mathbf{S}^r_R(\mathscr{P}))}+R^2\|\mathbf{E}^{\epsilon,r}_{\omega^{-\tau},\mu}G\|^2_{L^p(\mathbf{S}^r_R(\mathscr{P}))}\Big),\end{aligned}$$

where $q\in(1,2)$, $\gamma=\frac{p-2}{p}$, and c is a constant independent of $\epsilon,\omega,\mu,r,p,q,R,\mathscr{P}$. Let us define $\phi\equiv|\mathbf{E}^{\epsilon,r}_{\omega^{2-\tau},\mu}\nabla\Psi|^q$, $s\equiv\frac{2}{q}>1$, and

$$\zeta(y)\equiv|\mathbf{E}^{\epsilon,r}_{\omega^{-\tau},\mu}Q|^q(y)+\|\mathbf{E}^{\epsilon,r}_{\omega^{-2},\mu}Q\|^q_{C^\gamma(\mathbf{S}^r_R(\mathscr{P}))}+\|\mathbf{E}^{\epsilon,r}_{\omega^{-\tau},\mu}G\|^q_{L^p(\mathbf{S}^r_R(\mathscr{P}))}.$$

We have

$$\fint_{\mathbf{S}^r_{R/2}(\mathscr{P})}\phi^s\,dy\le c\Big(\fint_{\mathbf{S}^r_R(\mathscr{P})}\phi\,dy\Big)^s+\frac{1}{4}\fint_{\mathbf{S}^r_R(\mathscr{P})}\phi^s\,dy+c\fint_{\mathbf{S}^r_R(\mathscr{P})}\zeta^s\,dy.$$

By [16, Proposition 1.1], there is a $\lambda_* > 2$ such that if $\lambda \in (2, \lambda_*)$, then

$$\left(\fint_{\mathbf{S}^{r}_{R/2}(\mathscr{P})} |\mathbf{E}^{\epsilon,r}_{\omega^{2-\tau},\mu}\nabla\Psi|^{\lambda}dy\right)^{1/\lambda} \leq c\left(\fint_{\mathbf{S}^{r}_{R}(\mathscr{P})} |\mathbf{E}^{\epsilon,r}_{\omega^{2-\tau},\mu}\nabla\Psi|^{2}dy\right)^{1/2}$$
$$+c\left(R^{\frac{-2}{\lambda}}||\mathbf{E}^{\epsilon,r}_{\omega^{-\tau},\mu}Q||_{L^{\lambda}(\mathbf{S}^{r}_{R}(\mathscr{P}))} + ||\mathbf{E}^{\epsilon,r}_{\omega^{-2},\mu}Q||_{C^{\gamma}(\mathbf{S}^{r}_{R}(\mathscr{P}))} + ||\mathbf{E}^{\epsilon,r}_{\omega^{-\tau},\mu}G||_{L^{p}(\mathbf{S}^{r}_{R}(\mathscr{P}))}\right).$$

Lemma 4.12 follows from the inequality above. □

By Lemma 4.12, we have the following result.

Corollary 4.1. *Suppose that (A1)–(A3), $\frac{\epsilon}{r}, r, R \in (0, \frac{1}{2})$, $\mu \leq \left|\frac{\epsilon}{r}\right|^{\tau-1}$, $\mathscr{P} \in \Omega/r$, and Ψ is a weak solution of*

$$\begin{cases} -\nabla \cdot (\mathbf{E}^{\epsilon,r}_{\omega^2,\mu}\nabla\Psi) = G & \text{in } B_R(\mathscr{P}) \cap \Omega/r, \\ \Psi = \Psi_b & \text{on } B_R(\mathscr{P}) \cap \partial\Omega/r. \end{cases} \tag{4.49}$$

If $||\mathbf{E}^{\epsilon,r}_{1/\omega^{\tau},\mu}G||_{L^{p}(\mathbf{S}^{r}_{R}(\mathscr{P}))}$, $||\mathbf{E}^{\epsilon,r}_{\omega,\mu}\nabla\Psi||_{L^{2}(\mathbf{S}^{r}_{R}(\mathscr{P}))}$, and $||\Psi_b||_{C^{1,\gamma}(\mathbf{S}^{r}_{R}(\mathscr{P}))}$ are bounded independent of $\epsilon, \mu, \omega, r, R, \mathscr{P}$, then there is a sequence of solutions of (4.49) converging uniformly in $B_{R/2}(\mathscr{P}) \cap \Omega/r$. See Section 2 for $\mathbf{S}^{r}_{R}(\mathscr{P})$ and (A3) for p, τ, γ.

Proof. Set $\phi = \Psi - \Psi_b$ and $Q = \mathbf{E}^{\epsilon,r}_{\omega^2,\mu}\nabla\Psi_b$. Then

$$\begin{cases} -\nabla \cdot (\mathbf{E}^{\epsilon,r}_{\omega^2,\mu}\nabla\phi + Q) = G & \text{in } B_R(\mathscr{P}) \cap \Omega/r, \\ \phi = 0 & \text{on } B_R(\mathscr{P}) \cap \partial\Omega/r. \end{cases}$$

There is a $\lambda \in (2, \lambda_*)$ so that the right-hand side of (4.48) are bounded independent of $\epsilon, \mu, \omega, r, R, \mathscr{P}$. By Lemma 4.12 and Sobolev's embedding theorem [17], there is a sequence of solutions of (4.49) converging uniformly in $B_{R/2}(\mathscr{P}) \cap \Omega/r$. □

5. L^2-gradient convergence

This section shows a L^2-gradient convergence result for Eq (3.5) with "*uniform* L^1" (or "*uniform* $L^1 \setminus L^s$ *for* $s > 1$") external sources; that is, Lemma 5.2.

5.1. Preliminary

Let $H^1_{\#}(\mathbb{R}^2)$ denote the space containing local H^1 periodic functions with a period $\mathcal{Y}$. If $j \in \{1, 2\}$, we find $\mathbb{X}_{\omega,\mu,j} \in H^1_{\#}(\mathbb{R}^2)$ by solving

$$\begin{cases} \nabla \cdot \left(\mathbf{K}_{\omega^2,\mu}(\nabla\mathbb{X}_{\omega,\mu,j} + \vec{e}_j)\right) = 0 & \text{in } \mathcal{Y}, \\ (\mathbb{X}_{\omega,\mu,j})_{\mathcal{Y}} = 0, \end{cases} \tag{5.1}$$

where $\vec{e}_j$ is the unit vector in the j-th coordinate direction in $\mathbb{R}^2$. See (2.1) for $(\mathbb{X}_{\omega,\mu,j})_{\mathcal{Y}}$ and (4.1) for $\mathbf{K}_{\omega^2,\mu}$. By the Lax-Milgram Theorem and (A2), Eq (5.1) is uniquely solvable in $H^1_{\#}(\mathbb{R}^2)$ and

$$||\mathbf{K}_{\omega,\mu}\nabla\mathbb{X}_{\omega,\mu,j}||_{L^2(\mathcal{Y})} \leq c, \tag{5.2}$$

where c is independent of $\mu\,(=\frac{1}{\omega})$. If $y=(y_1,y_2)$ and $\Psi_j(y)=\mathbb{X}_{\omega,\mu,j}(y)+y_j$ for $j=1,2$, then $\nabla\cdot(\mathbf{K}_{\omega^2,\mu}\nabla\Psi_j)=0$ in $\mathcal{Y}$. Lemmas 4.1 and 4.7, and (5.2) imply

$$\begin{cases}\|\nabla\mathbb{X}_{\omega,\mu,j}+\vec{e}_j\|_{L^\infty(\mathcal{Y})}=\|\nabla\Psi_j\|_{L^\infty(\mathcal{Y})}\le c\|\nabla\Psi_j\|_{L^2(\mathcal{Y})}\le c,\\ \|\mathbf{K}_{\omega^2,\mu}(\nabla\mathbb{X}_{\omega,\mu,j}+\vec{e}_j)\|_{L^2(\mathcal{Y})}=\|\mathbf{K}_{\omega^2,\mu}\nabla\Psi_j\|_{L^2(\mathcal{Y})}\le c\|\nabla\Psi_j\|_{L^2(\mathcal{Y})}\le c,\end{cases}\tag{5.3}$$

where c is a constant independent of $\mu\,(=\frac{1}{\omega})$. The expressions in $(5.3)_1$ and $(5.1)_2$ imply

$$\|\mathbb{X}_{\omega,\mu,j}\|_{L^\infty(\mathcal{Y})}\le c\ (\text{independent of }\mu\,(=\omega^{-1})).\tag{5.4}$$

Define, for any $\nu>0$ and $j=1,2$,

$$\mathbb{X}^\nu_{\omega,\mu,j}(x)\equiv\nu\mathbb{X}_{\omega,\mu,j}(x/\nu),\qquad\qquad \mathbb{X}^\nu_{\omega,\mu}(x)\equiv\left(\mathbb{X}^\nu_{\omega,\mu,1}(x),\mathbb{X}^\nu_{\omega,\mu,2}(x)\right).\tag{5.5}$$

Let $\mathcal{K}_{\omega,\mu}$ denote a 2×2 matrix with (i,j)-entries as

$$\int_{\mathcal{Y}}\mathbf{K}_{\omega^2,\mu}(\delta_{i,j}+\partial_i\mathbb{X}_{\omega,\mu,j}(y))dy\quad\text{where}\quad \delta_{i,j}\equiv\begin{cases}1 & \text{if } i=j,\\ 0 & \text{if } i\ne j.\end{cases}\tag{5.6}$$

By the remark in [18, pages 43 and 44] and $(5.3)_2$, $\mathcal{K}_{\omega,\mu}$ is a positive scalar function depending only on $\mu\,(=\frac{1}{\omega})$ and satisfies

$$\begin{cases}0<\mathbf{d}_2\le\mathcal{K}_{\omega,\mu}\le\mathbf{d}_3,\\ \mathcal{K}_{\omega,\mu}\text{ is continuous in }\{\mu|\ \mu\in(0,1)\},\end{cases}\tag{5.7}$$

where $\mathbf{d}_2,\mathbf{d}_3$ are constants independent of $\mu\,(=\frac{1}{\omega})$.

5.2. L^2-gradient convergence

Lemma 5.1. *If $\epsilon,\mu_\epsilon\in(0,1)$, $\omega_\epsilon=\frac{1}{\mu_\epsilon}$, and $\widetilde{\mathcal{K}}=\lim_{\epsilon\to0}\mathcal{K}_{\omega_\epsilon,\mu_\epsilon}$, then $\Psi_\epsilon(x)\ (\equiv\mathbf{K}^\epsilon_{\omega^2_\epsilon,\mu_\epsilon}(x)\nabla(x+\mathbb{X}^\epsilon_{\omega_\epsilon,\mu_\epsilon}(x)))$ converge to $\widetilde{\mathcal{K}}I$ weakly in $L^2(\Omega)$. Here I is the identity; see (5.5) and (5.6) for $\mathbb{X}^\epsilon_{\omega_\epsilon,\mu_\epsilon}$, $\mathcal{K}_{\omega_\epsilon,\mu_\epsilon}$.*

Proof. To prove Lemma 5.1, we need to show the following, by [12, Proposition 1.46]:

$$\begin{cases}\|\Psi_\epsilon\|_{L^2(\Omega)}\le c\ (\textit{independent of }\epsilon),\\ \displaystyle\int_{\mathbf{D}\cap\Omega}\Psi_\epsilon(x)\,dx\xrightarrow[\epsilon\to0]{}\int_{\mathbf{D}\cap\Omega}\widetilde{\mathcal{K}}I\,dx=|\mathbf{D}\cap\Omega|\widetilde{\mathcal{K}}I,\end{cases}\tag{5.8}$$

for any rectangle $\mathbf{D}$. Here, $(5.3)_2$ implies $(5.8)_1$. By (2.9) in [12] and $(5.3)_2$,

$$\int_{\mathbf{D}\cap\Omega}\Psi_\epsilon(x)\,dx=\sum_{\mathbf{j}\in A(\epsilon)}\epsilon^2\mathcal{K}_{\omega_\epsilon,\mu_\epsilon}I+\sum_{\mathbf{j}\in A'(\epsilon)}\int_{\epsilon(\mathcal{Y}+\mathbf{j})\cap\Omega}\Psi_\epsilon(x)\,dx\xrightarrow[\epsilon\to0]{}|\mathbf{D}\cap\Omega|\widetilde{\mathcal{K}}I,$$

where $A(\epsilon)\equiv\{\mathbf{j}|\ \epsilon(\mathcal{Y}+\mathbf{j})\Subset\mathbf{D}\cap\Omega\}$ and $A'(\epsilon)\equiv\{\mathbf{j}|\ \overline{\epsilon(\mathcal{Y}+\mathbf{j})}\cap\partial(\mathbf{D}\cap\Omega)\ne\emptyset\}$. □

Lemma 5.2. *Consider the following problem:*

$$\begin{cases} -\nabla\cdot(\mathbf{E}^{\epsilon,r}_{\omega^2,\mu}\nabla\mathbb{V}) = \mathbb{G} & \text{in } B_2(0)\cap\Omega/r,\\ \mathbb{V} = \mathbb{V}_b & \text{on } B_2(0)\cap\partial\Omega/r. \end{cases} \tag{5.9}$$

Suppose that a sequence $\{\epsilon,\omega_\epsilon,\mu_\epsilon,r_\epsilon,\mathbb{V}_\epsilon,\mathbb{G}_\epsilon,\mathbb{V}_{b,\epsilon},\mathcal{K}_{\omega_\epsilon,\mu_\epsilon}\}$ *satisfies (A1)–(A3), (5.9), and*

$$\begin{cases} \epsilon, \frac{\epsilon}{r_\epsilon}\to 0, \quad r_\epsilon\to r\in[0,1], \quad \|\mathbf{E}^{\epsilon,r_\epsilon}_{1/\omega^\tau_\epsilon,\mu_\epsilon}\mathbb{G}_\epsilon\|_{L^p(\mathbf{S}^{r_\epsilon}_2(0))}\to 0,\\ \widetilde{\mathcal{K}} = \lim_{\epsilon\to 0}\mathcal{K}_{\omega_\epsilon,\mu_\epsilon},\\ \mathbb{V}_{b,\epsilon}(0) = \partial_T\mathbb{V}_{b,\epsilon}(0) = 0,\\ \|\mathbb{V}_\epsilon\|_{L^\infty(\mathbf{S}^{r_\epsilon}_2(0))},\ [\nabla\mathbb{V}_{b,\epsilon}]_{C^\gamma(\mathbf{S}^{r_\epsilon}_2(0))}\le 1. \end{cases} \tag{5.10}$$

See (A3) for p,τ,γ *and Section 2 for* $\mathbf{S}^r_R(0)$*;* $\widetilde{\mathcal{K}}$ *is a scalar constant (see (5.7)); and* $\partial_T\mathbb{V}_{b,\epsilon}$ *is the tangential derivative of* $\mathbb{V}_{b,\epsilon}$*. Then the following hold.*

(S1) $\|\mathbf{E}^{\epsilon,r_\epsilon}_{\omega_\epsilon,\mu_\epsilon}\nabla\mathbb{V}_\epsilon\|_{L^2(\mathbf{S}^{r_\epsilon}_1(0))}$ *are bounded independent of* $\epsilon,\omega_\epsilon,\mu_\epsilon,r_\epsilon$*;*
(S2) *There is a subsequence of* $\{\mathbb{V}_\epsilon\}$ *(with same notation for the subsequence) so that*

$$\begin{cases} \mathbb{V}_\epsilon\to\mathbb{V} & \text{weakly in } H^1(\mathbf{S}^r_1(0)),\\ \mathbf{E}^{\epsilon,r_\epsilon}_{\omega^2_\epsilon,\mu_\epsilon}\nabla\mathbb{V}_\epsilon\to\zeta & \text{weakly star in Radon measures}, \quad \text{as } \frac{\epsilon}{r_\epsilon}\to 0;\\ \mathbb{V}_{b,\epsilon}\to\mathbb{V}_b & \text{weakly in } C^{1,\gamma}(\mathbf{S}^r_1(0)), \end{cases}$$

(S3) $-\nabla\cdot\zeta = 0$ *in* $\mathbf{S}^r_1(0)$ *in the distribution sense;*
(S4) $\zeta = \widetilde{\mathcal{K}}\nabla\mathbb{V}$.

Proof. **Step I.** As $\frac{\epsilon}{r_\epsilon}\to 0$, by Hölder's inequality, $(5.10)_1$, (A2), and (A3),

$$\|\mathbb{G}_\epsilon\|_{L^1(\mathbf{S}^{r_\epsilon}_2(0))}\le\|\mathbf{E}^{\epsilon,r_\epsilon}_{1/\omega^\tau_\epsilon,\mu_\epsilon}\mathbb{G}_\epsilon\|_{L^p(\mathbf{S}^{r_\epsilon}_2(0))}\|\mathbf{E}^{\epsilon,r_\epsilon}_{\omega^\tau_\epsilon,\mu_\epsilon}\|_{L^{p'}(\mathbf{S}^{r_\epsilon}_2(0))}\to 0. \tag{5.11}$$

Let $\eta\in C^\infty_0(B_2(0))$ and $\eta = 1$ in $B_1(0)$. Test (5.9) against $(\mathbb{V}_\epsilon-\mathbb{V}_{b,\epsilon})\eta^2$, and apply $(5.10)_4$ and (5.11) to get **(S1)**. **(S2)** is from **(S1)** and [12, Proposition 1.48].

Step II. If $\psi\in C^\infty_0(\mathbf{S}^r_2(0))$, then $supp(\psi)\cap\partial\Omega/r_\epsilon = \emptyset$ when ϵ is small. Test (5.9) against ψ to get

$$\int_{\mathbf{S}^{r_\epsilon}_2(0)}\mathbf{E}^{\epsilon,r_\epsilon}_{\omega^2_\epsilon,\mu_\epsilon}\nabla\mathbb{V}_\epsilon\nabla\psi\,dx = \int_{\mathbf{S}^{r_\epsilon}_2(0)}\mathbb{G}_\epsilon\psi\,dx.$$

As $\epsilon\to 0$, **(S2)** and (5.11) imply $-\nabla\cdot\zeta = 0$ in $\mathbf{S}^r_1(0)$. So we get **(S3)**.

Step III. Recall (5.5) for $\mathbb{X}^{\epsilon/r_\epsilon}_{\omega_\epsilon,\mu_\epsilon,j}$ and **Step II** for ψ. Consider the identity

$$\int_{\mathbf{S}^{r_\epsilon}_1(0)}\mathbf{E}^{\epsilon,r_\epsilon}_{\omega^2_\epsilon,\mu_\epsilon}\nabla\mathbb{V}_\epsilon\,\psi\,\nabla(x_j+\mathbb{X}^{\epsilon/r_\epsilon}_{\omega_\epsilon,\mu_\epsilon,j})\,dx = \int_{\mathbf{S}^{r_\epsilon}_1(0)}\mathbf{E}^{\epsilon,r_\epsilon}_{\omega^2_\epsilon,\mu_\epsilon}\nabla(x_j+\mathbb{X}^{\epsilon/r_\epsilon}_{\omega_\epsilon,\mu_\epsilon,j})\,\psi\,\nabla\mathbb{V}_\epsilon\,dx. \tag{5.12}$$

By (5.4), (5.5), $(5.9)_1$, (5.11), **(S2)**, and **(S3)**, the left-hand side of (5.12) satisfies

$$\int_{\mathbf{S}^{r_\epsilon}_1(0)}\mathbf{E}^{\epsilon,r_\epsilon}_{\omega^2_\epsilon,\mu_\epsilon}\nabla\mathbb{V}_\epsilon\,\psi\nabla(x_j+\mathbb{X}^{\epsilon/r_\epsilon}_{\omega_\epsilon,\mu_\epsilon,j}) = -\int_{\mathbf{S}^{r_\epsilon}_1(0)}\mathbf{E}^{\epsilon,r_\epsilon}_{\omega^2_\epsilon,\mu_\epsilon}\nabla\mathbb{V}_\epsilon\,(x_j+\mathbb{X}^{\epsilon/r_\epsilon}_{\omega_\epsilon,\mu_\epsilon,j})\nabla\psi + \int_{\mathbf{S}^{r_\epsilon}_1(0)}\mathbb{G}_\epsilon(x_j+\mathbb{X}^{\epsilon/r_\epsilon}_{\omega_\epsilon,\mu_\epsilon,j})\psi$$

$$\xrightarrow[\frac{\epsilon}{r_\epsilon}\to 0]{} -\int_{\mathbf{S}_1^{;r}(0)} \zeta\, x_j \nabla\psi = \int_{\mathbf{S}_1^{;r}(0)} \zeta\, \vec{\mathbf{e}}_j \psi. \tag{5.13}$$

Next, we check the right-hand side of (5.12). By (4.1), (5.1), and Green's formula,

$$\begin{aligned}\int_{\mathbf{S}_1^{;r_\epsilon}(0)} \mathbf{E}^{\epsilon,r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon} \nabla(x_j + \mathbb{X}^{\epsilon/r_\epsilon}_{\omega_\epsilon,\mu_\epsilon,j})\, \psi\, \nabla\mathbb{V}_\epsilon\, dx &= -\int_{\mathbf{S}_1^{;r_\epsilon}(0)} \mathbf{K}^{\epsilon/r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon} \nabla(x_j + \mathbb{X}^{\epsilon/r_\epsilon}_{\omega_\epsilon,\mu_\epsilon,j})\mathbb{V}_\epsilon \nabla\psi\, dx \\ &+ \int_{\mathbf{S}_1^{;r_\epsilon}(0)} (\mathbf{E}^{\epsilon,r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon} - \mathbf{K}^{\epsilon/r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon})\nabla(x_j + \mathbb{X}^{\epsilon/r_\epsilon}_{\omega_\epsilon,\mu_\epsilon,j})\, \psi\, \nabla\mathbb{V}_\epsilon\, dx.\end{aligned} \tag{5.14}$$

Since $supp(\psi) \cap \partial\mathbf{S}_1^{;r}(0) = \emptyset$, $(\mathbf{E}^{\epsilon,r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon} - \mathbf{K}^{\epsilon/r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon})\psi = 0$ in $\mathbf{S}_1^{;r_\epsilon}(0)$ if $\frac{\epsilon}{r_\epsilon}$ is small. So

$$\int_{\mathbf{S}_1^{;r_\epsilon}(0)} (\mathbf{E}^{\epsilon,r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon} - \mathbf{K}^{\epsilon/r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon})\nabla(x_j + \mathbb{X}^{\epsilon/r_\epsilon}_{\omega_\epsilon,\mu_\epsilon,j})\, \psi\, \nabla\mathbb{V}_\epsilon\, dx \xrightarrow[\frac{\epsilon}{r_\epsilon}\to 0]{} 0. \tag{5.15}$$

Note that

$$\begin{aligned}-\int_{\mathbf{S}_1^{;r_\epsilon}(0)} \mathbf{K}^{\epsilon/r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon} \nabla(x_j + \mathbb{X}^{\epsilon/r_\epsilon}_{\omega_\epsilon,\mu_\epsilon,j})\mathbb{V}_\epsilon \nabla\psi\, dx &= -\int_{\mathbf{S}_1^{;r_\epsilon}(0)} \mathbf{K}^{\epsilon/r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon} \nabla(x_j + \mathbb{X}^{\epsilon/r_\epsilon}_{\omega_\epsilon,\mu_\epsilon,j})\mathbb{V} \nabla\psi\, dx \\ &- \int_{\mathbf{S}_1^{;r_\epsilon}(0)} \mathbf{K}^{\epsilon/r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon} \nabla(x_j + \mathbb{X}^{\epsilon/r_\epsilon}_{\omega_\epsilon,\mu_\epsilon,j})(\mathbb{V}_\epsilon - \mathbb{V}) \nabla\psi\, dx.\end{aligned}$$

Lemma 5.1 and $(5.10)_2$ imply

$$\begin{aligned}-\int_{\mathbf{S}_1^{;r_\epsilon}(0)} \mathbf{K}^{\epsilon/r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon} \nabla(x_j + \mathbb{X}^{\epsilon/r_\epsilon}_{\omega_\epsilon,\mu_\epsilon,j})\mathbb{V} \nabla\psi\, dx &\xrightarrow[\frac{\epsilon}{r_\epsilon}\to 0]{} -\int_{\mathbf{S}_1^{;r}(0)} \widetilde{\mathcal{K}}\vec{\mathbf{e}}_j\, \mathbb{V}\nabla\psi\, dx \\ &= \int_{\mathbf{S}_1^{;r}(0)} \widetilde{\mathcal{K}}\vec{\mathbf{e}}_j \nabla\mathbb{V}\, \psi\, dx = \int_{\mathbf{S}_1^{;r}(0)} \vec{\mathbf{e}}_j \psi\, \widetilde{\mathcal{K}}\nabla\mathbb{V}\, dx.\end{aligned} \tag{5.16}$$

Moreover, $(5.3)_2$, **(S2)**, and Hölder's inequality imply

$$\left| \int_{\mathbf{S}_1^{;r_\epsilon}(0)} \mathbf{K}^{\epsilon/r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon} \nabla(x_j + \mathbb{X}^{\epsilon/r_\epsilon}_{\omega_\epsilon,\mu_\epsilon,j})(\mathbb{V}_\epsilon - \mathbb{V}) \nabla\psi\, dx \right| \xrightarrow[\frac{\epsilon}{r_\epsilon}\to 0]{} 0, \tag{5.17}$$

and (5.12)–(5.17) imply $\int_{\mathbf{S}_1^{;r}(0)} \zeta\, \vec{\mathbf{e}}_j \psi\, dx = \int_{\mathbf{S}_1^{;r}(0)} \widetilde{\mathcal{K}}\nabla\mathbb{V}\, \vec{\mathbf{e}}_j \psi\, dx$. This proves $\zeta = \widetilde{\mathcal{K}}\nabla\mathbb{V}$, since ψ and j are arbitrary. This proves **(S4)**. □

6. The Lipschitz estimate

With the convergence results (that is, Lemmas 4.4 and 5.2, Remark 4.2, and Corollary 4.1) from Sections 4 and 5, we consider the Lipschitz estimate for strongly elliptic equations. This is done by a three-step compactness argument [2, 3]. The interior Lipschitz estimate for the solutions of strongly elliptic equations is given in Subsection 6.1; the boundary Lipschitz estimate is in Subsection 6.2.

6.1. Interior Lipschitz estimate

Assume $B_1(0) \Subset \Omega$.

Lemma 6.1. *Under (A2) and (A3), there are constants* $\theta, \epsilon_0 \in (0,1)$ *such that if*

$$\begin{cases} -\nabla \cdot (\mathbf{E}^{\nu}_{\omega^2,\mu} \nabla \mathbb{V}) = \mathbb{G} & \text{in } B_1(0), \\ \nu, \mu \in (0, \epsilon_0), \\ \|\mathbb{V}\|_{L^\infty(B_1(0))},\ \epsilon_0^{-1} \|\mathbf{E}^{\nu}_{1/\omega^\tau,\mu} \mathbb{G}\|_{L^p(B_1(0))} \le 1, \end{cases} \tag{6.1}$$

then

$$\sup_{z \in B_\theta(0)} \left| \mathbb{V}(z) - \mathbb{V}(0) - (z + \mathbb{X}^{\nu}_{\omega,\mu}(z)) \mathbf{b}_{\omega,\mu,\nu} \right| \le \theta^{1+\gamma}, \tag{6.2}$$

where $\mathbf{b}_{\omega,\mu,\nu} \equiv \mathcal{K}^{-1}_{\omega,\mu} (\mathbf{E}^{\nu}_{\omega^2,\mu} \nabla \mathbb{V})_{B_\theta(0)}$ *and* $\mathcal{K}^{-1}_{\omega,\mu}$ *is the inverse matrix of* $\mathcal{K}_{\omega,\mu}$*. See (A3) for* τ, p, p', γ *and (2.1) for* $(\mathbf{E}^{\nu}_{\omega^2,\mu} \nabla \mathbb{V})_{B_\theta(0)}$.

Proof. Consider $-\Delta \mathbb{V} = 0$ in $B_{4/5}(0)$. By [17, Theorem 4.6], a small $\theta \in (0,1)$ exists such that, for some $\widehat{\gamma} \in (\gamma, 1)$,

$$\sup_{z \in B_\theta(0)} \left| \mathbb{V}(z) - \mathbb{V}(0) - z (\nabla \mathbb{V})_{B_\theta(0)} \right| \le \theta^{1+\widehat{\gamma}} \|\mathbb{V}\|_{L^\infty(B_{4/5}(0))}. \tag{6.3}$$

We claim (6.2). If not, there is a sequence $\{\nu, \omega_\nu, \mu_\nu, \mathbb{V}_\nu, \mathbb{G}_\nu, \mathcal{K}_{\omega_\nu,\mu_\nu}\}$ satisfying (6.1) and

$$\begin{cases} \nu, \mu_\nu \to 0, \qquad \|\mathbf{E}^{\nu}_{1/\omega^\tau_\nu,\mu_\nu} \mathbb{G}_\nu\|_{L^p(B_1(0))} \to 0, \\ \widetilde{\mathcal{K}} = \lim_{\nu \to 0} \mathcal{K}_{\omega_\nu,\mu_\nu}, \\ \sup_{z \in B_\theta(0)} \left| \mathbb{V}_\nu(z) - \mathbb{V}_\nu(0) - (z + \mathbb{X}^{\nu}_{\omega_\nu,\mu_\nu}(z)) \mathbf{b}_{\omega_\nu,\mu_\nu,\nu} \right| > \theta^{1+\gamma}. \end{cases} \tag{6.4}$$

See (5.6) for $\mathcal{K}_{\omega,\mu}$; $(6.4)_2$ is due to $(5.7)_1$. By (A2) and (A3), Lemma 5.2, and $(6.4)_1$,

$$\begin{cases} \|\mathbb{G}_\nu\|_{L^1(B_1(0))} \le \|\mathbf{E}^{\nu}_{1/\omega^\tau_\nu,\mu_\nu} \mathbb{G}_\nu\|_{L^p(B_1(0))} \|\mathbf{E}^{\nu}_{\omega^\tau_\nu,\mu_\nu}\|_{L^{p'}(B_1(0))} \xrightarrow[\nu \to 0]{} 0, \\ \|\mathbf{E}^{\nu}_{\omega^2_\nu,\mu_\nu} \nabla \mathbb{V}_\nu\|_{L^1(B_{4/5}(0))} \le c \|\mathbf{E}^{\nu}_{\omega_\nu,\mu_\nu} \nabla \mathbb{V}_\nu\|_{L^2(B_{4/5}(0))} \le c, \end{cases} \tag{6.5}$$

where c is a constant independent of ν, ω_ν, μ_ν. The functions $\mathbf{E}^{\nu}_{\omega^2_\nu,\mu_\nu} \nabla \mathbb{V}_\nu$ are equi-integrable. That is, for any $\delta > 0$, by Hölder's inequality, (A2), and $(6.5)_2$, we have

$$\|\mathbf{E}^{\nu}_{\omega^2_\nu,\mu_\nu} \nabla \mathbb{V}_\nu\|_{L^1(B_\delta(0))} \le c \|\mathbf{E}^{\nu}_{\omega_\nu,\mu_\nu} \nabla \mathbb{V}_\nu\|_{L^2(B_{4/5}(0))} \delta \le c\, \delta, \tag{6.6}$$

where c is independent of $\nu, \omega_\nu, \mu_\nu, \delta$. Note that ν is the *periodic size*. By (6.1), (6.4), (6.5), and Lemma 5.2, there is a subsequence of $\{\nu, \omega_\nu, \mu_\nu, \mathbb{V}_\nu, \mathbb{G}_\nu, \mathcal{K}_{\omega_\nu,\mu_\nu}\}$ (with the same notation for the subsequence) satisfying

$$\begin{cases} \mathbb{V}_\nu \to \mathbb{V} & \text{weakly in } H^1(B_{4/5}(0)), \\ \mathbf{E}^{\nu}_{\omega^2_\nu,\mu_\nu} \nabla \mathbb{V}_\nu \to \widetilde{\mathcal{K}} \nabla \mathbb{V} & \text{weakly star in Radon measures}, \qquad \text{as } \nu \to 0, \\ \Delta \mathbb{V} = 0 & \text{in } B_{4/5}(0), \end{cases} \tag{6.7}$$

where $\widetilde{\mathcal{K}}$ is a positive constant. For the sequence $\{\nu, \omega_\nu, \mu_\nu, \mathbb{V}_\nu, \mathbb{G}_\nu, \mathcal{K}_{\omega_\nu,\mu_\nu}\}$ in (6.7), either (i) an infinite number of μ_ν satisfy $\mu_\nu \le \nu^{\tau-1}$ or (ii) an infinite number of μ_ν satisfy $\mu_\nu > \nu^{\tau-1}$. In either case, the sequence in (6.7) has a uniformly convergent subsequence. The reason is as follows.

- *For Case (i).* A sequence $\{\nu, \omega_\nu, \mu_\nu, \mathbb{V}_\nu, \mathbb{G}_\nu, \mathcal{K}_{\omega_\nu,\mu_\nu}\}$ in (6.7) satisfies the assumptions of Corollary 4.1. So Corollary 4.1 implies that a $\mathbb{V} \in H^1(B_{4/5}(0)) \cap C(B_{4/5}(0))$ and a subsequence of $\{\mathbb{V}_\nu\}$ in (6.7) (with the same notation for the subsequence) exist such that $\mathbb{V}_\nu \to \mathbb{V}$ in $C(B_{4/5}(0))$ as $\nu \to 0$.
- *For Case (ii).* Note that $\lim\limits_{\nu\to 0} \nu^{\frac{4}{\varkappa}} |\ln \mu_\nu| = 0$ for any $\varkappa > 2$ with $0 < \frac{6}{\varkappa} < \tau$. The assumptions of Lemma 4.4 and Remark 4.2 are satisfied by the sequence $\{\nu, \omega_\nu, \mu_\nu, \mathbb{V}_\nu, \mathbb{G}_\nu, \mathcal{K}_{\omega_\nu,\mu_\nu}\}$ in (6.7). Lemma 4.4 and Remark 4.2 imply that there is a $\mathbb{V} \in H^1(B_{4/5}(0)) \cap C(B_{4/5}(0))$ and a subsequence of $\{\mathbb{V}_\nu\}$ in (6.7) (with the same notation for the subsequence) such that $\mathbb{V}_\nu \to \mathbb{V}$ in $C(B_{4/5}(0))$ as $\nu \to 0$.

Therefore, there is a $\mathbb{V} \in H^1(B_{4/5}(0)) \cap C(B_{4/5}(0))$ and a subsequence of $\{\mathbb{V}_\nu\}$ in (6.7) (with the same notation for the subsequence) so that

$$\begin{cases} \mathbb{V}_\nu \to \mathbb{V} & \text{in } C(B_{4/5}(0)), \\ \|\mathbb{V}\|_{L^\infty(B_1(0))} \le 1, \\ \mathbf{E}^\nu_{\omega_\nu^2,\mu_\nu} \nabla \mathbb{V}_\nu \to \widetilde{\mathcal{K}} \nabla \mathbb{V} & \text{weakly star in Radon measures}, \\ -\Delta \mathbb{V} = 0 & \text{in } B_{4/5}(0). \end{cases} \tag{6.8}$$

For any $\delta \in (0, \theta)$, let $\eta_\delta \in C_0^\infty(B_\theta(0))$ be a non-negative function with $\eta_\delta \in [0, 1]$ and $\eta_\delta(x) = 1$ if $x \in B_{\theta-\delta}(0)$. By $(6.8)_{2,3}$ and (6.6), we see

$$\begin{aligned} &\lim_{\nu\to 0} \left| \int_{B_\theta(0)} (\mathbf{E}^\nu_{\omega_\nu^2,\mu_\nu} \nabla \mathbb{V}_\nu - \widetilde{\mathcal{K}} \nabla \mathbb{V})\, dx \right| \\ &\le \lim_{\nu\to 0} \left| \int_{B_\theta(0)} (\mathbf{E}^\nu_{\omega_\nu^2,\mu_\nu} \nabla \mathbb{V}_\nu - \widetilde{\mathcal{K}} \nabla \mathbb{V}) \eta_\delta\, dx \right| + \lim_{\nu\to 0} \left| \int_{B_\theta(0)} (\mathbf{E}^\nu_{\omega_\nu^2,\mu_\nu} \nabla \mathbb{V}_\nu - \widetilde{\mathcal{K}} \nabla \mathbb{V})(1 - \eta_\delta)\, dx \right| \\ &\le \lim_{\nu\to 0} \int_{B_\theta(0)\setminus B_{\theta-\delta}(0)} \left| \mathbf{E}^\nu_{\omega_\nu^2,\mu_\nu} \nabla \mathbb{V}_\nu \right| + \left| \widetilde{\mathcal{K}} \nabla \mathbb{V} \right| dx \le c\, \delta^{1/2}. \end{aligned} \tag{6.9}$$

Moreover, $(6.4)_2$ and (6.9) imply

$$\lim_{\nu\to 0} \mathbf{b}_{\omega_\nu,\mu_\nu,\nu} = \lim_{\nu\to 0} \mathcal{K}^{-1}_{\omega_\nu,\mu_\nu} \fint_{B_\theta(0)} \mathbf{E}^\nu_{\omega_\nu^2,\mu_\nu} \nabla \mathbb{V}_\nu\, dx = (\nabla \mathbb{V})_{B_\theta(0)}. \tag{6.10}$$

By (5.4), (5.5), (6.3), $(6.4)_3$, (6.8), and (6.10),

$$\begin{aligned} \theta^{1+\gamma} &\le \lim_{\nu\to 0} \sup_{z\in B_\theta(0)} \left| \mathbb{V}_\nu(z) - \mathbb{V}_\nu(0) - (z + \mathbb{X}^\nu_{\omega_\nu,\mu_\nu}(z)) \mathbf{b}_{\omega_\nu,\mu_\nu,\nu} \right| \\ &= \sup_{z\in B_\theta(0)} \left| \mathbb{V}(z) - \mathbb{V}(0) - z(\nabla \mathbb{V})_{B_\theta(0)} \right| \le \theta^{1+\widehat{\gamma}} \|\mathbb{V}\|_{L^\infty(B_{4/5}(0))}. \end{aligned} \tag{6.11}$$

We get a contradiction from (6.11). So (6.2) is true. □

Lemma 6.2. *Under (A2) and (A3), there are constants* $\theta, \epsilon_0 \in (0, 1)$ *such that if*

$$\begin{cases} -\nabla \cdot (\mathbf{E}^\epsilon_{\omega^2,\mu} \nabla V) = G & \text{in } B_1(0), \\ \epsilon, \mu \in (0, \epsilon_0), \\ k \in \mathbb{N} \text{ satisfying } \frac{\epsilon}{\epsilon_0} < \theta^k, \end{cases} \tag{6.12}$$

then we have the constants $\mathbf{a}^{\epsilon,k}_{\omega,\mu}$, $\mathbf{b}^{\epsilon,k}_{\omega,\mu}$ *satisfying*

$$\begin{cases} |\mathbf{a}^{\epsilon,k}_{\omega,\mu}| + |\mathbf{b}^{\epsilon,k}_{\omega,\mu}| \le cJ, \\ \sup_{z\in B_{\theta^k}(0)} \left| V(z) - V(0) - \epsilon\mathbf{a}^{\epsilon,k}_{\omega,\mu} - \left(z + \mathbb{X}^{\epsilon}_{\omega,\mu}(z)\right)\mathbf{b}^{\epsilon,k}_{\omega,\mu} \right| \le \theta^{k(1+\gamma)}J, \end{cases} \tag{6.13}$$

where $J \equiv \|V\|_{L^\infty(B_1(0))} + \epsilon_0^{-1}\|\mathbf{E}^{\epsilon}_{1/\omega^\tau,\mu}G\|_{L^p(B_1(0))}$ *and c is independent of* $\epsilon, \mu\, (= \frac{1}{\omega}), k$. *See (A3) for* τ, p, γ.

Proof. For $k = 1$, (6.13) is from Lemma 6.1 with $\nu = \epsilon, \mathbb{V} = \frac{V}{J}, \mathbb{G} = \frac{G}{J}$. In this case, $\mathbf{a}^{\epsilon,1}_{\omega,\mu} = 0$ and $\mathbf{b}^{\epsilon,1}_{\omega,\mu} = \mathcal{K}^{-1}_{\omega,\mu}\,(\mathbf{E}^{\epsilon}_{\omega^2,\mu}\nabla V)_{B_\theta(0)}$. If (6.13) is true for some $k \in \mathbb{N}$ satisfying $\frac{\epsilon}{\epsilon_0} < \theta^k$, we define

$$\begin{cases} \mathbb{V}(z) \equiv \dfrac{V(\theta^k z) - V(0) - \epsilon\mathbf{a}^{\epsilon,k}_{\omega,\mu} - \left(\theta^k z + \mathbb{X}^{\epsilon}_{\omega,\mu}(\theta^k z)\right)\mathbf{b}^{\epsilon,k}_{\omega,\mu}}{\theta^{k(1+\gamma)}J} \\ \mathbb{G}(z) \equiv \dfrac{G(\theta^k z)}{\theta^{k(\gamma-1)}J} \end{cases} \quad \text{in } B_1(0). \tag{6.14}$$

By induction, (5.1), and (5.5), $\mathbb{V}$ and $\mathbb{G}$ satisfy (6.1) with $\nu = \epsilon/\theta^k$. Apply Lemma 6.1 and $(6.14)_1$ to obtain

$$\sup_{z\in B_\theta(0)} \Big| V(\theta^k z) - V(0) - \left(\theta^k z + \mathbb{X}^{\epsilon}_{\omega,\mu}(\theta^k z)\right)\mathbf{b}^{\epsilon,k}_{\omega,\mu} + \epsilon\mathbb{X}^{1}_{\omega,\mu}(0)\mathbf{b}^{\epsilon,k}_{\omega,\mu} \\ -\theta^{k\gamma}J\left(\theta^k z + \mathbb{X}^{\epsilon}_{\omega,\mu}(\theta^k z)\right)\mathbf{b}_{\omega,\mu,\epsilon/\theta^k} \Big| \le \theta^{(k+1)(1+\gamma)}J, \tag{6.15}$$

where $\mathbf{b}_{\omega,\mu,\epsilon/\theta^k} \equiv \mathcal{K}^{-1}_{\omega,\mu}\,(\mathbf{E}^{\epsilon/\theta^k}_{\omega^2,\mu}\nabla\mathbb{V})_{B_\theta(0)}$. Define

$$\mathbf{a}^{\epsilon,k+1}_{\omega,\mu} \equiv -\mathbb{X}^{1}_{\omega,\mu}(0)\mathbf{b}^{\epsilon,k}_{\omega,\mu} \qquad \text{and} \qquad \mathbf{b}^{\epsilon,k+1}_{\omega,\mu} \equiv \mathbf{b}^{\epsilon,k}_{\omega,\mu} + J\theta^{k\gamma}\mathbf{b}_{\omega,\mu,\epsilon/\theta^k}. \tag{6.16}$$

By $(5.7)_1$ and an argument like that of (6.5), $|\mathbf{b}_{\omega,\mu,\epsilon/\theta^k}|$ is bounded uniformly in ϵ, ω, μ, k. Then $(6.13)_1$ follows from (5.4) and (6.16); $(6.13)_2$ is obtained by employing (6.16) and by substituting $\theta^k z$ with z in (6.15). □

Lemma 6.3. *Under (A2) and (A3), there is an* $\epsilon_0 \in (0,1)$ *such that any solution of* $(6.12)_1$ *satisfies, for any* $\epsilon, \mu \in (0, \epsilon_0)$,

$$\|\nabla V\|_{L^\infty(B_{1/2}(0))} \le c\big(\|V\|_{L^\infty(B_1(0))} + \|\mathbf{E}^{\epsilon}_{1/\omega^\tau,\mu}G\|_{L^p(B_1(0))}\big), \tag{6.17}$$

where c is a constant independent of $\epsilon, \mu\, (= \frac{1}{\omega})$. *See (A3) for* τ, p.

Proof. Let $\gamma, \theta, \epsilon_0, J$ be same as Lemma 6.2 and let c be a constant independent of ϵ, ω, μ. Suppose that $k \in \mathbb{N}$ satisfies $\theta^{k+1} \le \frac{\epsilon}{\epsilon_0} < \theta^k$, then by Lemma 6.2,

$$\sup_{z\in B_{\epsilon/\epsilon_0}(0)} \left| V(z) - V(0) - \epsilon\mathbf{a}^{\epsilon,k}_{\omega,\mu} - \left(z + \mathbb{X}^{\epsilon}_{\omega,\mu}(z)\right)\mathbf{b}^{\epsilon,k}_{\omega,\mu} \right| \le c\left|\frac{\epsilon}{\epsilon_0}\right|^{1+\gamma} J. \tag{6.18}$$

Define

$$\begin{cases} \mathbb{V}(z) \equiv \dfrac{V(\epsilon z) - V(0) - \epsilon\mathbf{a}^{\epsilon,k}_{\omega,\mu} - (\epsilon z + \epsilon\mathbb{X}^{1}_{\omega,\mu}(z))\mathbf{b}^{\epsilon,k}_{\omega,\mu}}{\epsilon^{1+\gamma}J} \\ \mathbb{G}(z) \equiv \dfrac{G(\epsilon z)}{\epsilon^{\gamma-1}J} \end{cases} \quad \text{in } B_{1/\epsilon_0(0)}.$$

Then $\mathbb{V}, \mathbb{G}$ satisfy $(6.1)_1$ with $\nu = 1$; $\|\mathbb{V}\|_{L^\infty(B_{1/\epsilon_0}(0))} + \|\mathbf{E}^1_{1/\omega^\tau,\mu}\mathbb{G}\|_{L^p(B_{1/\epsilon_0}(0))} \le c$ by (6.18). Lemma 4.1 implies

$$\|\nabla\mathbb{V}\|_{L^\infty(B_{1/2\epsilon_0}(0))} \le c, \tag{6.19}$$

and (5.3), $(6.13)_1$, and (6.19) imply (6.17). □

Remark 6.1. *Let ϵ_0 be the same as in Lemma 6.3. If $\mu \in [\epsilon_0, 1]$, (A2) implies that Eq $(6.12)_1$ is a uniform elliptic equation. By [27], we know the following:*

Under (A2), (A3), and $\mu \in [\epsilon_0, 1]$, any solution of $(6.12)_1$ satisfies (6.17).

By Lemma 4.1,

Under (A2), (A3), and $\epsilon \in [\epsilon_0, 1]$, any solution of $(6.12)_1$ satisfies (6.17).

Together with Lemma 6.3, we conclude that

Under (A2) and (A3), any solution of $(6.12)_1$ satisfies (6.17).

6.2. Boundary Lipschitz estimate

In this subsection, (4.13) is assumed and we let $\mathcal{R}_{\mathbf{d}_4,\mathbf{d}_5}(0) \equiv [-\mathbf{d}_4, \mathbf{d}_4] \times [-\mathbf{d}_5, \mathbf{d}_5]$ be a rectangle with $\mathbf{d}_4, \mathbf{d}_5 \in [3, 4]$ so that $|\mathcal{R}_{\mathbf{d}_4,\mathbf{d}_5}(0) \cap \frac{\epsilon}{r}(\mathcal{Y} + \mathbf{j})|$ for $\mathbf{j} \in \mathcal{I}_\epsilon$ is 0 or $|\frac{\epsilon}{r}|^2$ for some $0 < \frac{\epsilon}{r} \le 1$. Note that $\mathbf{d}_4, \mathbf{d}_5$ depend on $\frac{\epsilon}{r}$. If $0 < \frac{\epsilon}{r} \le 1$ and $\mathbf{d}_4, \mathbf{d}_5 \in [3, 4]$, define the following:

$$\begin{cases} \mathcal{Q}^{\cdot,r} \equiv \mathcal{R}_{\mathbf{d}_4,\mathbf{d}_5}(0) \cap \Omega/r, \\ \mathcal{Q}^{\cdot,r}_\dagger \equiv \displaystyle\bigcup_{\frac{\epsilon}{r}(\mathcal{Y}+\mathbf{j}) \subset \mathcal{R}_{\mathbf{d}_4,\mathbf{d}_5}(0);\ \mathbf{j}\in\mathcal{I}_\epsilon} \frac{\epsilon}{r}(\mathcal{Y} + \mathbf{j}). \end{cases}$$

If $r = 0$, define $\Omega/r \equiv \{(x_1, x_2) |\ x_2 > 0\}$. Then $\mathcal{Q}^{\cdot,r}$ is a bounded Lipschitz domain. Let $\eta \in C_0^\infty(\mathcal{R}_{\mathbf{d}_4,\mathbf{d}_5}(0))$ be a bell-shaped function with $\eta \in [0, 1]$ and $\eta = 1$ in $[-2, 2]^2$. Then $\eta = 0$ on $\partial\mathcal{Q}^{\cdot,r} \cap \partial\mathcal{Q}^{\cdot,r}_\dagger$. If $0 < \frac{\epsilon}{r} \le 1$, find $\mathbb{W}^{\epsilon,r}_{\omega,\mu,2} \in H^1(\mathcal{Q}^{\cdot,r})$ satisfying

$$\begin{cases} -\nabla \cdot \left(\mathbf{E}^{\epsilon,r}_{\omega^2,\mu}(\nabla\mathbb{W}^{\epsilon,r}_{\omega,\mu,2} + \vec{e}_2)\right) = 0 & \text{in } \mathcal{Q}^{\cdot,r}, \\ \mathbb{W}^{\epsilon,r}_{\omega,\mu,2} = (1 - \eta)\mathbb{X}^{\epsilon/r}_{\omega,\mu,2} & \text{on } \partial\mathcal{Q}^{\cdot,r}, \end{cases} \tag{6.20}$$

where $\vec{e}_2$ is the unit vector in the second coordinate direction. See (5.5) for $\mathbb{X}^{\epsilon/r}_{\omega,\mu,2}$ and Section 2 for $\mathbf{E}^{\epsilon,r}_{\omega^2,\mu}$. Recall [29, Lemma 6.9] to see the following result.

Lemma 6.4. *Under (A1), (A2), and $\frac{\epsilon}{r}, r \in (0, 1]$, a solution of (6.20) exists uniquely in $H^1(\mathcal{Q}^{\cdot,r})$. There is a constant c (independent of $\epsilon, \mu\, (= \frac{1}{\omega}), r, \mathbf{d}_4, \mathbf{d}_5$) such that*

$$\sup_{z\in\mathcal{Q}^{\cdot,r}} \left|\mathbb{W}^{\epsilon,r}_{\omega,\mu,2}(z)\right| \le c\frac{\epsilon}{r}.$$

Lemma 6.5. *Let θ, ϵ_0 be the same as in Lemma 6.1. Under (A1)–(A3) and $r \in (0, 1]$, constants $\widetilde{\theta}\ (< \theta)$, $\widetilde{\epsilon_0}\ (< \epsilon_0)$ exist such that if*

$$\begin{cases} -\nabla \cdot (\mathbf{E}^{\epsilon,r}_{\omega^2,\mu}\nabla\mathbb{V}) = \mathbb{G} & \text{in } B_1(0) \cap \Omega/r, \\ \mathbb{V} = \mathbb{V}_b & \text{on } B_1(0) \cap \partial\Omega/r, \\ \frac{\epsilon}{r}, \mu \in (0, \widetilde{\epsilon_0}), \end{cases} \tag{6.21}$$

and

$$\begin{cases} \mathbb{V}_b(0) = \partial_T \mathbb{V}_b(0) = 0, \\ \|\mathbb{V}\|_{L^\infty(\mathbf{S}_1^{;r}(0))},\ \frac{1}{\epsilon_0}\|\mathbf{E}^{\epsilon,r}_{1/\omega^\tau,\mu}\mathbb{G}\|_{L^p(\mathbf{S}_1^{;r}(0))},\ [\nabla \mathbb{V}_b]_{C^\gamma(\mathbf{S}_1^{;r}(0))} \le 1, \end{cases} \tag{6.22}$$

then

$$\sup_{z\in \mathbf{S}^{;r}_{\widetilde{\theta}}(0)} \left| \mathbb{V}(z) - (z_2 + \mathbb{W}^{\epsilon,r}_{\omega,\mu,2}(z))\mathbf{d}_{\omega,\epsilon,r} \right| \le \widetilde{\theta}^{1+\frac{\gamma}{2}}. \tag{6.23}$$

Here, $z = (z_1, z_2)$*;* $\partial_T \mathbb{V}_b$ *(or* $\partial_1 \mathbb{V}_b$*) is the tangential derivative of* $\mathbb{V}_b$*;* $\mathbf{d}_{\omega,\epsilon,r}$ *is the second component of* $\mathcal{K}^{-1}_{\omega,\mu}(\mathbf{E}^{\epsilon,r}_{\omega^2,\mu}\nabla\mathbb{V})_{\mathbf{S}^{;r}_{\widetilde{\theta}}(0)}$*; and* $\mathcal{K}^{-1}_{\omega,\mu}$ *is the inverse matrix of* $\mathcal{K}_{\omega,\mu}$*. See Section 2 for* $\mathbf{S}^{;r}_R(0)$ *and* $(\mathbf{E}^{\epsilon,r}_{\omega^2,\mu}\nabla\mathbb{V})_{\mathbf{S}^{;r}_{\widetilde{\theta}}(0)}$*, and see (A3) for* τ, p, p', γ*.*

Proof. The proof is similar to Lemma 6.1. Suppose that $r \in [0, 1]$ and $\mathbb{V}, \mathbb{V}_b$ satisfy

$$\begin{cases} -\Delta \mathbb{V} = 0 & \text{in } B_{4/5}(0) \cap \Omega/r, \\ \mathbb{V} = \mathbb{V}_b & \text{on } B_{4/5}(0) \cap \partial\Omega/r, \\ \mathbb{V}_b \in C^{1,\gamma}(\Omega/r) \text{ with } \mathbb{V}_b(0) = \partial_T \mathbb{V}_b(0) = 0. \end{cases} \tag{6.24}$$

By [17, Theorem 4.16] and (4.13), $\widetilde{\theta} \in (0, \frac{4}{5})$ and $\widehat{\gamma} \in (\frac{\gamma}{2}, \gamma)$ exist such that

$$\sup_{z\in \mathbf{S}^{;r}_{\widetilde{\theta}}(0)} \left| \mathbb{V}(z) - z_2\, (\partial_2 \mathbb{V})_{\mathbf{S}^{;r}_{\widetilde{\theta}}(0)} \right| \le \widetilde{\theta}^{1+\widehat{\gamma}} \left(\|\mathbb{V}\|_{L^\infty(\mathbf{S}^{;r}_{4/5}(0))} + [\nabla \mathbb{V}_b]_{C^\gamma(\mathbf{S}^{;r}_{4/5}(0))} \right), \tag{6.25}$$

where ∂_2 is the partial derivative with respect to the z_2 variable.

We claim (6.23). If not, there is a sequence $\{\epsilon, \omega_\epsilon, \mu_\epsilon, r_\epsilon, \mathbb{V}_\epsilon, \mathbb{G}_\epsilon, \mathbb{V}_{b,\epsilon}, \mathcal{K}_{\omega_\epsilon,\mu_\epsilon}\}$ satisfying (6.21), (6.22), and

$$\begin{cases} \frac{\epsilon}{r_\epsilon}, \mu_\epsilon \to 0, \qquad r_\epsilon \to r \in [0, 1], \qquad \|\mathbf{E}^{\epsilon,r_\epsilon}_{1/\omega^\tau_\epsilon,\mu_\epsilon}\mathbb{G}_\epsilon\|_{L^p(\mathbf{S}_1^{;r_\epsilon}(0))} \to 0, \\ \widetilde{\mathcal{K}} = \lim\limits_{\epsilon\to 0} \mathcal{K}_{\omega_\epsilon,\mu_\epsilon}, \\ \sup\limits_{z\in \mathbf{S}^{;r_\epsilon}_{\widetilde{\theta}}(0)} \left| \mathbb{V}_\epsilon(z) - (z_2 + \mathbb{W}^{\epsilon,r_\epsilon}_{\omega_\epsilon,\mu_\epsilon,2}(z))\mathbf{d}_{\omega_\epsilon,\epsilon,r_\epsilon} \right| > \widetilde{\theta}^{1+\frac{\gamma}{2}}. \end{cases} \tag{6.26}$$

See (5.6) for $\mathcal{K}_{\omega,\mu}$; $(6.26)_2$ is due to $(5.7)_1$. By (A2) and (A3), Lemma 5.2, and $(6.26)_1$,

$$\begin{cases} \|\mathbb{G}_\epsilon\|_{L^1(\mathbf{S}_1^{;r_\epsilon}(0))} \le \|\mathbf{E}^{\epsilon,r_\epsilon}_{1/\omega^\tau_\epsilon,\mu_\epsilon}\mathbb{G}_\epsilon\|_{L^p(\mathbf{S}_1^{;r_\epsilon}(0))} \|\mathbf{E}^{\epsilon,r_\epsilon}_{\omega^\tau_\epsilon,\mu_\epsilon}\|_{L^{p'}(\mathbf{S}_1^{;r_\epsilon}(0))} \xrightarrow[\frac{\epsilon}{r_\epsilon}\to 0]{} 0, \\ \|\mathbf{E}^{\epsilon,r_\epsilon}_{\omega^2_\epsilon,\mu_\epsilon}\nabla\mathbb{V}_\epsilon\|_{L^1(\mathbf{S}^{;r_\epsilon}_{4/5}(0))} \le c\|\mathbf{E}^{\epsilon,r_\epsilon}_{\omega_\epsilon,\mu_\epsilon}\nabla\mathbb{V}_\epsilon\|_{L^2(\mathbf{S}^{;r_\epsilon}_{4/5}(0))} \le c, \end{cases} \tag{6.27}$$

where c is a constant independent of $\epsilon, \omega_\epsilon, \mu_\epsilon, r_\epsilon$. Functions $\mathbf{E}^{\epsilon,r_\epsilon}_{\omega^2_\epsilon,\mu_\epsilon}\nabla\mathbb{V}_\epsilon$ are equi-integrable. That is, for any $\delta > 0$, by Hölder's inequality, (A2), and $(6.27)_2$, we have

$$\|\mathbf{E}^{\epsilon,r_\epsilon}_{\omega^2_\epsilon,\mu_\epsilon}\nabla\mathbb{V}_\epsilon\|_{L^1(B_\delta(0)\cap\mathbf{S}^{;r_\epsilon}_{4/5}(0))} \le c\|\mathbf{E}^{\epsilon,r_\epsilon}_{\omega_\epsilon,\mu_\epsilon}\nabla\mathbb{V}_\epsilon\|_{L^2(\mathbf{S}^{;r_\epsilon}_{4/5}(0))}\delta \le c\,\delta, \tag{6.28}$$

where c is independent of $\epsilon, \omega_\epsilon, \mu_\epsilon, r_\epsilon, \delta$. Then (6.21), (6.22), (6.26), (6.27), and Lemma 5.2 imply that there is a subsequence of $\{\epsilon, \omega_\epsilon, \mu_\epsilon, r_\epsilon, \mathbb{V}_\epsilon, \mathbb{G}_\epsilon, \mathbb{V}_{b,\epsilon}, \mathcal{K}_{\omega_\epsilon,\mu_\epsilon}\}$ (with the same notation for the subsequence) satisfying

$$\begin{cases} \mathbb{V}_\nu \to \mathbb{V} & \text{weakly in } H^1(\mathbf{S}^{;r}_{4/5}(0))), \\ \mathbf{E}^{\epsilon,r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon} \nabla \mathbb{V}_\epsilon \to \widetilde{\mathcal{K}} \nabla \mathbb{V} & \text{weakly star in Radon measures,} \quad \text{as } \frac{\epsilon}{r_\epsilon} \to 0, \\ \Delta \mathbb{V} = 0 & \text{in } \mathbf{S}^{;r}_{4/5}(0)), \end{cases} \tag{6.29}$$

where $\widetilde{\mathcal{K}}$ is a positive number. For the sequence $\{\epsilon, \omega_\epsilon, \mu_\epsilon, r_\epsilon, \mathbb{V}_\epsilon, \mathbb{G}_\epsilon, \mathbb{V}_{b,\epsilon}, \mathcal{K}_{\omega_\epsilon,\mu_\epsilon}\}$ in (6.29), either (i) an infinite number of μ_ϵ satisfy $\mu_\epsilon \le \left|\frac{\epsilon}{r_\epsilon}\right|^{\tau-1}$ or (ii) an infinite number of μ_ϵ satisfy $\mu_\epsilon > \left|\frac{\epsilon}{r_\epsilon}\right|^{\tau-1}$. In either case, the sequence in (6.29) has a uniformly convergent subsequence. The reason is as follows:

- *For Case (i).* The sequence $\{\epsilon, \omega_\epsilon, \mu_\epsilon, r_\epsilon, \mathbb{V}_\epsilon, \mathbb{G}_\epsilon, \mathbb{V}_{b,\epsilon}, \mathcal{K}_{\omega_\epsilon,\mu_\epsilon}\}$ in (6.29) satisfies the assumptions of Corollary 4.1. By Corollary 4.1, a $\mathbb{V} \in H^1(\mathbf{S}^{;r}_{4/5}(0)) \cap C(\mathbf{S}^{;r}_{4/5}(0))$ and a subsequence of $\{\mathbb{V}_\epsilon\}$ in (6.29) (with the same notation for the subsequence) exist such that $\|\mathbb{V}_\epsilon - \mathbb{V}\|_{L^\infty(B_{4/5}(0)\cap\Omega/r_\epsilon\cap\Omega/r)} \to 0$ as $\frac{\epsilon}{r_\epsilon} \to 0$.
- *For Case (ii).* Note that $\lim_{\frac{\epsilon}{r_\epsilon}\to 0} \left|\frac{\epsilon}{r_\epsilon}\right|^{\frac{4}{\varkappa}} \left|\ln \mu_\epsilon\right| = 0$ for any $\varkappa > 2$ with $0 < \frac{6}{\varkappa} < \tau$. The assumptions of Lemma 4.4 and Remark 4.2 are satisfied by the sequence $\{\epsilon, \omega_\epsilon, \mu_\epsilon, r_\epsilon, \mathbb{V}_\epsilon, \mathbb{G}_\epsilon, \mathbb{V}_{b,\epsilon}, \mathcal{K}_{\omega_\epsilon,\mu_\epsilon}\}$ in (6.29). By Lemma 4.4 and Remark 4.2, there is a function $\mathbb{V} \in H^1(\mathbf{S}^{;r}_{4/5}(0)) \cap C(\mathbf{S}^{;r}_{4/5}(0))$ and a subsequence of $\{\mathbb{V}_\epsilon\}$ in (6.29) so that $\|\mathbb{V}_\epsilon - \mathbb{V}\|_{L^\infty(B_{4/5}(0)\cap\Omega/r_\epsilon\cap\Omega/r)} \to 0$ as $\frac{\epsilon}{r_\epsilon} \to 0$.

So there is a $\mathbb{V} \in H^1(\mathbf{S}^{;r}_{4/5}(0)) \cap C(\mathbf{S}^{;r}_{4/5}(0))$ and a subsequence of $\{\mathbb{V}_\epsilon\}$ in (6.29) (with the same notation for the subsequence) so that, as $\frac{\epsilon}{r_\epsilon} \to 0$, we have

$$\begin{cases} \|\mathbb{V}_\epsilon - \mathbb{V}\|_{L^\infty(B_{4/5}(0)\cap\Omega/r_\epsilon\cap\Omega/r)} \to 0, \\ \|\mathbb{V}_{b,\epsilon} - \mathbb{V}_b\|_{C^1(B_{4/5}(0)\cap\Omega/r_\epsilon\cap\Omega/r)} \to 0, \\ \mathbf{E}^{\epsilon,r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon} \nabla \mathbb{V}_\epsilon \to \widetilde{\mathcal{K}} \nabla \mathbb{V} \ \text{ weakly star in Radon measures,} \\ \|\mathbb{V}\|_{L^\infty(\mathbf{S}^{;r}_1(0))}, \ [\nabla \mathbb{V}_b]_{C^\gamma(\mathbf{S}^{;r}_1(0))} \le 1, \\ \text{(6.24) holds.} \end{cases} \tag{6.30}$$

For any $\delta \in (0, \widetilde{\theta})$, let $\eta_\delta \in C_0^\infty(\mathbf{S}^{;r_\epsilon}_{\widetilde{\theta}}(0))$ be a non-negative function with $\eta_\delta \in [0,1]$ and $\eta_\delta(x) = 1$ in $\{x \in \mathbf{S}^{;r_\epsilon}_{\widetilde{\theta}}(0) |\ dist(x, \partial \mathbf{S}^{;r_\epsilon}_{\widetilde{\theta}}(0)) > \delta\}$. By $(6.30)_{3,4}$ and (6.28),

$$\begin{aligned} &\lim_{\frac{\epsilon}{r_\epsilon}\to 0} \left| \int_{\mathbf{S}^{;r_\epsilon}_{\widetilde{\theta}}(0)} (\mathbf{E}^{\epsilon,r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon} \nabla \mathbb{V}_\epsilon - \widetilde{\mathcal{K}} \nabla \mathbb{V})\, dx \right| \\ &\le \lim_{\frac{\epsilon}{r_\epsilon}\to 0} \left| \int_{\mathbf{S}^{;r_\epsilon}_{\widetilde{\theta}}(0)} (\mathbf{E}^{\epsilon,r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon} \nabla \mathbb{V}_\epsilon - \widetilde{\mathcal{K}} \nabla \mathbb{V}) \eta_\delta\, dx \right| + \lim_{\frac{\epsilon}{r_\epsilon}\to 0} \left| \int_{\mathbf{S}^{;r_\epsilon}_{\widetilde{\theta}}(0)} (\mathbf{E}^{\epsilon,r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon} \nabla \mathbb{V}_\epsilon - \widetilde{\mathcal{K}} \nabla \mathbb{V})(1-\eta_\delta)\, dx \right| \\ &\le \lim_{\frac{\epsilon}{r_\epsilon}\to 0} \int_{\{x \in \mathbf{S}^{;r_\epsilon}_{\widetilde{\theta}}(0) |\ dist(x, \partial \mathbf{S}^{;r_\epsilon}_{\widetilde{\theta}}(0)) \le \delta\}} \left| \mathbf{E}^{\epsilon,r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon} \nabla \mathbb{V}_\epsilon \right| + \left| \widetilde{\mathcal{K}} \nabla \mathbb{V} \right| dx \le c\, \delta^{1/2}. \end{aligned} \tag{6.31}$$

Here, (6.31) and $(6.26)_2$ imply

$$\lim_{\frac{\epsilon}{r_\epsilon}\to 0} \mathbf{d}_{\omega_\epsilon,\epsilon,r_\epsilon} = \lim_{\frac{\epsilon}{r_\epsilon}\to 0} \mathcal{K}^{-1}_{\omega_\epsilon,\mu_\epsilon} \fint_{\mathbf{S}^{;r_\epsilon}_{\widetilde{\theta}}(0)} \mathbf{E}^{\epsilon,r_\epsilon}_{\omega_\epsilon^2,\mu_\epsilon} \partial_2 \mathbb{V}_\epsilon\, dx = (\partial_2 \mathbb{V})_{\mathbf{S}^{;r}_{\widetilde{\theta}}(0)}. \tag{6.32}$$

So $(6.26)_3$, $(6.30)_1$, Lemma 6.4, (6.32), and (6.25) imply

$$\widetilde{\theta}^{1+\frac{\gamma}{2}} \le \lim_{\frac{\epsilon}{r_\epsilon}\to 0} \sup_{z\in \mathbf{S}^{r_\epsilon}_{\widetilde{\theta}}(0)} \left| \mathbb{V}_\epsilon(z) - (z_2 + \mathbb{W}^{\epsilon, r_\epsilon}_{\omega_\epsilon,\mu_\epsilon,2}(z))\mathbf{d}_{\omega_\epsilon,\epsilon,r_\epsilon} \right|$$
$$= \sup_{z\in \mathbf{S}^{r}_{\widetilde{\theta}}(0)} \left| \mathbb{V}(z) - z_2(\partial_2 \mathbb{V})_{\mathbf{S}^{r}_{\widetilde{\theta}}(0)} \right| \le \widetilde{\theta}^{1+\widehat{\gamma}}(\|\mathbb{V}\|_{L^\infty(\mathbf{S}^r_{4/5}(0))} + [\nabla \mathbb{V}_b]_{C^\gamma(\mathbf{S}^r_{4/5}(0))}).$$

We get a contradiction if $\widetilde{\theta}$ is small enough. So (6.23) is true. □

Lemma 6.6. *Let $\widetilde{\theta}, \widetilde{\epsilon}_0$ be the same as in Lemma 6.5. Under (A1)–(A3), if*

$$\begin{cases} -\nabla\cdot(\mathbf{E}^\epsilon_{\omega^2,\mu}\nabla V) = G & \text{in } B_1(0)\cap\Omega, \\ V = 0 & \text{on } B_1(0)\cap\partial\Omega, \\ \epsilon,\mu \in (0,\widetilde{\epsilon}_0), \\ k\in\mathbb{N} \text{ satisfying } \frac{\epsilon}{\widetilde{\epsilon}_0} < \widetilde{\theta}^k, \end{cases} \tag{6.33}$$

then we have the constants $\mathbf{d}^{\epsilon,k-1}_{\omega,\mu}$ satisfying

$$\begin{cases} |\mathbf{d}^{\epsilon,k-1}_{\omega,\mu}| \le c\widetilde{J}, \\ \displaystyle\sup_{z\in B_{\widetilde{\theta}^k}(0)\cap\Omega} \left| V(z) - \sum_{j=0}^{k-1} \widetilde{\theta}^{\frac{j\gamma}{2}} \Big(z_2 + \widetilde{\theta}^j \mathbb{W}^{\epsilon,\widetilde{\theta}^j}_{\omega,\mu,2}(\frac{z}{\widetilde{\theta}^j}) \Big) \mathbf{d}^{\epsilon,j}_{\omega,\mu} \right| \le \widetilde{\theta}^{k(1+\frac{\gamma}{2})}\widetilde{J}, \end{cases} \tag{6.34}$$

where $z=(z_1,z_2)$; $\widetilde{J} \equiv \|V\|_{L^\infty(B_1(0)\cap\Omega)} + \frac{1}{\widetilde{\epsilon}_0}\|\mathbf{E}^\epsilon_{1/\omega^\tau,\mu} G\|_{L^p(B_1(0)\cap\Omega)}$; c is a constant independent of $\epsilon, \mu\,(=\frac{1}{\omega}), k$. See (A2) and (A3) for ω,μ,τ,p,γ.

Proof. If $k=1$, (6.34) holds by Lemma 6.5 with $r=1$, $\mathbb{V}=\frac{V}{\widetilde{J}}$, and $\mathbb{G}=\frac{G}{\widetilde{J}}$. $\mathbf{d}^{\epsilon,0}_{\omega,\mu}$ is the second component of $\mathcal{K}^{-1}_{\omega,\mu}(\mathbf{E}^\epsilon_{\omega^2,\mu}\nabla V)_{B_{\widetilde{\theta}}(0)\cap\Omega}$ and $|\mathbf{d}^{\epsilon,0}_{\omega,\mu}| \le c\widetilde{J}$ (see (6.27)). Suppose that (6.34) is true for some $k\in\mathbb{N}$ satisfying $\frac{\epsilon}{\widetilde{\epsilon}_0} < \widetilde{\theta}^k$, we define, in $B_1(0)\cap\Omega/\widetilde{\theta}^k$,

$$\begin{cases} \mathbb{V}(z) \equiv \dfrac{1}{\widetilde{J}\ \widetilde{\theta}^{k(1+\frac{\gamma}{2})}} \Big(V(\widetilde{\theta}^k z) - \displaystyle\sum_{j=0}^{k-1} \widetilde{\theta}^{\frac{j\gamma}{2}} \Big(\widetilde{\theta}^k z_2 + \widetilde{\theta}^j \mathbb{W}^{\epsilon,\widetilde{\theta}^j}_{\omega,\mu,2}(\frac{\widetilde{\theta}^k z}{\widetilde{\theta}^j}) \Big) \mathbf{d}^{\epsilon,j}_{\omega,\mu} \Big), \\ \mathbb{G}(z) \equiv \dfrac{G(\widetilde{\theta}^k z)}{\widetilde{J}\ \widetilde{\theta}^{k(\frac{\gamma}{2}-1)}}, \\ \mathbb{V}_b(z) \equiv \dfrac{-1}{\widetilde{J}\ \widetilde{\theta}^{k(1+\frac{\gamma}{2})}} \displaystyle\sum_{j=0}^{k-1} \widetilde{\theta}^{\frac{j\gamma}{2}}\ \widetilde{\theta}^k\ z_2\, \mathbf{d}^{\epsilon,j}_{\omega,\mu}. \end{cases}$$

Note that $[\nabla\mathbb{V}_b]_{C^\gamma(B_1(0)\cap\Omega/\widetilde{\theta}^k)} = 0$. The functions $\mathbb{V}, \mathbb{G}, \mathbb{V}_b$ satisfy (6.21) and (6.22) with $r=\widetilde{\theta}^k$. Apply Lemma 6.5 to get

$$\sup_{z\in B_{\widetilde{\theta}}(0)\cap\Omega/\widetilde{\theta}^k} \left| \mathbb{V}(z) - \Big(z_2 + \mathbb{W}^{\epsilon,\widetilde{\theta}^k}_{\omega,\mu,2}(z) \Big) \mathbf{d}_{\omega,\epsilon,\widetilde{\theta}^k} \right| \le \widetilde{\theta}^{1+\frac{\gamma}{2}}, \tag{6.35}$$

where $\mathbf{d}_{\omega,\epsilon,\widetilde{\theta}^k}$ is the second component of $\mathcal{K}^{-1}_{\omega,\mu}(\mathbf{E}^{\epsilon,\widetilde{\theta}^k}_{\omega^2,\mu}\nabla\mathbb{V})_{B_{\widetilde{\theta}}(0)\cap\Omega/\widetilde{\theta}^k}$. By $(5.7)_1$ and an argument like (6.27), $|\mathbf{d}_{\omega,\epsilon,\widetilde{\theta}^k}|$ are bounded uniformly in $\epsilon, \mu\,(=\frac{1}{\omega}), k$. Rewrite (6.35) in terms of V in $B_{\widetilde{\theta}^{k+1}}(0)$ to obtain

$$\sup_{z\in B_{\widetilde{\theta}^{k+1}}(0)\cap\Omega} \left| V(z) - \sum_{j=0}^{k-1} \widetilde{\theta}^{\frac{j\gamma}{2}} \Big(z_2 + \widetilde{\theta}^j \mathbb{W}^{\epsilon,\widetilde{\theta}^j}_{\omega,\mu,2}(\frac{z}{\widetilde{\theta}^j}) \Big) \mathbf{d}^{\epsilon,j}_{\omega,\mu} - \widetilde{\theta}^{\frac{k\gamma}{2}}\widetilde{J}\Big(z_2 + \widetilde{\theta}^k \mathbb{W}^{\epsilon,\widetilde{\theta}^k}_{\omega,\mu,2}(\frac{z}{\widetilde{\theta}^k}) \Big) \mathbf{d}_{\omega,\epsilon,\widetilde{\theta}^k} \right| \le \widetilde{\theta}^{(k+1)(1+\frac{\gamma}{2})}\widetilde{J}.$$

If $\mathbf{d}^{\epsilon,k}_{\omega,\mu} \equiv \widetilde{J}\,\mathbf{d}_{\omega,\epsilon,\widetilde{\theta}^k}$, then (6.34) holds for $k+1$. □

Lemma 6.7. *Let $\widetilde{\epsilon}_0$ be the same as in Lemma 6.6. Suppose (A1)–(A3) and $\epsilon,\mu \in (0,\widetilde{\epsilon}_0)$, any solution of (6.33)$_{1,2}$ satisfies*

$$\|\nabla V\|_{L^\infty(B_{1/2}(0)\cap\Omega)} \le c\big(\|V\|_{L^\infty(B_1(0)\cap\Omega)} + \|\mathbf{E}^{\epsilon}_{1/\omega^\tau,\mu}G\|_{L^p(B_1(0)\cap\Omega)}\big), \tag{6.36}$$

where c is a constant independent of $\epsilon,\mu\,(=\frac{1}{\omega})$. See (A2) and (A3) for ω,μ,τ,p.

Proof. From (4.13), $0\in\partial\Omega$ and there is a $\gamma_*>0$ and a $C^{1,\gamma}$ function $\Upsilon:\mathbb{R}\to\mathbb{R}$ such that $B_{\gamma_*}(0)\cap\Omega = B_{\gamma_*}(0)\cap\big\{(z_1,z_2)\big|\, z_1\in\mathbb{R}, z_2>\Upsilon(z_1)\big\}$. For convenience of presentation, assume $\gamma_*=1$. We claim the following:

$$\sup_{(0,z_2)\in B_{1/2}(0)\cap\Omega} |\nabla V(0,z_2)| \le c\big(\|V\|_{L^\infty(B_1(0)\cap\Omega)} + \|\mathbf{E}^{\epsilon}_{1/\omega^\tau,\mu}G\|_{L^p(B_1(0)\cap\Omega)}\big). \tag{6.37}$$

Proof of the claim. Let c be a constant independent of ϵ,μ,ω; let $\widetilde{\theta},\widetilde{J},\gamma$ be the same as in Lemma 6.6; and k is a number satisfying $\widetilde{\theta}^{k+1}\le\frac{\epsilon}{\widetilde{\epsilon}_0}<\widetilde{\theta}^k$. If $z\equiv(0,z_2)\in B_{1/2}(0)\cap\Omega$, then either (i) $\frac{1}{2}\widetilde{\theta}^{m+1}\le z_2<\frac{1}{2}\widetilde{\theta}^m$ for $0\le m\le k$ or (ii) $0\le z_2\le\frac{1}{2}\widetilde{\theta}^{k+1}$.

For Case (i). In this case,

$$z\equiv(0,z_2),\qquad 2^{-1}\widetilde{\theta}^{m+1}\le z_2<2^{-1}\widetilde{\theta}^m\quad\text{for } 0\le m\le k. \tag{6.38}$$

Let $\xi^z\equiv dist(z,\partial\Omega)$ be the distance from z to $\partial\Omega$. By (4.13)$_4$ and (6.38),

$$\begin{cases}\xi^z\approx z_2\approx\widetilde{\theta}^m \quad (see\ Section 1\ for\ \approx),\\ \epsilon\le\widetilde{\epsilon}_0\,\widetilde{\theta}^k\le\widetilde{\epsilon}_0\widetilde{\theta}^m\le c\widetilde{\epsilon}_0\,\xi^z,\end{cases} \tag{6.39}$$

where c is independent of ϵ,μ,ω,m,z. By Lemma 6.6,

$$\sup_{x=(x_1,x_2)\in B_{\widetilde{\theta}^m}(0)\cap\Omega}\left|V(x)-\sum_{j=0}^{m-1}\widetilde{\theta}^{\frac{j\gamma}{2}}\Big(x_2+\widetilde{\theta}^j\mathbb{W}^{\epsilon,\widetilde{\theta}^j}_{\omega,\mu,2}(\frac{x}{\widetilde{\theta}^j})\Big)\mathbf{d}^{\epsilon,j}_{\omega,\mu}\right|\le c\,\widetilde{\theta}^{m(1+\frac{\gamma}{2})}\widetilde{J}, \tag{6.40}$$

where c is independent of $\epsilon,\mu\,(=\frac{1}{\omega}),m$. By (6.34)$_1$, Lemma 6.4, (6.39), and (6.40),

$$\sup_{B_{\widetilde{\theta}^m}(0)\cap\Omega}|V|\le c\widetilde{J}\Big(\widetilde{\theta}^{m(1+\frac{\gamma}{2})}+(\widetilde{\theta}^m+\epsilon)\Big)\le c\,\xi^z\widetilde{J}. \tag{6.41}$$

By (6.38) and (6.41), we see

$$\sup_{B_{\xi^z/2}(z)}|V|\le c\,\xi^z\widetilde{J}, \tag{6.42}$$

where c is independent of ϵ,ω,μ,m,z. Define

$$\begin{cases}\mathbb{E}^{\epsilon,\xi^z}_{\omega^2,\mu}(y)\equiv\mathbf{E}^{\epsilon}_{\omega^2,\mu}(z+\xi^z y)\\ \mathbb{V}(y)\equiv\dfrac{V(z+\xi^z y)}{\xi^z\,\widetilde{J}}\\ \mathbb{G}(y)\equiv\dfrac{\xi^z\,G(z+\xi^z y)}{\widetilde{J}}\end{cases}\qquad\text{in } B_{1/2}(0).$$

Then $\mathbb{V},\mathbb{G}$ satisfy, by (6.42),

$$\begin{cases}-\nabla\cdot(\mathbb{E}^{\epsilon,\xi^z}_{\omega^2,\mu}\nabla\mathbb{V})=\mathbb{G} & \text{in } B_{1/2}(0),\\ \|\mathbb{V}\|_{L^\infty(B_{1/2}(0))}+\left\|\mathbb{E}^{\epsilon,\xi^z}_{1/\omega^\tau,\mu}\mathbb{G}\right\|_{L^p(B_{1/2}(0))}\le c.\end{cases}$$

By $(6.39)_2$, $\epsilon/\xi^z\le c\widetilde{\epsilon}_0$. Remark 6.1 implies that $\|\nabla\mathbb{V}\|_{L^\infty(B_{1/4}(0))}\le c$. That is,

$$\|\nabla V\|_{L^\infty(B_{\xi^z/4}(z))}\le c\widetilde{J}.$$

This proves (6.37) for **Case (i)**.

For Case (ii). In this case, $0\le z_2\le\frac{1}{2}\widetilde{\theta}^{k+1}$ and $\widetilde{\theta}^k\approx\epsilon$. By Lemma 6.6,

$$\sup_{x=(x_1,x_2)\in B_{\widetilde{\theta}^k}(0)\cap\Omega}\left|V(x)-\sum_{j=0}^{k-1}\widetilde{\theta}^{\frac{j\gamma}{2}}\left(x_2+\widetilde{\theta}^j\mathbb{W}^{\epsilon,\widetilde{\theta}^j}_{\omega,\mu}(\frac{x}{\widetilde{\theta}^j})\right)\mathbf{d}^{\epsilon,j}_{\omega,\mu}\right|\le\widetilde{\theta}^{k(1+\frac{\gamma}{2})}\widetilde{J}.$$

Lemma 6.4 and $(6.34)_1$ imply

$$\sup_{B_{\widetilde{\theta}^k}(0)\cap\Omega}|V|\le c\epsilon\widetilde{J}. \tag{6.43}$$

Define

$$\begin{cases}\mathbb{V}(y)\equiv\dfrac{V(\epsilon y)}{\epsilon\widetilde{J}} & \\ \mathbb{G}(y)\equiv\dfrac{\epsilon\, G(\epsilon y)}{\widetilde{J}} & \end{cases}\quad\text{in } B_1(0)\cap\Omega/\epsilon.$$

By (6.43),

$$\|\mathbb{V}\|_{L^\infty(B_1(0)\cap\Omega/\epsilon)}+\left\|\mathbf{E}^{\epsilon,\epsilon}_{1/\omega^\tau,\mu}\mathbb{G}\right\|_{L^p(B_1(0)\cap\Omega/\epsilon)}\le c.$$

Then $\mathbb{V},\mathbb{G}$ satisfy

$$\begin{cases}-\nabla\cdot(\mathbf{E}^{\epsilon,\epsilon}_{\omega^2,\mu}\nabla\mathbb{V})=\mathbb{G} & \text{in } B_1(0)\cap\Omega/\epsilon,\\ \mathbb{V}=0 & \text{on } B_1(0)\cap\partial\Omega/\epsilon.\end{cases} \tag{6.44}$$

Lemma 4.1 and the classical regularity results [17] imply $\|\nabla\mathbb{V}\|_{L^\infty(B_{1/2}(0)\cap\Omega/\epsilon)}\le c$. This proves (6.37) for **Case (ii)**. Therefore, the claim (6.37) is proved.

Next, we repeat the same argument for (6.37) (i.e., (6.40)–(6.44)) by varying the origin along the boundary $B_1(0)\cap\partial\Omega$ and by adjusting the constant c. Then we see that (6.36) is true. □

Remark 6.2. *Let $\widetilde{\epsilon}_0$ be the same as in Lemma 6.7. If $\mu\in[\widetilde{\epsilon}_0,1]$, (A2) implies that Eq $(6.33)_1$ is a uniform elliptic equation. By [27], we know that*

Under (A1)–(A3) and $\mu\in[\widetilde{\epsilon}_0,1]$, any solution of $(6.33)_{1,2}$ satisfies (6.36).

By Lemma 4.1 and the classical regularity results [17],

Under (A1)–(A3) and $\epsilon\in[\epsilon_0,1]$, any solution of $(6.33)_{1,2}$ satisfies (6.36).

Together with Lemma 6.7, we conclude the following:

Under (A1)–(A3), any solution of $(6.33)_{1,2}$ satisfies (6.36).

7. Proof of Lemma 3.2

Equation (3.1) can be written as

$$\begin{cases} -\nabla\cdot(\mathbf{E}^{\epsilon}_{\omega^2,\mu}\nabla V) = G - \beta^2\mathbf{E}^{\epsilon}_{\omega^2,\mu}V & \text{in } \Omega, \\ V = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.1}$$

By Remarks 6.1 and 6.2, the solution of (7.1) satisfies

$$\|\nabla V\|_{L^\infty(\Omega)} \le c\big(\|V\|_{L^\infty(\Omega)} + \|\mathbf{E}^{\epsilon}_{1/\omega^\tau,\mu}G\|_{L^p(\Omega)} + \|\beta^2\mathbf{E}^{\epsilon}_{\omega^{2-\tau},\mu}V\|_{L^p(\Omega)}\big). \tag{7.2}$$

See (A3) for τ, p. Since $\beta > 1$ (the assumption of Lemma 3.2), the solution of (7.1) satisfies, by Lemma 3.1 and (A3),

$$\begin{cases} \|\beta^2\mathbf{E}^{\epsilon}_{\omega^{2-\tau},\mu}V\|_{L^p(\Omega)} = \left\|\beta^2\mathbf{E}^{\epsilon}_{\omega^{2/p},\mu}V\right\|_{L^p(\Omega)} \le c\left\|\mathbf{E}^{\epsilon}_{1/\omega^\tau,\mu}G\right\|_{L^p(\Omega)}, \\ \left\|\mathbf{E}^{\epsilon}_{\omega,\mu}\nabla V\right\|_{L^2(\Omega)} \le c\left\|\mathbf{E}^{\epsilon}_{1/\omega,\mu}G\right\|_{L^2(\Omega)}. \end{cases} \tag{7.3}$$

Next we estimate $\|V\|_{L^\infty(\Omega)}$ as follows:

- *For* $\mu \le \epsilon^{\tau-1}$. By Lemma 4.12, Sobolev's embedding theorem [17], and (7.3), the solution of (7.1) satisfies

$$\begin{aligned} \|V\|_{L^\infty(\Omega)} &\le c(\|\mathbf{E}^{\epsilon}_{\omega,\mu}\nabla V\|_{L^2(\Omega)} + \|\mathbf{E}^{\epsilon}_{1/\omega^\tau,\mu}G\|_{L^p(\Omega)} + \|\beta^2\mathbf{E}^{\epsilon}_{\omega^{2-\tau},\mu}V\|_{L^p(\Omega)}) \\ &\le c(\|\mathbf{E}^{\epsilon}_{1/\omega,\mu}G\|_{L^2(\Omega)} + \|\mathbf{E}^{\epsilon,r}_{1/\omega^\tau,\mu}G\|_{L^p(\Omega)}). \end{aligned} \tag{7.4}$$

- *For* $\mu > \epsilon^{\tau-1}$. Note that $\epsilon^{\frac{4}{\varkappa}}|\ln\mu|$ are bounded independent of ϵ, ω, μ for any $\varkappa > 2$ with $0 < \frac{6}{\varkappa} < \tau$. By Lemma 4.3 and (7.3), the solution of (7.1) satisfies

$$\|V\|_{L^\infty(\Omega)} \le c\big(\|\mathbf{E}^{\epsilon}_{1/\omega^\tau,\mu}G\|_{L^p(\Omega)} + \|\beta^2\mathbf{E}^{\epsilon}_{\omega^{2-\tau},\mu}V\|_{L^p(\Omega)}\big) \le c\|\mathbf{E}^{\epsilon}_{1/\omega^\tau,\mu}G\|_{L^p(\Omega)}. \tag{7.5}$$

Lemma 3.2 follows from (7.2)–(7.5).

8. Proof of Lemma 4.1

Let $\Gamma(z-y)$ denote the fundamental solution of the Laplace equation in $\mathbb{R}^2$ [17] and let $B_r \subset \mathbb{R}^2$ denote a disc centered at 0 with a radius r. Define a single-layer and a double-layer potentials as, for any smooth function ζ on the boundary ∂B_r,

$$\begin{cases} \mathscr{S}_{\partial B_r}(\zeta)(z) \equiv \int_{\partial B_r} \Gamma(z-y)\,\zeta(y)\,d\sigma_y \\ \mathscr{D}_{\partial B_r}(\zeta)(z) \equiv \int_{\partial B_r} \nabla_y\Gamma(z-y)\vec{\mathbf{n}}_y\,\zeta(y)\,d\sigma_y \end{cases} \quad \text{for } z \in \partial B_r,$$

where $\vec{\mathbf{n}}_y$ is the unit vector outward normal to ∂B_r. By [13, pages 148–151] and [11, pages 226 and 227] and by following the proof of Lemma 3.2 [27], we know the following.

Lemma 8.1. *For any $\alpha \in (0,1)$, the linear operators*

$$\begin{cases} \mathscr{S}_{\partial B_1} : C^{\alpha}(\partial B_1) \to C^{1,\alpha}(\partial B_1) \\ \mathscr{D}_{\partial B_1} : C^{\alpha}(\partial B_1) \to C^{1,\alpha}(\partial B_1) \end{cases}$$

are bounded, $I - \mathbf{d}_1\mathscr{D}_{\partial B_1}$ for $\mathbf{d}_1 \in [-2,2]$ are invertible in $C^{1,\alpha}(\partial B_1)$, and

$$\|\zeta\|_{C^{1,\alpha}(\partial B_1)} \le c\|(I - \mathbf{d}_1\mathscr{D}_{\partial B_1})(\zeta)\|_{C^{1,\alpha}(\partial B_1)},$$

where I is the identity operator and c is a constant independent of $\mathbf{d}_1$.

If $\vec{\mathbf{n}}$ is the unit vector outward normal to $\partial\mathcal{Y}_{\mu,m}$, define, for any $y \in \partial\mathcal{Y}_{\mu,m}$ and any function ζ on $\mathcal{Y}$,

$$\begin{cases} \zeta_{,\pm}(y) \equiv \lim_{t\to 0^+} \zeta(y \pm t\vec{\mathbf{n}}), & \lfloor\zeta\rfloor(y) \equiv \zeta_{,+}(y) - \zeta_{,-}(y), \\ \partial^{\pm}_{\mathbf{n}}\zeta \equiv \nabla\zeta_{,\pm}\cdot\vec{\mathbf{n}}, & \lfloor\partial_{\mathbf{n}}\zeta\rfloor(y) \equiv \partial^{+}_{\mathbf{n}}\zeta(y) - \partial^{-}_{\mathbf{n}}\zeta(y). \end{cases} \tag{8.1}$$

Proof of Lemma 4.1. Let $\widehat{J} \equiv \|\Psi\|_{L^2(\mathcal{Y}\setminus B_{1/4})} + \|Q\|_{C^{\gamma}(\mathcal{Y})} + \|\mathbf{K}_{1/\omega^2,\mu}G\|_{L^p(\mathcal{Y})}$, $\gamma \equiv \frac{p-2}{p}$, and let c be a constant independent of ω,μ. By [17, Theorem 4.15], any solution Ψ of (4.2) satisfies

$$\|\Psi\|_{C^{1,\gamma}(B_{9/20}\setminus B_{7/20})} \le c\widehat{J}. \tag{8.2}$$

Next, we find $\zeta \in C^{1,\gamma}(B_{2/5})$ by solving

$$\begin{cases} -\nabla\cdot(\nabla\zeta + Q) = \mathbf{K}_{1/\omega^2,\mu}G & \text{in } B_{2/5}, \\ \zeta = \Psi & \text{on } \partial B_{2/5}. \end{cases}$$

By [17, Theorems 4.15 and 4.16],

$$\|\zeta\|_{C^{1,\gamma}(B_{2/5})} \le c\widehat{J}. \tag{8.3}$$

Recall $\mathcal{Y}_{\mu,m} = B_{\mu/4}(0)$. Define $\Phi \equiv \Psi - \zeta$ in $B_{2/5}$ and $\widehat{\Phi}(y) \equiv \Phi(\frac{\mu}{4}y)$, $\widehat{\zeta}(y) \equiv \zeta(\frac{\mu}{4}y)$, $\widehat{Q}(y) \equiv \frac{\mu}{4}Q(\frac{\mu}{4}y)$ in $B_{8/5\mu}$. Then

$$\begin{cases} -\Delta\widehat{\Phi} = 0 & \text{in } B_{\frac{8}{5\mu}} \setminus \partial B_1, \\ \lfloor\widehat{\Phi}\rfloor = 0 & \text{on } \partial B_1, \\ \lfloor\mathbf{K}^{4/\mu}_{\omega^2,\mu}\nabla\widehat{\Phi}\rfloor\cdot\vec{\mathbf{n}}_y = \widehat{\mathbb{G}} & \text{on } \partial B_1, \\ \widehat{\Phi} = 0 & \text{on } \partial B_{\frac{8}{5\mu}}, \end{cases} \tag{8.4}$$

where $\vec{\mathbf{n}}_y$ is the unit vector normal to ∂B_1 and $\widehat{\mathbb{G}} = -\lfloor\mathbf{K}^{4/\mu}_{\omega^2,\mu}(\nabla\widehat{\zeta} + \widehat{Q})\rfloor\cdot\vec{\mathbf{n}}_y$. See (4.1) for $\mathbf{K}^{4/\mu}_{\omega^2,\mu}$ and (8.1) for $\lfloor\widehat{\Phi}\rfloor$, $\lfloor\mathbf{K}^{4/\mu}_{\omega^2,\mu}\nabla\widehat{\Phi}\rfloor$. Note, by (8.2), (8.3), and definition of the single-layer potential, we have

$$\begin{cases} \|\widehat{\mathbb{G}}\|_{C^{\gamma}(\partial B_1)} \le c\omega^2\mu\widehat{J}, \\ \left\|\mathscr{S}_{\partial B_{8/5\mu}}(\partial_{\mathbf{n}}\widehat{\Phi}|_{\partial B_{8/5\mu}})\right\|_{C^{1,\gamma}(\partial B_1)} \le c\widehat{J}|\ln\mu|, \end{cases} \tag{8.5}$$

where $\partial_{\mathbf{n}}\widehat{\Phi}|_{\partial B_{8/5\mu}}$ is the normal derivative of $\widehat{\Phi}$ on $\partial B_{8/5\mu}$. By Green's formula, (8.4), and [13, pages 148–151],

$$\begin{cases} \frac{\widehat{\Phi}}{2} + \mathscr{D}_{\partial B_1}(\widehat{\Phi}) = \mathscr{S}_{\partial B_1}(\nabla\widehat{\Phi}_{,-}\cdot\vec{\mathbf{n}}_y|_{\partial B_1}) \\ \frac{\widehat{\Phi}}{2} - \mathscr{D}_{\partial B_1}(\widehat{\Phi}) = -\mathscr{S}_{\partial B_1}(\nabla\widehat{\Phi}_{,+}\cdot\vec{\mathbf{n}}_y|_{\partial B_1}) + \mathscr{S}_{\partial B_{8/5\mu}}(\partial^{-}_{\mathbf{n}}\widehat{\Phi}|_{\partial B_{8/5\mu}}) \end{cases} \quad \text{on } \partial B_1.$$

Therefore,

$$\Big(I - \frac{2(1-\omega^2)}{1+\omega^2}\mathscr{D}_{\partial B_1}\Big)\widehat{\Phi} = \frac{2}{1+\omega^2}\Big(\mathscr{S}_{\partial B_1}(-\widehat{\mathbb{G}}) + \mathscr{S}_{\partial B_{8/5\mu}}(\partial_{\mathbf{n}}^{-}\widehat{\Phi}|_{\partial B_{8/5\mu}})\Big),$$

where I is the identity matrix. Apply (8.5) and Lemma 8.1 to see

$$\|\widehat{\Phi}\|_{C^{1,\gamma}(\partial B_1)} \leq \frac{c}{\omega^2}\left(\|\widehat{\mathbb{G}}\|_{C^{\gamma}(\partial B_1)} + \left\|\mathscr{S}_{\partial B_{8/5\mu}}(\partial_{\mathbf{n}}\widehat{\Phi}|_{\partial B_{8/5\mu}})\right\|_{C^{1,\gamma}(\partial B_1)}\right) \leq c\widehat{J}(\mu + \omega^{-2}|\ln\mu|). \tag{8.6}$$

By the maximal principle, (8.4), and (8.6),

$$\|\widehat{\Phi}\|_{W^{1,\infty}(B_1)} + \|\widehat{\Phi}\|_{W^{1,\infty}(B_{8/5\mu}\setminus B_1)} \leq c\widehat{J}(\mu + \omega^{-2}|\ln\mu|). \tag{8.7}$$

By the assumptions, (8.7), and the definition of $\widehat{\Phi}$, we see

$$\|\nabla\Phi\|_{L^\infty(B_{2/5})} \leq c\widehat{J}. \tag{8.8}$$

By (8.3) and (8.8), $\|\Psi\|_{L^\infty(B_{2/5})} \leq \|\nabla\Phi\|_{L^\infty(B_{2/5})} + \|\nabla\zeta\|_{L^\infty(B_{2/5})} \leq c\widehat{J}$. So Lemma 4.1 is proved. □

9. Homogenized macroscopic equations

This section recalls the homogenized macroscopic equations of the problem (1.1). The homogenization of quasilinear elliptic equations is studied in [4]. When $p = 2$ and the Dirichlet boundary condition are concerned, the quasilinear elliptic equations are reduced to the following linear elliptic equations:

$$\begin{cases} -\nabla\cdot(\mathbf{E}^{\epsilon}_{\omega^2,\mu}\nabla U_\epsilon) = F & \text{in } \mathfrak{D}, \\ U_\epsilon = U_b & \text{on } \partial\mathfrak{D}. \end{cases} \tag{9.1}$$

As $\epsilon \to 0$, under (A1), $\omega > 1$, $\omega^2|\mathfrak{D}^{\epsilon}_{\mu,m}| \approx 1$, and $\epsilon^2|\ln\mu| \approx 1$, the homogenized macroscopic equations of (9.1) are, by [4, Theorem A], and [8, Theorem 3],

$$\begin{cases} -\Delta U + \theta_3\,(U - V) = F & \text{in } \mathfrak{D}, \\ -\theta_4\,\partial^2_{x_3} V + \theta_3\,(V - U) = 0 & \text{in } \mathfrak{D}, \\ U = U_b & \text{on } \partial\mathfrak{D}, \\ V(x', 0) = U_b(x', 0), & \\ V(x', \mathbf{L}) = U_b(x', \mathbf{L}), & \end{cases} \tag{9.2}$$

where θ_3, θ_4 are two positive constants. If $U_b = 0$, the homogenized macroscopic Eq (9.2) is

$$\begin{cases} -\Delta U + \theta_3\,(U - V) = F & \text{in } \mathfrak{D}, \\ -\theta_4\,\partial^2_{x_3} V + \theta_3\,(V - U) = 0 & \text{in } \mathfrak{D}, \\ U = 0 & \text{on } \partial\mathfrak{D}, \\ V(x', 0) = V(x', \mathbf{L}) = 0. & \end{cases} \tag{9.3}$$

The solution V of

$$\begin{cases} -\theta_4\,\partial^2_{x_3} V + \theta_3\,(V - U) = 0 & \text{in } \mathfrak{D}, \\ V(x', 0) = V(x', \mathbf{L}) = 0 & \end{cases}$$

can be written as

$$V(x', x_3) = \int_0^{\mathbf{L}} \mathcal{G}(x', x_3, z) U(x', z)\, dz, \tag{9.4}$$

where the kernel $\mathcal{G}(x', x_3, z)$ can be computed (see [4]). By (9.4), the solution U of (9.3) satisfies

$$\begin{cases} -\Delta U + \theta_3\, U - \theta_3 \displaystyle\int_0^{\mathbf{L}} \mathcal{G}(x', x_3, z) U(x', z)\, dz = F & \text{in } \mathfrak{D}, \\ U = 0 & \text{on } \partial\mathfrak{D}. \end{cases} \tag{9.5}$$

Equations (9.2), (9.3), and (9.5) are the homogenized macroscopic equations with a nonlocal term. For more details, see [4, Theorem A], [7, Eq (1.2)], and [8, Theorem 3].

Author contributions

Li-Ming Yeh: Writing original draft, Formal analysis; Chiu-Yao He: Formal analysis; Shiah-Sen Wang: Formal analysis. All authors have read and approved the final version of the manuscript for publication.

Acknowledgments

The authors would like to thank the anonymous referee's valuable suggestions for improving this manuscript. The third author is supported by the grant number NSC 110-2115-M-A49-001-MY2 from the research program of National Science Council of Taiwan.

Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this manuscript.

References

1. A. Arkhipova, O. Erlhamahmy, Regularity of solutions to a diffraction-type problem for nondiagonal linear elliptic systems in the Campanato space, *Journal of Mathematical Sciences*, **112** (2002), 3944–3966. https://doi.org/10.1023/A:1020093606080
2. M. Avellaneda, F. Lin, Compactness methods in the theory of homogenization, *Commun. Pur. Appl. Math.*, **40** (1987), 803–847. https://doi.org/10.1002/cpa.3160400607
3. M. Avellaneda, F. Lin, L^p bounds on singular integrals in homogenization, *Commun. Pur. Appl. Math.*, **44** (1991), 897–910. https://doi.org/10.1002/cpa.3160440805
4. M. Bellieud, G. Bouchitté, Homogenization of elliptic problems in a fiber reinforced structure. Nonlocal effect, *Ann. Scuola Norm.-Sci.*, **26** (1998), 407–436.
5. D. Bergman, Y. Strelniker, Strong-field magnetotransport in a two-constituent columnar composite medium where the constituents have comparable resistivity tensors, *Phys. Rev. B*, **86** (2012), 024414. https://doi.org/10.1103/PhysRevB.86.024414

6. G. Bouchitte, M. Bellieud, Homogenization of a soft elastic material reinforced by fibers, *Asymptotic Anal.*, **32** (2002), 153–183. https://doi.org/10.3233/ASY-2002-531
7. M. Briane, Nonlocal effects in two-dimensional conductivity, *Arch. Rational Mech. Anal.*, **182** (2006), 255–267. https://doi.org/10.1007/s00205-006-0427-4
8. M. Briane, Y. Capdeboscq, L. Nguyen, Interior regularity estimates in high conductivity homogenization and application, *Arch. Rational Mech. Anal.*. **207** (2013), 75–137. https://doi.org/10.1007/s00205-012-0553-0
9. M. Briane, N. Tchou, Fibered microstructures for some nonlocal Dirichlet forms, *Ann. Scuola Norm.-Sci.*, **30** (2001), 681–711.
10. M. Briane, J. Casado-Díaz, Asymptotic behaviour of equicoercive diffusion energies in dimension two, *Calc. Var.*, **29** (2007), 455–479. https://doi.org/10.1007/s00526-006-0074-5
11. G. Chen, J. Zhou, *Boundary element methods with applications to nonlinear problems*, Paris: Atlantis Press, 2010. https://doi.org/10.2991/978-94-91216-27-5
12. D. Cioranescu, P. Donato, *An introduction to homogenization*, Oxford: Oxford University Press, 1999. https://doi.org/10.1093/oso/9780198565543.001.0001
13. L. Escauriaza, M. Mitrea, Transmission problems and spectral theory for singular integral operator on Lipschitz domains, *J. Funct. Anal.*, **216** (2004), 141–171. https://doi.org/10.1016/j.jfa.2003.12.005
14. L. Evans, *Weak convergence methods for nonlinear partial differential equations*, Providence: American Mathematical Society, 1990.
15. J. Geng, Hardy spaces and the Neumann problem in L^p for elliptic equation with periodic high-contrast coefficients in Lipschitz domains, *Math. Ann.*, **386** (2023), 1641–1693. https://doi.org/10.1007/s00208-022-02446-0
16. M. Giaquinta, *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, Princeton: Princeton University Press, 1983.
17. D. Gilbarg, N. Trudinger, *Elliptic partial differential equations of second order*, Berlin: Springer-Verlag, 1983. https://doi.org/10.1007/978-3-642-61798-0
18. V. Jikov, S. Kozlov, O. Oleinik, *Homogenization of differential operators and integral functions*, Berlin: Springer-Verlag, 1994. https://doi.org/10.1007/978-3-642-84659-5
19. O. Ladyzenskaja, N. Ural'ceva, *Linear and quasi-linear elliptic equations*, New York: Academic Press, 1968.
20. J. Park, D. Kang, B. Paulson, T. Nazari, K. Oh, Liquid core photonic crystal fiber with lowrefractive-index liquids for optofluidic applications, *Opt. Express*, **22** (2014), 17320. https://doi.org/10.1364/OE.22.017320
21. B. Schweizer, Uniform estimates in two periodic homogenization problems, *Commun. Pur. Appl. Math.*, **53** (2000), 1153–1176. https://doi.org/10.1002/1097-0312(200009)53:9¡1153::AID-CPA4¿3.0.CO;2-R
22. Z. Shen, $W^{1,p}$ estimates for elliptic homogenization problems in nonsmooth domains, *Indiana Univ. Math. J.*, **57** (2008), 2283–2298. https://doi.org/10.1512/iumj.2008.57.3344

23. Z. Shen, Large-scale Lipschitz estimates for elliptic systems with periodic high-contrast coefficients, *Commun. Part. Diff. Eq.*, **46** (2021), 1027–1057. https://doi.org/10.1080/03605302.2020.1858098
24. Z. Shen, Homogenization of boundary value problems in perforated Lipschitz domains, *J. Differ. Equations*, **376** (2023), 283–339. https://doi.org/10.1016/j.jde.2023.09.005
25. A. Sili, Homogenization of a nonlinear monotone problem in an anisotropic medium, *Math. Mod. Meth. Appl. S.*, **14** (2004), 329–353. https://doi.org/10.1142/S0218202504003258
26. Q. Xu, Uniform regularity estimates in homogenization theory of elliptic sysyems with lower order terms on the Neumann boundary problem, *J. Differ. Equations*, **261** (2016), 4368–4423. https://doi.org/10.1016/j.jde.2016.06.027
27. L. Yeh, Elliptic equations in highly heterogeneous porous media, *Math. Method. Appl. Sci.*, **33** (2010), 198–223. https://doi.org/10.1002/mma.1163
28. L. Yeh, Non-uniform elliptic equations in convex Lipschitz domains, *Nonlinear Anal.-Theor.*, **118** (2015), 63–81. https://doi.org/10.1016/j.na.2015.01.019
29. L. Yeh, Strongly elliptic equations with periodic coefficients in two-dimensional space, *Z. Anal. Anwend.*, in press. https://doi.org/10.4171/ZAA/1785